OPERATIONS RESEARCH GROUP

Columbia University

School of Engineering and Applied Science

NEW YORK, N. Y. 10027

Operations Research Group
Columbia University
New York, New York

A PRIMAL METHOD FOR MINIMAL COST FLOWS WITH APPLICATIONS TO THE ASSIGNMENT AND TRANSPORTATION PROBLEMS

by

Morton Klein

Technical Report Number 32
June 1, 1966

This research was supported by the Army, Navy, Air Force and N.A.S.A. under a contract administered by the Office of Naval Research

Contract Number Nonr 266(55)
Task Number NR 042-099

A PRIMAL METHOD FOR MINIMAL COST FLOWS WITH APPLICATIONS TO THE ASSIGNMENT AND TRANSPORTATION PROBLEMS

by

Morton Klein

Columbia University

1. INTRODUCTION

Suppose we have a network with vertices $V = \{1, \ldots, n\}$, directed edges $E = \{(i,j) \in V \times V\}$, a non-negative integral valued function k giving the maximum allowable flow k_{ij} over every edge, and a non-negative cost function a giving the cost a_{ij} associated with a unit of flow over any edge. A flow X of value v is an integral valued function defined on E satisfying

$$(1.1) \qquad 0 \leq x_{ij} \leq k_{ij} \qquad , \quad (i,j) \in E$$

This research is supported by the Army, Navy, Air Force, and N.A.S. under a contract administered by the Office of Naval Research; contract Nonr 266(55). Reproduction in whole or in part is permitted for any purpose of the United States Government.

$$(1.2) \qquad \sum_{j=1}^{n} (x_{ij} - x_{ji}) = \begin{cases} v \; , & i = 1 \\ 0 \; , & i = 2, \ldots, n-1 \\ -v \; , & i = n. \end{cases}$$

Vertices 1 and n are, respectively, called the flow source and sink.

The minimal cost flow problem is to find, among all flows X of value v, one which minimizes

$$(1.3) \qquad Q(X) = \sum_{(i,j) \in E} x_{ij} \, a_{ij} \; .$$

The main computational procedures developed to date for solving this problem are the primal-dual type algorithms of Ford and Fulkerson [6], Busacker and Gowen (described in [3]), and Jewell [9]. These are dual methods in which feasible flows (those satisfying (1.1) and (1.2)) become available when the computations terminate. A similar method for convex cost flow problems has been given by Hu [8]. Fulkerson's "Out-of-Kilter" algorithm (described in [6]) is essentially a primal method in that it can be started with a feasible flow or one becomes available at an early stage. Another primal approach for problems with convex costs has also been suggested recently by Menon [11].

The assignment and transportation problems can be thought of as special minimal cost flow problems in the sense that their networks have a particular bipartite form. Kuhn's "Hungarian Method" [10], a primal-dual type algorithm, and two variants: ([12] and one given in [6]) provide the most popular methods for solving these problems. Other methods are described by Flood [5a] and Hoffman and Markowitz [7]. Primal methods are also available: Dantzig's adaptation of the simplex method (described in [4]), the methods given by Beale [2], Flood [5b], and, most recently, by Balinski and Gomory [1].

The purpose of this paper is to suggest that one more primal method can be added to the above arsenal for both minimal cost flow and assignment-transportation problems. With slight modification this method can also be used for problems involving convex costs. It appears to be efficient in small hand-calculated examples, and has an appealing simplicity in explanation. Also, it shares, with other primal methods, the property that it can be started with a "good" solution and a better one is always available in case early termination of computations is required.

2. A MINIMAL COST FLOW ALGORITHM

In this section we give a procedure for solving the minimal cost flow problem. We assume a familiarity with the maximal flow and shortest route problems together with the Ford-Fulkerson [6] methods for solving them.

The method suggested here is to first find a flow satisfying (1.1) and (1.2) by, say, the Ford and Fulkerson maximum flow routine [6] (pp. 17-18). Given such a flow X, we then construct an associated network G(X) which has the same vertices as the original network and directed edges, as follows:

(2.1) $E(X)$: (i,j) , if $x_{ij} < k_{ij}$ and $x_{ji} = 0$

(j,i) , if $x_{ij} > 0$,

with revised capacities:

(2.2) k' : $k'_{ij} = k_{ij} - x_{ij}$, if $x_{ij} < k_{ij}$ and $x_{ji} = 0$

$k'_{ji} = x_{ij}$, if $x_{ij} > 0$

and with revised edge costs:

(2.3) a' : $a'_{ij} = a_{ij}$, if $x_{ij} < k_{ij}$ and $x_{ji} = 0$

$a'_{ji} = -a_{ij}$, if $x_{ij} > 0$.

These revised edge costs are simply those associated with increasing or cancelling the flow by one unit on these edges: The revised capacities indicate the extent to which this can be done.

Now, we use a result proved in Busacker and Saaty [3] (pp. 256-257).

<u>Theorem</u>: X is a minimal cost flow <u>if and only if</u> there is no directed cycle C in G(X) such that the sum of the costs around C's edges are negative. (A directed cycle is a sequence of distinct directed edges of the form $\{(i_0, i_1)(i_1, i_2) \ldots (i_p, i_q)(i_q, i_0)\}$ involving distinct vertices.)

An immediate consequence of this theorem is that a test of the optimality of the flow X is at hand if G(X) can be checked for the existence of a negative cost directed cycle. Further, if the method also locates such a cycle, one gets an improved flow by simply sending a positive unit flow around this cycle. Such a flow alteration obviously leads to a lower total cost and also leaves the flow value v unchanged.

Fortunately, there are known methods for locating negative cost directed cycles. The Fulkerson "Out-of-Kilter" procedure [6] (pp. 162-169) is one. Various shortest route algorithms may also be used; of these, we shall use the index reduction procedure (with which we are most familiar) described by Ford and Fulkerson [6] (pp. 131-132). As these authors have pointed out, this method requires the assumption of non-negative directed cycle costs. It is easy to show that it breaks down at precisely the point at which such a cycle is encountered; we take advantage of this property.

We need only put the above assortment of results together to obtain the following.

<u>Algorithm for Minimal Cost Flows</u>

1. Use the Ford-Fulkerson maximal flow routine to find a flow X of value v.

2. Form the associated network G(X) according to (2.1), (2.2), and (2.3).

3. Test for the existence of a negative cost cycle C by using, say, the index reduction shortest-route algorithm to find a least cost route from the source to every vertex in G(X). Then, we get either

<u>Case a</u>: there are no negative cost directed cycles in G(X) and the flow X is optimal,

or

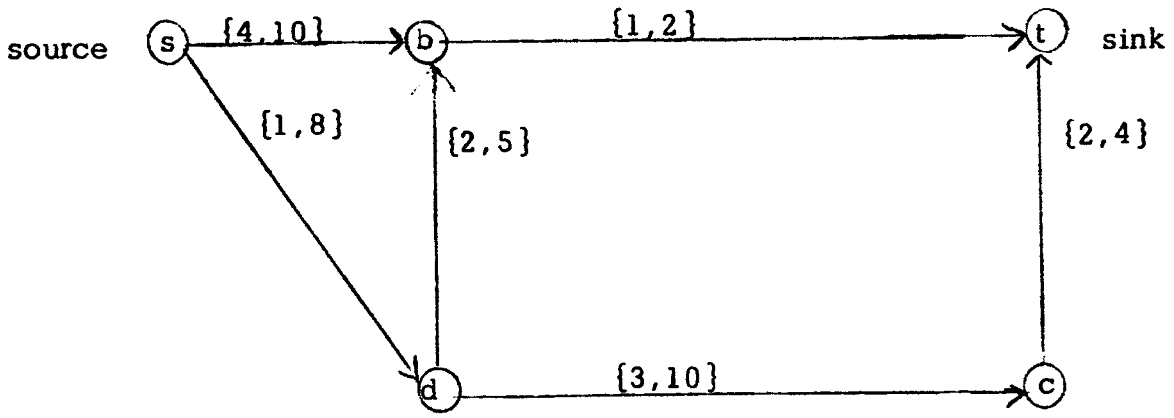
<u>Case b</u>: a negative cost directed cycle C is located and a new flow X' can be determined according to

$$x'_{ij} = \begin{cases} x_{ij} & , \quad \text{if } (i,j) \notin C \\ x_{ij} - \delta & , \quad \text{if } (j,i) \in C \text{ and } a'_{ji} \leq 0 \\ x_{ij} + \delta & ; \quad \text{if } (i,j) \in C \text{ and } a'_{ij} \geq 0 \end{cases}$$

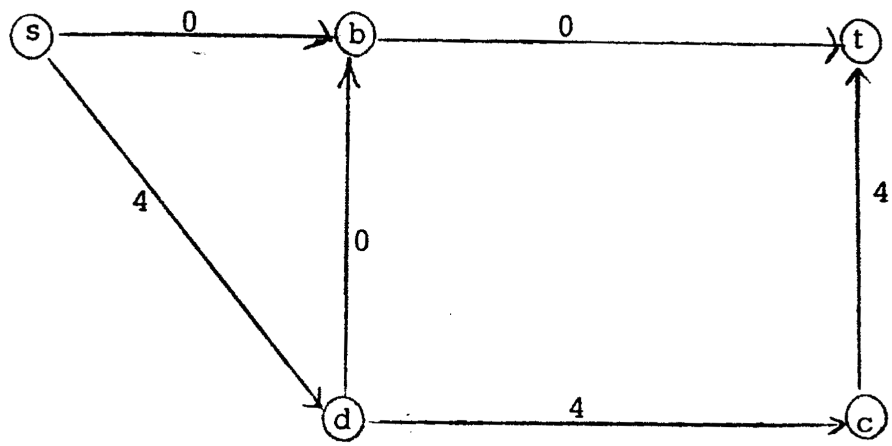
where $\delta = \min_{(i,j) \in C} \{k'_{ij}\}$, i.e., δ is the largest amount of flow which can be sent around C.

Return to Step 2.

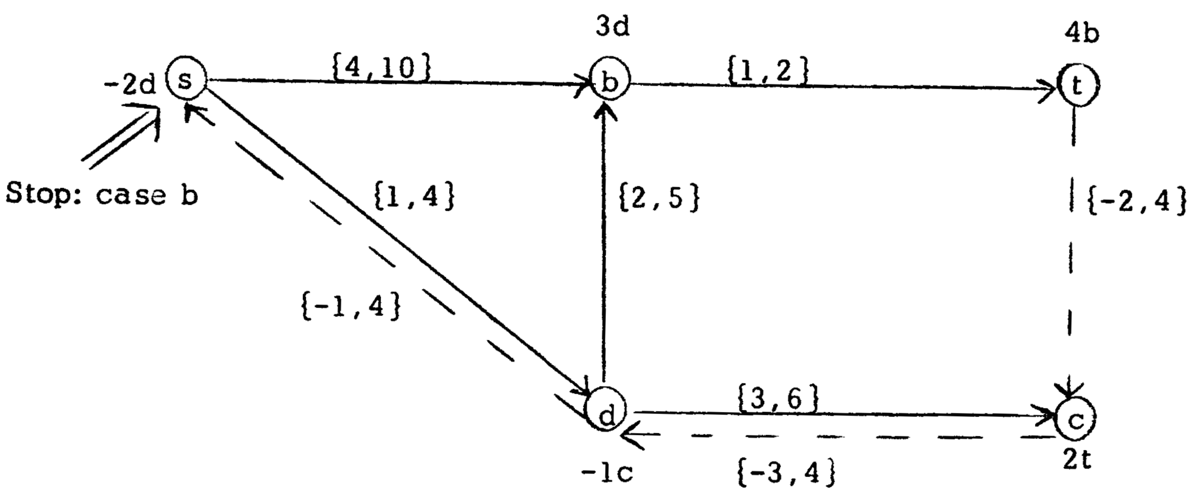
Example: Suppose a minimal cost flow of value 4 is to be imposed on the following network from vertex s to vertex t. Costs and capacities are indicated according to the legend {cost, capacity} on each edge.



We start with an arbitrary flow X of value 4, indicated below, (omitting Step 1 for a problem of this size).



Steps 2-3: G(X) is then



where negative cost edges are indicated by dashed lines and the vertex index reduction shortest-route procedure has yielded the sequence of labels adjoining each vertex. A negative cycle is indicated as soon as the source s receives a negative label (Case b).

A path is then traced backwards from s to locate the cycle; e.g., s, d, c, t, b, d. Thus,

$$C = \{(d,b), (b,t), (t,c), (c,d)\} ,$$

$$\delta = \min \{5, 2, 4, 4\} = 2 ,$$

and the new flow X' is

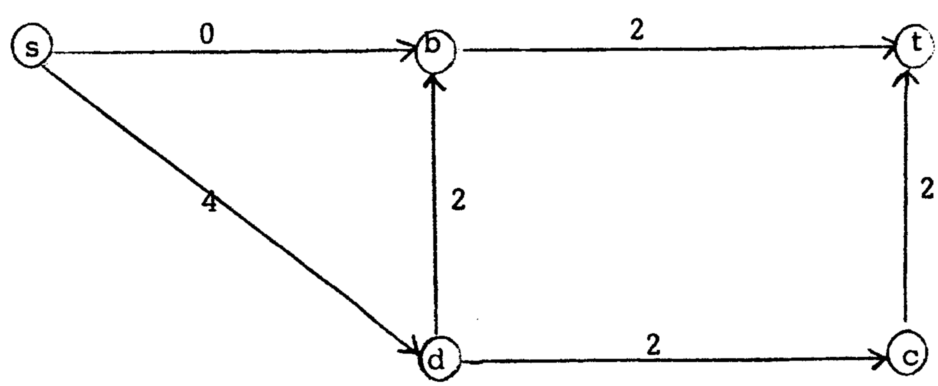
$$x'_{db} = 0 + 2 = 2$$

$$x'_{bt} = 0 + 2 = 2$$

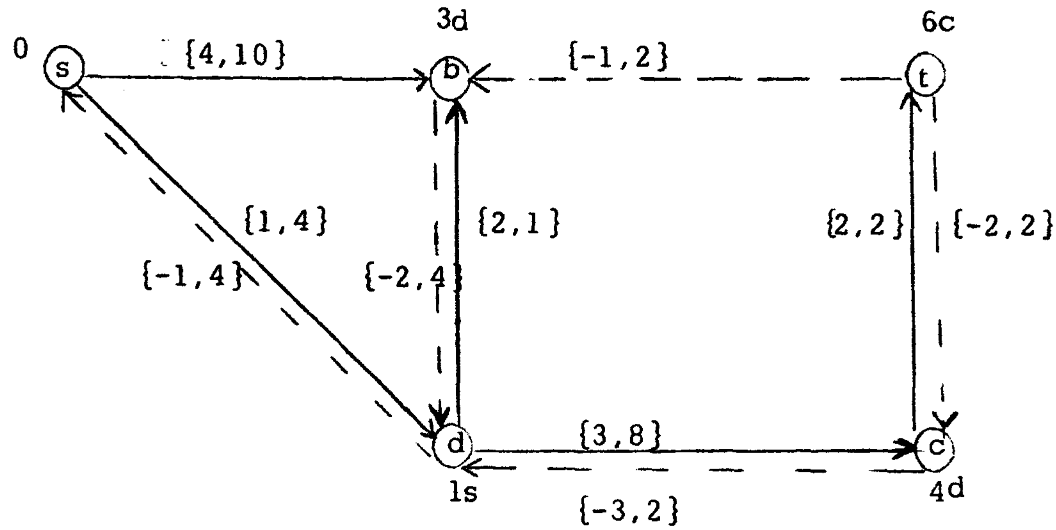
$$x'_{ct} = 4 - 2 = 2$$

$$x'_{dc} = 4 - 2 = 2 .$$

All other edge flow values are unchanged. The new flow is shown below.



Now G(X'), with vertex labels, is



Since the source s has a non-negative label, Case a has been attained and X' is optimal.

3. APPLICATION TO THE ASSIGNMENT PROBLEM

In this section we show how the method described above specializes for solving the assignment problem.

The assignment problem is to fill n jobs by as many men at least total cost. If a_{ij} represents the cost of using man i in job j, then a mathematical statement of the problem is to find a permutation matrix $X = (x_{ij})$ of order n, which minimizes the total cost $Q(X) = \Sigma x_{ij} a_{ij}$.

The equivalent minimum cost flow problem (following Ford and Fulkerson) is illustrated for the case v = n = 3 below in Figure 1.

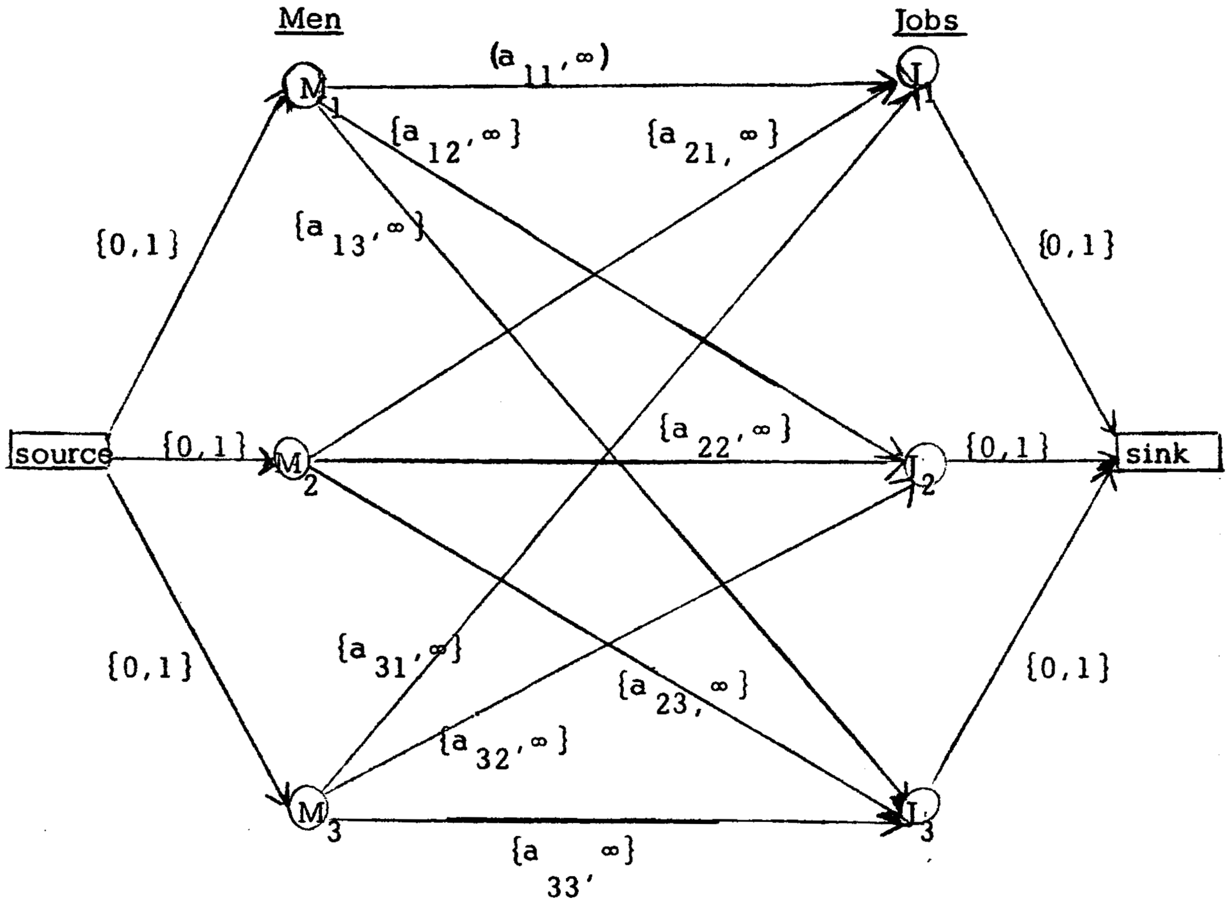


Figure 1: <u>Network Representation of an Assignment Problem</u>

As before, we arbitrarily designate one vertex of the network as the point, called the <u>initial point</u>, from which all least cost routes are computed. Because of the special "bipartite" form of the network (the source and sink can be dropped, since their associated edges have zero cost and unit capacity) calculations are more conveniently made using a matrix.

Let X be a trial solution (e.g., $x_{ii} = 1$, $i=1,\ldots,n$; $x_{ij} = 0$, $i \neq j$) and let $A'(X)$ be an associated matrix with elements

$$a'_{ij} = \begin{cases} -a_{ij}, & \text{if } x_{ij} > 0 \\ a_{ij}, & \text{if } x_{ij} = 0 \end{cases} \qquad (3.1)$$

The index reduction procedure can be carried out by means of a series of alternate row and column labelings on A with each successive row (column) label giving the cost associated with following a traceable route from the initial point to the column (row) from which the current cost is measured.

Let (M_i, d_i) be a label assigned to column j of A. Then the implication is that there is a directed route from the initial point to M_i and a directed edge to (job) column j such that the total cost is d_i. The row labels (J_j, d_j) are defined similarly.

Details of the proposed computational procedure are given below. A numerical example is carried along for illustrative purposes.

A convenient notational device we employ is to write a^-_{ij} if $a'_{ij} \leq 0$ and $x_{ij} > 0$.

<u>Initial Labels</u>

Let M_1 be the initial point. Label row 1 $(-,0)$.

1. Label each column $(M_1, |a'_{1j}|)$
2. Label each row $i \neq 1$, $(J_j, d_j + a^-_{ij})$

<u>Example</u> [1]: Let the first trial solution be $x_{ii}=1$, $i=1,\ldots,4$, $x_{ij}=0$, $i \neq j$, then $A'(X)$ is shown below with the initial labels indicated in the margins.

Jobs

		J_1	J_2	J_3	J_4		
Initial point	M_1	-2	3	1	1	$-,0$	
	M_2	5	-8	3	2	$J_2,-5$	Step 2
Men	M_3	4	9	-5	1	$J_3,-4$	Step 2
	M_4	8	7	8	-4	$J_4,-3$	Step 2
Step 1		$M_1,2$	$M_1,3$	$M_1,1$	$M_1,1$		

Label Reductions

3. Label each column $(M_{\underline{i}}, \min_i (d_i + |a'_{ij}|))$, where $\underline{i}$ is any index at which the minimum is attained

4. Label each row $(J_j, d_j + a^-_{ij})$

5. Continue Steps 3 and 4 until obtaining, either

Case a: Two successive sets of row or column labels are the same, or

Case b: The initial points label becomes negative.

In Case a the trial solution is optimal and the procedure terminates.

If Case b occurs, a negative cost cycle, C, can be found by tracing a path, P, backwards from the initial point according to the succession of adjacent row and column labels until a label is encountered for the second time. Then, an improved trial solution X' is given by

$$(3.2) \qquad x'_{ij} = \begin{cases} x_{ij}, & \text{if } (M_i, J_j) \not\in C \\ 0, & \text{if } |M_i, J_j| \in C \text{ and } x_{ij} = 1 \\ 1, & \text{if } |M_i, J_j| \in C \text{ and } x_{ij} = 0, \end{cases}$$

where the notation $|M_i, J_j|$ indicates that either (M_i, J_j) of (J_j, M_i) is an element of C.

Example Con't:

	J_1	J_2	J_3	J_4		
M_1	-2*	3*	1	1	-, 0	J_1, -2 ← Stop: case b
M_2	5*	-8*	3	2	J_2, -5	
M_3	4	9	-5	1	J_3, -4	
M_4	8	7	8	-4	J_4, -3	↖ Step 4
	M_1, 2	M_1, 3	M_1, 1	M_1, 1		
Step 3→	M_2, 0	M_1, 3	M_2, -2	M_2, -3		

$P = (M_1, J_1, M_2, J_2, M_1)$ and the elements of C are (M_1, J_1), (J_1, M_2), (M_2, J_2), (J_2, M_1), these are marked with asterisks (*) in A'. The new trial solution is, from (3.2)

$x'_{33} = 1$,

$x'_{44} = 1$,

$x'_{12} = 1$,

$x'_{21} = 1$,

all other $x'_{ij} = 0$,

with total cost 17.

The example is continued. The new A' matrix together with the successive row and column labels is

	J_1	J_2	J_3	J_4				
M_1	2	-3	1	1	–, 0	J_2, 0	J_2, -3	← Stop: Case b
M_2	-5*	8	3*	2	J_1, -3	J_1, -5		
M_3	4*	9	-5*	1	J_3, -4	J_3, -5		
M_4	8	7	8	-4	J_4, -3	J_4, -7		
	M_1, 2	M_1, 3	M_1, 1	M_1, 1				
	M_3, 0	M_1, 3	M_2, 0	M_3, -3				
	M_3, -1	M_4, 0	M_2, -2	M_3, -4				

$$P = (M_1, J_2, M_4, J_4, \overbrace{\underbrace{M_3, J_3, M_2, J_1, M_3}})$$

and

$$C = \{(M_3, J_3), (J_3, M_2), (M_2, J_1), (J_1, M_3)\} .$$

Applying (3.2), the next trial solution is

$$x'_{12} = 1 ,$$

$$x'_{44} = 1 ,$$

$$x'_{31} = 1 ,$$

$$x'_{23} = 1 ,$$

all other $x'_{ij} = 0$,

with total cost 14.

The associated A' matrix is

	J_1	J_2	J_3	J_4			
M_1	2*	3*	1	1	–, 0	J_2, 0	J_2, –1 ← Stop: Case b
M_2	5	8	–3	2	J_3, –2	J_3, –2	
M_3	–4*	9	5	1*	J_1, –2	J_1, –2	
M_4	8	7*	8	–4*	J_4, –3	J_4, –5	
	M_1, 2	M_1, 3	M_1, 1	M_1, 1			
	M_1, 2	M_1, 3	M_1, 1	M_3, –1			
	M_1, 2	M_4, 2	M_1, 1	M_3, –1			

$$P = (M_1, J_2, M_4, J_4, M_3, J_1, M_1)$$

and

$$C = \{(M_1, J_2), (J_2, M_4), (M_4, J_4), (J_4, M_3), (M_3, J_1), (J_1, M_1)\}$$

Again, applying (3.2) the new trial solution is

$$x'_{23} = 1 ,$$

$$x'_{11} = 1 ,$$

$$x'_{34} = 1 ,$$

$$x'_{42} = 1 ,$$

$$\text{all other } x'_{ij} = 0 ,$$

with total cost 13.

The associated matrix A' is

	J_1	J_2	J_3	J_4		
M_1	-2	3	1	1	–, 0	J_1, 0
M_2	5	8	-3	2	J_3, -2	J_3, -2
M_3	4	9	5	-1	J_4, 0	J_4, -1
M_4	8	-7	8	4	J_2, -4	J_2, -4
	M_1, 2	M_1, 3	M_1, 1	M_1, 1		
	M_1, 2	M_1, 3	M_1, 1	M_4, 0		
	M_1, 2	M_1, 3	M_1, 1	M_4, 0	← Stop: Case a	

Since Case a has occurred, this trial solution is optimal.

4. AN ALGORITHM FOR THE TRANSPORTATION PROBLEM

The method given here for the transportation problem is, with slight alteration, the same as that suggested for the assignment problem. The reason for this computational similarity is, of course, the well-known relationship between the two problems: each is a special case of the other.

We suppose that the transportation problem involves shipments of a single commodity from n plants $P_1, \ldots, P_n$, with capacities $c_1, \ldots, c_n$ to m warehouses $W_1, \ldots, W_m$ with requirements $r_1, \ldots, r_m$. If a_{ij} represents the cost of shipping a unit from P_i to W_j, and x_{ij} the quantity shipped from P_i to W_j, the problem is to minimize the total cost

$$(4.1) \qquad Q(X) = \sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij} a_{ij}$$

constrained by

$$
\left\{
\begin{array}{ll}
\sum_{i=1}^{n} x_{ij} = r_j & , \quad j=1,\ldots,m \; ; \\
\sum_{j=1}^{m} x_{ij} = c_i & , \quad i=1,\ldots,n \; ; \\
x_{ij} = 0,1,\ldots\ldots, &
\end{array}
\right. \tag{4.2}
$$

where we assume that the c_i's and r_j's are positive integers, and $\Sigma c_i = \Sigma r_j$.

We omit the network formulation of the problem and go directly to the computational procedure. The terminology is the same as that used for the assignment problem except that we speak of plants and warehouses instead of men and jobs.

Trial Solutions: Although the computational procedure can be written so that one can work with any trial solution satisfying (4.2), it is convenient to restrict ourselves to the "basic feasible solutions", possibly degenerate, which are used in Dantzig's adaptation of the simplex method for transportation problems. These solutions contain, at most, m+n-1 positive entries. They also have the property that no cycles can be formed by their positive entries. These two properties may also be justified by simple combinatorial arguments [5b].

Suppose X is a trial solution (i.e., it satisfies (4.2) and the above). We again define the associated matrix $A'(X)$ by

$$a'_{ij} = \begin{cases} -a_{ij}, & \text{if } x_{ij} > 0 \\ a_{ij}, & \text{if } x_{ij} = 0 \end{cases}$$

and write $\bar{a}_{ij}$ if $a'_{ij} \leq 0$ and $x_{ij} > 0$.

Initial Labels

Let P_1 be the initial point. Label its row (-,0)

1. Label column j $(P_1, |a_{1j}|)$
2. Label each row $i \neq 1$ $(W_{\underline{j}}, \min_j \{d_j + \bar{a}_{ij}\})$ where $\underline{j}$ is the column at which the minimum is attained.

Example: Consider the following transportation array [1] and a first trial solution:

	W_1	W_2	W_3	W_4	W_5	
P_1	3	6	3	1	1	4
P_2	2	4	3	2	7	5
P_3	1	1	2	1	2	6
	2	2	3	4	4	

Transportation Array

	W_1	W_2	W_3	W_4	W_5
P_1				2	2
P_2			3		2
P_3	2	2		2	

Trial Solution

(Here we have attempted to find a "good" trial solution by trying to ship as much as possible along the cheapest routes.)

Initial Labels:

		W_1	W_2	W_3	W_4	W_5	
Initial point	P_1	3	6	-3	-1	-1	–, 0
	P_2	2	4	-3	2	-7	W_5, -6
	P_3	-1	-1	2	-1	2	W_4, 0
		$P_{\underline{1}}$, 3	$P_{\underline{1}}$, 6	$P_{\underline{1}}$, 3	$P_{\underline{1}}$, 1	P_1, 1	

Label Changes:

3. Label each column $\left(P_{\underline{i}},\ \min_{i} (d_i + |a'_{ij}|)\right)$

4. Label each row $\left(\underline{W_j},\ \min_j (d_j + a^-_{ij})\right)$

5. Continue steps 3 and 4 until either

   a. two successive sets of row or column labels are the same (in which case the trial solution is optimal), or

   b. the initial point's label becomes negative (indicating the existence of a negative cost directed cycle C).

<u>Example Con't:</u>

	W_1	W_2	W_3	W_4	W_5		
P_1	3	6	3	-1*	-1*	−, 0	W_4, -5 ← Stop: Case b
P_2	2	4	-3	2*	-7*	W_5, -6	
P_3	-1	-1	2	-1	2	W_4, 0	
	P_1, 3	P_1, 6	P_1, 3	P_1, 1	P_1, 1		
	P_2, -4	P_2, -2	P_2, -3	P_2, -4	P_1, 1		

If case b occurs (as in the above) C is located by tracing a path P back from the initial point until a row or column label is encountered for the second time; e.g.,

$$P = (P_1, W_4, P_2, W_5, P_1)$$

and

$$C = \{(P_1, W_4)\ (W_4, P_2)\ (P_2, W_5)\ (W_5, P_1)\}\ \text{(asterisked above)}$$

C is more conveniently written with the subscripts in the usual row, column order. Thus, we write

$$C = \{(P_1, W_4),\ (P_2, W_4),\ (P_2, W_5),\ (P_1, W_5)\}\ .$$

Let X(C) be the entries of X whose subscripts correspond to those of C, i.e., if $|P_i, W_j| \in C$ then $x_{ij} \in X(C)$, and index these entries (with superscripts) as follows: assign the index 1 to any entry whose value is zero. (There is at least one such entry by virtue of our use of "basic feasible" trial solutions.) Now, following the cycle (in either direction) continue with successive positive integers: 2,3,...,k until all elements in X(C) have been indexed. Note that the index k is an even number.

A new improved trial solution X' is defined by

$$x'_{ij} = \begin{cases} x_{ij}\ , & \text{if } x_{ij} \notin X(C) \\ x_{ij} + \delta\ , & \text{if } x_{ij}^{(t)} \in X(C) \text{ and } t \text{ is odd} \\ x_{ij} - \delta\ , & \text{if } x_{ij}^{(t)} \in X(C) \text{ and } t \text{ is even}\ , \end{cases}$$

where δ is the value of the smallest even indexed element of X(C). It is easy to see that X' is also a "basic feasible" solution.

<u>Example Con't</u>: $\delta = 2$ and X' is

	W_1	W_2	W_3	W_4	W_5
P_1				2-2=0 (2)	2+2=4 (3)
P_2			3	0+2=2 (1)	2-2=0 (4)
P_3	2	2		2	

with the index numbers shown. The total cost is 23.

The remaining calculations are given below.

	W_1	W_2	W_3	W_4	W_5		
P_1	3	6	3	1	-1	–, 0	W_5, 0
P_2	2	4	-3	-2	7	W_4, -1	W_4, -1
P_3	-1	-1	2	-1	2	W_4, 0	W_4, 0
	P_1, 3	P_1, 6	P_1, 3	P_1, 1	P_1, 1		
	P_2, 1	P_3, 1	P_3, 2	P_3, 1	P_1, 1		

↑ Stop: Case a

Since case a has occurred, the trial solution is optimal.

5. CONVEX COST FLOWS

As indicated by Hu [8], the primal-dual type algorithms for minimal (linear) cost flow problems can be adapted to handle cases in which the edge cost functions are positive, non-decreasing, and convex. The basic notion, used here also, is to use the (changing) marginal costs associated with possible unit flow alterations in the associated graph. In our case, this means that cyclical flow changes will involve only one unit of flow (i.e., $\delta = 1$) and the edge costs for the associated graph will depend somewhat more on the current flow values on each edge than they did in the linear cost problem.

If we represent each edge cost function by b_{ij}, the problem is to find the minimum of

$$(5.1) \qquad Q(X) = \sum_{(i,j)\in E} b_{ij}(x_{ij})$$

constrained by (1.1) and (1.2).

All we need to do is to note that the network G(X) associated with a feasible flow X, is defined as before, except that the revised (marginal) edge costs b'_{ij} are given by

$$b'_{ij} = \left[b_{ij}(x_{ij}+1) - b_{ij}(x_{ij})\right] \qquad \text{if } x_{ij} < k_{ij} \text{ and } x_{ji}=0;$$

and

$$b'_{ji} = -\left[b_{ij}(x_{ij}) - b_{ij}(x_{ij}-1)\right] \qquad \text{if } x_{ij} > 0 \quad .$$

This, plus our previous observation that $\delta = 1$, enables the use of the minimal cost flow routine. Similar alterations can be made to handle the transportation problem with convex costs.

6. REFERENCES

1. Balinski, M.L. and Gomory, R.E., "A Primal Method for the Assignment and Transportation Problems", Management Science, Vol. 10, No. 3 (1964), pp. 578-593.

2. Beale, E.M.L., "An Algorithm for Solving the Transportation Problem When the Shipping Cost Over Each Route is Convex", Naval Research Logistics Quarterly, Vol. 6, No. 1 (1959), pp. 43-56.

3. Busacker, R.G. and Saaty, T.L., Finite Groups and Networks, McGraw-Hill, New York, N.Y., 1965.

4. Dantzig, G.B., Linear Programming and Extensions, Princeton University Press, Princeton, N.J., 1963.

5a. Flood, M.M., "An Alternative Proof of a Theorem of Konig as an Algorithm for the Hitchcock Distribution Problem", in Combinatorial Analysis, American Mathematical Society, Providence, Rhode Island (1960) pp. 299-307.

b. "On the Hitchcock Distribution Problem", Pacific Journal of Mathematics, Vol. 3, No. 2 (1953) pp. 369-386.

6. Ford, L.R., Jr. and Fulkerson, D.R., Flows in Networks, Princeton University Press, Princeton, N.J., 1962.

7. Hoffman, A.J. and Markowitz, H.M., "A Note on Shortest Path, Assignment, and Transportation Problems", Naval Research Logistics Quarterly, Vol. 10, No. 4 (1963) pp. 375-379.

8. Hu, T.C., "Minimum Convex-Cost Flows in Networks", The RAND Corporation, Memorandum RM-4265-PR, November 1964.

9. Jewell, W.S , "Optimal Flow Through Networks With Gains", Operations Research, Vol. 10, No. 4, (1962) pp. 476-499.

10. Kuhn, H.W., "The Hungarian Method for the Assignment Problem", Naval Research Logistics Quarterly, Vol. 2, Nos. 1 and 2 (1955) pp. 83-97.

11. Menon, V.V., "The Minimum Cost Flow Problem With Convex Costs", Naval Research Logistics Quarterly, Vol. 12, No. 2 (1965) pp. 163-172.

12. Munkres, J., "Algorithms for the Assignment and Transportation Problems", Journal of the Society for Industrial and Applied Mathematics, Vol. 5, No. 1 (1957) pp. 32-38.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Columbia University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A PRIMAL METHOD FOR MINIMAL COST FLOWS WITH APPLICATIONS TO THE ASSIGNMENT AND TRANSPORTATION PROBLEMS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

KLEIN, Morton

6. REPORT DATE June 1966	7a. TOTAL NO. OF PAGES 30	7b. NO. OF REFS 12

8a. CONTRACT OR GRANT NO. Nonr 266(55) b. PROJECT NO. c. Task No. NR 042-099 d.	9a. ORIGINATOR'S REPORT NUMBER(S) Operations Research Group, Columbia University, School of Engineering and Applied Science 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

Releasable without limitations on dissemination.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch, Office of Naval Research, Washington, D.C., 20360

13. ABSTRACT

A simple procedure is given for solving minimal cost flow problems in which feasible flows are maintained throughout. It specializes to give primal algorithms for the assignment and transportation problems. Convex cost problems can also be handled.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Network flows Linear programming Assignment problem Transportation problem Graph theory						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
____________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
____________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
____________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.