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OPTIMUM SECONDARY IMPULSE FOR INTERPLANETARY TRANSFERS

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ABSTRACT

Exact analysis is developed with the aid of patched conic theory which permits determination of the optimum two-impulse velocity requirements for interplanetary transfers. Plane changes are accomplished by adding impulses normal to the instantaneous plane of motion. A single transcendental equation is developed in the relative inclination of the optimum orbital transfer planes. Solution of the transcendental equation will also yield the true anomaly difference from the departure terminal at which the secondary impulse is to be applied.

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I. INTRODUCTION

Interplanetary transfer from a given planet to a specified objective planet is a problem of increasing importance [2]. Usually, the analysis involved in the solution of such problems must be attacked through direct numerical methods. At times, however, the assumption of patched conic sections permits certain useful analytical solutions to be developed. The analysis contained within this paper assumes that patched conic theory can be utilized to yield the desired transfer trajectory.

The purpose of this paper is to obtain analytical expressions for the out-of-plane velocity increments that must be added normal to the orbital transfer plane in order to yield an "optimum" transfer maneuver between two terminals. It is assumed that the problem is restricted to two impulses and therefore two separate planes of motion exist between the departure and arrival terminals. It is further assumed that the space vehicle is captured at the second or arrival terminal by the local gravitational forces of the target planet.

The analysis con ned herein foll s the lines of development of Fimple [1]. However, ll small ngle ap roximations and singularities have been removed f the an ysis.

II. INITIAL COORDINATE SYSTEM

The position and velocity vectors of a given planet are usually available in terms of osculating mean elements or perhaps more accurate parameters relative to the ecliptic coordinate system [4].

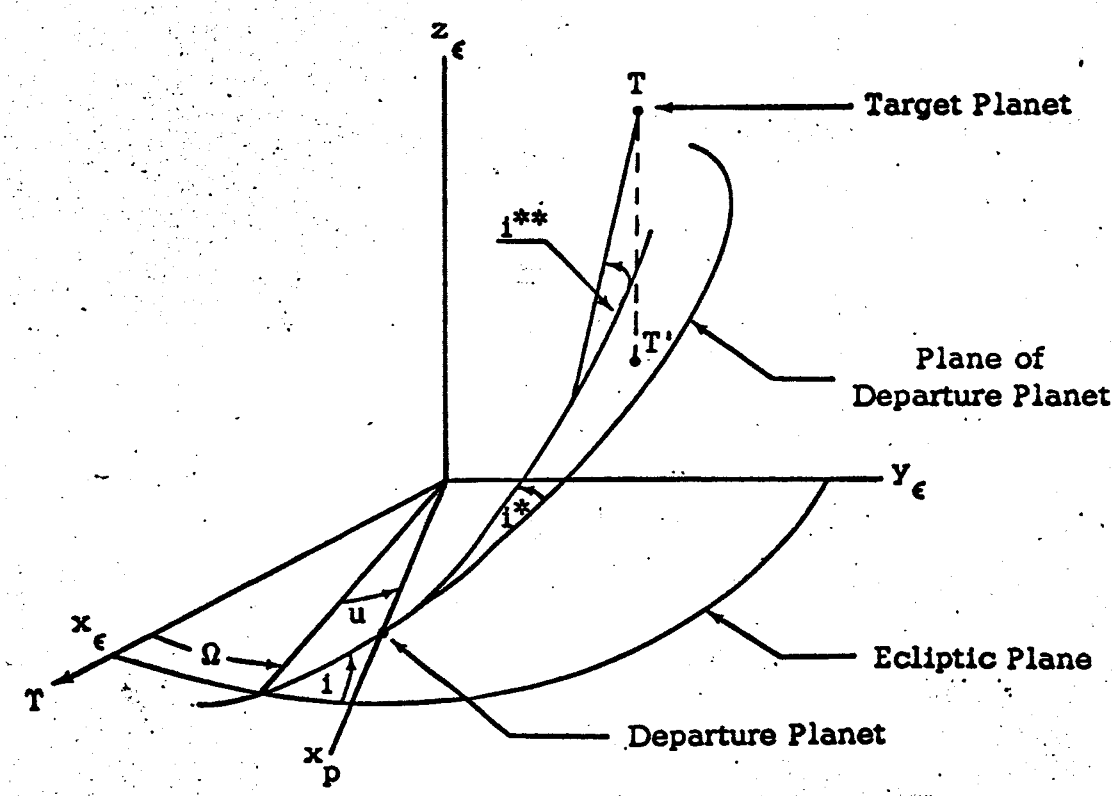

Figure 1

Planet Orientation Parameters

For a given Julian Date, corresponding to initiation of the optimal transfer trajectory, it is assumed that the position and velocity vectors of the departure planet are known in the heliocentric ecliptic coordinate system denoted by the x_ϵ, y_ϵ, z_ϵ axes of Figure 1. In the analysis developed herein, it will be beneficial to adopt a coordinate system that is fixed to the departure planet at time of initial launch into the heliocentric transfer maneuver. Hence, the principal axis x_p is taken along the radius vector of the departure planet at time of launch. The y_p axis is advanced to x_p by a right angle and lies in the instantaneous plane of motion of the departure planet, thus defining the fundamental plane. Lastly, z_p is picked such that x_p, y_p and z_p form a right-handed system. If it is assumed that Ω, ω, and i, that is, the longitude of the ascending node, the argument of perigee, and the orbital inclination of the departure planet are not varying with time, then the position and velocity mappings from the ecliptic to the planetary coordinate system are given directly by

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = [M] \begin{bmatrix} x_\epsilon \\ y_\epsilon \\ z_\epsilon \end{bmatrix}, \quad \begin{bmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{z}_p \end{bmatrix} = [M] \begin{bmatrix} \dot{x}_\epsilon \\ \dot{y}_\epsilon \\ \dot{z}_\epsilon \end{bmatrix} + [\dot{M}] \begin{bmatrix} x_\epsilon \\ y_\epsilon \\ z_\epsilon \end{bmatrix}, \quad (1)$$

where the time dependent transformation matrix is:

$$[M] = \begin{bmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{bmatrix} \tag{2}$$

with

$$U_x = \cos u \cos\Omega - \sin u \sin\Omega \cos i$$

$$U_y = \cos u \sin\Omega + \sin u \cos\Omega \cos i$$

$$U_z = \sin u \sin i$$

$$V_x = -\sin u \cos\Omega - \cos u \sin\Omega \cos i$$

$$V_y = -\sin u \sin\Omega + \cos u \cos\Omega \cos i$$

$$V_z = \cos u \sin i$$

$$W_x = \sin\Omega \sin i$$

$$W_x = -\cos\Omega \sin i$$

$$W_z = \cos i .$$

The angle u, the argument of latitude, is defined in terms of true anomaly v by

$$u = v + \omega ,$$

and its rate of change can be shown to be [4]:

$$\dot{u} = \dot{v} + \dot{\omega} = \dot{v} = \frac{(\bar{\mu} p)^{\frac{1}{2}}}{r^2} , \quad (3)$$

where p is the semiparameter of the departure planet and $\bar{\mu}$ is the sum of the masses of the Sun and departure planet. To reiterate then, for any given universal or ephemeris launch time t_o, all pertinent position and velocity vectors may be rotated into the x_p, y_p, z_p frame by matrix [M] and [$\dot{M}$] through Eqs. (1) .

III. TRANSFER PROCESS

Examination of the solar system shows that the relative inclinations of all the planets to each other are fairly small. Hence, the angle i^*, the relative inclination between the departure planet's plane (plane 1) and the first transfer plane (plane 2), also will be small. Similarly i^{**}, the inclination of the second transfer plane to the first transfer plane, will again be quite small. Imagine that a one-impulse transfer or orbit is found by any of the standard two-position vector and time interval methods [4] between terminal 1 and the projection of terminal 2 on plane 1. In other words, a heliocentric transfer orbit is placed between the position vector of the departure planet and the projection of the radius vector of the arrival planet upon the plane of motion of the original departure planet [3]. As may be obvious, if the target planet happened to lie in the plane of the departure planet upon targeting, absolutely no out-of-plane impulse need be applied to effect the three-dimensional transfer maneuver.

For the immediate future, let it be assumed that a one-impulse transfer is utilized to satisfy the x_p and y_p coordinates, call them x_{pT} , y_{pT} , at the final terminal of the target planet. Hence, for an assumed launch date and time differential, any of the methods developed in Ref. [4] yield the fictitious transfer orbit constrained to lie in plane 1, containing the departure point and the projection of T, i.e., T', of Figure 1. Satisfaction of the true transfer will now require that impulses be applied normal to the departure plane in order to obtain the correct z_{pT} or final objective point.

The analysis developed herein will limit the transfer maneuver to two impulses, one applied at departure to attain a transfer in plane 2 at angle i^* to plane 1, and a second impulse applied at point I, an intermediate point, to obtain an orbit in plane 3, inclined by an angle i^{**} to plane 2. In the process of performing the maneuver, the total out-of-plane velocity increment magnitude will be minimized and point I determined. Plane 3 will be constrained to contain the final point T.

In closing this section it is well to note that if the projection of the target planet upon plane 1 is dropped to a point somewhere between T' and T in Figure 1, a very slightly different transfer maneuver will be realized. Actually, if T is not projected at all, then a true single impulse transfer will be placed between the respective terminals. As will be seen later, at times a single impulse transfer is better than a double impulse transfer. It may be required to repeat the analysis developed herein, as will be discussed later, for various projections, T' lying between point T and plane 1, to obtain the truly optimum two-impulse transfer. This does not pose any particular problems to the analysis. In this case, plane 1 should be redefined to be the instantaneous plane at the launch time defined by $\underline{r}_1$ and $\dot{\underline{r}}_1$, the

position vector of the first terminal or departure planet position and the required heliocentric velocity vector obtained from the two-position vector and time interval method.

IV. ANALYTICAL EXPRESSIONS FOR ELEMENT CHANGE

Consider a vehicle moving in plane 1, as illustrated in Figure 2, with position and velocity vectors $\underline{r}_1$ and $\dot{\underline{r}}_1$. Notice that $\dot{\underline{r}}_1$ is the heliocentric transfer velocity in plane 1.

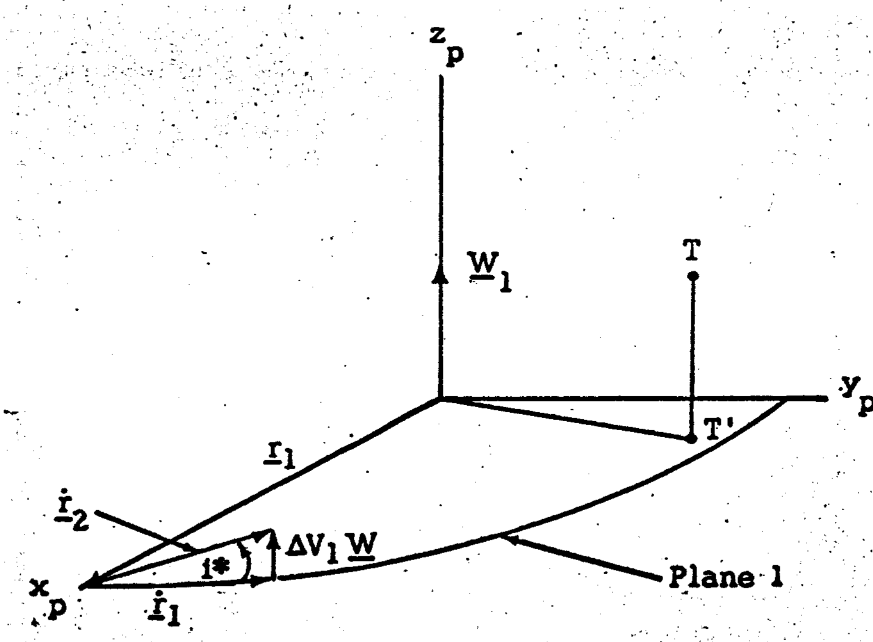

Figure 2
Plane Change Geometry

To effect a transfer maneuver from plane 1 into plane 2, a velocity increment must be added to $\dot{\underline{r}}_1$ normal to plane 1, i.e., $\Delta V_1 \underline{W}$. The result of this vector addition can be expressed as $\dot{\underline{r}}_2$, the new velocity vector, where:

$$\dot{\underline{r}}_2 = \dot{\underline{r}}_1 + \Delta V_1 \underline{W}_1 \quad . \tag{4}$$

or from Figure 2,

$$\dot{\underline{r}}_2 = \dot{\underline{r}}_1 + V_1 \tan i^* \underline{W}_1 \quad . \tag{5}$$

Notice that V_1 is the magnitude of velocity vector $\dot{\underline{r}}_1$. In passing, it is well to note that $\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 = V_1^2$ while $\dot{r}_1 \neq \dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1$. In the process of impulsively adding the velocity requirement for a plane rotation through the angle i^*, the radius vector $\underline{r}_1$ is unchanged, i.e., $\underline{r}_2 = \underline{r}_1$. Since the orbital elements of the orbit contained in plane 2 are functions of $\underline{r}_2$ and $\dot{\underline{r}}_2$, relationships will now be sought between the rates of change of the orbital elements for given changes in i^*. Consider first the computation of V_2 from

$$\frac{V_2^2}{\mu} = \frac{\dot{\underline{r}}_2 \cdot \dot{\underline{r}}_2}{\mu} = \frac{V_1^2 \sec^2 i^*}{\mu} \quad , \tag{6}$$

where μ is the sum of the masses of the transfer vehicle and Sun.

The vis-viva equation may be expressed as:

$$\frac{1}{a_2} = \frac{2}{r_2} - \frac{V_1^2 \sec^2 i^*}{\mu} \qquad (7)$$

and solved for a_2, the semimajor axis of orbit 2, to yield

$$a_2 = \left[\frac{2}{r_2} - \frac{V_1^2 \sec^2 i^*}{\mu} \right]^{-1} \qquad (8)$$

and thus

$$\frac{\partial a_2}{\partial i^*} = \frac{2}{\mu} (a_2 V_1)^2 \sec^2 i^* \tan i^* . \qquad (9)$$

Similarly, the standard two-body formulas for the orbital eccentricity e, and true anomaly v:

$$D = (\underline{r} \cdot \dot{\underline{r}}) / (\mu)^{\frac{1}{2}}$$

$$e^2 = \left(1 - \frac{r}{a}\right)^2 + \frac{1}{a} D^2$$

$$p = a(1 - e^2)$$

$$S_v \equiv e \sin v = \frac{D}{r}(p)^{\frac{1}{2}}$$

$$C_v \equiv e \cos v = \frac{p}{r} - 1 \tag{10}$$

can be used to derive

$$\frac{\partial D_2}{\partial i^*} = 0 \tag{11}$$

$$\frac{\partial p_2}{\partial i^*} = \frac{2a_2 V_1^2}{\mu}\left[p_2 + D_2^2 - 2r_2\left(1 - \frac{r_2}{a_2}\right)\right]\sec^2 i^* \tan i^* \tag{12}$$

$$\frac{\partial S_{v2}}{\partial i^*} = \frac{1}{2}\,\frac{D_2}{r_2(p_2)^{\frac{1}{2}}}\,\frac{\partial p_2}{\partial i^*} \tag{13}$$

$$\frac{\partial C_{v2}}{\partial i^*} = \frac{1}{r_2}\,\frac{\partial p_2}{\partial i^*}\quad, \tag{14}$$

where the parameters e and p as functions of i^* are

$$D_2 = \frac{\underline{r}_2 \cdot \underline{\dot{r}}_2}{\mu^{\frac{1}{2}}} = (x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1)\Big/(\mu)^{\frac{1}{2}} \tag{15}$$

$$e_2^2 = \left(1 - \frac{r_2}{a_2}\right)^2 + \frac{1}{a_2} D_2^2 \qquad (16)$$

$$p_2 = a_2(1 - e_2^2) \qquad (17)$$

with a_2 obtained from Eq. (8) and

$$S_{v2} = \frac{D_2}{r_2} (p_2)^{\frac{1}{2}} , \quad C_{v2} = \frac{p_2}{r_2} - 1 . \qquad (18)$$

The unit vector $\underline{W}_1 = (W_{x1} , W_{y1} , W_{z1})$ is directly computable from the known position and velocity vectors $\underline{r}_1$ and $\dot{\underline{r}}_1$ from:

$$\underline{W}_1 = \frac{\underline{r}_1 \times \dot{\underline{r}}_1}{(\mu p_1)^{\frac{1}{2}}} . \qquad (19)$$

An auxiliary set of unit vectors $\underline{U}_2$ and $\underline{V}_2$ will be introduced and defined by:

$$\underline{U}_2 = \frac{\underline{r}_2}{r_2} \qquad (20)$$

$$\underline{V}_2 = \left[r_2 \dot{\underline{r}}_2 - \left(D_2 \sqrt{\mu} \right) \underline{U}_2 \right] \Big/ (\mu p_2)^{\frac{1}{2}} \quad . \tag{21}$$

Geometrically, $\underline{U}_2$ is a unit vector pointing towards the vehicle, and $\underline{V}_2$, likewise, is a unit vector advanced from $\underline{U}_2$ by a right angle, which lies in the plane of motion. As might be evident, $\underline{U}_2$ is not functionally dependent on i^*, whereas $\underline{V}_2$ is directly linked to i^* through:

$$\underline{V}_2 = \left[r_1 \left(\dot{\underline{r}}_1 + V_1 \tan i^* \underline{W}_1 \right) - \left(\underline{r}_1 \cdot \dot{\underline{r}}_1 \right) \underline{U}_2 \right] \Big/ (\mu p_2)^{\frac{1}{2}} . \tag{22}$$

V. POSITION AND VELOCITY VECTORS AT SECONDARY IMPULSE

In terms of the unit vectors $\underline{U}_{20}$ and $\underline{V}_{20}$, evaluated at time t_o, the position and velocity vectors at any future time, as denoted by the secondary subscript I, may be shown to be [4]:

$$\underline{r}_{2I} = x_{vI}\,\underline{U}_{20} + y_{vI}\,\underline{V}_{20} \tag{23}$$

$$\underline{\dot{r}}_{2I} = \dot{x}_{vI}\,\underline{U}_{20} + \dot{y}_{vI}\,\underline{V}_{20} \;, \tag{24}$$

where

$$x_{vI} \equiv r_I \cos v^* \;, \quad y_{vI} \equiv r_I \sin v^*$$

$$\dot{x}_{vI} \equiv \left(\frac{\mu}{p_2}\right)^{\frac{1}{2}} \left[S_{v2} - \sin v^*\right] \;, \quad \dot{y}_{vI} \equiv \left(\frac{\mu}{p_2}\right)^{\frac{1}{2}} \left[C_{v2} + \cos v^*\right] \tag{25}$$

with

$$v^{*} = v_I - v_1 \ .$$

In order to effect the second plane change, a velocity increment will be added normal to plane 1 and thus force the incremented velocity vector to lie in plane 3. The reason for not adding the second velocity increment, call it ΔV_2, normal to plane 2 is because this maneuver would cause a side velocity increment

$$\Delta V_2 \sin i^{*} \tag{26}$$

to occur parallel to plane 1. This situation, under the construction employed herein, would cause an error in the interception coordinates x_{pT} and y_{pT}. Actually, adding the velocity increment normal to plane 2 causes the error $\Delta V_2 \sin i^{*}$ to be incurred, which must be corrected. Firing normal to plane 1 causes a loss of velocity increment equal to $\Delta V_2 \sin i^{*}$. Apparently, no true loss is incurred by adding ΔV_2 normal to plane 1 so that it is possible to form the new velocity vector as:

$$\dot{\underline{r}}_{3I} = \dot{\underline{r}}_{2I} + \Delta V_2 \underline{W}_1 \ . \tag{27}$$

The magnitude of ΔV_2 is still to be determined.

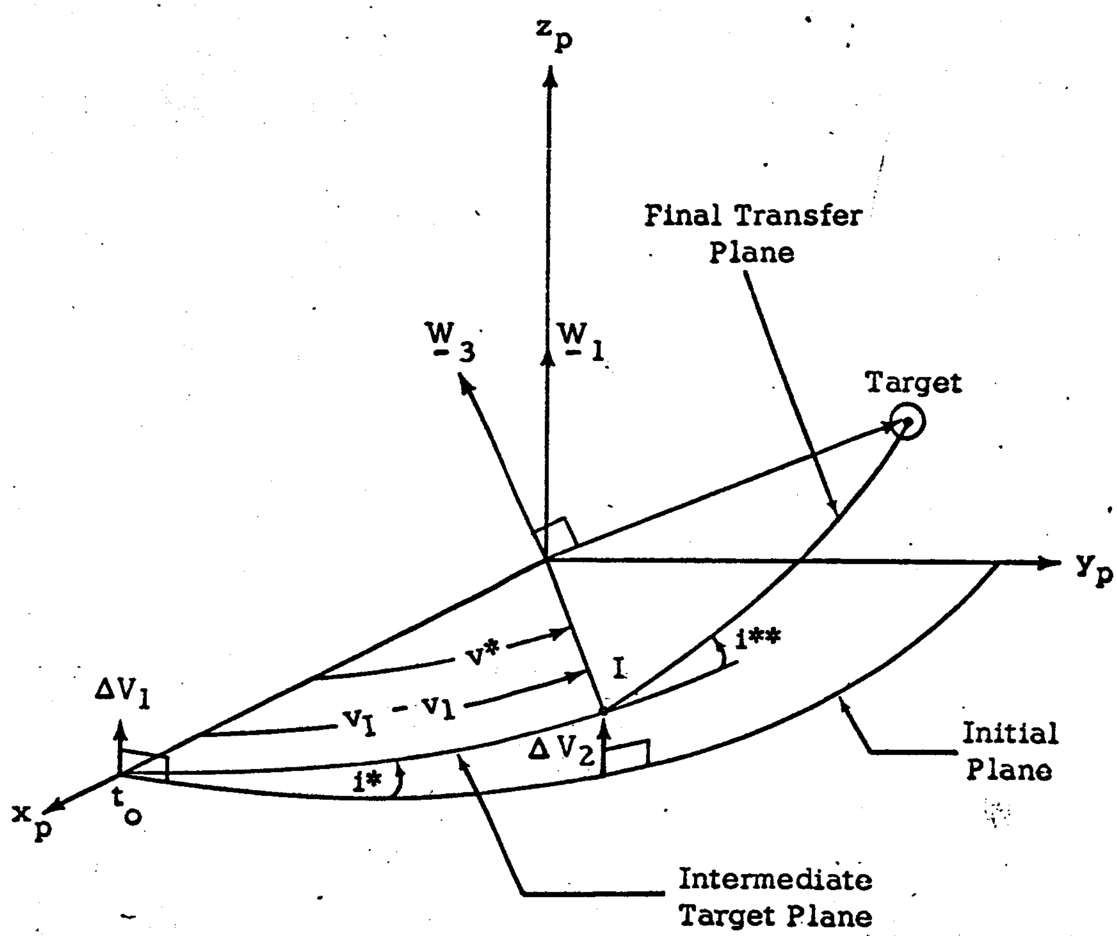

Figure 3

Impulse Application Points

VI. EXPRESSION FOR THE PRIMARY IMPULSE

From the previous sections, it is assumed that a heliocentric transfer orbit has been obtained which passes through the planet located at terminal 1 of the transfer orbit and the projection of T on plane 1. Hence, $\underline{r}_1$, $\dot{\underline{r}}_1$, the initial heliocentric velocity and position vectors are known.

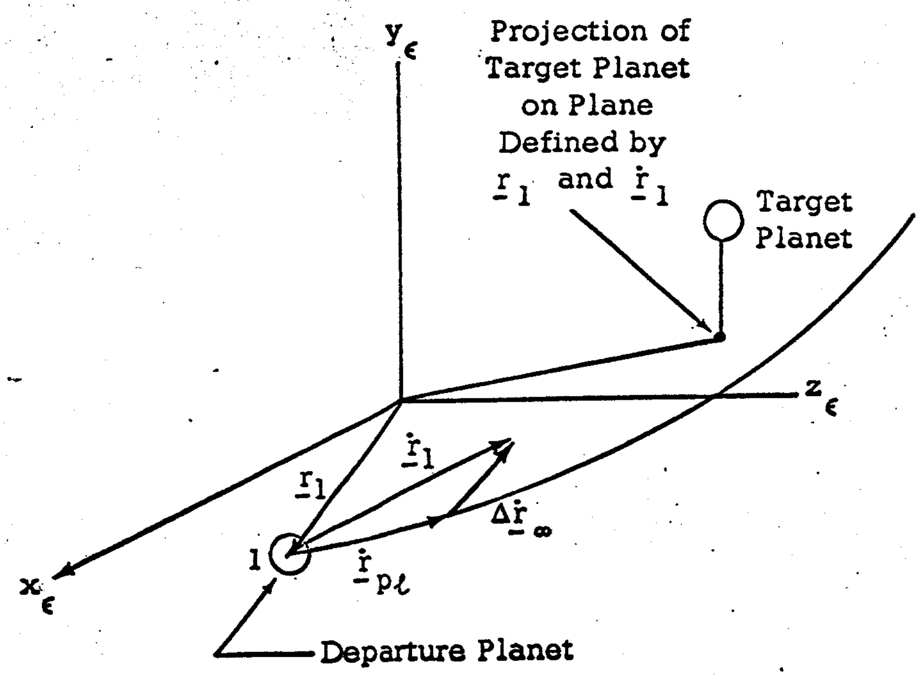

Figure 4

Velocity and Position Vectors at Takeoff

As can be seen from Figure 4, addition of $\Delta \dot{\underline{r}}_\infty$ to the planetary velocity at $\underline{r}_1 = \underline{r}_{p\ell}$, where the $p\ell$ subscript denotes the initial planet, results in the required transfer trajectory. It should be noticed that $\Delta \dot{\underline{r}}_\infty$ is obtained by the vector subtraction of the heliocentric and planetary velocity vectors at the first terminal.

Now, since the launch velocity, denoted by the subscript o, at terminal 1 is equal to the sum of the squares of the escape velocity and residual velocity at infinity,

$$V_o = \left(V_{par}^2 + \left| \Delta \dot{\underline{r}}_\infty \right|^2 \right)^{\frac{1}{2}} , \qquad (28)$$

where V_{par} is the parabolic velocity and $\Delta \dot{\underline{r}}_\infty$ is the velocity at a very remote point, say for purposes herein, infinity. Parabolic velocity is defined by setting the semimajor axis of the vis-viva equation to a very large number such that for all practical purposes as $\frac{1}{a} \rightarrow 0$,

$$V_{par}^2 = \frac{2\mu'}{r_o} , \qquad (29)$$

where r_o is the planetocentric position vector magnitude of the vehicle about to commence the transfer maneuver and μ' is the sum of the masses of the vehicle and escape planet. From the previous discussion, it follows that the additional velocity increment is:

$$\Delta V_1 = \left\{ V_{par}^2 + (\dot{x}_{p\ell} - \dot{x}_1)^2 + (\dot{y}_{p\ell} - \dot{y}_1)^2 + (\dot{z}_{p\ell} - \dot{z}_1)^2 + V_1^2 \tan^2 i^* \right\}^{\frac{1}{2}}$$

$$- \mu' \left(\frac{2}{r_o} - \frac{1}{a} \right)^{\frac{1}{2}} , \qquad (30)$$

where the subscript 1 connotes heliocentric transfer velocity, the subscript $p\ell$ denotes planetary velocity, and the last term on the right-hand side of Eq. (30) is the orbital speed at point r_o, the launch point. It should be noted that, the contribution of $\dot{z}_{p\ell}$ and $\dot{z}_1$ when T is projected to T', is zero due to the coordinate system employed. The contribution $V_1 \tan i^*$ is the velocity required for the out-of-plane maneuver.

VII. EXPRESSION FOR SECONDARY IMPULSE

Consider the determination of a third normal unit vector, $\underline{W}_3$, perpendicular to the target planet plane of motion. The normal vector $\underline{W}_3$ is easily found at the desired intercept time from the known position and velocity vectors $\underline{r}_3$, $\dot{\underline{r}}_3$ from:

$$\underline{W}_3 = \underline{r}_3 \times \dot{\underline{r}}_3 / (\mu p_3)^{\frac{1}{2}} \quad . \tag{31}$$

It should be noticed that $\underline{W}_3$ is independent of i^* and v^*. In order to constrain $\dot{\underline{r}}_3$ or the final velocity vector to lie in plane 3, the following condition may be imposed:

$$\dot{\underline{r}}_{3I} \cdot \underline{W}_3 = 0 \tag{32}$$

or from Eq. (27),

$$\left(\dot{\underline{r}}_{2I} + \Delta V_2 \underline{W}_1 \right) \cdot \underline{W}_3 = 0 \tag{33}$$

which yields:

$$\Delta V_2 = - \frac{\dot{\underline{r}}_{2I} \cdot \underline{W}_3}{\underline{W}_1 \cdot \underline{W}_3} \quad . \tag{34}$$

Using Eqs. (24) and (26), the following reduction is possible:

$$\Delta V_2 = - \frac{\left[(\mu p_2)^{\frac{1}{2}} \dot{x}_{vI} \, \omega_1 + \dot{y}_{vI} \, (\omega_2 + \omega_3 \tan i^*) \right]}{(\mu p_2)^{\frac{1}{2}} \omega_4} \quad , \tag{35}$$

where

$$\omega_1 \equiv \underline{U}_{20} \cdot \underline{W}_3$$

$$\omega_2 \equiv r_1 (\dot{\underline{r}}_1 \cdot \underline{W}_3) - (\underline{r}_1 \cdot \dot{\underline{r}}_1)(\underline{U}_{20} \cdot \underline{W}_3)$$

$$\omega_3 \equiv r_1 V_1 (\underline{W}_1 \cdot \underline{W}_3)$$

$$\omega_4 \equiv \underline{W}_1 \cdot \underline{W}_3$$

and

It should be noticed that the ω_i coefficients are known directly from the initial conditions and auxiliary data of the problem. Equation (35) may be thrown into its true functional dependence by employing Eqs. (25), i.e.,

$$\Delta V_2 = - \frac{\left[\left\{ S_{v2} - \sin v^* \right\} \mu \, \omega_1 + \left(\frac{\mu}{p_2} \right)^{\frac{1}{2}} \left\{ C_{v2} + \cos v^* \right\} \left\{ \omega_2 + \omega_3 \tan i^* \right\} \right]}{(\mu p_2)^{\frac{1}{2}} \, \omega_4} . \qquad (36)$$

The functional dependence of S_{v2}, C_{v2} and p_2 upon i^* are given by Eqs. (15), (16), (17) and (18).

VIII. TOTAL VELOCITY INCREMENT MINIMIZATION

If the assumption is made that the planetary vehicle or probe is captured by the objective or target planet, the total mission velocity increment requirement is given by:

$$\Delta = \Delta V_1 + \left| \Delta V_2 \right| . \qquad (37)$$

More explicitly, using Eqs. (30 and (36),

$$\Delta = \left\{ \beta_1 + \beta_2 \tan^2 i^* \right\}^{\frac{1}{2}} - \beta_3 + \left| - \left[\left\{ S_{v2} - \sin v^* \right\} \mu \omega_1 + \left(\frac{\mu}{p_2} \right)^{\frac{1}{2}} \left\{ C_{v2} + \cos v^* \right\} \left\{ \omega_2 + \omega_3 \tan i^* \right\} \right] \Big/ \left[(\mu p_2)^{\frac{1}{2}} \omega_4 \right] \right| .$$

(38)

where:

$$\beta_1 \equiv V_{par}^2 + (\dot{x}_{p\ell} - \dot{x}_1)^2 + (\dot{y}_{p\ell} - \dot{y}_1)^2 + (\dot{z}_{p\ell} - \dot{z}_1)^2$$

$$\beta_2 \equiv V_1^2$$

$$\beta_3 \equiv \mu' \left(\frac{2}{r_o} - \frac{1}{a}\right)^{\frac{1}{2}} .$$

As can be seen, Δ is a function of i^* and v^* so that setting $\frac{\partial \Delta}{\partial i^*}$, $\frac{\partial \Delta}{\partial v^*}$ equal to zero will result in the desired stationary condition. Hence, by formal differentiation, $\partial \Delta / \partial i^*$ yields

$$F_1 \equiv \beta_2 \tan i^* \sec^2 i^* \left\{ \beta_1 + \beta_2 \tan^2 i^* \right\}^{-\frac{1}{2}}$$

$$\pm \left[\frac{1}{\gamma_2} \left(\left(\frac{\mu}{p_2} \right)^{\frac{1}{2}} \frac{C\gamma_1}{2p_2} \frac{\partial p_2}{\partial i^*} - \left(\frac{\mu}{p_2} \right)^{\frac{1}{2}} \left(C\,\omega_3 \sec^2 i^* + \gamma_1 \frac{\partial C_{v2}}{\partial i^*} \right. \right. \right.$$

$$\left. \left. - \mu\,\omega_1 \frac{\partial S_{v2}}{\partial i^*} \right) + \frac{1}{2p_2\,\gamma_2} \left(\left(\frac{\mu}{p_2} \right)^{\frac{1}{2}} C\gamma_1 + \mu\,\omega_1 S \right) \frac{\partial p_2}{\partial i^*} \right] = 0 \quad (39)$$

with:

$$\gamma_1 \equiv \omega_2 + \omega_3 \tan i^*$$

$$\gamma_2 \equiv (\mu p_2)^{\frac{1}{2}} \omega_4$$

$$C \equiv C_{v2} + \cos v^*$$

$$S \equiv S_{v2} - \sin v^*$$

and $\partial \Delta / \partial v^*$ results in

$$F_2 \equiv \mu \omega_1 \cos v^* + \left(\frac{\mu}{p_2}\right)^{\frac{1}{2}} \gamma_1 \sin v^* = 0 \quad . \tag{40}$$

The F_1 , F_2 equations represent a system of two equations in the unknown optimal variables i^* , v^* . These two equations are very complicated analytic functions of i^* and v^*, functionally dependent upon the partial derivatives and elements

$$\frac{\partial S_{v2}}{\partial i^*} , \quad \frac{\partial C_{v2}}{\partial i^*} , \quad \frac{\partial p_2}{\partial i^*} , \quad S_{v2} , \quad C_{v2} , \quad p_2 ,$$

which have already been obtained analytically, Eqs. (12) through (18) .

To reduce Eq. (39) to an equation in i^* , consider introduction of the auxiliary angle σ defined through:

$$\cos\sigma = \frac{\mu\omega_1}{\left[(\mu\omega_1)^2 + \frac{\mu}{p_2}\gamma_1^2\right]^{\frac{1}{2}}} , \quad \sin\sigma = \frac{\left(\frac{\mu}{p_2}\right)^{\frac{1}{2}}\gamma_1}{\left[(\mu\omega_1)^2 + \frac{\mu}{p_2}\gamma_1^2\right]^{\frac{1}{2}}} \tag{41}$$

so that Eq. (40) becomes

$$\cos(\sigma - v^*) = 0 \tag{42}$$

which implies

$$\sin v^* = \pm \cos\sigma . \tag{43}$$

Expansion of Eq. (42) also yields

$$\cos v^* = -\sin v^* \tan\sigma , \tag{44}$$

so that for secondary impulses applied less than or equal to π radians apart from the primary impulse, the sign of Eq. (43) should be chosen such that $\sin v^* \geq 0$. As may be evident, the last two equations reduce Eq. (39) to an analytic function of i^*, since C and S are uniquely determined.

Solution of Eq. (39), an exact equation in i^*, can be rapidly and conveniently handled by the standard numerical Newton procedure. In all cases, a small value of i^*, say a few degrees, will converge rapidly to the desired zero. Of the two zeros inherent to Eq. (39), a positive value of i^* is chosen if the target planet is above the initial departure plane, and similarly, a negative i^* is the solution of the transfer problem when the target planet is below the original departure plane.

IX. EXAMINATION OF THE VELOCITY INCREMENT FUNCTION

Equation (37), for constant v^*, is an analytic function with certain properties. Certainly the minimum value of ΔV_1, from Eq. (38), is:

$$\left(\Delta V_1\right)_{min} = (\beta_1)^{\frac{1}{2}} - \beta_3 \qquad (45)$$

for $i^* = 0$, and increases positively for any other values of i^*. The absolute function $\left|\Delta V_2\right|$ is, of course, always positive and vanishes at:

$$i_{\ell}^{*} = \tan^{-1}\left[-\frac{\left\{S_{v2} - \sin v^*\right\}(\mu p_2)^{\frac{1}{2}}\,\omega_1}{\left\{C_{v2} + \cos v^*\right\}\omega_3} - \frac{\omega_2}{\omega_3}\right]. \qquad (46)$$

A little thought will reveal that ΔV_1 is a tangent function increasing to the one-half power so that curve ab of Figure 5 is produced. The second impulse function must increase from both sides of i^*_ℓ and yield curve c i^*_ℓ d of Figure 5. Addition of ΔV_1 and $|\Delta V_2|$ results in Δ , the total velocity increment function. Furthermore, since ΔV_1 increases in (o, i^*_ℓ) and $|\Delta V_2|$ decreases in the same interval, it is conceivable that a minimum exists, call it i^*_m . As may be evident from the graphical construction, the maximum value of i^*_m , the zero of Eq. (39) will, for optimum transfer, never exceed i^*_ℓ, since this would correspond to an increased velocity increment and no minimum can exist for $i^* > i^*_\ell$.

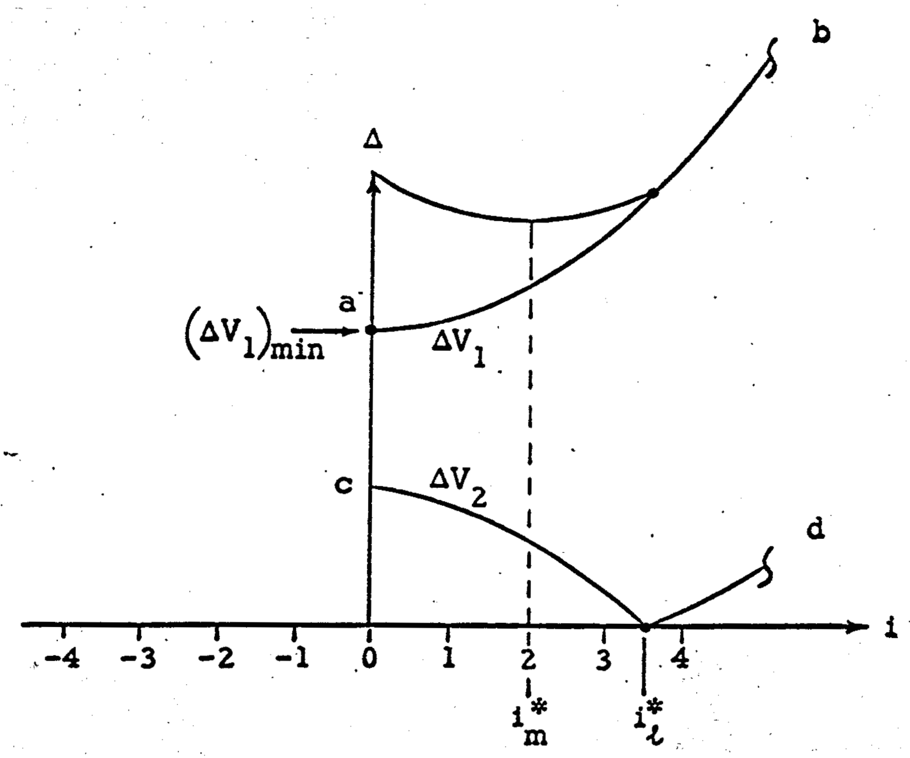

Figure 5

Graphical Representation
Velocity Increment Equation

Hence, for a given zero of Eq. (39) corresponding to v^*, the limiting i_ℓ^* is found from Eq. (46). Then, if

$$i_m^* = i_\ell^* ,$$

a single impulse transfer will be better than a double impulse transfer. In opposite fashion, if

$$0 \leq i_m^* \leq i_\ell^* ,$$

the double impulse transfer will be optimum.

X. TARGET PROJECTION EQUATIONS

The preceding analysis tacitly assumed that the original relative departure velocity vector was contained in the plane of the departure planet. Thus, a heliocentric orbit was placed between the radius vector at the departure planet and the projection of the target planet upon the initial plane of motion of the departure planet. If T in Figure 1 had not been projected upon plane 1, it is evident that a single impulse transfer would have resulted between the initial and final terminals. It appears feasible to expect that an optimum projection length of T' towards T or z^*_{p3} exists which, in combination with the previous analysis, will yield a true optimum. Hence, the previous analysis should be performed for $z^*_{p3} = 0$, with no change to the equations developed, and for $z^*_{p3} = n \Delta z^*_{p3}$ where n is a constant such that it divides the length T'T into even increments. In essence then, the initial one-impulse heliocentric transfer trace is taken slightly inclined to the instantaneous departure plane of the planet at terminal 1.

The angle of inclination between plane 1 and the new departure plane, call it 0, will be given in terms of the vectors with components $\underline{r}_1 = (x_1, y_1, z_1)$ and $\underline{r}_{p3} = (x_{p3}, y_{p3}, z^*_{p3})$ by:

$$i_o = \cos^{-1}\left[(x_1 y_{p3} - x_{p3} y_1)/(r_1 r_p \sin(v_3 - v_1))\right] ,$$

$$0 < i_o \leq \pi , \qquad (47)$$

where

$$\sin(v_3 - v_1) = \pm\left[1 - (\underline{r}_1 \cdot \underline{r}_p/\underline{r}_1 r_p)^2\right]^{\frac{1}{2}}$$

with the positive sign holding if the target planet is less than or equal to π radians apart from the first terminal and the negative sign controlling for transfers $> \pi$ apart.

A rotation from the x_p, y_p, z_p coordinate system to the newly oriented x'_p, y'_p, z'_p system is given by:

$$\begin{bmatrix} x'_p \\ y'_p \\ z'_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i_o & \sin i_o \\ 0 & -\sin i_o & \cos i_o \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} , \qquad (48)$$

with the same matrix holding for velocity transformations. Therefore, the previous analysis may now be repeated with $\dot{\underline{r}}'_1$, $\underline{r}'_1$ to see if the total velocity requirements are diminished. The optimally gain will in most cases be very small.

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