SEMI-ANNUAL PROGRESS REPORT OF RESEARCH
PERFORMED UNDER NASA RESEARCH GRANT
NGR 17-003-004 FOR THE PERIOD
15 FEBRUARY 1966 TO 15 AUGUST 1966

SOME FREE VIBRATION AND DYNAMIC RESPONSE PROBLEMS ASSOCIATED WITH CENTRIFUGALLY STABILIZED DISK AND SHELL STRUCTURES

SUBMITTED TO:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
APPLIED MATHEMATICS BRANCH
NASA HEADQUARTERS
WASHINGTON, D. C.

by the

AERONAUTICAL ENGINEERING DEPARTMENT
SCHOOL OF ENGINEERING
WICHITA STATE UNIVERSITY

AUGUST, 1966

Approved: Walter Eversman

Walter Eversman
Associate Professor

SUMMARY OF EFFORT TO DATE

This semi-annual progress report covers the calendar period 15 February 1966 to 15 August 1966. An active research effort was carried out only during the period 15 February 1966 to 1 June 1966, which corresponds to the spring academic semester at Wichita State University. The research was conducted primarily by the principal investigator, Dr. Walter Eversman. Near the end of the active period, a master's degree candidate, Mr. Thomas Gilley, began to participate in the research effort. Mr. Gilley is a part time graduate student and will not be on the grant budget. The formal research program was suspended during the summer period. It is anticipated that a full time graduate student will participate on the grant in the fall and will remain actively associated with it until its scheduled completion at the end of the summer in 1967.

The research to date has consisted of two parts. The principal effort has been directed toward the determination of the natural frequencies and normal vibration modes for the free vibrations of flat spinning annular membrane disks. The hub configurations which are currently being studied are the completely clamped case in which both radial and transverse displacements of the disk are constrained, and the loosely clamped case in which only the transverse displacements are constrained. An intermediate case in which partial clamping permits partially constrained radial displacements should prove to present no features significantly different from the other two cases.

A concurrent program has been carried out to extend the analysis to shallow membrane shells of revolution which are centrifugally stabilized. To date this effort has consisted entirely of a literature survey for the purpose of establishing the appropriate equations of motion.

ANALYTICAL EFFORT

The major portion of the analytical effort to date has been expended in the formulation of the equations of motion and solution techniques for the flat spinning annular elastic membrane which is completely clamped by a rigid hub. The axisymmetric case for this hub configuration has been studied by Simmonds [1]. He found that the governing differential equation was of the hypergeometic type subject to boundary conditions of zero deflection at the hub and finiteness of the solution at the free outer edge. His solutions were found to differ by a simple transformation of variables from the corresponding case for the loosely clamped membrane with no hole [2].

The central purpose of the current research has been to extend Simmonds' work to the cases when there are nodal diameters. In this case the governing differential equation for the radial dependence of the mode shapes is

$$x(x-1)(x+\delta^2)\frac{d^2w}{dx^2}+\left[2x^2-(1-\delta^2)x\right]\frac{dw}{dx}$$

$$+\frac{1}{4}\left\{\left[(1-\delta^2)-\frac{\delta^2}{x}\right]s^2-\mu^2 x\right\}w=0 \qquad (1)$$

where

$$\mu^2 = \frac{8}{3+\nu}\left[\frac{\lambda^2}{m\omega^2}+\frac{1+3\nu}{8}s^2\right]$$

$$x = \left(\frac{r}{b}\right)^2$$

$$\delta^2 = \frac{1-\nu}{3+\nu}\left(\frac{a}{b}\right)^2\left[\frac{(3+\nu)b^2-(1+\nu)a^2}{(1+\nu)b^2+(1-\nu)a^2}\right]$$

$$\lambda^2 = p^2 m$$

and

a = hub radius

b = disk radius

r = radial co-ordinate of disk element

w = transverse displacement of disk element

m = membrane mass per unit volume

ν = Poisson's ratio

p = natural frequency of vibration

s = number of nodal diameters

ω = spin augular speed

The boundary conditions to be satisfied require that there be no transverse deflection at the hub and that the deflection be finite at the outer edge, i.e.

$$w[(a/b)^2] = 0$$

$$w[1] = \text{Finite}$$

The specification of complete clamping at $x = (a/b)^2$ introduces a second parameter not present in the studies of Simmonds [2] and Eversman [3]. In these studies it was found that the frequency spectrum for any membrane disk could be specified as a function of the ratio of the clamping radius to the disk radius (annulus radius ratio in [3]). Poisson's ratio entered conveniently as part of a non-dimensional frequency. In the present case Poisson's ratio does not enter in this simple way.

Differential equation (1) is of the Fuchsian type with three finite regular singular points at $x = 0$, $x = 1$, and $x = -\delta^2$, and a regular singular point at infinity. It can be transformed into Heun's equation in at least two ways involving linear fractional transformations of the independent variable and a subsequent change of the dependent variable [4]. In the general case neither of these transformations generates a simpler mathematical problem. One requires a transformation of the dependent variable which involves complex arithmetic and the other generates a transformed space in which the singularities are arranged in such a way that known guarantees of the convergence properties of the solutions are insufficient. Since solutions to this problem will involve substantial computational effort in whatever form the differential equation is cast, it has been decided to forego the transformation to Heun's equation.

Solutions to Eqn. (1) are obtained by expansion in a power series about the singular point at the disk free edge, x = 1. This is accomplished by introducing the origin shift

$$\zeta = \frac{x-1}{(a/b)^2-1} = -\alpha^2(x-1) \qquad (2)$$

to obtain the differential equation

$$\zeta\,(\zeta-\alpha^2)^2\left[\zeta-\alpha^2(1+\delta^2)\right]\frac{d^2w}{d\zeta^2}$$

$$+(\zeta-\alpha^2)^2\left\{\zeta+\left[\zeta-\alpha^2(1+\delta^2)\right]\right\}\frac{dw}{d\zeta}$$

$$+\frac{1}{4}\left\{\left[-\alpha^2(1-\delta^2)(\zeta-\alpha^2)-\alpha^4\delta^2\right]s^2-\mu^2(\zeta-\alpha^2)^2\right\}w = 0 \qquad (3)$$

Solutions are sought in the form

$$w = \sum_{n=0}^{\infty} C_n\zeta^{n+\rho} \qquad (4)$$

The roots of the indicial equation are found to be

$$\rho_1 = 0$$

$$\rho_2 = 0$$

so that only one linearly independent solution of the form of Eqn. (4) exists. A second linearly independent solution exists but involves a logarithmic singularity at $\zeta = 0$ and is therefore discarded because of the boundary requirement of finiteness at the outer edge.

With the above observations the problem has been reduced to one of finding coefficients of the assumed series solution

$$w = \sum_{n=o}^{\infty} C_n \zeta^n \tag{6}$$

such that the deflection vanishes at the hub radius. Because of the origin shift the hub radius now occurs at $\zeta = 1$. The coefficients in the series solution are determined by the recurrence relations

$$C_o = \text{arbitrary} \tag{7}$$

$$\delta_1 C_1 + \lambda_o C_o = 0 \tag{8}$$

$$\delta_2 C_2 + \lambda_1 C_1 + \beta_o C_o = 0 \tag{9}$$

$$\delta_{n+1} C_{n+1} + \lambda_n C_n + \beta_{n-1} C_{n-1} + \alpha_{n-2} C_{n-2} = 0;\ n>2 \tag{10}$$

where

$$\alpha_{n-2} = (n-2)(n-1) - \frac{\mu^2}{4}$$

$$\beta_{n-1} = -\alpha^2 \left\{ (5 + \delta^2)(n-1) + (3 + \delta^2)(n-1)(n-2) - \frac{1}{4}\left[2\mu^2 - (1 - \delta^2) s^2\right] \right\}$$

$$\lambda_n = \alpha^4 \left\{ (4 + 2\delta^2) n + (3 + 2\delta^2) n(n-1) + \frac{1}{4}\left[(1 - 2\delta^2) s^2 - \mu^2\right] \right\}$$

$$\delta_{n+1} = -\alpha^6 (1 + \delta^2)(n+1)^2$$

The parameter μ^2 plays the role of an eigenvalue which is to be determined so that solutions of Eqn. (3), defined by Eqns. (6) - (10), can be found which satisfy the boundary condition

$$w(1) = \sum_{n=o}^{\infty} C_n = F\left(\mu^2\right) = 0 \tag{11}$$

Equations (7) through (11) define a transcendental function in μ^2 whose zeros define the required values of μ^2. A numerical routine has been written in PDQ fortran for the IBM 1620 digital computer for the purpose of finding the roots of the eigenvalue equation. This routine searches for sign changes in $F(\mu^2)$ and then applies a simple secant method for root iteration. Initial guesses are made for the desired roots. These guesses are based on results of previous cases and the knowledge that enlarging the hub to disk radius ratio will increase the vibration natural frequency.

Concurrent with the development of the eigenvalue program, a similar program was written to evaluate the natural frequencies for the symmetric case studied by Simmonds [1]. The results from this program were compared to the results obtained from the more general program in the case when the number of nodal dimeters s is zero. This comparison has generated confidence in the correctness of the calculated results. A similar check will be made for the case of one nodal diameter. This case can be reduced to Heun's equation for a substantial reduction in computational effort and will provide a second useful check of the more general case.

To date no "production" computation for the purpose of generating final data has been carried out. It is hoped that a graduate student will be available for this task in the fall.

The progress so far reported by Mr. Gilley in his study of the case of the loosely clamped spinning annular membrane has been mainly of an exploratory nature. It has been verified that the governing differential equation for the radial mode dependence is the same as the one studied by Eversman [3]. The boundary conditions are the same as for the fully clamped case discussed above. No difficulty is foreseen in treating this case by the above methods.

The analysis of the intermediate case of partial clamping has not been initiated. Bulkeley and Savage [5] have studied this hub condition for the case of symmetric vibrations.

EXTENSION TO MEMBRANE SHELLS

The application of the principle of centrifugally stabilized structures to devices such as parabolic radar dishes poses the problem of the analysis of the vibrations of axisymmetric shallow membrane shells which are centrifugally loaded. This extension of the analysis being carried out under the grant is presently in the exploratory phase. A literature survey has been carried out to aid in the formulation of the appropriate equations of motion and boundary conditions.

Reissner [6] formulates the equations of equilibrium for small displacements of shallow spherical shells. His theory includes both bending and membrane stresses. A subsequent paper by Reissner [7] presents solutions to specific cases of static loading, including an inertia load on a shell spinning about its symmetry axis. This case is not fully developed but would appear to offer an alternative approach to that taken by Cohen [8,9] for the purpose of calculating the stresses in spinning shallow paraboloidal radar dishes.

A number of papers have appeared which treat the vibrations of shallow axisymmetrical shells. Certain of the better known of these are listed in references [10] through [13]. For the purpose of the present investigation these all lack the treatment of the case of vibration about an initially stressed equilibrium state. In addition, important acceleration couplings between transverse and in-surface vibrations which arise due to spin are not treated.

Johnson [14] treats the dynamics of shallow elastic membranes. He develops the governing differential equations using a slightly different co-ordinate system than that used by Reissner. His paper treats the case of the initially curved membrane which is deflected to a flat plate because of the spin. This includes the case of the initially flat membrane studied by Simmonds [2]. Johnson also treats the case of vibrations about an equilibrium configuration which is a spherical shell. In the development of the spinning shell equilibrium equations he arrives at the conclusion that the radial stress vanishes if the shell has curvature while it is known that it doesn't vanish if the shell is a flat disk. This is disturbing on physical grounds since it would seem that an extremely shallow spinning shell should experience nearly the same stresses as a flat disk. This presents an interesting area of research which should be carried out using Reissner's theory including bending and membrane theory.

The first attempt at a general formulation of the equations of equilibrium for rotationally symmetric membranes under uniform centrifugal loading appears to be due to Simmonds [15]. His work [16] on the coupling between radial and circumferential vibrations for flat spinning disks is of importance for the extension to spinning shells in which case all vibrations are coupled. DiTaranto and Lessen [17] examine this coupling in the case of a spinning circular cylinder.

The equations of motion which govern the vibrations of shallow spinning membrane shells about a stressed equilibrium position will necessarily exhibit the characteristics developed by Novozhilov [18] in the case of small strains but finite rotations.

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