ON SOLUTION STRUCTURE OF THE RADIAL HEAT PROBLEM WITH SINGULAR DATA

By

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1. Introduction. Let μ be a real parameter and let $\Delta_\mu \equiv D_r^2 + [(\mu - 1)/r]\, D_r$, the radial Laplacian operator. In this paper, we will be concerned with the structure and behavior of the solutions of the initial value problem

$$(1.1)\begin{cases} \text{(a)} \quad u_t(r,t) = \Delta_\mu u(r,t),\ r > 0,\ t > 0 \\ \text{(b)} \quad u(r,0) = \varphi(r),\ r > 0 \end{cases}$$

for various choices of the data function $\varphi(r)$. Particular emphasis will be given to solutions corresponding to functions $\varphi(r)$ that have poles or logarithmic singularities at $r = 0$ but which are otherwise analytic.

In recent years, extensive research has been carried out relating to partial differential equations that have singular points. The Euler-Poisson-Darboux equation is one of the most notable examples of such equations. During the last decade, A. Weinstein and his associates [9] (other references are given here) developed a broad body of significant results on the solution structure of the equation. Weinstein and P. C. Rosenbloom [8] have given theories for a more general class of the equations of the form

$$u_{tt}(x,t) + \frac{k}{t} u_t(x,t) + P(x, t, D_x, u) = 0.$$

It is clear that the equation (1.1) does not fit into this class.

A detailed study has been made about solutions and expansions of solutions of (1.1) when φ (r) is an entire function of r^2 of suitable growth ([1], [5], [6]). More recently, the author has related the particular form of solution of (1.1) used in [1] to Laplace transforms and their inverses [2] and has applied this to data involving distributions. Through an examination of these transforms for elementary functions along with an application of the referred to expansion theory, we can obtain solutions of (1.1), valid in the large, in the form of convolutions when φ (r) has poles or logarithmic singularities (or the product of a pole and a logarithmic singularity) at r = 0. This will permit us to read off properties of the solution function. It is assumed that φ (r) has, at most, exponential growth of order $0\ (e^{\alpha r^2})$ as $r \to \infty$. For functions in this class, our results will show that there exist solutions of (1.1) provided $\int_{0+}^{\beta} \varphi(r)\, r^{\mu-1}\, dr$ exists for any positive β.

In section 2, we summarize, for the reader's convenience, a portion of the basic definitions, notations, and results from [1,2,5,6]. At the same time, we introduce a pair of solution forms (in addition to the one appearing in [2]) corresponding to data functions having poles or logarithmic singularities at r = 0. Relations among solutions of (1.1) and properties of these solutions are developed in section 3 and 4 when φ (r) is entire or has poles but no logarithmic singularities. In one of these forms, the pole dissipates when $t > 0$ while in the other form, the pole remains intact. For μ an even integer with $\mu \geqq 4$, it will be shown in section 5 that the solutions corresponding to the function $r^{2-\mu+2m}$, with $0 \leq m \leq [\mu/2 - 2]$, take on a particularly simple form. The last two sections relate to solutions corresponding to data of the form $\varphi(r) = \Psi(r) \log r$, firstly when $\Psi(r)$ is entire in r^2 and secondly when $\Psi(r)$ has a pole. Again, the dissipation property holds for one form when $\mu > 2$ but not necessarily for the other. Elementary examples are provided throughout to illustrate theorems or properties of solutions.

2. Preliminary Remarks. We now introduce some of the basic ideas and known results that will be needed in the ensuing development. At the same time, a notation will be introduced that will permit us to easily distinguish between the different forms of solutions of (1.1).

Definition (2.1). The basic source solution of (1.1) is given by $S_\mu(r,t) = (4\pi t)^{-\mu/2} e^{-r^2/4t}$.

Theorem 2.1 [2]. Let $u_1^\mu(r,t;\varphi(r))$ denote a solution of (1.1). Let

$$T_\mu(p,t) = \int_0^\infty e^{-\{1/4t - 1/p\} x} x^{\mu/2 - 1} \varphi(x^{1/2})\, dx. \tag{2.1}$$

Then

$$u_1^\mu(r, t; \varphi(r)) = \pi^{\mu/2} S_\mu(r,t) (r^2/16t^2)^{1-\mu/2} L_p^{-1}\{p^{-\mu/2} T_\mu(p,t)\} \tag{2.2}$$

in which the variable in this inverse Laplace transform is replaced by $a = r^2/16t^2$.

The definition of a given in theorem 2. 1 will be used throughout this paper.

Definition (2.2) ([1], [6]). The radial heat polynomials $\{R_j^\mu(r,t)\}_{j=0}^\infty$ are defined by $R_j^\mu(r,t) = u_1^\mu(r,t; r^{2j})$, $j = 0, 1, 2, \ldots$.

They satisfy the generating relation

$$(1 - 4\lambda t)^{-\mu/2} e^{\lambda r^2/(1-4\lambda t)} = \sum_{j=0}^\infty \frac{\lambda^2}{j!} R_j^\mu(r,t)$$

and are given by $R_j^{\mu}(r,t) = j!\,(4t)^j\, L_j^{(\mu/2-1)}(-r^2/4t)$ where $L_j^{\nu}(x)$ denotes the generalized Laguerre polynomial of degree j and index ν.

Definition (2.3). An entire function $\varphi(z) = \sum_{j=o}^{\infty} a_j z^j$ is of growth (ρ,τ) iff $\limsup_{j\to\infty} (j/e\rho)\,|a_j|^{\rho/j} \leqq \tau$.

Theorem 2.2 [1]. Let $\varphi(r) = \sum_{j=o}^{\infty} a_j r^{2j}$ be an entire function of growth $(1,\sigma)$ in r^2. Then the series $\sum_{j=o}^{\infty} a_j R_j^{\mu}(r,t)$ converges to the solution function $u_1^{\mu}(r,t;\varphi(r))$ of (1.1) in the time strip $|t| < 1/(4\sigma)$ and satisfies a Huygens' principle there provided $\mu > 1$.

The solution function $u_1^{\mu}(r,t;\varphi(r))$ defined above is not the only form of solution related to the problem (1.1). For, under the transformation $u(r,t) = r^{2-\mu} v(r,t)$, we find that $v(r,t)$ satisfies

$$(2.4)\begin{cases} v_t(r,t) = v_{rr}(r,t) + [(4-\mu) - 1]\, r^{-1} v_r(r,t) \\ v(r,o) = r^{2-\mu}\varphi(r). \end{cases}$$

The above equation has the same form as (1.1 a) with μ replaced by $4-\mu$. Denote the solution of (1.1) corresponding to (2.4) by $u_2^{\mu}(r,t;\varphi(r))$. An examination of theorem (2.1) shows that

Theorem 2.3. Let $\bar{T}(p,t) = \int_o^{\infty} e^{-\{1/4t - 1/p\}x}\,\varphi(x^{1/2})\,dx$.

Then

$$(2.5)\quad u_2^{\mu}(r,t;\varphi(r)) = \pi^{\mu/2} S_{\mu}(r,t)\, L_p^{-1}\{p^{\mu/2-2}\,\bar{T}(p,t)\}$$

in which the variable in this inverse Laplace transform is replaced by $a = r^2/16\,t^2$.

Observe that $u_1^{2}(r,t;\varphi(r)) = u_2^{2}(r,t;\varphi(r))$. We will obtain other connections between u_1^{μ} and u_2^{μ} in section 3. Symbolically, we can write $u_1^\mu(r,t;\varphi(r)) = e^{t\Delta_\mu}\varphi(r)$ and $u_2^\mu(r,t;\varphi(r)) = r^{2-\mu} e^{t\Delta_{4-\mu}}\{r^{\mu-2}\varphi(r)\}$.

By the interpretation $e^{t\Delta_\mu}\varphi(r) = \sum_{j=o}^{\infty} \frac{t^j}{j!} \Delta_\mu^j \varphi(r)$, we may be able to attach a meaning to u_1 even if the integrals in theorem 2.1 fail to exist. In particular, if m is an integer, then

$$(2.6) \qquad e^{t\Delta_{-2m}} r^{2j} = R_j^{-2m}(r,t), \; j = o, 1, \ldots .$$

Finally, let us suppose that the data function $\varphi(r)$ in (1.1) has the form $\Psi(r) \log r$ in which $\Psi(r)$ is entire of growth $(1,\sigma)$ in r^2. Under the transformation

$$(2.7) \qquad u(r,t) = v(r,t) \log r + w(r,t)$$

we can require that $v(r,t)$ and $w(r,t)$ satisfy the problem

$$(2.8)\begin{cases} \text{(a)} \quad v_t(r,t) = \Delta_\mu v(r,t), \; v(r,0) = \Psi(r) \\ \text{(b)} \quad w_t(r,t) = \Delta_\mu w(r,t) + F(r,t,v), \; w(r,o) = 0 \end{cases}$$

with $F(r,t,v) = 2r^{-1} v_r + (\mu - 2) r^{-2} v$. A solution of (a) is given either by $u_1^\mu(r,t;\Psi(r))$ or $u_2^\mu(r,t;\Psi(r))$. Then by Duhamel's principle [4], we find that

$$(2.9)\begin{cases} \text{(a)} \quad w_{1i}^\mu(r,t) = \int_o^t u_i^\mu(r,t-\eta; F_i^\mu(r,\eta))\, d\eta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2 \\ \text{(b)} \quad w_{2i}^\mu(r,t) = \int_o^t r^{2-\mu} u_i^{4-\mu}(r,t-\eta; r^{\mu-2}F_i^\mu(r,\eta))\, d\eta \end{cases}$$

with $F_i^\mu(r,\eta) = F(r,\eta; v_i^\mu(r,\eta))$. The integrals in (2.9) are not always defined and appropriate pairings must be made according to the value of μ. We will later discuss the integrals (2.9) with the F_1^μ function. The analysis for the other case is analogous but more tedious.

3. Relations Among the Solution Forms. It is evident that the solution forms u_i^{μ} $(r,t;\ \varphi(r))$, $i = 1,2$, defined in section 2 are basic to most of our developments relating to the behavior of solutions of (1.1). For this reason, it is useful to obtain some of the relationships pertaining to these forms. It will be convenient to call upon some elements of Laplace transforms as well as the theorem 2.2. The symbol $*$ will denote convolution, i.e. $f(a)*g(a) = \int_0^a f(\xi)g(a-\xi)\, d\xi$.

Theorem 3.1. $u_2^{\mu}(r,t;\ r^{2-\mu}\ \Psi(r)) = r^{2-\mu}\ u_1^{4-\mu}\ (r,t;\ \Psi(r))$ whenever either side of this expression is defined.

Proof. When $\mu < 4$, this is immediate by comparing the integrals in theorems 2.1 and 2.3. If $\mu \geqq 4$, this can be regarded as a continuation formula if either side of the expression can be defined symbolically.

Theorem 3.2.

$$(3.1)\quad u_2^{\mu}(r,t;r^{2j}) = \begin{cases} \text{(a)}\quad u_1^{\mu}(r,t;\ r^{2j}),\ \mu \text{ even integer and } \mu \geqq 2. \\[2ex] \text{(b)}\quad \dfrac{(4t)^{1-\mu/2}}{\Gamma(1-\mu/2)} \displaystyle\int_0^a e^{-4(a-\xi)t}\, R_j^{\,2}\,(4t\sqrt{\xi},t)\ (a-\xi)^{-\mu/2}\, d\xi,\ \mu < 2 \\[2ex] \text{(c)}\quad \dfrac{(4t)^{[\mu/2]+1-\mu/2}}{\Gamma\{[\mu/2]+1-\mu/2\}} \displaystyle\int_0^a (a-\xi)^{[\mu/2]-\mu/2}\, e^{-4(a-\xi)t} R_j^{2\{[\mu/2]+1\}}(4t\sqrt{\xi},t)\,d\xi \end{cases}$$

if $\mu > 2$ but is not an even integer.

Proof. We derive (3.1) with the aid of a generating function. Select $\varphi(r) = e^{\lambda r^2}$ in theorem 2.3. Then it follows that

$$(3.2)\quad p^{\mu/2-2}\ \bar{T}\ (p,t) = \frac{4t}{1-4\lambda t}\ \frac{p^{\mu/2-1}}{\{p-4t\ /(1-4\lambda t)\}}$$

(a). If μ is an even integer with $\mu \geq 2$, then $L_p^{-1}\{p^{\mu/2-1}\} = \delta^{(\mu/2-1)}(a)$ where $\delta(a)$ denotes the Dirac distribution. By the convolution theorem

$$L_p^{-1}\{p^{\mu/2-2}\,\bar{T}(p,t)\} = \frac{4t}{1-4\lambda t}\,\delta^{(\mu/2-1)}(a) * e^{4at/(1-4\lambda t)}$$

$$= \left(\frac{4t}{1-4\lambda t}\right)^{\mu/2} e^{4at/(1-4\lambda t)} = (4t)^{\mu/2}\, e^{4at}\, \frac{e^{16at^2\lambda/(1-4\lambda t)}}{(1-4\lambda t)^{\mu/2}}$$

$$= (4t)^{\mu/2}\, e^{4at} \sum_{j=o}^{\infty} \frac{\lambda^j}{j!}\, R_j^{\mu}(4t\sqrt{a},t),$$

the last step following by the generating relation in definition (2.2). The stated result (a) then follows by introducing the multiplier $\pi^{\mu/2}\, S_{\mu}(r,t)$ in theorem 2.3 and comparing coefficients of $\lambda^j/j!$.

(b). If $\mu < 2$ in (3.2), an application of the convolution theorem gives

$$L_p^{-1}\{p^{\mu/2-2}\,\bar{T}(p,t)\} = \frac{4t}{1-4\lambda t}\,\frac{1}{\Gamma(1-\mu/2)}\int_o^a (a-\xi)^{-\mu/2} e^{4\xi t/1-4\lambda t}\, d\xi$$

$$= \frac{4t}{\Gamma(1-\mu/2)}\int_o^a (a-\xi)^{-\mu/2}\, e^{4\xi t}\left\{\frac{e^{16\xi t^2\lambda/(1-4\lambda t)}}{1-4\lambda t}\right\} d\xi.$$

The result (b) follows by expanding the generating function in brackets, introducing the multiplier $\pi^{\mu/2} S_{\mu}(r,t)$, and comparing coefficients of $\lambda^j/j!$.

(c). If $\mu > 2$ but is not an even integer, then (3.2) can be written in the form

$$\left\{\frac{4t}{1-4\lambda t}\right\}\; p^{[\mu/2]}\; \frac{1}{\{p-4t/(1-4\lambda t)\}}\; \frac{1}{p^{[\mu/2]+1-\mu/2}}$$

in which $[\mu/2]$ denotes the greatest integer function. Taking inverses, we get

$$(3.3)\quad L_p^{-1}\{p^{\mu/2-2}\,\bar{T}(p,t)\} = \frac{\{4t/(1-4\lambda t)\}}{\Gamma\{[\mu/2]+1-\mu/2\}}\,\delta^{[\mu/2]}(a) * e^{4at/1-4\lambda t} * a^{[\mu/2]-\mu/2}$$

Since $\delta^{[\mu/2]}(a) * e^{4at/(1-4\lambda t)} = (4t/(1-4\lambda t))^{[\mu/2]}\, e^{4at/(1-4\lambda t)}$, (3.3) reduces to

$$L_p^{-1}\{p^{\mu/2-2}\,\bar{T}(p,t)\} = \frac{\left(\frac{4t}{1-4\lambda t}\right)^{[\mu/2]+1}}{\Gamma\{[\mu/2]+1-\mu/2\}} \int_0^a e^{4\xi t/(1-4\lambda t)} (a-\xi)^{[\mu/2]-\mu/2}\, d\xi.$$

The stated result (c) now follows by the argument used in (b).

<u>Theorem 3.3</u> Let $\varphi(r) = \sum_{j=0}^{\infty} a_j r^{2j}$ be an entire function of growth $(1,\sigma)$ in r^2. Then, for $0 \le 1/4\sigma$

$$(3.4)\quad u_2^{\mu}(r,t,\varphi(r)) = \begin{cases} \text{(a)} \quad u_1^{\mu}(r,t;\varphi(r)),\ \mu \text{ even},\ \mu \geqq 2 \\ \text{(b)} \quad \dfrac{(4t)^{1-\mu/2}}{\Gamma(1-\mu/2)} \displaystyle\int_0^a (a-\xi)^{-\mu/2} e^{-4(a-\xi)t} u_1^2(4t\sqrt{\xi},t;\varphi)\, d\xi,\ \mu < 2 \\ \text{(c)} \quad \dfrac{(4t)^{[\mu/2]+1-\mu/2}}{\Gamma\{[\mu/2]+1-\mu/2\}} \displaystyle\int_0^a (a-\xi)^{[\mu/2]-\mu/2} e^{-4(a-\xi)t} u_1^{2\{[\mu/2]+1\}}(4t\sqrt{\xi},t;\varphi)\, d\xi\ \ \mu>2 \end{cases}$$

μ not an even integer.

<u>Proof.</u> The validity of (3.4a) follows trivially by the (a) part of theorem 3.2 and theorem 2.2. We need only examine the proof of part (b) of the theorem since (c) follows by similar reasoning.

The series $\sum_{j=0}^{\infty} a_j u_2^{\mu}(r,t;r^{2j})$ converges to the solution function $u_2^{\mu}(r,t;\varphi(r))$ for $0 \leqq t < 1/4\sigma$ if the series

$$\sum_{j=0}^{\infty} a_j \int_0^a (a-\xi)^{-\mu/2} e^{-4(a-\xi)t} u_1^2(4t\sqrt{\xi},t;r^{2j})\, d\xi \text{ converges.}$$

By interchanging summation and integration, this last series reduces to $\int_o^a (a-\xi)^{-\mu/2} e^{-4(a-\xi)t} u_1^2 (4t\sqrt{\xi}, t; \varphi(r))\, d\xi$. This interchange step is valid if the following series converges:

$$(3.5) \quad \sum_{j=o}^{\infty} |a_j| \int_o^a (a-\xi)^{-\mu/2} e^{-4(a-\xi)t} |u_1^2(4t\sqrt{\xi}, t; r^{2j})|\, d\xi.$$

Select $t \leqq t_o < 1/4\sigma$ and $o < \delta < 1/4\sigma - t_o$. It follows that

$$|u_1^2 (4t\sqrt{\xi}, t; r^{2j})| \leqq (1 + {}^{to}/\delta) \left[\frac{4j\,(t_o + \delta)}{e}\right]^j e^{4t^2\xi/\delta}\, t^{1-\mu/2} \quad \text{for}$$

$o \leq t \leq to$, $o \leq r < \infty$ (see [1], p. 276). Since, by hypothesis, $|a_j| \leqq A \left(\frac{e}{j}\right)^j \sigma^j$ for some constant A, the series (3.5) is dominated by

$$A\,(1+{}^{to}/\delta) \sum_{j=o}^{\infty} [4\sigma\,(t_o + \delta)\,]^j \{t^{1-\mu/2} \int_o^a (a-\xi)^{-\mu/2} e^{-4(a-\xi)t} e^{4t^2\xi/\delta}\, d\xi\}$$

and, by the ratio test, this converges if the bracketed integral exists. With the change of variables $4(a-\xi)\,t = \zeta$, it is easy to show that this integral is bounded by $\Gamma\,(1-\mu/2)\, e^{r^2/4\delta}$. Since t_o was arbitrary, this completes the proof.

4. <u>General Relations on Poles</u>. An examination of theorem 3.1 shows that if the data function has a pole at $r = o$, then the u_2^{μ} solution is readily obtainable if one can attached a meaning to $u_1^{4-\mu}(r,t; \Psi)$, analytically or symbolically. For example, if $\Psi(r) = 1$, then $u_1^{4-\mu}(r,t; 1) = e^{t\Delta_{4-\mu}} \cdot 1 = 1$ and $u_2^{\mu}(r,t; r^{2-\mu}) = r^{2-\mu}$ for all real μ.

However, even if $\Psi(r)$ is entire in r^2 of growth $(1,\sigma)$, the existence of $u_1^{(4-\mu)}(r,t;\Psi)$ is not assured unless $\mu < 3$. In order to obtain a better understanding of solutions of (1.1) when $\varphi(r)$ possesses a pole at $r = o$, we examine the u_1^{μ} solution function in more detail.

For this purpose, select $\varphi_\lambda(r) = r^{2-\mu-2\alpha+2m} e^{\lambda r^2}$ as a generating function with $\alpha < 1$ and m a non-negative integer. A simple computation shows that for this $\varphi_\lambda(r)$:

$$p^{-\mu/2} T_\mu(p,t) = \frac{\Gamma(m+1-\alpha)}{\{p - 4t/(1-4\lambda t)\}^{m+1-\alpha}} \quad \frac{1}{p^{\mu/2+\alpha-m-1}} \quad \left(\frac{t}{1-4\lambda t}\right)^{m+1-\alpha}$$

Assume that $\mu/2 + \alpha - m - 1 > o$. Then, by the convolution theorem:

$$L_p^{-1}\{p^{-\mu/2} T_\mu(p,t)\}$$

$$(4.1) = \left(\frac{4t}{1-4\lambda t}\right)^{m+1-\alpha} \frac{a^{\mu/2+\alpha-m-2}}{\Gamma(\mu/2+\alpha-m-1)} * a^{m-\alpha} e^{4at/(1-4\lambda t)}$$

$$= \frac{(4t)^{m+1-\alpha}}{\Gamma(\mu/2+\alpha-m-1)} \int_o^a \xi^{m-\alpha} e^{4\xi t} (a-\xi)^{\mu/2+\alpha-m-2} \left\{\frac{e^{16\xi t^2\lambda/(1-4\lambda t)}}{(1-4\lambda t)^{m+1-\alpha}}\right\} d\xi .$$

An application of definition 2.2 followed by an introduction of the required multiplier in theorem 2.1 gives:

$$(4.2) \qquad u_1^{\mu}(r,t;\, r^{2-\mu-2\alpha+2m+2j})$$

$$= \pi^{\mu/2} S_\mu(r,t) \left(\frac{r^2}{16t^2}\right)^{1-\mu/2} \frac{(4t)^{m+1-\alpha}}{\Gamma(\mu/2+\alpha-m-1)} \int_o^a \xi^{m-\alpha} (a-\xi)^{\mu/2+\alpha-m-2} e^{4\xi t} R_j^{2(m+1-\alpha)}(4t\sqrt{\xi},t)\, d\xi.$$

An examination of this solution shows that if α is close to but less than 1 and $\mu > 1$, the choices $m = o$ and $j = o$ lead to a meaningful definition of u_1^{μ} $(r,t;\ r^{2-\mu-2\alpha})$. We are not in the position to apply the expansion theorem 2.2; however, since the choice m = o gives rise to radial heat polynomials of index < 1 for α near 1. If we select m = 1 in (4.2) and require that $\mu/2 + \alpha > 2$, the radial heat polynomials entering the integrand of (4.2) have index at least 2. Upon combining these observations along with theorem 2.2 and the method of proof of theorem (3.3b), we have the following result:

<u>Theorem 4.1</u> Let $\varphi(r) = r^{2-\mu-2\alpha}\ \{\beta + r^2\ \Psi(r)\}$ in which α is less than but close to 1, $^{\mu}/2 + \alpha > 2$, and $\Psi(r)$ is an entire function in r^2 of growth $(1,\sigma)$. Then, for $o \leq t < 1/4\sigma$,

$$(4.3)\left\{\begin{aligned}&u_1^{\mu}\ (r,t;\ \varphi(r)) = \\ &\pi^{\mu/2}\ s_{\mu}\ (r,t)\ \left(r^2/16t^2\right)^{1-\mu/2}\ \left\{\frac{\beta(4t)^{1-\alpha}}{\Gamma(\mu/2+\alpha-1)}\int_o^a \xi^{-\alpha}(a-\xi)^{\mu/2+\alpha-2}e^{4\xi t}\,d\xi\ +\right.\\ &\left.\frac{(4t)^{2-\alpha}}{\Gamma(\mu/2+\alpha-2)}\int_o^a \xi^{1-\alpha}(a-\xi)^{\mu/2+\alpha-3}\,e^{4\xi t}\,u_1^{4-2\alpha}(4t\sqrt{\xi},t;\ \Psi)\,d\xi\ \right\}\end{aligned}\right.$$

This result shows that $\varphi(r)$ can behave, at worst, as $r^{-\mu+\epsilon}$, for arbitrary $\epsilon > o$, and still give rise to a classical solution of (1.1).

A further examination of (4.2) shows that if $o \leq \alpha < 1/2$ and m=o, then the integral there converges if $\mu > 2$ and defines the set $\{u_1^{\mu}(r,t;\ r^{2-\mu-2\alpha+2j})\}_{j=0}^{\infty}$ Moreover, the radial heat polynomials entering the right member of (4.2) have

have index > 1. Consequently, we have

Theorem 4.2. Let $\varphi(r) = r^{2-\mu-2\alpha}\Psi(r)$ where $o \leq \alpha < 1/2$ and $\Psi(r)$ is entire in r^2 of growth $(1,\sigma)$. Then, for $o \leq t < 1/4\sigma$ and $\mu > 2$,

(4.4)
$$\begin{cases} u_1^{\mu}\,(r,t;\;\varphi(r)) = \\ \pi^{\mu/2} S_{\mu}(r,t)\left(\dfrac{r^2}{16t^2}\right)^{1-\mu/2} \dfrac{(4t)^{1-\alpha}}{\Gamma(\mu/2+\alpha-1)} \displaystyle\int_o^a \xi^{-\alpha}\,(a-\xi)^{\mu/2+\alpha-2} e^{4\xi t} u_1^{2-2\alpha}(4t\sqrt{\xi},t;\;\Psi)\,d\xi \end{cases}$$

The choice $\alpha = o$ in theorem (4.2) leads to the case in which the most badly behaved pole entering the data is precisely the potential function for the operator Δ_{μ}, namely $r^{2-\mu}$. The choice $\mu = 4$ in (4.4) gives $u_1^4(r,t;r^{-2}) = r^{-2}(1-e^{-r^2/4t})$ while $u_2^4\,(r,t;\;r^{-2}) = r^{-2}$. Thus, in the first case, the pole in the data dissipates in the solution while in the second case it remains intact. This dispersion property for the u_1^{μ} solution always holds. For example, if we make the change of variables $\xi = a\,\zeta$ in the first integral in (4.3), we get

$$u_1^{\mu}\,(r,t;\;r^{2-\mu-2\alpha}) = \frac{(4t)^{1-\alpha-\mu/2} e^{-4at}}{\Gamma(\mu/2+\alpha-1)} \int_o^1 \zeta^{-\alpha}(1-\zeta)^{\mu/2+\alpha-2} e^{4a\zeta t} d\zeta\,.$$ Then

for $t > o$, $$\lim_{r \to o} u_1^{\mu}\,(r,t;\;r^{2-\mu-2\alpha}) = \frac{\Gamma(1-\alpha)}{\Gamma(\mu/2)}\,(4t)^{1-\alpha-\mu/2}.$$

We shall give some further special results about the case $\alpha = o$ in section 5.

Finally, we record, without proof, a result for odd data functions. The proof follows the lines of theorems 4.1 and 4.2.

Theorem 4.3. Let $\varphi(r) = r\Psi(r)$ in which $\Psi(r)$ is an entire function in r^2 of growth $(1,\sigma)$. Then for $o \leq t < 1/4\sigma$ and $\mu > 2$,

$$u_1^{\mu}(r,t;\;\varphi(r)) = \frac{r^{2-\mu}e^{-4at}}{\sqrt{\pi}(4t)^{5/2-\mu}} \int_o^a (a-\xi)^{-1/2}\xi^{\frac{\mu-3}{2}}\, u_1^{\mu-1}\,(4t\sqrt{\xi},t;\;\Psi)\,d\xi.$$

This theorem coupled with the theorem 2.2 permits the treatment of solutions of (1.1) corresponding to the most general type of entire function of r.

Example. Take $\mu = 3$ and $\varphi(r) = r\, J_o(r)$. Now

$u_1^2(r,t;\ J_o(r)) = e^{-t} J_o(r)$ so that

$$u_1^3(r,t;\ rJ_o(r)) = \sqrt{\frac{4t}{\pi r^2}}\, e^{-t-4at} \int_o^a (a-\xi)^{-1/2}\, e^{4\xi t}\, J_o\,(4t\sqrt{\xi}\,)\, d\xi.$$

5. Special Pole Properties. In the preceding section, we have given some general properties of solutions of (1.1) when the data function $\varphi(r)$ possesses a pole at r=o. There is some interesting structure of these solutions that is not quite brought out by these theorems, particularly when α=o in theorem 4.2. The purpose of this section will be to examine this case and to show what reductions take place when μ is an even integer with $\mu \geqq 4$.

If we select $m = \alpha = o$ in (4.1) and make the change of variables $a-\xi = (1-4\lambda t)\,\sigma/4t$, we can write

$u_1^{\mu}(r,t;\ r^{2-\mu}\, e^{\lambda r^2}) =$

$$\frac{r^{2-\mu} e^{\lambda r^2/(1-4\lambda t)}}{\Gamma(\mu/2-1)\ (1-4\lambda t)^{2-\mu/2}} \int_o^{\frac{4at}{1-4\lambda t}} \xi^{\mu/2-2}\, e^{-\sigma}\, d\sigma \tag{5.1}$$

The integral in the last member of this is precisely the incomplete gamma function $\gamma\,(\mu/2-1,\ 4at/(1-4\lambda t))$ [7, p. 95]. Making use of the defining relations for $\gamma\,(\alpha,x)$ in [7], it follows that

$$u_1^{\mu}(r,t;\ r^{2-\mu}\, e^{\lambda r^2}) = \frac{e^{-4at}}{(4t)^{\mu/2-1}} \sum_{n=o}^{\infty} \frac{(4at)^n}{\Gamma(\mu/2+n)}\ (1-4\lambda t)^{-n-1}$$

From this we find

$$u_1^{\mu}(r,t;\ r^{2-\mu+2j}) = D_{\lambda}^{\ j}\ u_1(r,t;\ r^{2-\mu}\ e^{\lambda r^2})\ \big|_{\lambda=o} =$$

(5.2)

$$\frac{\pi^{\mu/2}}{\Gamma(\mu/2)}\ 4t\ S_{\mu}(r,t)\ \{j!\ (4t)^j\ {}_1F_1\ (j+1;\ \mu/2;\ 4at)\}.$$

It is interesting to note that the bracketed team here differs from the radial heat polynomials by replacing the Laguerre polynomials by the ${}_1F_1$ functions.

Now, let us examine the function ${}_1F_1\ (j+1;m;\ z)$ for m an integer. Let $I\ f(z) = \int_o^z f(\eta)d\eta$. Then we find that

$$I^j\ {}_1F_1\ (j+1;\ m;\ Z) = \frac{Z^{j+1-m}}{j!}\ \Gamma(m) \sum_{n=o}^{\infty} \frac{Z^{n+m-1}}{(n+m-1)!} =$$

$$\frac{\Gamma(m)}{j!}\ Z^{j+1-m}\ \{e^Z - \sum_{k=o}^{m-1} Z^k/k!\ \}\ \text{and}$$

$${}_1F_1\ (j+1;\ m;\ Z) = \frac{\Gamma(m)}{j!}\ \{D_Z^{\ j}[e^Z Z^{j+1-m}] - D_Z^{\ j} \sum_{k=o}^{m-1} \frac{Z^{j+1-m+k}}{k!}\ \}$$

Through an application of the Rodrigues formula for Laguerre polynomials [7, p.85], the right member of this last expression reduces to

$${}_1F_1(j+1;m;Z) = \begin{cases} \Gamma(m)\ e^Z\ Z^{1-m}\ L_j^{(1-m)}\ (-Z) & ,\ j \geq m-1 \\ \Gamma(m)\ e^Z\ Z^{1-m}\ L_j^{(1-m)}\ (-Z)\ + & \\ \frac{(-1)^{j+1}}{j!} \sum_{k=o}^{m-2-j} \frac{(m-2-k)!}{k!\,(m-2-j-k)!}\ Z^{1-m+k} & ,\ o \leq j \leq m-2. \end{cases}$$

Substituting this evaluation for ${}_1F_1\ (j+1;m;Z)$ into (5.2) and using definition (2.2) we get

Theorem 5.1. Let $\mu=2m$ be an even integer with $\mu \geq 4$. Then

$$(5.3)\qquad u_1^{2m}(r,t;r^{2-2m+2j}) = \begin{cases} \text{(a)} \quad r^{2-2m}\, R_j^{4-2m}(r,t), \; j \geqq m-1 \\ \text{(b)} \quad r^{2-2m} R_j^{4-2m}(r,t) + \\ \quad (-1)^{j+1}\, e^{-4at} \sum_{k=o}^{m-2-j} \frac{(m-2-k)!}{k!\,(m-2-j-k)!}\, r^{2(1-m+k)}(4t)^{j-k}, o \leq j \leq m-1. \end{cases}$$

It should be observed that (5.3a) can be replaced by R^{2m}_{j+1-m} (r,t). This results from an application of theorem 3.1 followed by theorem 3.2a. The result (2.6) is used in (5.3b).

The above results permit a simple treatment of (1.1) if μ is even and $\mu \geqq 4$ and $\alpha = o$ in theorem 4.2. We merely decompose the data into a portion having pole type terms (finite in number) and a portion which is entire. Then apply (5.3b) to the pole terms and theorem 2.2 to the entire part.

6. Logarithmic Singularities. The results relating to pole type terms now permit us to examine solutions of (1.1) corresponding to data of the form $\Psi(r)$ log r with $\Psi(r)$ entire in r^2 of growth $(1,\sigma)$. The case in which $\Psi(r)$ has a pole will be deferred until the last section.

A solution of (1.1) is sought in the form (2.7). This form is not meant to suggest that the $w(r,t)$ function does not involve or lead to logarithmic terms in r. This function may indeed contain such terms except in the case $\mu = 2$. The form used is merely to relate the solution function u_1^{μ} $(r,t;\ \Psi(r))$to the function log r in the final solution function.

As noted in section 2, we restrict ourselves to making use of the function $u_1^{\mu}(r,t;\ \Psi(r))$ in the equations (2.8) and (2.9). The problem then reduces to one of exhibiting the existence of at least one of

$w_{11}^{\mu}(r,t)$ or $w_{21}^{\mu}(r,t)$ in the relations (2.9). Set $v_1^{\mu}(r,t) = u_1^{\mu}(r,t;\ \Psi)$ and write this in the form $\{v_1^{\mu}(r,t) - v_1^{\mu}(o,t)\}$. Since $v_1^{\mu}(r,t)$ is an even function of r, this last decomposition can be expressed in the form $v_1^{\mu}(r,t) = r^2 Z(r,t) + v_1^{\mu}(o,t)$. It follows from this $F_1^{\mu}(r,t;\ v_1^{\mu})$ takes the form $\bar{Z}(r,t) + (\mu-2)\, v_1^{\mu}(o,t)\, r^{-2}$ with $\bar{Z}(r,t) = 2/r\ \partial/\partial r\ \{r^2 Z(r,t)\} + (\mu-2) Z(r,t)$. From the entireness property of $\Psi(r)$, it follows that $\bar{Z}(r,t)$ is entire in r and analytic in t in $|t| < 1/4\sigma$. Moreover, $\bar{Z}(r,t)$ grows no more rapidly than Ke^{Ar^2} (K,A constants) in this time strip [1, p. 284].

If $\mu=2$, the existence of $w_{11}^{2}(r,t) = w_{21}^{2}(r,t)$ is assured by the properties of $\bar{Z}(r,t)$. Moreover, this term contains no logarithmic terms. If $\mu \neq 2$, we must exhibit the existence of at least one of the following integrals:

(a) $\int_o^t r^{2-\mu}\, u_1^{4-\mu}(r,t-\eta;\ r^{\mu-2}\, \bar{Z}(r,\eta) + (\mu-2)\, u_1^{\mu}(o,\eta)\, r^{\mu-4})\, d\eta$

(6.1) (b) $\int_o^t u_1^{\mu}(r,t-\eta;\ \bar{Z}(r,\eta) + (\mu-2)\, u_1^{\mu}(o,\eta)\, r^{-2})\, d\eta$

Case a. If $\mu \geqq 4$ and μ is an even integer, the term $u_1^{4-\mu}(r,t;r^{\mu-4})$ can be given as $R^{4-\mu}_{\frac{1}{2}(\mu-4)}(r,t-\eta)$ by (2.6). By theorem (3.1) we can write the term in the integral of (6.1a) involving $\bar{Z}(r,\eta)$ as $u_2^{\mu}(r,t-\eta;\ \bar{Z}(r,\eta))$. This is just $u_1^{\mu}(r,t-\eta;\ \bar{Z}(r,\eta))$ by theorem 3.3a for μ even. Consequently, the integral (6.1a) exists and leads to non-logarithmic terms.

Case b. If $\mu > 2$, an application of theorem 2.1 shows that $u_1^{\mu}(r,t-\eta;\ r^{-2}) = \int_o^{r^2/4(t-\eta)} e^{-\sigma}\, \{1 - \frac{4(t-\eta)}{r^2}\sigma\}^{\mu/2-2}\, d\sigma$. Moreover, $\lim_{\eta \to t-} u_1^{\mu}(r,t-\eta;\ r^{-2}) = r^{-2}$ which exists for $r \neq o$. Since $v(o,\eta)$ is continuous it results that, for $r > o$, the integral

$\int_o^t u_1^\mu(r,t-\eta;\ v_1^\mu\ (o,\eta)\ r^{-2})\ d\eta$ exists. This term may contribute logarithmic terms. The part of the integral (6.1b) that involves $\bar{Z}\ (r,\eta)$ exists by the arguments for (a) above. Summarizing these conclusions, we have

<u>Theorem 6.1</u>. Let $\varphi(r) = \Psi(r) \log r$ with $\Psi(r)$ entire of growth $(1,\sigma)$ in r^2. For $\mu \geqq 2$ and $o \leq t < 1/4\sigma$, there exists a solution of (1.1) in the form

$$(6.2)\quad u_1^\mu\ (r,t;\varphi) = u_1^\mu(r,t;\Psi)\ \log r + w_{11}^\mu(r,t).$$

For $\mu\ \geqq 4$ and an even integer, there exists a solution of (1.1) of the form

$$(6.3)\quad u_1^\mu\ (r,t;\ \varphi) = u_1^\mu(r,t;\ \Psi)\ \log r + w_{21}^\mu\ (r,t)$$

The function $w_{21}^\mu\ (r,t)$ contains no logarithmic terms.

<u>Example</u>. Take $\mu = 4$ and $\varphi(r) = \log r$. Then $u_1^4\ (r,t;\ 1) = 1$ and $F_1^4\ (r,t) = 2r^{-2}$. From (5.3), the solution corresponding to (6.2) is given by $u_1^4(r,t;\ \log r) = \log r + 2t/r^2 - (2/r^2) \int_o^t e^{-r^2/4(t-\eta)}\ d\eta$ while the solution corresponding to (6.3) is just $\log r + 2t/r^2$. With the change of variables $t-\eta = \frac{r^2}{4}\ \sigma^{-1}$ and repeated integration by parts, the first of the above solutions can be written as

$$u_1^4\ (r,t;\ \log r) = \{1-e^{-r^2/4t}\}\ \{\log r + 2t/r^2\} + \tfrac{1}{2}\ \log(4t)e^{-r^2/4t} + \tfrac{1}{2} \int_{r^2/4t}^\infty e^{-\xi} \log \xi d\xi.$$

From this, we observe that $\lim_{\substack{r \to o \\ t\ > o}}\ u_1^4\ (r,t;\ \log r) = \tfrac{1}{2}\ \{1 + \log 4t + \Gamma'(1)\}$

which says the pole dissipates. In the second solution, the logarithmic term remained and a pole type term was added. We will exhibit the dissipation property for the $u_1^{\mu}(r,t)$ solution in the last section.

7. <u>Coupled Singularities</u>. We now give some results for the case in which the initial data function $\varphi(r)$ in (1.1) contains terms of the form $r^{2-\mu-2\alpha} \log r^2$ with $\alpha < 1$. Because of the rather complicated expressions that arise, no attempt is made to list results of a general nature such as are contained in theorems 4.1 - 4.3 for poles alone. For terms of this type in the data, it is easier to apply theorem 2.1 directly to these terms rather than use the procedure of section 6. We list these results (without derivation) for the u_1^{μ} function and note the simplification that occurs when $\alpha = o$, that is, when the pole term is the potential function for Δ_{μ}.

We have, after appropriate changes of variables for $\mu > 2$ and $\alpha < 1$

$$u_1^{\mu}(r,t;\, r^{2-\mu-2\alpha} \log r^2) =$$

(7.1) $$(4t)^{1-\alpha-\mu/2} \left\{\frac{\Gamma'(1-\alpha) + \Gamma(1-\alpha) \log 4t}{\Gamma(1-\alpha)\, \Gamma(\mu/2+\alpha-1)}\right\} \int_o^1 \sigma^{-\alpha}(1-\sigma)^{\mu/2+\alpha-2} e^{-4(1-\sigma)at}\, d\sigma \; +$$

$$\frac{(4t)^{1-\alpha-\mu/2}}{\Gamma(\mu/2+\alpha-1)} \int_o^1 \{e^{-4a(1-\sigma)t}(1-\sigma)^{\mu/2+\alpha-2}\, \sigma^{-\alpha} - e^{-4a\sigma t}(1-\sigma)^{-\alpha} \sigma^{\mu/2+\alpha-2}\} \log \sigma \, d\sigma \; +$$

$$\frac{(4t)^{1-\alpha-\mu/2}}{\Gamma(\mu/2+\alpha-1)} \int_o^1 \left\{\frac{\Gamma'(\mu/2+\alpha-1)}{\Gamma(\mu/2+\alpha-1)}\, e^{-4a\sigma t}(1-\sigma)^{-\alpha} \sigma^{\mu/2+\alpha-2} \; - \right.$$

$$\left. \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)}\, e^{-4a(1-\sigma)t}(1-\sigma)^{\mu/2+\alpha-2}\, \mu^{-\alpha}\right\} d\sigma$$

When $\alpha = o$, this simplifies to

$$u_1^{\mu}(r,t;\ r^{2-\mu}\log r^2) =$$

(7.2) $$(4t)^{1-\mu/2}\left\{\frac{\log 4t}{\Gamma(\mu/2-1)} + \frac{\Gamma'(\mu/2-1)}{[\Gamma(\mu/2-1)]^2}\right\}\int_o^1 \sigma^{\mu/2} e^{-4a\sigma t}\, d\sigma \; +$$

$$\frac{(4t)^{1-\mu/2}}{\Gamma(\mu/2-1)}\int_o^1 \{e^{-4a(1-\sigma)t}\,(1-\sigma)^{\mu/2-2} - e^{-4at\sigma}\,\sigma^{\mu/2-2}\}\log\sigma\, d\sigma.$$

The relation (7.1) coupled with theorem 4.1 verifies the remark about the worst allowable behavior of $\varphi(r)$ mentioned in the introduction. Finally, it follows directly from (7.1) that for $t > o$, $\lim_{r \to o} u_1^{\mu}(r,t;r^{2-\mu-2a}\log r^2)$ exists. Simply set a =o in the integrand. This shows the pole dissipation property for the u_1^{μ} solution.

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