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AERO-ASTRONAUTICS REPORT NO. 26

NGR-44-006-045

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N66 34633
(ACCESSION NUMBER) (THRU)
29 (PAGES) 1 (CODE)
CR 77291 (NASA CR OR TMX OR AD NUMBER) 01 (CATEGORY)
FACILITY FORM 602

RICE UNIVERSITY

1966

GPO PRICE $

CFSTI PRICE(S) $

Hard copy (HC) $2.00

Microfiche (MF) .50

ff 653 July 65

THREE-DIMENSIONAL, LIFTING WINGS OF MINIMUM DRAG IN HYPERSONIC FLOW(*)

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ANGELO MIELE(**) and DAVID G. HULL(***)

34633

SUMMARY

The problem of minimizing the drag of a three-dimensional, slender, flat-top wing of given span in hypersonic flow is considered under the assumptions that the pressure coefficient is modified Newtonian and the skin-friction coefficient is constant. The indirect methods of the calculus of variations in two independent variables are employed, and the minimum drag problem is solved for (a) given lift and (b) given lift and volume.

If the lift is the only given quantity, the optimum wing has a constant chordwise slope and a trailing edge thickness distribution similar to the chord distribution. While the planform area is uniquely determined, the chord distribution is not. In

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(*) This research was supported by the Langley Research Center of the National Aeronautics and Space Administration under Grant No. NGR-44-006-045.

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other words, there exist an infinite number of chord distributions yielding the same maximum value of the lift-to-drag ratio.

If the lift and the volume are given, two solutions are possible depending on the value of the volume-lift parameter, a parameter directly proportional to the volume and inversely proportional to the lift squared. If the volume-lift parameter is greater than a certain critical value, the optimum wing is identical with that of case (a). If the volume-lift parameter is smaller than the critical value, the optimum wing has a constant chord and a constant trailing edge thickness. Also, the chordwise slope is constant in the spanwise sense but not in the chordwise sense. Finally, the maximum lift-to-drag ratio decreases as the volume-lift parameter decreases.

1. INTRODUCTION

In previous papers (Refs. 1 through 3), the problem of minimizing the zero-lift drag of a three-dimensional wing in hypersonic flow was considered under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is zero or constant. Various conditions were imposed on the volume, the planform shape, and the thickness distribution on the periphery of the planform.

In this paper, the problem investigated in Refs. 1 through 3 is considered once more in connection with a wing designed to produce a specified lift. For simplicity, the analysis is limited to the class of flat-top wings whose upper surface is parallel to the undisturbed flow direction (Ref. 4). For these wings, the minimal problem consists of extremizing a surface integral with a variable boundary subjected to several constraints of the isoperimetric type.

In the following sections, the necessary conditions for the extremum are derived in general according to the mathematical treatment presented in Chapter 3 of Ref. 5. Then, several particular cases are studied and solved in detail. The hypotheses employed are as follows: (a) a plane of symmetry exists between the left-hand and right-hand parts of the wing; (b) the upper surface is flat; (c) the free-stream velocity is parallel to the line of intersection of the plane of symmetry and the flat top; (d) the wing is slender in both the chordwise and spanwise senses, that is, the squares of both the chordwise and spanwise slopes are small with respect to one; (e) the pressure coefficient is proportional to the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (f) the skin-friction coefficient is constant and equal to some suitably chosen average value; (g) the base drag is neglected;

and (h) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces.

2. FUNDAMENTAL EQUATIONS

In order to relate the drag and the lift of a wing to its geometry, we consider the following Cartesian coordinate system OXYZ (Fig. 1): the origin O is the apex of the wing; the X-axis is the intersection of the plane of symmetry and the flat-top plane, positive toward the trailing edge; the Z-axis is contained in the plane of symmetry, perpendicular to the X-axis, and positive downward; and the Y-axis is such that the XYZ-system is right-handed. We assume that the planform geometry and the thickness distribution on the periphery of the planform are given by (Fig. 2)

$$\begin{aligned} &\underline{\text{Leading edge}} \qquad X = m(Y) \ , \quad Z = 0 \\ &\underline{\text{Trailing edge}} \qquad X = n(Y) \ , \quad Z = t(Y) \end{aligned} \tag{1}$$

where the function m(Y) is arbitrarily prescribed and the functions n(Y), t(Y) are free. We observe that, because of hypotheses (a) through (h), the drag D and the lift L can be written as (Ref. 4)

$$\begin{aligned} D/4nq_\infty &= \int_0^{b/2} \int_{m(Y)}^{n(Y)} (P^3 + K)\, dX dY \\ L/4nq_\infty &= \int_0^{b/2} \int_{m(Y)}^{n(Y)} P^2\, dX dY \end{aligned} \tag{2}$$

where

$$K = C_f/n \tag{3}$$

In the above equations, q_∞ is the free-stream dynamic pressure, b the given wing span, n a factor modifying the Newtonian pressure distribution(*), C_f the skin-friction

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(*) The pressure coefficient employed in Eqs. (2) is $C_p = 2nP^2$.

coefficient, Z(X, Y) the function describing the geometry of the lower surface, and $P \equiv \partial Z/\partial X$ the chordwise slope. The associated volume is given by

$$V/2 = \int_0^{b/2} \int_{m(Y)}^{n(Y)} Z\,dX\,dY \qquad (4)$$

3. COORDINATE TRANSFORMATION

We introduce the coordinate transformation

$$x = X - m(Y) \ , \quad y = Y \ , \quad z = Z \tag{5}$$

in which x denotes a modified abscissa measured from the leading edge along the line defined by y = Const and z = 0. In this new coordinate system, the planform geometry and the thickness distribution on the periphery of the planform are given by (Fig. 2)

$$\begin{array}{lll} \underline{\text{Leading edge}} & x = 0 \ , & z = 0 \\ \underline{\text{Trailing edge}} & x = c(y) \ , & z = t(y) \end{array} \tag{6}$$

where

$$c(y) = n(y) - m(y) \tag{7}$$

denotes the chord distribution and where the functions c(y), t(y) are free. Furthermore, the drag, the lift, and the volume become

$$\begin{aligned} D/4nq_\infty &= \int_0^{b/2}\int_0^{c(y)} (p^3 + K)\,dxdy \\ L/4nq_\infty &= \int_0^{b/2}\int_0^{c(y)} p^2\,dxdy \\ V/2 &= \int_0^{b/2}\int_0^{c(y)} z\,dxdy \end{aligned} \tag{8}$$

where b is the given wing span, z(x, y) the function describing the geometry of the lower surface, and $p \equiv \partial z/\partial x$ the chordwise slope.

4. MINIMUM DRAG PROBLEM

At this point, we formulate the following problem: "In the class of functions $z(x, y)$ which satisfy the isoperimetric constraints (8-2) and (8-3) as well as the boundary conditions (6), find that particular function which minimizes the integral (8-1)." This is a problem of the isoperimetric type with a variable boundary and involves the independent variables x, y and the dependent variable z. According to Lagrange multiplier theory (see, for instance, Chapter 3 of Ref. 5), we study the minimization of the functional form

$$I = \int_0^{b/2} \int_0^{c(y)} F(z, p, \lambda_1, \lambda_2)\, dx\, dy \tag{9}$$

subject to the constraints (8-2) and (8-3) as well as the boundary conditions (6) with the understanding that the fundamental function is defined as

$$F = p^3 + K - \lambda_1 p^2 + \lambda_2 z \tag{10}$$

where λ_1 and λ_2 denote constant Lagrange multipliers. This fundamental function is characterized by the first partial derivatives

$$F_p = 3p^2 - 2\lambda_1 p \ , \quad F_q = 0 \ , \quad F_z = \lambda_2 \tag{11}$$

and the second partial derivatives

$$\begin{aligned} &F_{pp} = 2(3p - \lambda_1) \ , \quad F_{qq} = 0 \ , \quad F_{pq} = 0 \\ &F_{zz} = 0 \ , \quad F_{zp} = 0 \ , \quad F_{zq} = 0 \end{aligned} \tag{12}$$

in which $p \equiv \partial z/\partial x$ and $q \equiv \partial z/\partial y$ denote the chordwise slope and the spanwise slope of the extremal surface.

5. NECESSARY CONDITIONS

The function z(x, y) extremizing the integral (9) must be a solution of the Euler equation

$$\partial F_p/\partial x - F_z = 0 \tag{13}$$

which, in the light of Eqs. (11), can be rewritten as

$$\partial(3p^2 - 2\lambda_1 p)/\partial x - \lambda_2 = 0 \tag{14}$$

Therefore, upon integrating in the x-direction, we see that the following first integral is valid:

$$3p^2 - 2\lambda_1 p - \lambda_2 x = f(y) \tag{15}$$

where f(y) is an arbitrary function of the spanwise coordinate.

The boundary conditions for the Euler equation are partly of the prescribed type and partly of the natural type. The latter are to be determined from the transversality condition

$$(F - pF_p)\delta x - (\dot{x}F + qF_p)\,\delta y + F_p\delta z = 0 \tag{16}$$

($\dot{x}$ denotes the derivative dx/dy evaluated on the boundary) which must be satisfied for every set of variations consistent with the conditions imposed on the planform shape and the thickness distribution on the periphery of the planform. For the leading edge, the following relations hold:

$$\delta x = \delta z = 0 \quad , \quad \dot{x} = \dot{z} = 0 \tag{17}$$

($\dot{z}$ denotes the derivative dz/dy evaluated on the boundary) so that Eq. (16) is satisfied. For the trailing edge, Eq. (16) is satisfied for every set of variations providing

$$F - pF_p = 0 \quad , \quad \dot{x}F + qF_p = 0 \quad , \quad F_p = 0 \tag{18}$$

that is, providing

$$F = 0 \quad , \quad F_p = 0 \tag{19}$$

Because of Eqs. (10) and (11), the conditions (19) take the form

<u>Trailing edge</u> $$p^3 + K - \lambda_1 p^2 + \lambda_2 t(y) = 0 \quad , \quad 3p^2 - 2\lambda_1 p = 0 \tag{20}$$

Once the solution of the Euler equation is obtained, it is necessary to verify that it minimizes the functional (9). In this connection, the Legendre condition

$$F_{pp} \geq 0 \tag{21}$$

must be satisfied and ensures a relative minimum with respect to weak variations. Because of Eq. (12-1), its explicit form is

$$3p - \lambda_1 \geq 0 \tag{22}$$

If strong variations of the slope are considered, the Legendre condition is to be replaced by the Weierstrass condition

$$F(z, p_c, \lambda_1, \lambda_2) - F(z, p, \lambda_1, \lambda_2) - F_p(z, p, \lambda_1, \lambda_2)(p_c - p) \geq 0 \tag{23}$$

where z and p are the ordinate and the slope of the extremal surface and p_c is the slope of the comparison surface. The explicit form of this inequality

$$(p_c - p)^2(p_c + 2p - \lambda_1) \geq 0 \tag{24}$$

must hold for every comparison slope consistent with the constraint

$$p_c \geq 0 \tag{25}$$

which expresses the limit of validity of the Newtonian pressure law. Consequently, the following inequality must be satisfied at each point of the extremal surface:

$$2p - \lambda_1 \geq 0 \tag{26}$$

and is more restrictive than Ineq. (22).

6. NONDIMENSIONAL QUANTITIES

In order to simplify the representation of the results for the particular cases, it is convenient to introduce the nondimensional coordinates

$$\xi = x/c(y) \quad , \quad \eta = 2y/b \quad , \quad \zeta = z/t(y) \tag{27}$$

and the nondimensional chord distribution

$$\gamma = c(y)/c(0) \tag{28}$$

Also, we define the thickness ratio of the root airfoil and the lift-to-drag ratio

$$\tau = t(0)/c(0) \quad , \quad E = L/D \tag{29}$$

as well as the parameters

$$\begin{aligned} \tau_* &= \tau K^{-1/3} \\ E_* &= E K^{1/3} \\ c_* &= c(0)\, b(nq_\infty/L)\, K^{2/3} \\ V_* &= Vb(nq_\infty/L)^2 K \end{aligned} \tag{30}$$

7. GIVEN LIFT

If the lift is prescribed while the volume is free, the relationship $\lambda_2 = 0$ holds so that the extremal surface is described by the first integral

$$3p^2 - 2\lambda_1 p = f(y) \tag{31}$$

the fixed end conditions (6-1), and the natural boundary conditions

Trailing edge $$p^3 + K - \lambda_1 p^2 = 0 \ , \quad 3p^2 - 2\lambda_1 p = 0 \tag{32}$$

Equations (32) admit the solutions

$$\lambda_1 = 3 \sqrt[3]{K/4} \tag{33}$$

and

Trailing edge $$p = \sqrt[3]{2K} \tag{34}$$

indicating that the chordwise slope is constant over the trailing edge. By combining Eqs. (31) and (32-2), we see that

$$f(y) = 0 \tag{35}$$

meaning that the chordwise slope is also constant over the entire extremal surface. As a consequence, the optimum wing is described by the partial differential equation

$$p = \sqrt[3]{2K} \tag{36}$$

which, in the light of the conditions (6-1), admits the particular integral

$$z = \sqrt[3]{2K}\ x \tag{37}$$

Next, the conditions (6-2) are applied to obtain the relationship

$$t(y) = \sqrt[3]{2K}\ c(y) \tag{38}$$

meaning that the trailing edge thickness distribution and the chord distribution are similar. Then, by forming the ratio of the above equations and introducing the dimensionless coordinates (27), we conclude that the optimum wing surface is given by

$$\zeta = \xi \tag{39}$$

Finally, the evaluation of the integrals (8) leads to

$$\begin{aligned} D &= 6nKq_\infty bc(o) \int_0^1 \gamma\, d\eta \\ L &= 2n(2K)^{2/3} q_\infty bc(o) \int_0^1 \gamma\, d\eta \\ V &= (K/4)^{1/3} bc^2(0) \int_0^1 \gamma^2\, d\eta \end{aligned} \tag{40}$$

so that, because of Eqs. (30) and (38), the optimum wing is characterized by the parameters

$$\begin{aligned} \tau_* &= \sqrt[3]{2} \\ E_* &= \sqrt[3]{4}/3 \\ c_* &= \left[2\sqrt[3]{4} \int_0^1 \gamma\, d\eta\right]^{-1} \\ V_* &= (1/16) \int_0^1 \gamma^2\, d\eta \left[\int_0^1 \gamma\, d\eta\right]^{-2} \end{aligned} \tag{41}$$

Equations (41-1) and (41-2) show that the thickness ratio and the lift-to-drag ratio of the extremal solution are uniquely determined. On the other hand, Eqs. (41-3) and (41-4) show that the chord distribution and the volume are not uniquely determined; in other words, there exist an infinite number of wings having the same maximum value of the lift-to-drag ratio (41-2).

8. GIVEN LIFT AND VOLUME

If the lift and the volume are given, the extremal solution is governed by the first integral

$$3p^2 - 2\lambda_1 p - \lambda_2 x = f(y) \tag{42}$$

the fixed end conditions (6-1), and the natural boundary conditions

Trailing edge $\quad p^3 + K - \lambda_1 p^2 + \lambda_2 t(y) = 0 \quad , \quad 3p^2 - 2\lambda_1 p = 0 \qquad (43)$

The analysis shows that these equations admit the following classes of solutions:

Class I $\quad \lambda_2 = 0$

Class II $\quad \lambda_2 \geq 0 \qquad (44)$

Solutions of Class I. For these solutions, which are characterized by $\lambda_2 = 0$, Eqs. (42) and (43) reduce to Eqs. (31) and (32). As a consequence, Eqs. (33) through (41) are valid here. Once more, the thickness ratio and the lift-to-drag ratio are uniquely determined while the chord distribution is not.

In order to determine the range of applicability of these solutions, we observe that the volume-lift parameter V_* is a known quantity. On the other hand, we can formulate an auxiliary extremal problem, that of minimizing the right-hand side of Eq. (41-4) conceived as a product of powers of integrals subjected to the initial condition $\gamma(0) = 1$. If this is done and if the theory of Ref. 6 is applied, we see that the extremum occurs for $\gamma = 1$ and that the minimum value of the volume-lift parameter is $V_* = 1/16$. In the light of this result, we conclude that the solutions of Class I are valid providing

$$V_* \geq 1/16 \tag{45}$$

Solutions of Class II. For these solutions, the natural boundary conditions (43) can be solved in the form

Trailing edge $$p = (2/3)\lambda_1 \quad , \quad t(y) = (1/\lambda_2)(4\lambda_1^3/27 - K) \tag{46}$$

indicating that the chordwise slope and the thickness are constant along the trailing edge.

If the first integral (42) is applied at the trailing edge and is combined with Eq. (46-1), it is seen that

$$f(y) = -\lambda_2 c(y) \tag{47}$$

and that

$$3p^2 - 2\lambda_1 p + \lambda_2(c - x) = 0 \tag{48}$$

This is an algebraic equation of the second degree in p which--in the light of the Legendre condition (22)--admits the solution

$$p = \lambda_1/3 + (1/3)\left[\lambda_1^2 - 3\lambda_2(c - x)\right]^{1/2} \tag{49}$$

If this partial differential equation is integrated in the x-direction and the conditions (6-1) are imposed, we obtain the relationship

$$z = (\lambda_1/3)x - (2/27\lambda_2)\left\{\left[\lambda_1^2 - 3\lambda_2 c\right]^{3/2} - \left[\lambda_1^2 - 3\lambda_2(c-x)\right]^{3/2}\right\} \tag{50}$$

which, at the trailing edge, becomes

$$t = (\lambda_1/3)c - (2/27\lambda_2)\left[(\lambda_1^2 - 3\lambda_2 c)^{3/2} - \lambda_1^3\right] \tag{51}$$

Since the trailing edge thickness is constant, Eq. (51) implies that the optimum wing has a constant chord. Therefore, if the following definitions are introduced:

$$\alpha = 3\lambda_2 c/\lambda_1^2 \tag{52}$$

and

$$G(\xi,\alpha) = 3\alpha\xi - 2\left\{[1 - \alpha]^{3/2} - [1 - \alpha(1 - \xi)]^{3/2}\right\} \tag{53}$$

Eqs. (50) and (51) can be rewritten as

$$\begin{aligned} z &= (\lambda_1^3/27\lambda_2)\, G(\xi,\alpha) \\ t &= (\lambda_1^3/27\lambda_2)\, G(1,\alpha) \end{aligned} \tag{54}$$

Consequently, in nondimensional form, the optimum shape is described by

$$\zeta = \frac{G(\xi,\alpha)}{G(1,\alpha)} \tag{55}$$

The next step is to relate the quantity α to the prescribed values of the lift and the volume as well as to determine the scaling factors t and c. To do this, we combine Eqs. (46-2) and (54-2) to obtain the relations

$$\lambda_1/\sqrt[3]{K} = 3[4 - G(1,\alpha)]^{-1/3}$$

$$\tau_* = [G(1,\alpha)/3\alpha]\,[4 - G(1,\alpha)]^{-1/3} \tag{56}$$

Furthermore, upon combining the shape equation (55) with the expressions (8) for the drag, the lift, and the volume and introducing the following definitions:

$$A(\alpha) = 2[4 - G(1,\alpha)]^{-1}\left\{4 - 3\alpha/2 + (2/5\alpha)[6 - (6 - \alpha)(1 - \alpha)^{3/2}]\right\} + 2$$

$$B(\alpha) = 2[4 - G(1,\alpha)]^{-2/3}\left\{2 - \alpha/2 + (4/3\alpha)[1 - (1 - \alpha)^{3/2}]\right\} \tag{57}$$

$$C(\alpha) = [4 - G(1,\alpha)]^{-1/3}\left\{1/2 - (2/15\alpha^2)[(2 + 3\alpha)(1 - \alpha)^{3/2} - 2]\right\}$$

we deduce that

$$D/nq_\infty bcK = A(\alpha)$$

$$L/nq_\infty bcK^{2/3} = B(\alpha) \tag{58}$$

$$V/bc^2K^{1/3} = C(\alpha)$$

so that

$$E_* = B(\alpha)/A(\alpha)$$

$$c_* = 1/B(\alpha) \tag{59}$$

$$V_* = C(\alpha)/B^2(\alpha)$$

The final step consists of eliminating the quantity α between Eqs. (55), (56-2), and (59) to obtain the relationships

$$\zeta = f_1(\xi, V_*) \tag{60}$$

and

$$\tau_* = f_2(V_*) \quad , \quad c_* = f_3(V_*) \quad , \quad E_* = f_4(V_*) \tag{61}$$

which are plotted in Figs. 3 through 6 and are valid providing

$$V_* \leq 1/16 \tag{62}$$

Remark. In the previous analysis, it has been assumed that the span b is prescribed. Should the span be free, all of the solutions would be of Class I and would be characterized by the thickness ratio (41-1) and the lift-to-drag ratio (41-2).

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LIST OF CAPTIONS

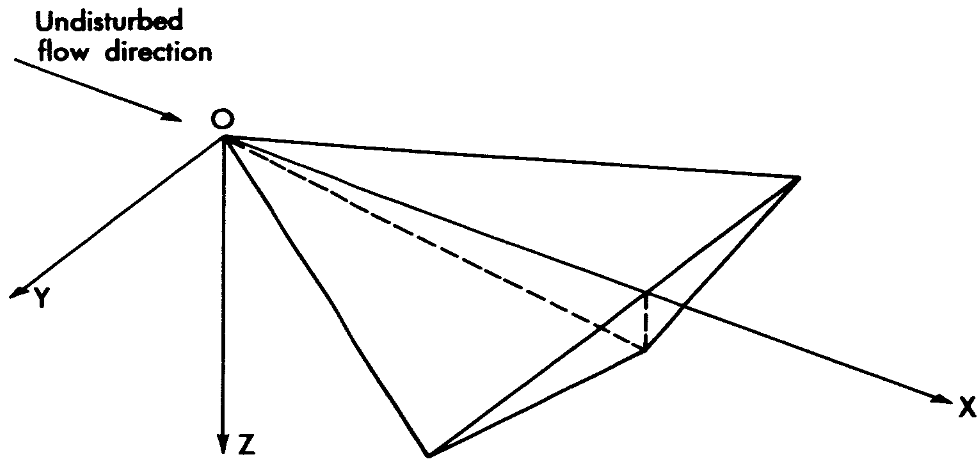

Fig. 1 Coordinate system.

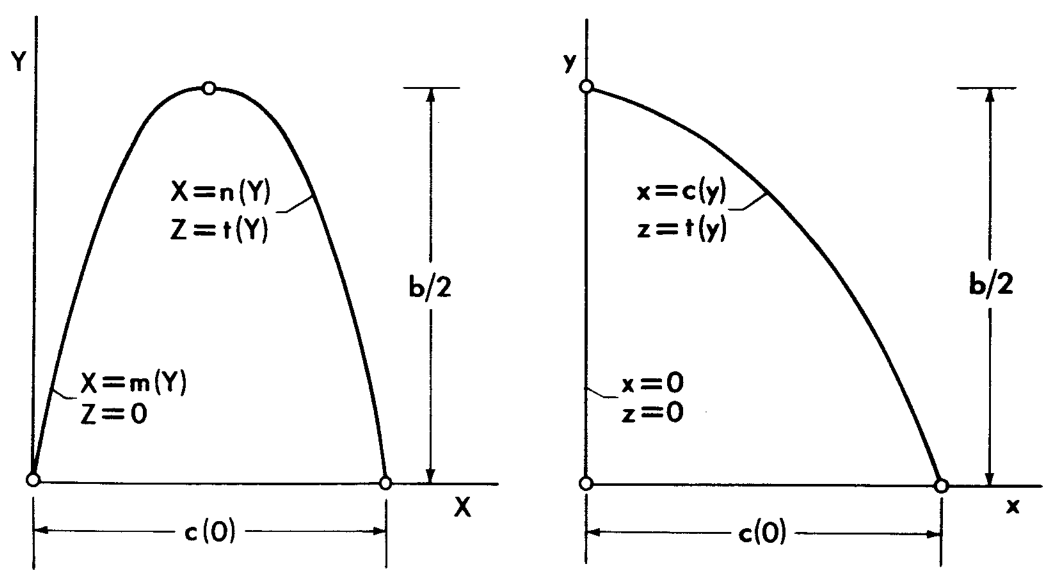

Fig. 2 Planform geometry.

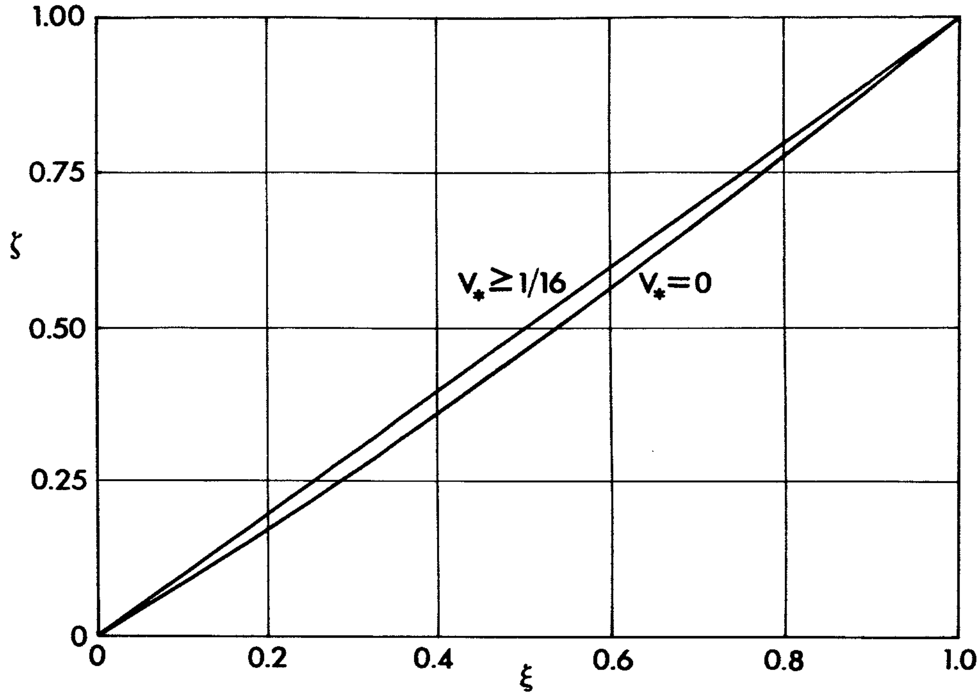

Fig. 3 Optimum shape.

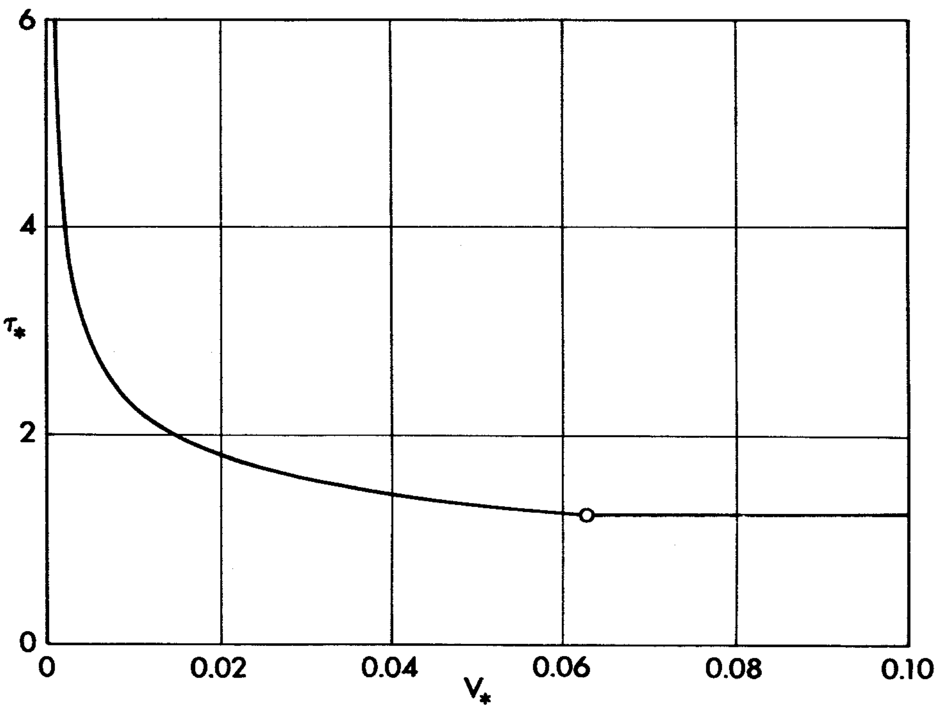

Fig. 4 Optimum thickness ratio.

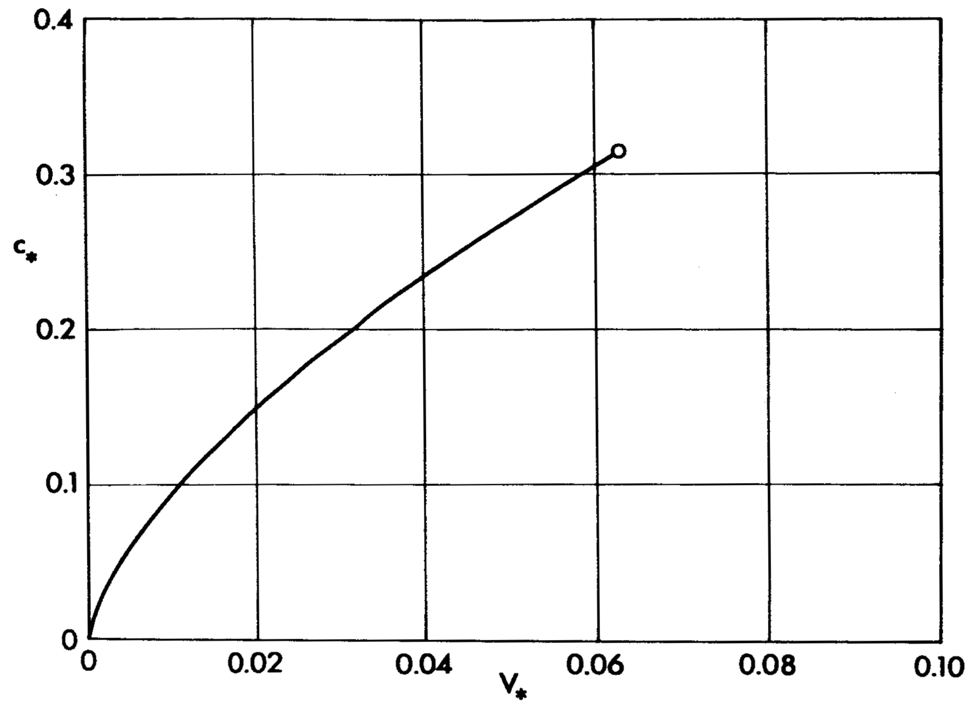

Fig. 5 Optimum chord.

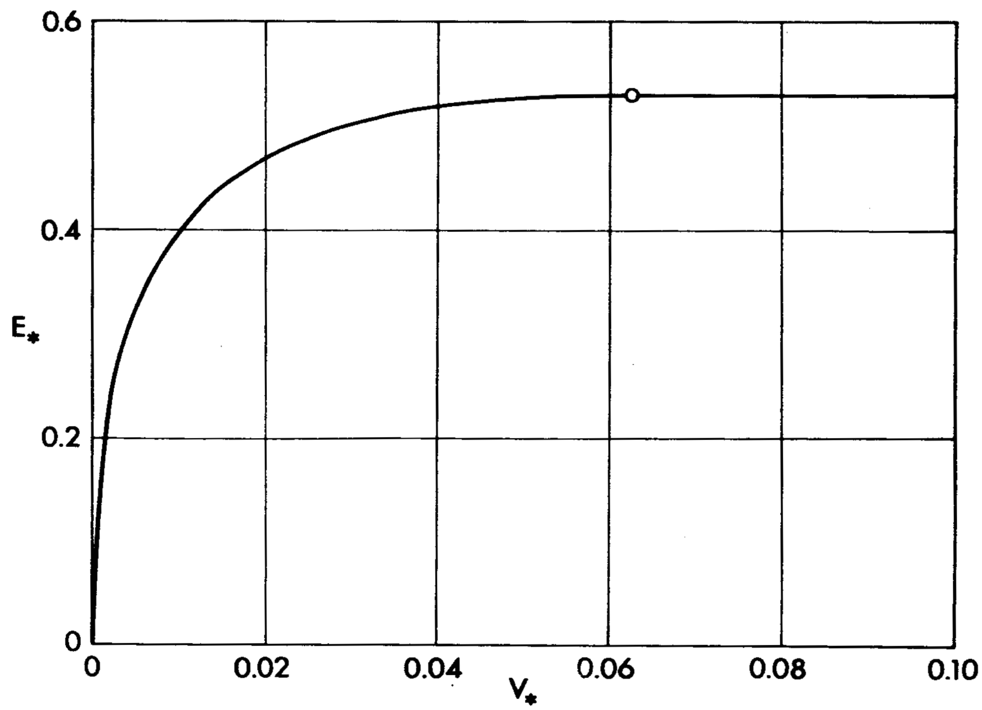

Fig. 6 Maximum lift-to-drag ratio.