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REFLECTION OF PLANE WAVES FROM THE FLAT BOUNDARY OF A MICROPOLAR ELASTIC HALFSPACE

by

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REFLECTION OF PLANE WAVES FROM THE FLAT BOUNDARY OF A MICROPOLAR ELASTIC HALFSPACE

Abstract

This paper is concerned with the investigation of the effect of microstructure in the solution of several problems of wave propagation. The propagation of plane waves in a micropolar elastic half-space and their reflections from a stress free flat surface are studied. It has been found that in a micropolar elastic solid six waves can exist traveling at four distinct speeds, three of which disappear below a critical frequency dependent upon the character of the medium. Reflection laws and amplitude ratios are presented for three specific problems.

Author

INTRODUCTION

The classical elasticity theory is believed to be inadequate for the treatment of deformations and motions of a material possessing granular structure. In particular, the affect of granular, or microstructure, becomes important in transmitting waves of small wavelength and/or high frequency. When the wavelength is comparable with the average grain size the motion of the grains must be taken into account. This introduces new types of waves not encountered in the classical theory.

The present paper is an attempt to study the effect of microstructure in the solution of several problems concerning wave propagations. To this end we use the theory of micropolar elasticity developed in a series of papers by Eringen and his coworkers [1] to [3].

The basic difference between the theory of micropolar elasticity and that of classical elasticity is the introduction of an independent microrotation vector. In classical elasticity all other quantities can be obtained from a knowledge of the three components of the displacement vector. In micropolar elasticity, in addition, we must have knowledge of the three components of the microrotation vector. The development of the general theory shows that such solids can support couple stresses and may be affected by the spin inertia.

In Chapter 1 we present a resume of the basic equations of micropolar elasticity necessary for the analysis of wave motion. A complete derivation and discussion of these equations were given in 1964 by Eringen and Suhubi [1], [2]. Recently Eringen [3] has recapitulated the micropolar elasticity and studied various questions on stability and uniqueness of the solutions of static and dynamic boundary value problems.

Using this theory we determine the types and speeds of plane waves in an infinite micropolar elastic solid. New dispersive microrotational waves are found in addition to those similar to the classical ones. The dispersion relations are discussed in detail resulting in several inequalities among the constitutive coefficients. We also find a cutoff frequency for several of these new waves below which they degenerate to a vibratory motion of the medium.

Chapter 2 is devoted to a discussion of reflection of plane, longitudinal displacement waves from a flat free surface. Reflection angles and amplitude ratios are obtained. Certain special cases are studied in detail and a few typical curves are sketched. In Chapter 3 the same program is carried out for the reflection of coupled transverse shear and microrotational waves and in Chapter 4 for those of longitudinal microrotational waves. Chapters 2 and 4 also contain an analysis of the limiting case when the incident waves are grazing parallel to the boundary. The first limiting analysis in Chapter 2 is similar to that presented by Goodier and Bishop [4] for classical elastic waves (see also Ewing, Jardetsky and Press [5]). An experimental verification of the findings of Goodier and Bishop has not been made though Kólsky [6] states that the generation of a transverse wave when a longitudinal wave runs parallel to a free surface has been observed experimentally. In Chapters 2 and 4 we also use a limiting analysis due to F. C. Roesler [7] the results of which in the classical case have been verified experimentally by D. G. Christie [8]. The content of Chapters 2, 3 and 4 is believed to be new.

CHAPTER I

PLANE WAVES IN AN INFINITE MICROPOLAR ELASTIC SOLID

This chapter is concerned with the discussion of the propagation of plane waves in a microelastic solid. The analysis is based on Eringen's theory of micropolar elasticity [2], [3]. Brief accounts on the subject are also to be found in an independent paper by Pal'mov [9] and one by Mindlin [10]. While the present work has certain similarities to the first of these papers, it differs from the second and, in fact, some of the results are in direct contradiction.

Basic Equations

Eringen's theory of micropolar elasticity is based upon the following equations:

Balance of momentum:

$$t_{k\ell,k} + \rho(f_\ell - \ddot{u}_\ell) = 0 \tag{1.1}$$

Balance of moment of momentum:

$$m_{rk,r} + \epsilon_{k\ell r}\, t_{\ell r} + \rho(\ell_k - j\, \ddot{\varphi}_k) = 0 \tag{1.2}$$

Conservation of energy:

$$\rho\, \dot{\epsilon} = t_{k\ell}(v_{\ell,k} - \epsilon_{k\ell r}\, \nu_r) + m_{k\ell}\, \nu_{\ell,k} \tag{1.3}$$

Constitutive equations:

$$t_{k\ell} = \lambda\, u_{r,r}\, \delta_{k\ell} + \mu(u_{k,\ell} + u_{\ell,k}) + \kappa(u_{\ell,k} - \epsilon_{k\ell r}\, \varphi_r) \tag{1.4}$$

$$m_{k\ell} = \alpha\, \varphi_{r,r}\, \delta_{k\ell} + \beta\, \varphi_{k,\ell} + \gamma\, \varphi_{\ell,k} \tag{1.5}$$

where λ , μ , κ , α , β and γ are the material moduli and

$t_{k\ell}$ ≡ stress tensor, $m_{k\ell}$ ≡ couple stress tensor

ρ ≡ density, f_ℓ ≡ body force

u_k ≡ displacement vector, φ_k ≡ microrotation vector

ℓ_k ≡ body couple, j ≡ microinertia

ϵ ≡ internal energy density, $v_k \equiv \dot{u}_k$, $\nu_k \equiv \dot{\varphi}_k$

$\epsilon_{k\ell m}$ ≡ permutation symbol ($\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{132} = -\epsilon_{321} = -\epsilon_{213} = 1$ and all other $\epsilon_{k\ell m} = 0$)

$\delta_{k\ell}$ ≡ Kronecker delta (= 1 when $k = \ell$ and zero otherwise)

Here we employ rectangular coordinates x_k ($k = 1,2,3$) or ($x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$) and the usual summation convention on repeated indices. Also indices following a comma indicate partial differentiation and a superposed dot indicates the time rate, e.g.,

$$u_{k,\ell} \equiv \frac{\partial u_k}{\partial x_\ell} \quad , \quad \dot{u}_k \equiv \frac{\partial u_k}{\partial t} \tag{1.6}$$

Eringen [3] has shown that the following inequalities among the material moduli are necessary and sufficient for the internal energy to be non-negative

$$\begin{array}{lll} 0 \leq 3\lambda + 2\mu + \kappa & 0 \leq \mu & 0 \leq \kappa \\ 0 \leq 3\alpha + 2\gamma & -\gamma \leq \beta \leq \gamma & 0 \leq \gamma \end{array} \tag{1.7}$$

Upon substituting (1.4) and (1.5) into (1.1) and (1.2), respectively, we obtain the field equations of the theory

$$(\lambda + \mu)\, u_{\ell,\ell k} + (\mu + \kappa)\, u_{k,\ell\ell} + \kappa\, \epsilon_{k\ell m}\, \varphi_{m,\ell} - \rho\, \ddot{u}_k = 0 \tag{1.8}$$

$$(\alpha + \beta)\, \varphi_{\ell,\ell k} + \gamma\, \varphi_{k,\ell\ell} + \kappa\, \epsilon_{k\ell m}\, u_{m,\ell} - 2\kappa\, \varphi_k - \rho j \ddot{\varphi}_k = 0 \qquad (1.9)$$

where we have also set $\mathfrak{f} = \boldsymbol{\ell} = 0$.

The boundary conditions on tractions and couples at a point of the surface $\mathcal{S}$ of the body $\mathcal{V} + \mathcal{S}$ are expressed as

$$\left.\begin{aligned} t_{k\ell}\, n_k &= t_\ell \\ m_{k\ell}\, n_k &= m_\ell \end{aligned}\right\} \text{ on } \mathcal{S}$$

where t_ℓ and m_ℓ are respectively the surface tractions and surface couples prescribed on $\mathcal{S}$ and n_k is the exterior normal to $\mathcal{S}$. With the use of (1.4) and (1.5) these read

$$\lambda\, u_{r,r}\, n_\ell + \mu(u_{k,\ell} + u_{\ell,k})n_k + \kappa(u_{\ell,k} - \epsilon_{k\ell m}\, \varphi_m)n_k = t_\ell \qquad (1.10)$$

$$\alpha\, \varphi_{r,r}\, n_\ell + \beta\, \varphi_{k,\ell}\, n_k + \gamma\, \varphi_{\ell,k}\, n_k = m_\ell \qquad (1.11)$$

It is convenient to express the field equations (1.8) and (1.9) in vector form

$$(c_1^2 + c_3^2)\, \nabla(\nabla\cdot \underset{\sim}{u}) - (c_2^2 + c_3^2)\, \nabla \times (\nabla \times \underset{\sim}{u}) + c_3^2\, \nabla \times \underset{\sim}{\varphi} = \ddot{\underset{\sim}{u}} \qquad (1.12)$$

$$(c_4^2 + c_5^2)\, \nabla(\nabla\cdot \underset{\sim}{\varphi}) - c_4^2\, \nabla \times (\nabla \times \underset{\sim}{\varphi}) + \omega_0^2\, \nabla \times \underset{\sim}{u} - 2\omega_0^2\, \underset{\sim}{\varphi} = \ddot{\underset{\sim}{\varphi}} \qquad (1.13)$$

where

$$\begin{aligned} c_1^2 &= \frac{\lambda + 2\mu}{\rho} & c_2^2 &= \frac{\mu}{\rho} & c_3^2 &= \frac{\kappa}{\rho} \\ c_4^2 &= \frac{\gamma}{\rho j} & c_5^2 &= \frac{\alpha + \beta}{\rho\, j} & \omega_0^2 &= \frac{c_3^2}{j} = \frac{\kappa}{\rho j} \end{aligned} \qquad (1.14)$$

We now decompose the vectors $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ into scalar and vector potentials as follows

$$\underset{\sim}{u} = \nabla \bar{u} + \nabla \times \underset{\sim}{U} \qquad \nabla \cdot \underset{\sim}{U} = 0 \tag{1.15}$$

$$\underset{\sim}{\varphi} = \nabla \bar{\varphi} + \nabla \times \underset{\sim}{\Phi} \qquad \nabla \cdot \underset{\sim}{\Phi} = 0 \tag{1.16}$$

Introduction of these potentials into equations (1.12) and (1.13) yields

$$\nabla[(c_1^2 + c_3^2)\nabla^2 \bar{u} - \ddot{\bar{u}}] - \nabla \times [(c_2^2 + c_3^2) \nabla \times (\nabla \times \underset{\sim}{U}) - c_3^2 \nabla \times \underset{\sim}{\Phi} + \ddot{\underset{\sim}{U}}] = 0 \tag{1.17}$$

$$\nabla[(c_4^2 + c_5^2) \nabla^2 \bar{\varphi} - 2\omega_0^2 \bar{\varphi} - \ddot{\bar{\varphi}}] - \nabla \times [c_4^2 \nabla \times (\nabla \times \underset{\sim}{\Phi}) - \omega_0^2 \nabla \times \underset{\sim}{U} + 2\omega_0^2 \underset{\sim}{\Phi} + \ddot{\underset{\sim}{\Phi}}] = 0 \tag{1.18}$$

These equations may be expressed as

$$\nabla a + \nabla \times \underset{\sim}{A} = \underset{\sim}{B} = 0 \;, \quad \nabla \cdot \underset{\sim}{A} = 0 \tag{1.19}$$

where the scalar potential a and the vector potential $\underset{\sim}{A}$ are defined as the appropriate quantities in the brackets. In particular, we can write, [11, p. 52],

$$a = \nabla \cdot \underset{\sim}{C} \qquad \text{and} \qquad \underset{\sim}{A} = \nabla \times \underset{\sim}{C}$$

where $\underset{\sim}{C}$ is defined by

$$\mathcal{C} = \int_V \frac{\mathcal{B}}{r}\, dv$$

In the present case $\mathcal{B} = 0$, hence $\mathcal{C} = 0$ so that

$$a = 0 \qquad \text{and} \qquad \mathcal{A} = 0$$

Thus the <u>necessary</u> <u>and</u> <u>sufficient</u> <u>conditions</u> that equations (1.17) and (1.18) be satisfied are that each quantity enclosed in brackets be identically zero. Hence

$$(c_1^2 + c_3^2)\, \nabla^2 \bar{u} = \ddot{\bar{u}} \tag{1.20}$$

$$(c_4^2 + c_5^2)\, \nabla^2 \bar{\varphi} - 2\omega_0^2\, \bar{\varphi} = \ddot{\bar{\varphi}} \tag{1.21}$$

$$(c_2^2 + c_3^2)\, \nabla^2 \mathcal{U} + c_3^2\, \nabla \times \mathcal{Q} = \ddot{\mathcal{U}} \tag{1.22}$$

$$c_4^2\, \nabla^2 \mathcal{Q} - 2\omega_0^2\, \mathcal{Q} + \omega_0^2\, \nabla \times \mathcal{U} = \ddot{\mathcal{Q}} \tag{1.23}$$

It may be observed that equations (1.20) and (1.21) are uncoupled for the scalar potentials $\bar{u}$ and $\bar{\varphi}$ while equations (1.22) and (1.23) constitute a coupled system for the determination of the vector potentials $\mathcal{U}$ and $\mathcal{Q}$.

<u>Plane Waves in Infinite Medium</u>

Plane waves advancing in the positive direction of the unit vector ν may be expressed as

$$\{\bar{u}, \bar{\varphi}, \mathcal{U}, \mathcal{Q}\} = \{a, b, \mathcal{A}, \mathcal{B}\} \exp[ik(\nu \cdot x - vt)] \tag{1.24}$$

where a , b are complex constants, $\mathcal{A}$, $\mathcal{B}$ may be complex constant vectors, v is the phase velocity, k is the wave number and $\mathbf{r}$ is the position vector. Thus

$$k = \frac{2\pi}{\ell} \quad , \quad \mathbf{r} = x\,\mathbf{I} + y\,\mathbf{J} + z\,\mathbf{K}$$

in which ℓ is the wavelength and $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are the unit cartesian base vectors.

Substitution of (1.24) into (1.20) yields

$$v_1^2 = c_1^2 + c_3^2 = (\lambda + 2\mu + \kappa)\rho^{-1} \tag{1.25}$$

Hence, if v_1 is to be real we must have

$$\lambda + 2\mu + \kappa \geq 0 \tag{1.26}$$

From equation (1.15) we obtain for the displacement vector

$$\mathbf{u} = ik_1 a\mathbf{v} \exp[ik_1(\mathbf{v} \cdot \mathbf{r} - v_1 t)]$$

which is in the direction of propagation. Hence these waves represent the counterpart of the classical dilatational waves. For $\kappa = 0$, (1.25) gives the classical wave speed. Since the displacement is in the direction of propagation for the waves traveling at speed v_1 we shall designate them longitudinal displacement waves.

Turning our attention now to equation (1.21), for the speed of propagation v_2 , we obtain

$$v_2^2 = (c_4^2 + c_5^2) + \frac{2\omega_0^2}{k^2} \tag{1.27}$$

Introducing the angular frequency ω by

$$\omega = 2\pi f = 2\pi \frac{v}{\ell} = \frac{2\pi}{\ell} v = kv \tag{1.28}$$

equation (1.27) may be written as

$$v_2^2 = \frac{c_4^2 + c_5^2}{\left(1 - \frac{2\omega_0^2}{\omega_2^2}\right)} = \frac{\alpha + \beta + \gamma}{\rho j\left(1 - \frac{2\kappa}{\rho j \omega_2^2}\right)} \tag{1.29}$$

Equation (1.29) shows that the speed of propagation v_2 depends on the frequency ω_2 . Hence these waves are dispersive. In particular if $\omega_2 > \sqrt{2}\,\omega_0$ and

$$\alpha + \beta + \gamma \geq 0 \tag{1.30}$$

then v_2 is real and the microrotation waves exist. The microrotation vector $\underset{\sim}{\varphi}$ is given by

$$\underset{\sim}{\varphi} = \nabla\bar{\varphi} = ik_2\, b\underset{\sim}{\nu} \exp[ik_2(\underset{\sim}{\nu} \cdot \underset{\sim}{r} - v_2 t)] \tag{1.31}$$

This expression also shows that the microrotation vector points in the direction of propagation. This is a new wave not encountered in classical elasticity theory and to distinguish it from other microrotational waves we shall call it a <u>longitudinal</u> <u>microrotational</u> <u>wave</u>.

If $\omega_2 = \sqrt{2}\,\omega_0 = \omega_c$ the wave has infinite velocity v_2 as given by equation (1.29) and the wave does not exist.

When $\omega_2 < \sqrt{2}\,\omega_0$ equation (1.29) shows that the speed v_2^2 is negative and v_2 is pure imaginary, that is,

$$v_2 = \pm\, i|v_2| \tag{1.32}$$

Hence, it appears as if $\sqrt{2}\,\omega_0$ acts as a cutoff frequency, below which the wave vanishes. Carrying the investigation a little further, since $\omega \geq 0$ equation (1.28) shows that

$$k_2 = \frac{\omega_2}{v_2} = \mp i \frac{\omega_2}{|v_2|} \tag{1.33}$$

where the upper and lower signs of (1.33) correspond to the upper and lower signs of (1.32) respectively. Considering first the upper signs, from (1.24) we may write

$$\bar{\varphi} = b \exp\{\frac{\omega_2}{|v_2|} \underset{\sim}{\nu} \cdot \underset{\sim}{r}\} \exp\{-i\,\omega_2 t\}$$

This represents a harmonic vibration of the medium the magnitude of which grows exponentially with distance and therefore it is unsatisfactory. For the lower signs in equations (1.32) and (1.33) we obtain

$$\bar{\varphi} = b \exp\{-\frac{\omega_2}{|v_2|} \underset{\sim}{\nu} \cdot \underset{\sim}{r}\} \exp\{-i\,\omega_2 t\} \tag{1.34}$$

which decays exponentially with distance. This is a physically acceptable motion. For $\omega_2 \ll 1$

$$\frac{\omega_2}{|v_2|} \simeq \frac{2\omega_0}{\sqrt{c_4^2 + c_5^2}}$$

and equation (1.34) approaches a finite limit as $\omega_2 \to 0$. In particular, $\bar{\varphi}$ and hence $\underset{\sim}{\varphi}$ becomes independent of time and reduces to a static microrotation.

For the investigation of vector waves we substitute (1.24) into (1.22) and (1.23). Hence

$$\alpha_A \underset{\sim}{A} + i\,\alpha_B \underset{\sim}{\nu} \times \underset{\sim}{B} = 0 \tag{1.35}$$

$$i\,\beta_A\,\boldsymbol{\nu}\times\mathbf{A}+\beta_B\,\mathbf{B} = 0 \tag{1.36}$$

where

$$\alpha_A = k^2(v^2 - c_2^2 - c_3^2) \quad , \quad \alpha_\beta = k\,c_3^2$$
$$\beta_A = k\,\omega_0^2 \quad , \quad \beta_B = k^2\left(v^2 - c_4^2 - \frac{2\omega_0^2}{k^2}\right) \tag{1.37}$$

Forming the scalar products of equations (1.35) and (1.36) with $\boldsymbol{\nu}$ it becomes apparent that

$$\boldsymbol{\nu}\cdot\mathbf{A} = \boldsymbol{\nu}\cdot\mathbf{B} = 0 \tag{1.38}$$

provided $\alpha_A \neq 0$, $\alpha_B \neq 0$, $\beta_A \neq 0$ and $\beta_B \neq 0$. Hence both vectors $\mathbf{A}$ and $\mathbf{B}$ lie in a common plane whose unit normal is $\boldsymbol{\nu}$. Solving for $\mathbf{B}$ from equation (1.36) we get

$$\mathbf{B} = -\,i\,\frac{\beta_A}{\beta_B}\,\boldsymbol{\nu}\times\mathbf{A} \tag{1.39}$$

from which we conclude that the three vectors $\boldsymbol{\nu}$, $\mathbf{A}$ and $\mathbf{B}$ are mutually perpendicular.

Equation (1.39) shows that if $\mathbf{A} \equiv 0$ then $\mathbf{B} \equiv 0$ making both $\mathbf{U}$ and $\boldsymbol{\Phi}$ vanish. Thus there would be no coupled waves propagated. A similar analysis from equation (1.35) holds for $\mathbf{B} \equiv 0$. These two waves cannot vanish unless they do so simultaneously, hence they are truly coupled waves.

If $\mathbf{U}$, and hence $\boldsymbol{\Phi}$, are non-zero, the second terms of $(1.15)_1$ and $(1.16)_1$ will show that $\mathbf{u}$ and $\boldsymbol{\varphi}$ are normal to each other and to the direction of propagation $\boldsymbol{\nu}$. Hence they are transverse waves. We call the wave associated with $\mathbf{U}$ a *transverse displacement wave* and the one associated with $\boldsymbol{\Phi}$ a *transverse microrotational wave*. The transverse displacement wave is similar to the classical shear wave and will reduce to it in the limit of classical elasticity. The appearance of a transverse microrotational wave coupled with it is new.

The velocities of propagation of these waves are determined by carrying (1.39) into (1.35) with the use of (1.38) for $\underset{\sim}{A} \neq \underset{\sim}{0}$ gives

$$a(v^2)^2 + bv^2 + c = 0 \tag{1.40}$$

where

$$\begin{aligned} a &= (1 - \frac{2\omega_0^2}{\omega^2}) \\ b &= -[c_4^2 + c_2^2(1 - \frac{2\omega_0^2}{\omega^2}) + c_3^2(1 - \frac{\omega_0^2}{\omega^2})] \\ c &= c_4^2(c_2^2 + c_3^2) \end{aligned} \tag{1.41}$$

Equation (1.40) is a quadratic equation in v^2 which yields two distinct speeds of propagation. These waves are dispersive. In the classical case the roots of (1.40) are $v = \pm 0$, $v = \pm\sqrt{\mu/\rho}$ of which the last one is the speed of shear waves given in the classical theory. In order to have two real velocities both roots of equation (1.40) for v^2 must be positive.

The positive roots of equation (1.40) are

$$v_3 = [\frac{1}{2a}(-b + \sqrt{b^2 - 4ac})]^{1/2} \tag{1.42}$$

$$v_4 = [\frac{1}{2a}(-b - \sqrt{b^2 - 4ac})]^{1/2} \tag{1.43}$$

By use of (1.41) we can eventually obtain

$$\sqrt{b^2 - 4ac} = \{[(c_4^2 - c_2^2 - c_3^2) + (c_2^2 + \frac{c_3^2}{2}) \frac{2\omega_0^2}{\omega^2}]^2 + 2c_3^2 c_4^2 \frac{2\omega_0^2}{\omega^2}\}^{1/2} \qquad (1.44)$$

which shows that the discriminant is greater than or equal to zero since $\kappa \geq 0$ and $\gamma \geq 0$ by (1.7) and $\omega \geq 0$*. From equation $(1.41)_1$ we see that if $\omega > \omega_c$ $a > 0$, if $\omega = \omega_c$ then $a = 0$, and finally if $\omega < \omega_c$, $a < 0$. Since the numerator of v_3 is always positive we conclude that v_3 is real for $\omega > \omega_c$, infinite for $\omega = \omega_c$, and imaginary $\omega < \omega_c$. The detailed analysis in the next section shows that v_4 remains real and finite for all ω. Hence, this critical frequency $\omega_c = \sqrt{2}\,\omega_0$ is again a cutoff frequency for one of the wave speeds.

In summary of this section, we find that there are six waves traveling at four distinct speeds in an infinite micropolar elastic solid: (a) A longitudinal displacement wave at speed v_1 similar to the dilatational wave of the classical theory; (b) a longitudinal microrotation wave traveling with a speed v_2 with its microrotation vector in the direction of propagation. This motion exists as a progressive wave only if the frequency ω is larger than the critical circular frequency $\omega_c = \sqrt{2}\,\omega_0$. Below this frequency the wave degenerates into sinusoidal vibrations decaying with distance from the source. (c) The remaining four are two sets of waves composed of two waves each. One set propagates at speed v_3 and the other at speed v_4. Each set consists of a transverse displacement wave coupled with a transverse microrotational wave. An analysis similar to that for the longitudinal microrotation wave shows

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*An investigation was made allowing $\omega^2 < 0$ and ω imaginary. No physically acceptable progressive wave solutions were obtained. This is not surprising, however, since the basic theory does not contain a mechanism for internal friction.

that the set of coupled waves traveling at speed v_3 exist only if $\omega > \omega_c$ degenerating into a distance decaying sinusoidal vibration otherwise. A more detailed analysis of the dispersion relations (1.29), (1.42) and (1.43) is carried out in the next section.

Analysis of Dispersion Relations

In this section we study the dependence of wave speeds on frequency. The wave speed v_1 is constant, thus we only need to study the characters of v_2 , and then v_3 and v_4 .

According to equation (1.29) we have

$$v_2^2 = (c_4^2 + c_5^2)(1 - x)^{-1} \quad , \quad x \equiv \frac{2\omega_0^2}{\omega^2} \tag{1.45}$$

A sketch of (1.45) is shown in Fig. 1.1. The sketch of v_2^2 versus ω is simply a reflection of the figure about the line $x = 1$ $(\omega = \omega_c = \sqrt{2}\,\omega_0)$ as shown in Figure 1.2 since x and ω are mutually reciprocal.

For the wave velocities v_3 and v_4 we have

$$v_{3,4}^2 = \frac{1}{2(1-x)} \{c_2^2 + c_3^2 + c_4^2 - (c_2^2 + \frac{c_3^2}{2})x \pm \sqrt{[(c_4^2-c_2^2-c_3^2) + (c_2^2+\frac{c_3^2}{2})x]^2 + 2c_3^2c_4^2x}\} \tag{1.46}$$

where the upper sign refers to v_3^2 and the lower one to v_4^2 in the above and in what follows. Letting $x \to 0$ $(\omega \to \infty)$ (1.46) yields the short wavelength values

$$v_3^2 = c_4^2 \quad , \quad v_4^2 = c_2^2 + c_3^2 \tag{1.47}$$

For $x \to \infty$ $(\omega \to 0)$ the long wavelength limits are obtained

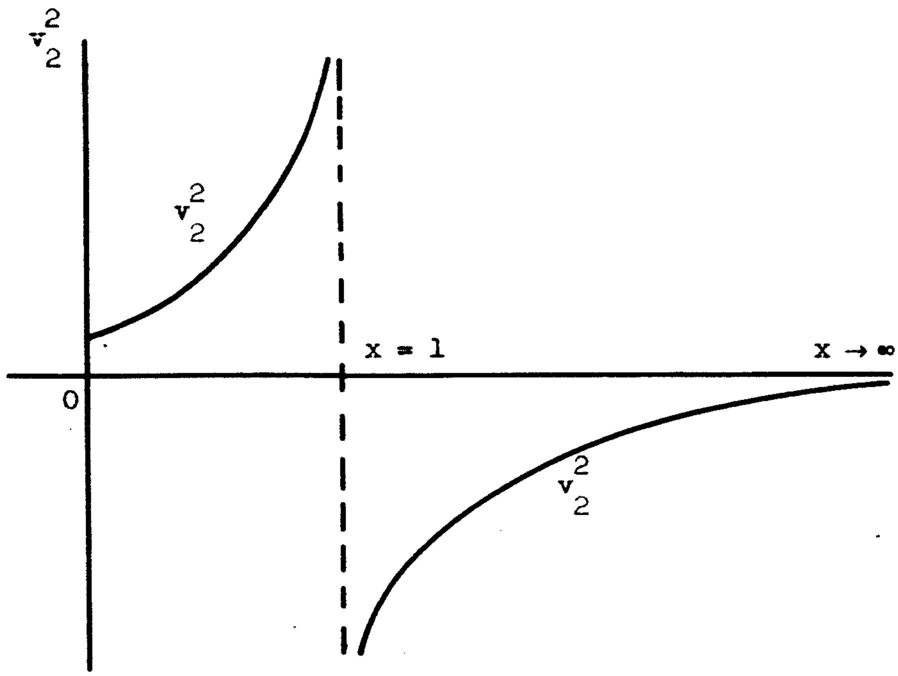

Figure 1.1. Sketch of v_2^2 versus x .

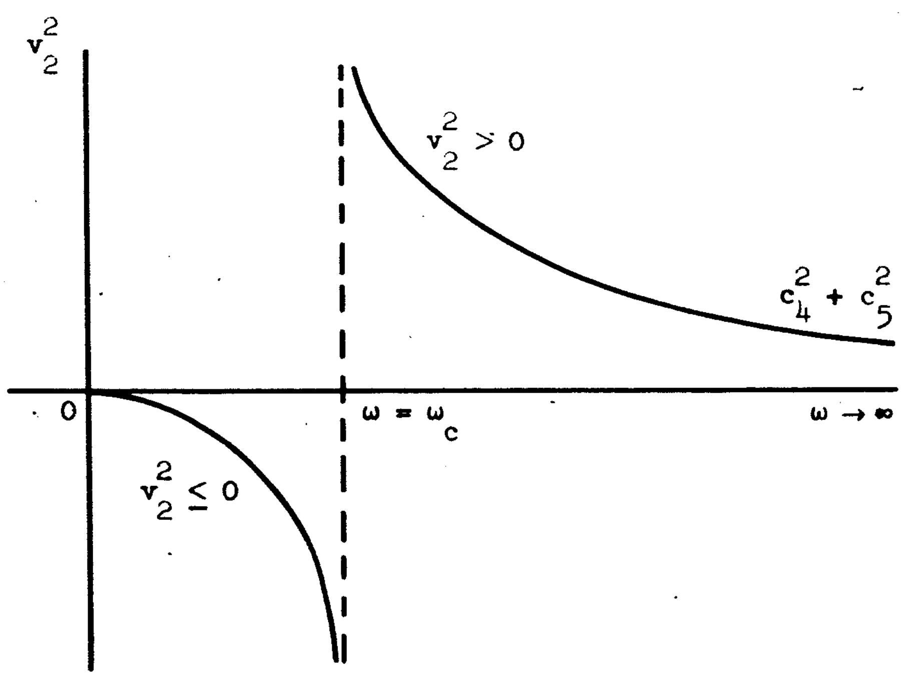

Figure 1.2. Sketch of v_2^2 versus ω .

$$v_3^2 = 0 \quad , \quad v_4^2 = c_2^2 + \frac{1}{2} c_3^2 \tag{1.48}$$

If we allow $\omega = \omega_c$ then $x = 1$ and the denominator of equation (1.46) vanishes. This leads to $v_3 = \infty$ for $\omega > \omega_c$, $v_3 = -\infty$ for $\omega < \omega_c$ and

$$\lim_{x \to 1} v_4^2 = c_4^2(c_2^2 + c_3^2)(c_4^2 + \frac{c_3^2}{2})^{-1} \tag{1.49}$$

We see that if $c_3^2 = 0$ then $v_4^2 = c_2^2 = \frac{\mu}{\rho}$, the classical speed, in all three cases. If $c_4^2 = 0$ also, then $v_3^2 = 0$ and we have the complete classical situation.

Comparing our knowledge of v_3^2 and v_4^2 at this point we see that at $\omega = \omega_c$ v_3^2 is infinite and v_4^2 is finite and they both approach finite values in the limit as ω tends to infinity. Hence, the question arises as to whether or not they intersect. If $v_4^2(\infty) > v_3^2(\infty)$ then the curves intersect, otherwise they do not. Assuming the curves intersect then there exists an ω such that $v_3^2 = v_4^2$. Using equations (1.42) and (1.43) we find that we must have $b^2 - 4ac = 0$. In an earlier discussion of equation (1.44) we concluded that

$$b^2 - 4ac \geq 0 \qquad \text{all } \omega$$

the equal sign being in the limit as $\omega \to \infty$. Hence we reach a contradiction and the curves do not intersect for any finite ω . They may, however, approach the same limit as $\omega \to \infty$. This means $v_3^2(\infty) = v_4^2(\infty)$ or $c_4^2 = c_2^2 + c_3^2$. In general then $v_3^2(\omega) > v_4^2(\omega)$, except possibly at infinity where the equal sign holds, hence $v_3^2 \geq v_4^2$ for $\omega > \omega_c$ which means

$$c_4^2 \geq c_2^2 + c_3^2 \tag{1.50}$$

Using equations (1.14) we obtain an additional inequality among the constitutive coefficients

$$\frac{\gamma}{j} \geq \mu + \kappa \tag{1.51}$$

which must be satisfied to have a consistent solution for v_3^2 and v_4^2 .

We can continue this analysis by taking the limits of the derivatives of v_3^2 and v_4^2 , etc. The results of all these computations are incorporated in Figure 1.3.

Next we investigate the relative magnitudes of v_4^2 at $\omega = 0$, $\omega = \omega_c$, and $\omega = \infty$. Equations $(1.47)_2$ and $(1.48)_2$ show immediately that if $c_3^2 \neq 0$

$$v_4^2(0) < v_4^2(\infty) \tag{1.52}$$

Likewise, comparison of equations $(1.47)_2$ and (1.49) shows that

$$v_4^2(\omega_c) < v_4^2(\infty) \tag{1.53}$$

To compare $v_4^2(0)$ and $v_4^2(\omega_c)$ we must compare equations $(1.48)_2$ and (1.49). From the inequality (1.50) we see that $c_2^2 + \frac{1}{2} c_3^2 \leq c_4^2$, or $v_4^2(0) \leq v_4^2(\omega_c)$. Consequently we conclude that

$$v_4^2(0) \leq v_4^2(\omega_c) \leq v_4^2(\infty) \tag{1.54}$$

and show these results in Figure 1.3.

Comparison of equations (1.25) and $(1.47)_2$ with the aid of (1.14) shows that

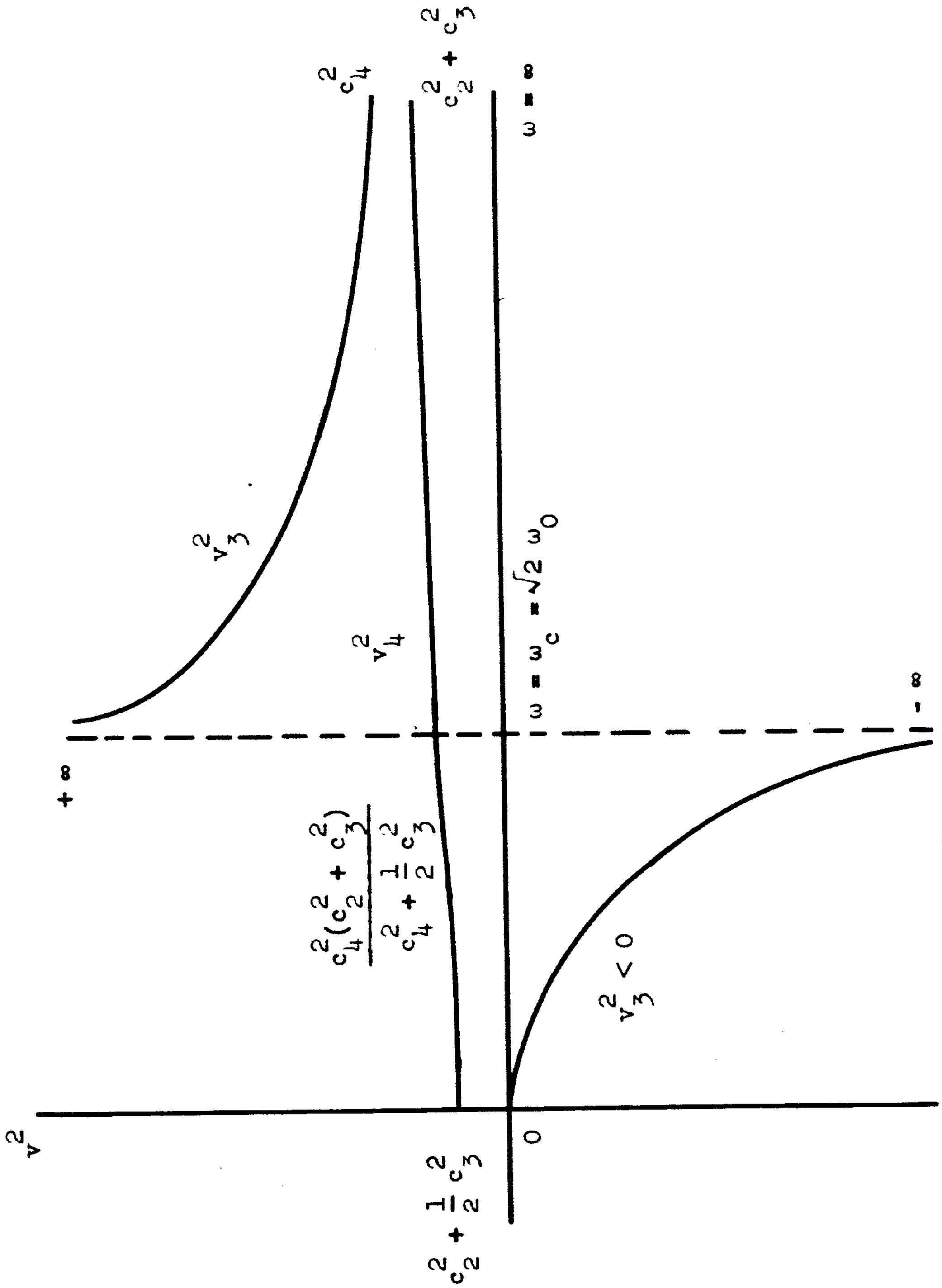

Figure 1.3. Sketch of v_3^2 and v_4^2 versus ω .

$$v_1^2 > v_4^2 \tag{1.55}$$

A similar comparison of (1.25) and $(1.47)_1$ for v_1^2 and v_3^2 is inconclusive. If

$$\frac{\gamma}{j} \geq \lambda + 2\mu + \kappa \tag{1.56}$$

then

$$v_3^2 \geq v_1^2 \qquad \omega > \omega_c \tag{1.57}$$

However, if the opposite is true then $v_3^2(\infty) < v_1^2$, and since $v_3^2(\omega)$ increases for decreasing ω and v_1^2 is constant there exists a frequency ω , say ω^* , where $\omega_c < \omega^* < \infty$ such that $v_3^2 = v_1^2$. That is, ω^* is found by equating equation (1.25) to equation (1.46) with the upper sign and solving for ω . Then we conclude that

$$\begin{aligned} v_3^2 &\geq v_1^2 \qquad \omega_c < \omega \leq \omega^* \\ v_1^2 &\geq v_3^2 \qquad \omega \geq \omega^* \end{aligned} \tag{1.58}$$

Since the reflection problems considered in the later chapters do not simultaneously involve longitudinal displacement waves and longitudinal microrotation waves a comparison of speeds v_1 and v_2 is not necessary.

We now examine the relative magnitudes of v_2^2 , v_3^2 and v_4^2 . Comparison of Figures 1.2 and 1.3 show at infinity

$$v_2^2(\infty) > v_3^2(\infty) > v_4^2(\infty) \tag{1.59}$$

and hence v_2^2 will be greater than v_3^2 for $\omega > \omega_c$ if $v_2^2 > v_3^2$ at $\omega = \omega_c$. To this end we determine the limit of v_3^2/v_2^2 as $\omega \to \omega_c$ $(x \to 1)$. At $x = 1$ this ratio simplifies to

$$\frac{v_3^2}{v_2^2} = (c_4^2 + \frac{1}{2} c_3^2)(c_4^2 + c_5^2)^{-1} \tag{1.60}$$

Now if we are to have $v_2^2 > v_3^2$ then $c_5^2 > \frac{1}{2} c_3^2$ from which (1.14) shows that the inequality

$$\alpha + \beta > \frac{1}{2} j\kappa \tag{1.61}$$

among the constitutive coefficients must be satisfied. If the opposite of (1.61) is satisfied then there exists a frequency ω^{**} such that $v_2^2(\omega^{**}) = v_3^2(\omega^{**})$ when $\omega_c < \omega^{**} < \infty$. In this case $v_3^2 \geq v_2^2$ when $\omega_c < \omega \leq \omega^{**}$ and $v_3^2 \leq v_2^2$ when $\omega^{**} \leq \omega$.

Briefly, if inequality (1.61) is satisfied then

$$v_2^2 > v_3^2 > v_4^2 \qquad \omega_c < \omega$$

holds. If the opposite inequality is satisfied there exists ω^{**} such that either

$$v_3^2 \geq v_2^2 \geq v_4^2 \qquad \omega_c < \omega \leq \omega^{**} \tag{1.62}$$

or

$$v_2^2 \geq v_3^2 \geq v_4^2 \qquad \omega^{**} \leq \omega \tag{1.63}$$

We have covered all possible cases without definite knowledge of the relative values of the constitutive coefficients. These relations are vital to the analyses in the remaining chapters.

CHAPTER 2

REFLECTION OF A LONGITUDINAL DISPLACEMENT WAVE

Formulation

This chapter is devoted to the study of reflection of a longitudinal displacement plane wave at a stress free plane surface of infinite length. The free surface is taken to be the x,y-plane with positive z pointing into the medium, Figure 2.1. An incident plane wave advancing in the direction of a unit vector ν_1 is reflected at the $z = 0$ plane. To satisfy the boundary conditions on tractions and couples at the boundary it is necessary to postulate the existence of reflected waves in three distinct directions, ν_2 , ν_3 and ν_4 . These are (1) a longitudinal displacement wave having speed v_1 in the direction ν_2 at an angle θ_2 , (2) at speed v_4 a transverse displacement wave coupled with a transverse microrotational wave in the direction of ν_4 , and (3) a similar set of coupled waves in the ν_3 direction at speed v_3 if $\omega > \omega_c$. If $\omega < \omega_c$ this last set of waves degenerates to a vibration of the medium as discussed in Chapter 1, thus we assume $\omega > \omega_c$ in the remainder of the chapter.

If the $x = 0$ plane is so selected as to make the incident displacement vector remain in the y, z plane then the reflected waves at the free surface $z = 0$ will also have their displacement fields in the same plane. Thus the study of the problem in two-dimensions is sufficient for the understanding of the three-dimensional problem. The nonvanishing components of the potentials are given by

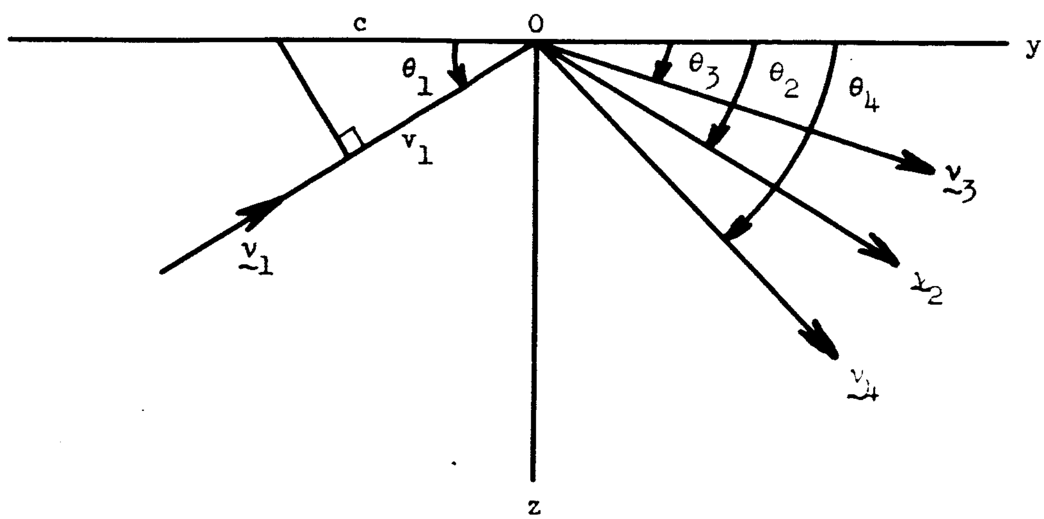

Figure 2.1. Reflection of a longitudinal displacement wave.

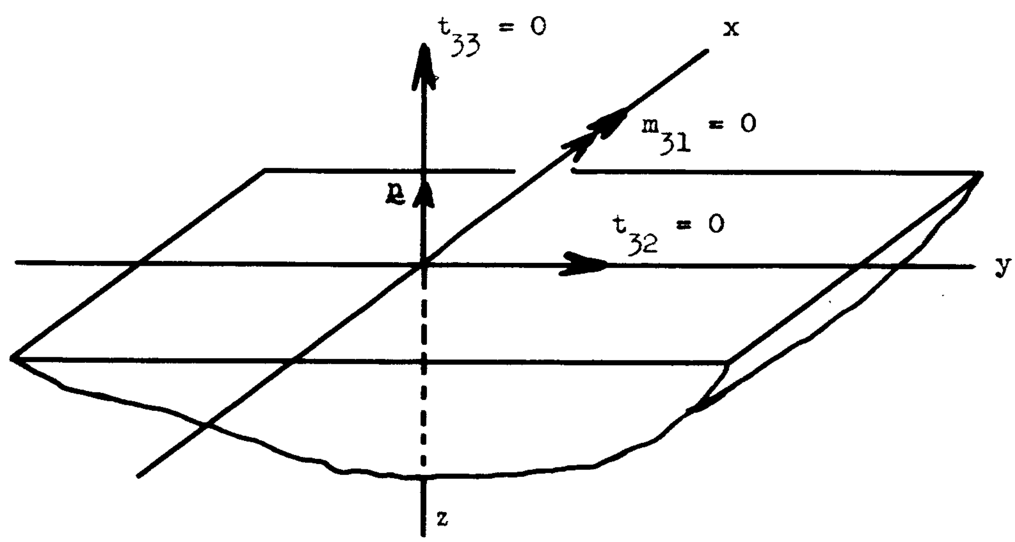

Figure 2.2. Boundary conditions on surface z = 0 .

$$\bar{u}_\alpha = a_\alpha \exp[i(k_1 \nu_\alpha \cdot r - \omega_1 t)]$$

$$U_\beta = I A_{\beta x} \exp[i(k_\beta \nu_\beta \cdot r - \omega_\beta t)] \tag{2.1}$$

$$\Phi_\beta = (B_{\beta y} J + B_{\beta z} K) \exp[i(k_\beta \nu_\beta \cdot r - \omega_\beta t)]$$

where $\omega = kv$, $(\alpha = 1, 2)$, $\beta = 3, 4)$ and the repeated indices are not summed. The coefficients A and B are related to each other by equation (1.39) so that

$$B_3 = - \frac{i\,\omega_0^2 A_{3x}}{k_3\left(v_3^2 - \frac{2\omega_0^2}{k_3^2} - c_4^2\right)} (\nu_{3z} J - \nu_{3y} K) \tag{2.2}$$

with a similar equation for B_4 .

With the boundary surface $z = 0$ being free from tractions and couples we must have $t_\ell = m_\ell = 0$. Thus through (1.10), (1.11) and the fact that $u_1 = \varphi_2 = \varphi_3 = 0$ we get

$$t_{33} = \lambda(\bar{u}_{,yy} + \bar{u}_{,zz}) + (2\mu+\kappa)(\bar{u}_{,zz} - U_{x,yz}) = 0 \tag{2.3}$$

$$t_{32} = \mu(\bar{u}_{,yz} - U_{x,yy}) + (\mu+\kappa)(\bar{u}_{,yz} + U_{x,zz}) + \kappa(\Phi_{z,y} - \Phi_{y,z}) = 0 \tag{2.4}$$

$$m_{31} = \gamma(\Phi_{z,yz} - \Phi_{y,zz}) = 0 \tag{2.5}$$

which must be satisfied at $z = 0$ for all y and t . The positive directions of surface tractions, couple and the exterior normal are shown in Figure 2.2.

Basic Solution

The solution of a reflection problem consists of determining the amplitudes and directions of the reflected waves when a known wave is incident on the boundary. The potentials (2.1) are used in the boundary conditions (2.3), (2.4) and (2.5) together with equation (2.2) and a similar equation for B_4 to determine the three amplitudes a_2, A_{3x} and A_{4x} in terms of a_1.

The potentials (2.1) satisfy the boundary condition (2.3) at $z = 0$ if

$$\omega_1 = \omega_3 = \omega_4 \tag{2.6}$$

$$k_1 \nu_{1y} = k_1 \nu_{2y} = k_3 \nu_{3y} = k_4 \nu_{4y} \tag{2.7}$$

$$k_1 \nu_{1x} = k_1 \nu_{2x} = k_3 \nu_{3x} = k_4 \nu_{4x} \tag{2.8}$$

and

$$\lambda a_1 k_1^2 + (2\mu+\kappa)\, a_1 k_1^2 \nu_{1z}^2 + \lambda a_2 k_1^2 + (2\mu+\kappa)\, a_2 k_1^2 \nu_{2z}^2$$

$$- (2\mu+\kappa) k_3^2\, \nu_{3y} \nu_{3z} A_{3x} - (2\mu+\kappa) k_4^2\, \nu_{4y} \nu_{4z} A_{4x} = 0 \tag{2.9}$$

Since the incident wave is in the y,z-plane $\nu_{1x} = 0$ and (2.8) yields

$$\nu_{2x} = \nu_{3x} = \nu_{4x} = 0$$

showing that all the waves lie in a y,z-plane as assumed earlier. Equation (2.6) states that all the frequencies are equal and equation (2.7) will allow us to determine the angles of reflection of the various waves for a given incident angle contained in ν_{1y}. Using the relation $\omega = kv$ we can write (2.7) as

$$\frac{v_{1y}}{v_1} = \frac{v_{2y}}{v_1} = \frac{v_{3y}}{v_3} = \frac{v_{4y}}{v_4} = \frac{1}{c} \tag{2.10}$$

where c has a simple physical meaning, namely the speed of the wave-front along the free surface. In terms of the angles, (2.10) may be written

$$\cos\theta_2 = \cos\theta_1 \quad , \quad \cos\theta_3 = \frac{v_3}{v_1}\cos\theta_1 \quad , \quad \cos\theta_4 = \frac{v_4}{v_1}\cos\theta_1 \tag{2.11}$$

showing $\theta_2 = \theta_1$.

Equation (2.9) is one of the three needed to determine the amplitude ratios a_2/a_1 , A_{3x}/a_1 and A_{4x}/a_1 . The other two boundary conditions (2.4) and (2.5) give the two additional equations needed. The three equations so obtained are

$$[\lambda + (2\mu+\kappa)v_{1z}^2]\, a_1 + [\lambda + (2\mu+\kappa)v_{2z}^2]a_2 - (2\mu+\kappa)v_{1y}^2 \frac{v_{3z}}{v_{3y}} A_{3x} - (2\mu+\kappa)v_{1y}^2 \frac{v_{4z}}{v_{4y}} A_{4x} = 0 \tag{2.12}$$

$$(2\mu+\kappa)v_{1y}v_{1z}a_1 + (2\mu+\kappa)v_{2y}v_{2z}a_2 + v_{1y}^2 \{(\mu+\kappa)\frac{v_{3z}^2}{v_{3y}^2} - \mu + \kappa\frac{\omega_0^2}{\omega_3^2}\frac{v_3^2}{v_{3y}^2}[v_3^2(1 - \frac{2\omega_0^2}{\omega_3^2}) - c_4^2]^{-1}\}A_{3x} + v_{1y}^2 \{(\mu+\kappa)\frac{v_{4z}^2}{v_{4y}^2} - \mu + \kappa\frac{\omega_0^2}{\omega_4^2}\frac{v_4^2}{v_{4y}^2}[v_4^2(1 - \frac{2\omega_0^2}{\omega_4^2}) - c_4^2]^{-1}\}A_{4x} = 0 \tag{2.13}$$

$$A_{3x} = -[v_3^2(1 - \frac{2\omega_0^2}{\omega_3^2}) - c_4^2][v_4^2(1 - \frac{2\omega_0^2}{\omega_4^2}) - c_4^2]^{-1} \frac{v_{3y}}{v_{3z}} \frac{v_{4z}}{v_{4y}} A_{4x} \tag{2.14}$$

In the special case of $v_{1y} = 0$ equation (2.12) reduces to $a_2 = -a_1$ and the other two reduce to a system of homogeneous equations for the determination of A_{3x} and A_{4x}. The determinant of the coefficients is nonzero, hence $A_{3x} = A_{4x} = 0$. This is similar to the classical case in that an incident wave normal to the surface results in a reflected wave of the same type normal to the surface with a phase change of 180^o. For $v_{1y} \neq 0$ we divide equations (2.12) and (2.13) by v_{1y}^2 and use (2.14) to eliminate A_{3x}. The solution for a_2/a_1 and A_{4x}/a_1 is found to be

$$\frac{a_2}{a_1} = \{[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1][(\mu+\kappa)\tan^2\theta_4 - \mu - (\mu+\kappa)Q_1^2\tan\theta_3\tan\theta_4 - Q_2^2 + (\mu Q_1^2+Q_3^2)\frac{\tan\theta_4}{\tan\theta_3}] + (2\mu+\kappa)^2\tan\theta_1\tan\theta_4(Q_1^2-1)\}$$

$$\times\{-[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1][(\mu+\kappa)\tan^2\theta_4 - \mu - (\mu+\kappa)Q_1^2\tan\theta_3\tan\theta_4 - Q_2^2 + (\mu Q_1^2+Q_3^2)\frac{\tan\theta_4}{\tan\theta_3}] + (2\mu+\kappa)^2(Q_1^2-1)\tan\theta_1\tan\theta_4\}^{-1} \tag{2.15}$$

and

$$\frac{A_{4x}}{a_1} = -2(2\mu+\kappa)[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1]\tan\theta_1$$

$$\times\{-[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_1][(\mu+\kappa)\tan^2\theta_4 - \mu - (\mu+\kappa)Q_1^2\tan\theta_3\tan\theta_4 - Q_2^2 + (\mu Q_1^2+Q_3^2)\frac{\tan\theta_4}{\tan\theta_3}] + (2\mu+\kappa)^2(Q_1^2-1)\tan\theta_1\tan\theta_4\}^{-1} \tag{2.16}$$

where we have written ω for the common value of $\omega_1 = \omega_3 = \omega_4$ and

$$Q_1^2 = [v_3^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2][v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

$$Q_2^2 = -\kappa \frac{\omega_0^2}{\omega^2} \frac{v_4^2}{v_{4y}^2} [v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1} \qquad (2.17)$$

$$Q_3^2 = -\kappa \frac{\omega_0^2}{\omega^2} \frac{v_3^2}{v_{3y}^2} [v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

The general solution for the amplitude ratios as given above can be transformed into different forms by using (2.11) to eliminate θ_3 and/or θ_4, however, since there is nothing to be gained by this we turn our attention to some special cases.

Special Cases

i. $A_{4x}/a_1 = 0$. Considering the numerator of equation (2.16) to be zero, at first glance there appear to be three values of θ_1 which make this amplitude ratio vanish. However, since the constitutive constant λ is positive two values of θ_1 are complex, thus the only real value of θ_1 is zero which is the case of a grazing incident wave.

Consider now an incident wave with $\theta_1 = 0$, i.e., $v_{1y} = 1$ and $v_{1z} = 0$. From (2.11) we conclude that $\theta_2 = 0$ but θ_3 and θ_4 are nonzero. Then (2.16) and (2.14), respectively, show that $A_{4x} = 0$ and $A_{3x} = 0$. Equation (2.13) is identically satisfied and (2.12) yields $a_2 = -a_1$. However, since the exponentials are the same when $\theta_1 = \theta_2 = 0$, $\bar{u}_2 = -\bar{u}_1$ and the motions cancel each other as in the classical case; hence we must resort to some form of limiting process as θ_1 tends to zero to obtain nonzero displacements. For this analysis see Special Cases v. and vi.

ii. $\theta_4 = 0$. This is the case of reflected waves at speed v_4 parallel to the surface. If $\theta_4 = 0$ then $v_{4z} = \sin\theta_4 = 0$ and $v_{4y} = \cos\theta_4 = 1$ which in turn yields

$$v_{1y} = \cos\theta_1 = \frac{v_1}{v_4}$$

from (2.10). Then equation (1.55) shows $v_1 > v_4$ giving $\cos\theta_1 > 1$ and θ_1 is imaginary. Since our incident wave angle θ_1 is between 0° and 90° and real this means we cannot have a reflected grazing wave with speed v_4 .

iii. $\theta_3 = 0$. In this case θ_1 will be real if $v_3 > v_1$ since

$$\cos\theta_1 = \frac{v_1}{v_3}$$

Two possibilities in this case are respectively given by (1.57) and (1.58) corresponding to the inequalities

$$\frac{\gamma}{j} \geq (\lambda + 2\mu + \kappa) \quad \text{and} \quad \frac{\gamma}{j} \leq (\lambda + 2\mu + \kappa)$$

Hence $v_3 \geq v_1$ and θ_1 is real if (1.57) or $(1.58)_1$ is satisfied.

Using (2.14) and $(2.17)_1$ we can write

$$A_{3x} = -Q_1^2 \frac{\tan\theta_4}{\tan\theta_3} A_{4x} \tag{2.18}$$

Multiplying the numerator and denominator of (2.15) by $\tan\theta_3/\tan\theta_4$ and letting $\theta_3 = 0$ we get $a_2/a_1 = -1$. If we carry (2.16) into (2.18) and let $\theta_3 = 0$ we get

$$\frac{A_{3x}}{a_1} = -2 \tan\theta_1(2\mu + \kappa)(\mu + \frac{Q_3^2}{Q_1^2})^{-1}$$

Also through (2.18) in this limit $A_{4x}/a_1 = 0$ when $\theta_4 \neq 0$. For $\theta_3 = 0$ these results take the forms

$$\frac{a_2}{a_1} = -1$$

$$\frac{A_{3x}}{a_1} = -2(2\mu+\kappa)\left(\frac{v_3^2}{v_1^2} - 1\right)^{1/2}\left\{\mu-\kappa\frac{\omega_0^2}{\omega^2}\frac{v_3^2}{v_{3y}^2}\left[v_3^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\right\}^{-1} \quad (2.19)$$

$$\frac{A_{4x}}{a_1} = 0 \quad , \quad \cos\theta_1^* = \frac{v_1}{v_3} \quad , \quad \cos\theta_4 = \frac{v_4}{v_3}$$

where the angle θ_1^* is given by $(2.11)_2$. This completes the solution for a set of reflected grazing waves at speed v_3.

The angle of incidence θ_1 and the amplitude ratios given by (2.19) are plotted against ω^2 for various values of parameters. The micro-inertia j is estimated on the basis of a polycrystalline metal whose grain size is approximately 0.0025 inches. Based on this grain size the lower limit for j is about 10^{-6} in.^{2} for one grain. For a microvolume of 1000 grains we get an average value of j approximately 10^{-3} in.^{2}. We also assume ρ to be the mass density of steel, approximately 10^{-3} lb.-sec.^{2}-in.$^{-4}$.

It is assumed that κ is small compared to λ and μ and that $\lambda = \mu$. We further assume that α, β and γ are small compared to λ and μ, and in particular, to simplify the calculations we let $\alpha = \beta = \gamma$ such that the inequality (1.51) is satisfied.

With these assumptions we use (2.19) along with several different values of the parameter $\gamma/j\lambda$. The results are sketched in Figures 2.3, 2.4 and 2.5. The curves in general show that the amplitude ratio A_{3x}/a_1 increases for increasing $\gamma/j\lambda$. If $\kappa \neq 0$ Figure 2.5 shows that the amplitude ratio for a fixed $\gamma/j\lambda$ is finite for large ω and becomes more negative for decreasing ω until we reach the neighborhood of ω_c^2 in which case it quickly turns to zero. The curve remains

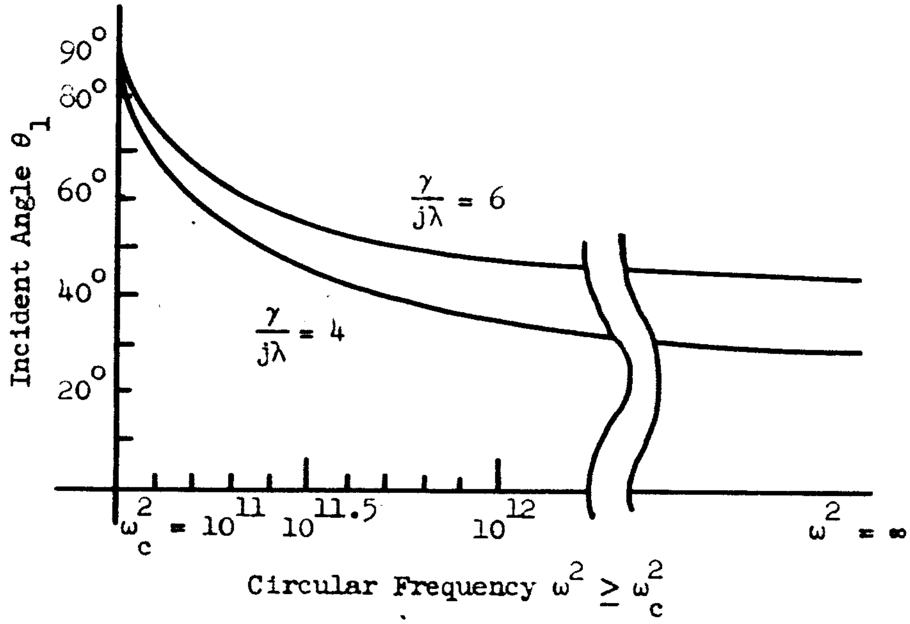

Figure 2.3. Incident angle θ_1 vs. frequency with $\kappa/\lambda = 0.05$.

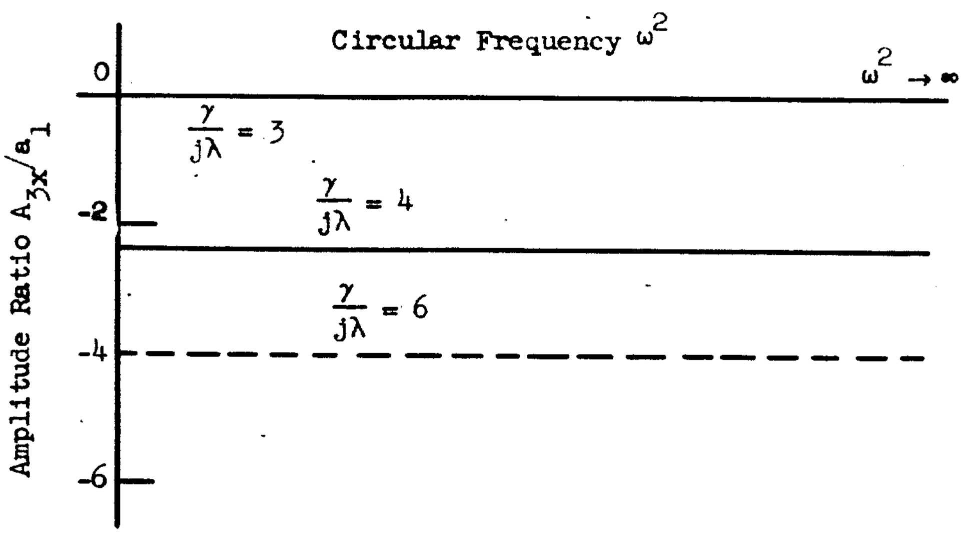

Figure 2.4. Amplitude ratio A_{3x}/a_1 vs. ω^2 with $\kappa = 0$.

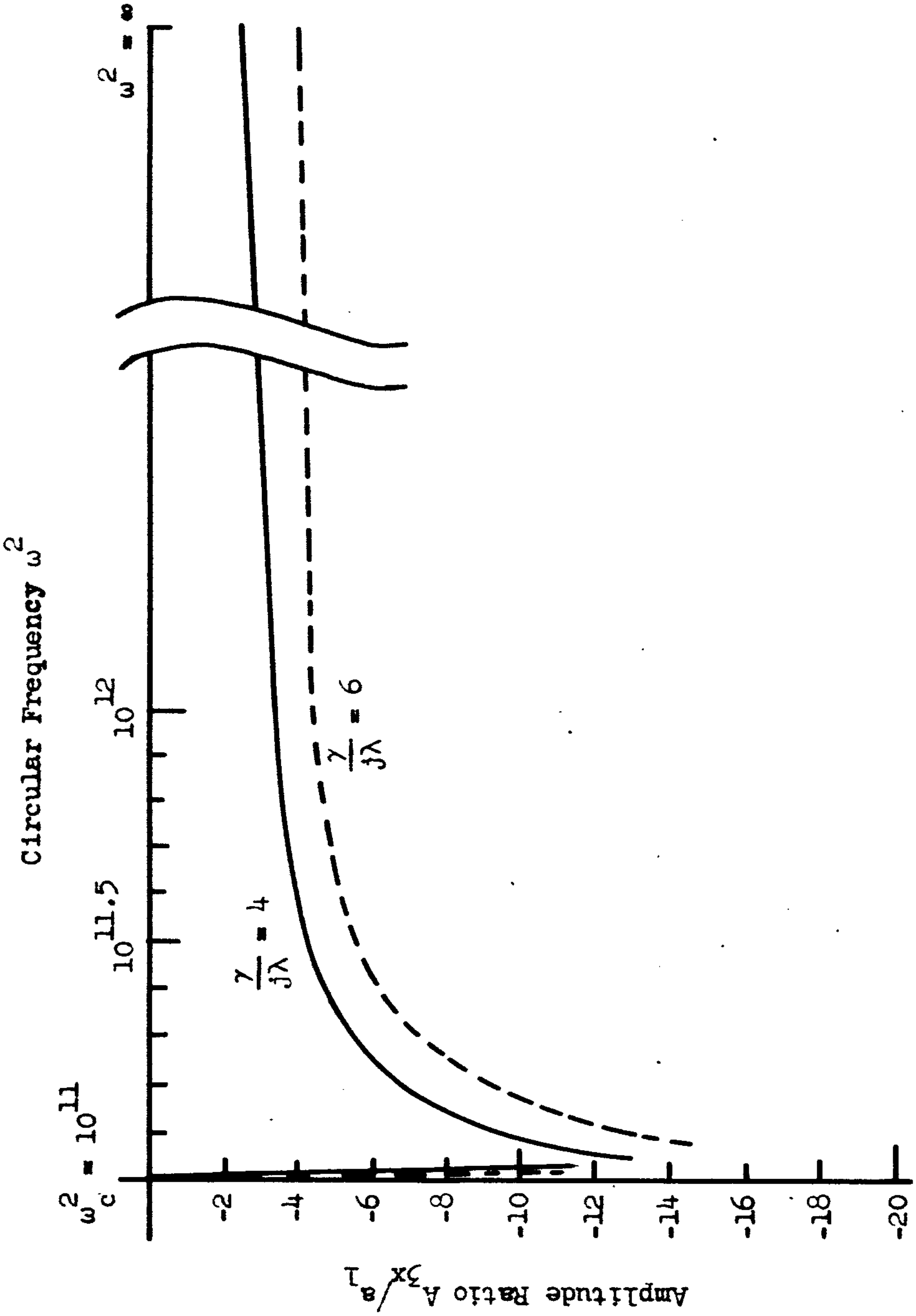

Figure 2.5. Amplitude ratio A_{3x}/a_1 vs. ω^2 with $\kappa/\lambda = 0.05$.

bounded for all finite values of the parameter $\gamma/j\lambda$; however, as this parameter goes to infinity the entire amplitude ratio curve moves out towards infinity. Physically, however, this ratio is bounded since $\gamma/\lambda \ll 1$ and j cannot equal zero since it is a function of grain size. For $j = 0$ the character of our differential equation changes so that no wave solution exists.

iv. θ_3 is complex. In the previous case we have seen that as θ_1 is decreased from 90° towards 0° the angle θ_3 of the reflected waves at speed v_3 decreases from 90° to 0° faster than θ_1 . In particular θ_3 goes to zero at $\theta_1 = \theta_1^* > 0$. As θ_1 is decreased below the critical angle θ_1^* equation $(2.11)_2$ shows that $(v_3/v_1)\cos\theta_1 > 1$ and hence $\cos\theta_3 > 1$ and θ_3 is complex. Substitution of

$$\sin\theta_3 = i\left(\frac{v_3^2}{v_1^2}\cos^2\theta_1 - 1\right)^{1/2} = i\,\eta \tag{2.20}$$

and $\cos\theta_3$ into the potential $(2.1)_2$ with $\beta = 3$ yields

$$\underline{U}_3 = \underline{I}\,A_{3x}\exp(-k_3\eta z)\exp[ik_3(\frac{v_3}{v_1}\cos\theta_1\ y - v_3 t)] \tag{2.21}$$

from which we see that the transverse waves at speed v_3 become a disturbance propagating along the boundary at the speed

$$\bar{v} = \frac{v_3}{\cos\theta_3} = \frac{v_1}{\cos\theta_1} = c \tag{2.22}$$

whose amplitude decays exponentially with distance z into the medium. This is similar to the classical case for an incident shear wave, but differs in the fact that a_2/a_1 is no longer equal to one and A_{4x}/a_1 is no longer zero. Thus as θ_1 is decreased beyond θ_1^* we have a disturbance along the surface, a reflected longitudinal wave at speed v_1 and angle θ_1 , and a set of coupled transverse waves at speed v_4

and angle θ_4. With θ_3 complex the amplitude ratios a_2/a_1 and A_{4x}/a_1 are also complex indicating phase shifts in the reflected waves.

At a first glance one is tempted to call the exponentially decaying surface wave a Rayleigh wave. On closer examination, however, we see that the wave speed varies continuously from $\bar{v} = v_3$ when $\theta_1 = \theta_1^*$ and $\bar{v} = v_1$ when $\theta_1 = 0$ as we decrease θ_1 whereas a Rayleigh wave has a fixed speed. The incident wave forces the surface disturbance to travel at a specified speed.

An independent approach can also be made by setting $a_1 = 0$ in (2.12) to (2.14). The three homogeneous equations may possess nonzero solutions for a_2, A_{3x} and A_{4x} if the determinant of the coefficients is zero. This leads to a polynomial equation for the determination of the Rayleigh wave speeds. The problem was studied by Suhubi and Eringen [2].

v. First limiting case for a zero angle of incidence. We now study the problem of reflection when the angle of incidence θ_1 tends to zero. From the previous case we know that θ_3 is complex unless $\omega \geq \omega^*$ and $(1.58)_2$ is satisfied in which case θ_3 is real and is given by $\cos\theta_3 = v_3/v_1$. The limiting angle θ_4 is likewise given by $\cos\theta_4 = v_4/v_1$ from $(2.11)_3$ with $\cos\theta_1 = 1$. The angle θ_2 is equal to θ_1 and tends to zero also. Thus the general solution reduces to zero motion as θ_1 goes to zero.

To obtain a nonzero solution we present briefly an analysis similar to the one given by Goodier and Bishop [4] for the classical case. The basic procedure is to expand the angle relations (2.11) in powers of θ_1 (Taylor's series about $\theta_1 = 0$) and similarly for the general solution given by (2.15) and (2.16). Afterwards the product $a_1\theta_1$ is assumed to remain finite as θ_1 tends to zero. The resulting solution is a nonzero motion. Carrying out this process we find for the scalar potential

$$\bar{u} = \bar{u}_1 + \bar{u}_2 = -a_1\theta_1 2\left(\frac{\delta^*}{\lambda\gamma^*} + ik_1 z\right)\exp[ik_1(y-v_1 t)] + O(\theta_1^2) \qquad (2.23)$$

where

$$\delta^* = (2\mu+\kappa)^2\,(Q_1^2-1)\left(\frac{v_1^2}{v_4^2} - 1\right)^{1/2}$$

$$\gamma^* = (\mu+\kappa)\left(\frac{v_1^2}{v_4^2} - 1\right) - \mu - Q_2^2 - (\mu+\kappa)Q_1^2\left(\frac{v_1^2}{v_4^2}-1\right)^{1/2}\left(\frac{v_1^2}{v_3^2}-1\right)^{1/2} \qquad (2.24)$$

$$+ (\mu Q_1^2 + Q_3^2)\left(\frac{v_1^2}{v_4^2} - 1\right)^{1/2}\left(\frac{v_1^2}{v_3^2} - 1\right)^{-1/2} .$$

If we allow θ_1 to go to zero and a_1 to tend to infinity such that $-2a_1\theta_1 = a_0$ = constant then equation (2.23) will yield a nonzero motion of the medium. The first term in (2.23) is constant and could represent the incident wave; however, the second term which represents the reflected longitudinal wave is proportional to the distance z. This is physically unacceptable because it becomes unbounded for increasing z. Also, as θ_1 tends to zero the incident wave and the reflected wave of the same type become indistinguishable from each other physically. Even though this theory has its flaw it does predict a reflected shear wave coupled with a microrotational wave at speed v_4.

vi. Second limiting process for a zero angle of incidence. In the case where θ_1 tends to zero it may be possible to obtain a physically meaningful solution following a method similar to the one used by Roesler [7] for the classical case.

The basic equations (2.12) and (2.13) are solved for the ratios

$$\frac{A_{4x}}{a_2+a_1} = -[\lambda + (\lambda+2\mu+\kappa)\ \tan^2\theta_1][(2\mu+\kappa)(Q_1^2-1)\ \tan\theta_4]^{-1} \tag{2.25}$$

$$\frac{A_{4x}}{a_2-a_1} = -(2\mu+\kappa)\ \tan\theta_1\ \{(\mu+\kappa)\ \tan^2\theta_4 - \mu - Q_2^2 - (\mu+\kappa)Q_1^2\ \tan\theta_3\tan\theta_4 + (\mu Q_1^2 + Q_3^2)\ \frac{\tan\theta_4}{\tan\theta_3}\}^{-1} \tag{2.26}$$

where (2.14) has again been used to eliminate A_{3x}. As θ_1 tends to zero the first ratio has a finite limit different from zero while the second one tends to zero. Experimentally the measurable quantity for the longitudinal displacement waves is $a_2 + a_1$ but neither a_2 nor a_1 alone since at $\theta_1 = 0$ the waves are indistinguishable. We then conclude that A_{4x} is finite and given by equation (2.25) since we assume the physically measurable quantity $a_2 + a_1$ to be finite. For finite A_{4x} (2.26) shows that $a_2 - a_1$ must be infinite. Hence if $a_1 + a_2$ is finite and $a_2 - a_1$ is infinite they must be of opposite signs and both infinite in magnitude. The potentials (2.1) for the displacements then have the values

$$\bar{u} = \bar{u}_1 + \bar{u}_2 = (a_1 + a_2)\ \exp[ik_1(y - v_1 t)] \tag{2.27}$$

and

$$\underline{U}_4 = \underline{I}\ A_{4x}\ \exp[ik_4(y\ \cos\theta_4 + z\ \sin\theta_4 - v_4 t)] \tag{2.28}$$

where A_{4x} is given by (2.25) and the angle θ_4 is given by $(2.11)_3$ as

$$\cos\theta_4 = \frac{v_4}{v_1} \tag{2.29}$$

This solution satisfies the wave equations and appropriate boundary conditions and has been verified experimentally for classical elasticity [9]. In the present case it predicts a set of coupled waves at speed v_4 and angle θ_4 in addition to the longitudinal wave at speed v_1 along the surface. As pointed out previously, under certain conditions we may have reflected a set of coupled waves at speed v_3 and angle θ_3 also.

Summary

In summarizing the analysis of this chapter, we have seen that an incident longitudinal displacement wave at a plane stress free boundary, in general, reflects as a wave of the same type and two sets of coupled transverse waves. One set of coupled waves travels at speed v_3 in the direction ν_3 and the other set travels at speed v_4 in the direction ν_4. For normal incidence $(\theta_1 = 90^\circ)$ it was shown that the waves at speed v_3 and v_4 vanish and the only wave reflected normal to the surface is a wave of the same type as the incident one. As the angle of incidence is decreased from 90° the general solution is given by equations (2.11), (2.14), (2.15) and (2.16).

The general solution prevails until $\theta_1 = \theta_1^*$ is reached at which time $\theta_3 = 0$, $\theta_1 \neq 0$, and $A_{4x} = 0$. At this angle of incidence (2.19) shows that we have a surface motion traveling at speed v_3 and a reflected longitudinal wave at speed v_1 and angle $\theta_2 = \theta_1 = \theta_1^*$. As θ_1 is decreased from θ_1^* to 0° the angle θ_3 becomes complex. The interpretation here is that we have reflected into the medium a longitudinal wave at angle θ_1 and a set of coupled transverse waves at speed v_4 and angle θ_4 as well as a set of coupled surface waves decaying with depth into the medium and traveling at a speed c where $v_3 \geq c \geq v_1$. Finally as θ_1 tends to zero we have the limiting solution of case vi.

CHAPTER 3

REFLECTION OF COUPLED TRANSVERSE SHEAR AND MICROROTATIONAL WAVES

Formulation

In this chapter we study the reflection of a set of coupled transverse shear and microrotational waves at speed v_4. The presentation parallels that of Chapter 2 for the longitudinal wave.

Assuming the set of coupled waves at speed v_4 to be in the direction ν_1 the boundary conditions will be satisfied if we have reflected a longitudinal displacement wave at speed v_1 and direction ν_2 and the two sets of coupled waves at speeds v_3 and v_4 as shown in Figure 3.1. For the incident waves we have the potentials

$$U_1 = I\, A_{1x} \exp[i(k_4 \nu_1 \cdot r - \omega_4 t)]$$
$$\Phi_1 = (B_{1y}\, J + B_{1z}\, K) \exp[i(k_4 \nu_1 \cdot r - \omega_4 t)] \tag{3.1}$$

and for the reflected waves we have

$$\bar{u}_2 = a_2 \exp[i(k_1 \nu_2 \cdot r - \omega_1 t)]$$
$$U_\beta = I\, A_{\beta x} \exp[i(k_\beta \nu_\beta \cdot r - \omega_\beta t)] \tag{3.2}$$
$$\Phi_\beta = (B_{\beta y}\, J + B_{\beta z}\, K) \exp[i(k_\beta \nu_\beta \cdot r - \omega_\beta t)]$$

where $\beta = 3,4$ and not summed. The coefficients of Φ_1, Φ_3 and Φ_4

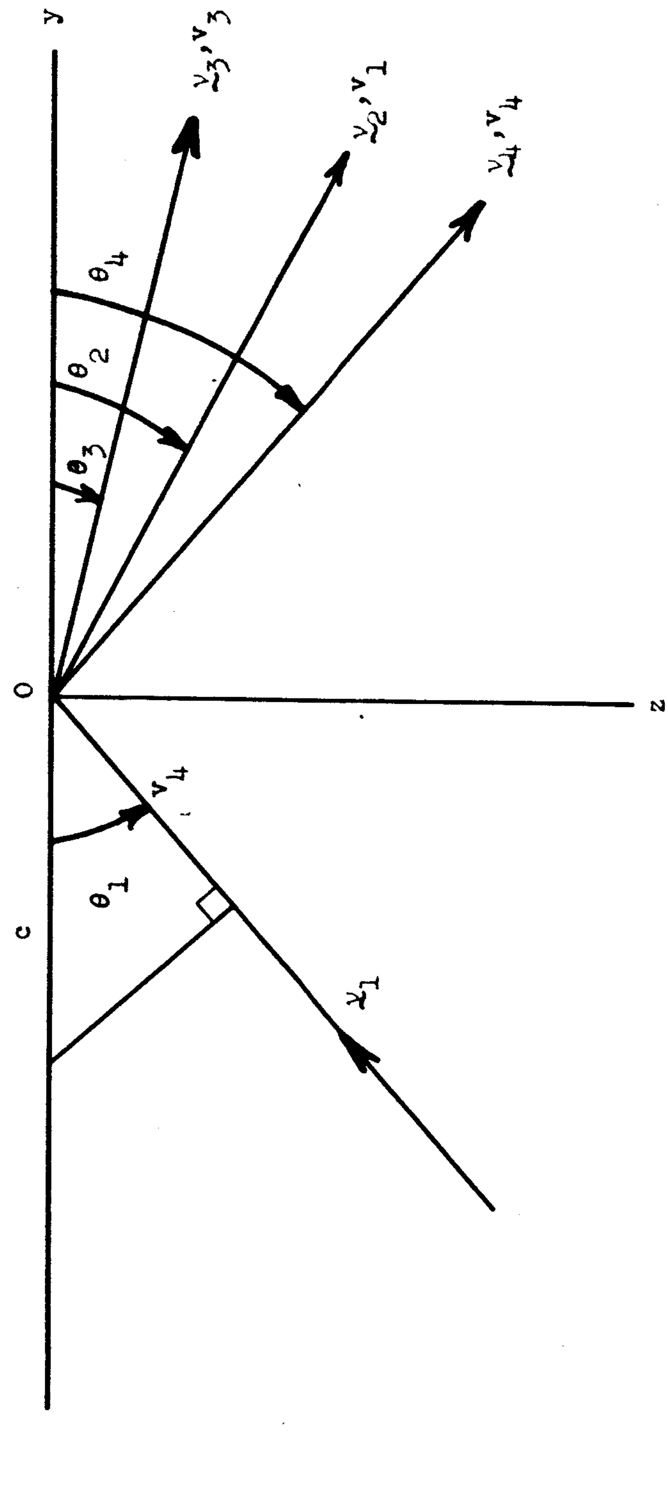

Figure 3.1. Reflection of coupled waves at speed v_4 .

are related to the respective U_1 coefficient through equations similar to (2.2). Assuming A_{1x} known we determine the amplitudes a_2, A_{3x} and A_{4x}.

The potentials (3.1) and (3.2) must satisfy the three conditions (2.3) to (2.5) for all y and t and $z = 0$. These three equations enable us to solve for the three amplitude ratios a_2/A_{1x}, A_{3x}/A_{1x} and A_{4x}/A_{1x} as well as the angles of reflection.

Basic Solution

Substitution of the potentials (3.1) and (3.2) into the boundary conditions (2.3) to (2.5) shows that they are satisfied if

$$\theta_4 = \theta_1 \quad , \quad \cos\theta_2 = \frac{v_1}{v_4}\cos\theta_1 \quad , \quad \cos\theta_3 = \frac{v_3}{v_4}\cos\theta_1 \tag{3.3}$$

and

$$[\lambda + (2\mu+\kappa)\nu_{2z}^2]k_1^2 \frac{a_2}{A_{1x}} - (2\mu+\kappa)k_3^2 \nu_{3y} \nu_{3z} \frac{A_{3x}}{A_{1x}}$$

$$- (2\mu+\kappa)k_4^2 \nu_{4y} \nu_{4z} \frac{A_{4x}}{A_{1x}} = (2\mu+\kappa)k_4^2 \nu_{1y} \nu_{1z} \tag{3.4}$$

$$(2\mu+\kappa)k_1^2\nu_{2y}\nu_{2z} \frac{a_2}{A_{1x}} + \{(\mu+\kappa)k_3^2 v_{3z}^2 - \mu k_3^2 v_{3y}^2 + \kappa\, \omega_0^2[v_3^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\}\frac{A_{3x}}{A_{1x}}$$

$$+ \{(\mu+\kappa)k_4^2 v_{4z}^2 - \mu k_4^2 v_{4y}^2 + \kappa\, \omega_0^2[v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\} \frac{A_{4x}}{A_{1x}}$$

$$= -\{(\mu+\kappa)k_4^2 v_{1z}^2 - \mu k_4^2 v_{1y}^2 + \kappa\, \omega_0^2[v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\} \tag{3.5}$$

and

$$k_3 \nu_{3z} [v_3^2 (1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1} \frac{A_{3x}}{A_{1x}} + k_4 \nu_{4z} [v_4^2 (1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1} \frac{A_{4x}}{A_{1x}}$$

$$= - k_4 \nu_{1z} [v_4^2 (1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1} \qquad (3.6)$$

Excluding the special case of $\nu_{1y} = 0$ which will be studied later, the three equations (3.4) to (3.6) yield for the amplitude ratios

$$\frac{a_2}{A_{1x}} = \frac{2}{\Delta}[(\mu+\kappa) \tan^2\theta_1 - \mu + Q_5^2] (2\mu+\kappa)(Q_1^{-2}-1) \tan\theta_1 \tan\theta_3 \qquad (3.7)$$

$$\frac{A_{3x}}{A_{1x}} = - \frac{2}{\Delta}[\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_2][(\mu+\kappa)\tan^2\theta_1 - \mu + Q_5^2]\tan\theta_1 \qquad (3.8)$$

$$\Delta\frac{A_{4x}}{A_{1x}} = [\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_2] \{\tan\theta_1[(\mu+\kappa)\tan^2\theta_3 - \mu + Q_4^2]$$

$$+ Q_1^{-2} \tan\theta_3[(\mu+\kappa)\tan^2\theta_1 - \mu + Q_5^2]\}$$

$$- (2\mu+\kappa)^2 (Q_1^{-2} - 1) \tan\theta_1 \tan\theta_2 \tan\theta_3 \qquad (3.9)$$

where Q_1^2 is given by $(2.17)_1$ and

$$Q_4^2 = \kappa \frac{\omega_0^2}{\omega^2} \frac{v_4^2}{\nu_{1y}^2} [v_3^2 (1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

$$Q_5^2 = \kappa \frac{\omega_0^2}{\omega^2} \frac{v_4^2}{\nu_{1y}^2} [v_4^2 (1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1} \qquad (3.10)$$

$$\Delta = [\lambda + (\lambda+2\mu+\kappa)\tan^2\theta_2]\ \{\tan\theta_1[(\mu+\kappa)\tan^2\theta_3 - \mu + Q_4^2]$$

$$- \tan\theta_3\ Q_1^{-2}[(\mu+\kappa)\tan^2\theta_1 - \mu + Q_5^2]\}$$

$$- (2\mu+\kappa)^2\ (Q_1^{-2} - 1)\ \tan\theta_1\ \tan\theta_2\ \tan\theta_3 \qquad (3.10)\ \text{cont.}$$

These equations give the amplitudes of the various wave potentials as functions of the incident wave amplitude and direction (θ_1). Since we lack definitive knowledge of the constitutive constants further discussion of the general solution is uninformative and we turn our attention to some special cases.

Special Cases

In this section we consider the special cases when an amplitude or an angle of the reflected waves is zero. In particular, we consider the case $v_{1y} = \cos\theta_1 = 0$ ($\theta_1 = 90^0$) and end this section with a discussion of grazing incident waves ($\theta_1 = 0$).

i. $a_2/A_{1x} = 0$. Since Q_1^2 is never 1, the numerator of (3.7) will equal zero if any one of the following three relations are satisfied

$$\tan\theta_1 = 0 \quad , \quad \tan\theta_3 = 0 \quad , \quad (\mu+\kappa)\tan^2\theta_1 - \mu + Q_5^2 = 0 \qquad (3.11)$$

The case $\theta_1 = 0$ is considered at the end of this section (the last case). If equation $(3.11)_2$ is satisfied then $\theta_3 = 0$ and (3.3) gives for θ_1, the angle the incident wave makes with the surface,

$$\cos\theta_1 = \frac{v_4}{v_3} \qquad (3.12)$$

Using this result along with equations (3.3) we can reduce the general solution to

$$\frac{a_2}{A_{1x}} = 0 \quad , \quad \frac{A_{4x}}{A_{1x}} = 1 \quad ,$$

$$\frac{A_{3x}}{A_{1x}} = -2\{(\mu+\kappa)(\frac{v_3^2}{v_4^2} - 1) - \mu + \kappa v_3^2 \frac{\omega_0^2}{\omega^2}[v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\}$$
$$\times \{-\mu + \kappa v_3^2 \frac{\omega_0^2}{\omega^2}[v_3^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\}^{-1} \qquad (3.13)$$

Thus, if the incident waves are at an angle θ_1 given by equation (3.12) then we have no wave motion associated with the potential $\bar{u}_2$, the reflected waves at speed v_3 are along the surface, and the coupled waves reflected into the media travel at the same speed and angle with the surface as the incident waves.

Now, if neither θ_1 nor θ_3 is zero, the amplitude a_2 can still vanish for a value of θ_1 satisfying $(3.11)_3$. Rearranging this equation we can write

$$\cos^2\theta_1 = (2\mu+\kappa)^{-1}\{\mu + \kappa - \kappa v_4^2 \frac{\omega_0^2}{\omega^2}[c_4^2 - v_4^2(1 - \frac{2\omega_0^2}{\omega^2})]^{-1}\} \qquad (3.14)$$

which yields a nonzero value of θ_1 for which a_2 vanishes.

We note that (3.8) shows that A_{3x} also vanishes when equation $(3.11)_3$ is satisfied. Thus for the angle θ_1 given by (3.14) the solution reduces to

$$a_2 = 0 \quad , \quad A_{3x} = 0 \quad , \quad \frac{A_{4x}}{A_{1x}} = 1 \qquad (3.15)$$

Hence for this value of θ_1 we simply have reflected a set of coupled waves of the same type making the same angle with the surface as the incident waves. Aside from the two extreme values of θ_1 (0^o and 90^o)

the only remaining special case is when $\theta_2 = 0$.

ii. $\theta_2 = 0$. When the angle θ_2 vanishes, θ_1 has the value

$$\cos \theta_1 = \frac{v_4}{v_1} \tag{3.16}$$

as given by equation $(3.3)_2$. This is a real angle since v_4 is less than v_1 . However, the angle θ_3 is complex for values of the frequency between ω_c and ω^* (see $(1.58)_1$) since v_3 is greater than v_1 and $\cos\theta_3 = v_3/v_1$ as given by (3.3). The amplitude ratios, which do not reduce greatly, are given by the general solution and are seen to be complex since θ_3 is complex.

Thus for an incident set of waves at the angle θ_1 given by (3.16) the reflected waves along the surface consist of two superposed sets. One set travels at speed $c = v_1$ and decays with depth into the medium. This set corresponds to the potentials $\mathfrak{U}_3$ and Φ_3 . The other surface wave at speed v_1 , corresponding to potential $\bar{u}_2$, has no decay factor, although any further decrease of θ_1 would result in a decaying wave at speed c , less than v_1 . There will also be a set of reflected waves of the same type as the incident waves. These waves will undergo a phase shift since the amplitude ratio A_{4x}/A_{1x} will be complex.

iii. $\theta_1 = 90^\circ$. Since we divided our equations by $\nu_{1y} = \cos\theta_1$ earlier, we now consider separately the situation if $\nu_{1y} = 0$ $(\theta_1 = 90^\circ)$. Then $\nu_{1z} = -1$ and (3.3) shows that $\nu_{2y} = \nu_{3y} = \nu_{4y} = 0$ and hence $\nu_{2z} = \nu_{3z} = \nu_{4z} = 1$. Using these relations in (3.4), (3.5) and (3.6) we obtain three equations which yield amplitude ratios

$$\frac{a_2}{A_{1x}} = 0 \tag{3.17}$$

$$\frac{A_{3x}}{A_{1x}} = -\frac{2k_4}{\Delta_1}\left[v_4^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\left\{(\mu+\kappa)k_4^2 + \kappa\,\omega_0^2\left[v_4^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\right\}$$

and (3.17) cont.

$$\Delta_1 \frac{A_{4x}}{A_{1x}} = k_4\left[v_4^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\left\{(\mu+\kappa)k_3^2 + \kappa\omega_0^2\left[v_3^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\right\} + k_3\left[v_3^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\left\{(\mu+\kappa)k_4^2 + \kappa\omega_0^2\left[v_4^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\right\}$$

where

$$\Delta_1 \equiv k_4\left[v_4^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\left\{(\mu+\kappa)k_3^2 + \kappa\omega_0^2\left[v_3^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\right\} - k_3\left[v_3^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\left\{(\mu+\kappa)k_4^2 + \kappa\omega_0^2\left[v_4^2\left(1-\frac{2\omega_0^2}{\omega^2}\right)-c_4^2\right]^{-1}\right\} \quad (3.18)$$

Hence, when the waves are incident normal to the boundary there are two sets of waves reflected normal to the boundary, one at speed v_4 and one at speed v_3. This is quite different from the classical case where an incident shear wave normal to the boundary results in a reflected shear wave (180° out of phase with the incident wave) normal to the boundary. For our solution to reduce to the classical case we set the constitutive coefficients $\kappa = \alpha = \beta = \gamma = 0$. With these coefficients zero v_3 vanishes and v_4 reduces to the classical shear wave velocity. At this point it appears that (3.5) and (3.6) yield two different solutions for A_{4x}/A_{1x}. This difficulty can be resolved by remembering that equation (3.6) comes from the couple stress boundary condition (2.5) which vanishes identically if $\alpha = \beta = \gamma = 0$. Thus equation (3.6) is nonexistent and the solution does indeed reduce to the classical one.

iv. $\theta_1 = 0$. An analysis similar to case vi of Chapter 1 shows that if $A_{1x} + A_{4x}$ is finite and we are to have a_2 and A_{3x} finite then $A_{4x} - A_{1x}$ must be infinite. However, both angles θ_2 and θ_3 are complex and we would have the surface shear wave represented by $A_{1x} + A_{4x}$ and the two exponentially decaying surface motions represented by a_2 and A_{3x} . Since all of these waves travel at the same speed $c = v_4$ it appears that separation of these effects experimentally would be extremely difficult. Hence it is postulated that we would indeed see a transverse surface wave at speed v_4 represented by $A_{1x} + A_{4x}$.

Summary

Summarizing, we see that in general an incident set of coupled waves at speed v_4 results in the reflection of two sets of coupled waves, one set at speed v_4 and the other at speed v_3 , and a longitudinal displacement wave at speed v_1 , assuming $\omega > \omega_c$. If $\omega < \omega_c$ the set of reflected waves at speed v_3 degenerates to a vibration of the medium and we have essentially the classical case.

If $\omega > \omega_c$ and $\theta_1 = 90^o$ we have two sets of coupled waves reflected normal to the boundary. As θ_1 is decreased the reflected waves move toward the surface and when θ_1 equals the critical value given by (3.12) $\theta_3 = 0$, the longitudinal wave vanishes, the set of waves at speed v_3 becomes parallel to the surface, and reflected into the medium is the set of coupled waves at speed v_4 similar to the incident waves and in phase with them. As θ_1 is decreased beyond this value the surface waves associated with $\underline{U}_3$ and $\underline{\Phi}_3$ decay with depth into the medium, and reflected into the medium we have a longitudinal wave at speed v_1 and a set of coupled waves similar to the incident waves at speed v_4 . The amplitude ratios become complex since θ_3 is complex indicating that the reflected waves have a phase shift. With further decrease of θ_1 we reach another value such that $\theta_2 = 0$. The only waves reflected into the medium now are the coupled waves of the same type as the incident waves.

Further decrease of θ_1 makes θ_2 complex as well as θ_3, hence the waves associated with $\bar{u}_2$, $\mathfrak{U}_3$ and $\mathfrak{L}_3$ all travel at speed c and decay with depth into the medium. Equations (3.9) and $(3.10)_3$ then show that the numerator of A_{4x}/A_{1x} is the complex conjugate of the denominator. This means $|A_{4x}/A_{1x}| = 1$ and the waves reflected into the medium are of the same type and magnitude as the incident waves but with a phase shift. This phenomenon is similar to total reflection in a refraction problem.

When $\theta_1 = 0$ we predict a set of coupled waves at speed v_4 along with the exponentially decaying surface motion associated with the potentials $\bar{u}_2$, $\mathfrak{U}_3$ and $\mathfrak{L}_3$. Experimentally these superposed motions appear to be inseparable.

At the value of the angle θ_1 given by (3.14) the amplitude a_2 vanishes. The numerator of (3.9) set equal to zero may give possible values of θ_1 such that A_{4x} would equal zero. In order to make an analysis of this case meaningful we must wait for an experimental determination of the constitutive coefficients involved.

CHAPTER 4

REFLECTION OF A LONGITUDINAL MICROROTATIONAL WAVE

Formulation

This chapter is devoted to the study of the reflection of a longitudinal microrotational wave at speed v_2 at a stress free surface of a half space. We recall that this wave has its microrotation vector parallel to the direction of propagation. Since this wave degenerates into a distance decaying vibration for $\omega < \omega_c$ the incident wave and its reflections disappear. Thus the analysis must be confined to the range $\omega > \omega_c$.

At first glance it appears that an incident longitudinal microrotational wave may reflect only another wave of the same type at the same angle. This would be similar to the reflection of horizontally polarized waves in the classical case. Unfortunately, this one reflected wave satisfies two boundary conditions only.

An investigation shows that horizontally polarized transverse displacement waves with particle motions only in the x-direction will provide a set of reflected waves consistent with the boundary conditions. Coupled with each transverse displacement wave there is a transverse microrotational wave whose microrotation vector is perpendicular to the direction of propagation and in this case the x axis also. This picture of reflection is self-consistent without the necessity of having a reflected longitudinal displacement wave at speed v_1 . Hence when we discuss longitudinal waves we mean the longitudinal microrotation wave at speed v_2 and when we mention coupled waves we mean the transverse microrotation wave coupled with the transverse displacement wave at speed v_3 and a similar set at speed v_4 .

In order that the transverse displacement waves have particle motion in the x-direction only U_x must equal zero. Moreover we have $\underset{\sim}{v} \cdot \underset{\sim}{A} = 0$ or $v_y A_y + v_z A_z = 0$. It follows that the vector potential must have the form

$$\underset{\sim}{U} = A_y(\underset{\sim}{J} - \frac{v_y}{v_z}\underset{\sim}{K}) \exp[ik(\underset{\sim}{v} \cdot \underset{\sim}{r} - vt)] \qquad (4.1)$$

Thus we see that the potential $\underset{\sim}{U}$ actually has only one coefficient A_y . Also it was shown that the coefficient $\underset{\sim}{B}$ of the potential $\underset{\sim}{\Phi}$ is given in terms of $\underset{\sim}{A}$ by

$$\underset{\sim}{B} = -i \frac{\beta_A}{\beta_B} \underset{\sim}{v} \times \underset{\sim}{A}$$

Since both $\underset{\sim}{v}$ and $\underset{\sim}{A}$ lie in the y,z plane we see that $\underset{\sim}{B}$ will be parallel to the x-axis. Thus there are three unknown coefficients: b_2 for the reflected longitudinal microrotational wave, A_{3y} for the coupled waves at speed v_3 , and A_{4y} for the coupled waves at speed v_4 .

We are therefore led to assume that an incident longitudinal microrotation wave at a plane stress-free boundary will result in reflected waves as shown in Figure 4.1. The incident and reflected longitudinal waves are represented by $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$, respectively. One set of coupled transverse waves is traveling at speed v_3 in the direction $\underset{\sim}{v}_3$ and the other set is traveling at speed v_4 in the direction $\underset{\sim}{v}_4$. This situation is confirmed with the satisfaction of boundary conditions.

The potentials representing the waves in this problem are now explicitly stated. The potential for the incident longitudinal wave is

$$\bar{\varphi}_1 = b_1 \exp[ik_2(v_{1y}y + v_{1z}z - v_2t)] \qquad (4.2)$$

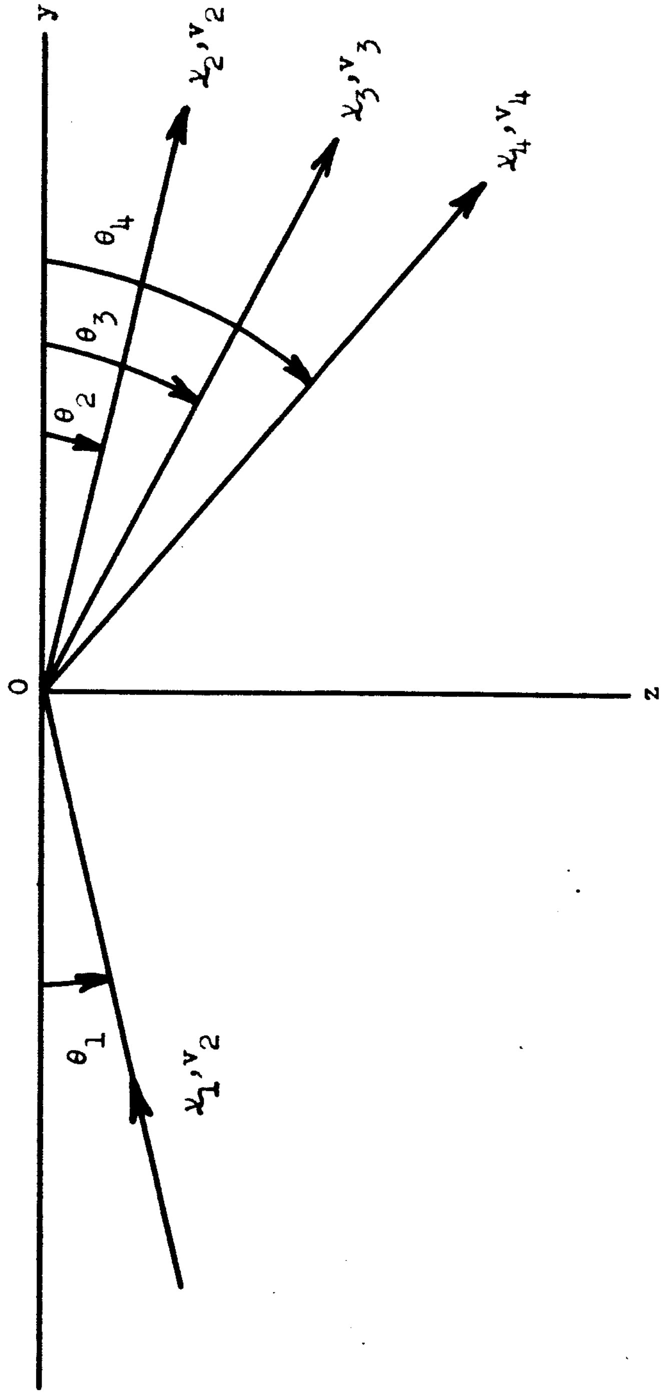

Figure 4.1. Reflection of a longitudinal microrotation wave.

and for the reflected longitudinal wave is

$$\bar{\varphi}_2 = b_2 \exp[ik_2(\nu_{2y}y + \nu_{2z}z - v_2t)] \tag{4.3}$$

The potentials for the transverse displacement and microrotation waves at speed v_3 are respectively given by

$$\underline{U}_3 = (A_{3y}\,\underline{J} - \frac{\nu_{3y}}{\nu_{3z}} A_{3y}\,\underline{K}) \exp[ik_3\nu_{3y}y + \nu_{3z}z - v_3t)] \tag{4.4}$$

$$\underline{\Phi}_3 = \underline{I}\, i \frac{\beta_{A3}}{\beta_{B3}} \frac{A_{3y}}{\nu_{3z}} \exp[ik_3(\nu_{3y}y + \nu_{3z}z - v_3t)] \tag{4.5}$$

and those at speed v_4 by

$$\underline{U}_4 = (A_{4y}\,\underline{J} - \frac{\nu_{4y}}{\nu_{4z}} A_{4y}\,\underline{K}) \exp[ik_4(\nu_{4y}y + \nu_{4z}z - v_4t)] \tag{4.6}$$

$$\underline{\Phi}_4 = \underline{I}\, i \frac{\beta_{A4}}{\beta_{B4}} \frac{A_{4y}}{\nu_{4z}} \exp[ik_4(\nu_{4y}y + \nu_{4z}z - v_4t)] \tag{4.7}$$

With the potentials as given above the three boundary conditions $t_{32} = t_{33} = m_{31} = 0$ for the free surface $z = 0$ are satisfied identically. The remaining three conditions are:

$$t_{31} = (\mu + \kappa)(U_{z,yz} - U_{y,zz}) - \kappa(\bar{\varphi}_{,y} + \Phi_{x,z}) = 0 \tag{4.8}$$

$$m_{32} = \beta(\bar{\varphi}_{,yz} - \Phi_{x,yy}) + \gamma(\bar{\varphi}_{,yz} + \Phi_{x,zz}) = 0 \tag{4.9}$$

$$m_{33} = (\alpha + \beta + \gamma)(\bar{\varphi}_{,zz} - \Phi_{x,yz}) + \alpha(\bar{\varphi}_{,yy} + \Phi_{x,yz}) = 0 \tag{4.10}$$

since $U_x = \Phi_y = \Phi_z = \frac{\partial}{\partial x} = 0$.

Basic Solution

As in the previous two problems we wish to determine the directions and amplitudes of the reflected waves when a known wave is incident on the boundary $z = 0$ at a known angle. Substituting the potentials into the boundary conditions (4.8) to (4.10), we find that they are all satisfied if

$$\begin{aligned} \theta_2 &= \theta_1 \\ \cos\theta_3 &= \frac{v_3}{v_2}\cos\theta_1 = \frac{v_3}{c} \\ \cos\theta_4 &= \frac{v_4}{v_2}\cos\theta_1 = \frac{v_4}{c} \end{aligned} \tag{4.11}$$

and

$$\kappa \frac{v_{1y}}{v_2} b_2 + i\{(\mu+\kappa)\frac{\omega}{v_3^2} + \kappa \frac{\omega_0^2}{\omega}[v_3^2(1-\frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\}A_{3y}$$
$$+ i\{(\mu+\kappa)\frac{\omega}{v_4^2} + \kappa \frac{\omega_0^2}{\omega}[v_4^2(1-\frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\}A_{4y} = -\kappa \frac{v_{1y}}{v_2} b_1 \tag{4.12}$$

$$(\beta + \gamma)k_2^2 v_{2z} v_{1y} b_2 - i(\beta \frac{v_{3y}^2}{v_{3z}^2} - \gamma)k_3 v_{3z}\omega_0^2[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1} A_{3y}$$
$$- i(\beta \frac{v_{4y}^2}{v_{4z}^2} - \gamma)v_{4z}k_4\omega_0^2[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}A_{4y} = -(\beta+\gamma)k_2^2 v_{1z}v_{1y}b_1 \tag{4.13}$$

$$k_2^2[(\alpha+\beta+\gamma)v_{2z}^2 + \alpha v_{2y}^2]b_2 - i(\beta+\gamma)v_{3y}k_3\omega_0^2[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}A_{3y}$$

$$-i(\beta+\gamma)k_4 v_{4y}\omega_0^2[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}A_{4y} = -k_2^2[(\alpha+\beta+\gamma)v_{1z}^2+\alpha v_{1y}^2]b_1 \qquad (4.14)$$

where we wrote $kv = \omega$. Again c represents the speed of the wave front along the surface $z = 0$.

Excluding the case $v_{1y} = 0$, studied later, the solution of the three equations (4.12), (4.13) and (4.14) for the amplitude ratios is

$$\Delta \frac{b_2}{b_1} = -\kappa(\beta+\gamma)v_2^2 \frac{\omega_0^2}{\omega^2}(\gamma \tan\theta_3 - \gamma \tan\theta_4 - \beta \cot\theta_3 + \beta \cot\theta_4)$$

$$+ \frac{v_2^2}{v_3^2} \frac{\{(\beta+\gamma)^2\tan\theta_1 + [\alpha + (\alpha+\beta+\gamma)\tan^2\theta_1][\gamma \tan\theta_4 - \beta \cot\theta_4]\}}{\{(\mu+\kappa)[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2] + \kappa v_3^2 \frac{\omega_0^2}{\omega^2}\}}$$

$$- \frac{v_2^2}{v_4^2} \frac{\{(\beta+\gamma)^2\tan\theta_1 + [\alpha + (\alpha+\beta+\gamma)\tan^2\theta_1][\gamma \tan\theta_3 - \beta \cot\theta_3]\}}{\{(\mu+\kappa)[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2] + \kappa v_4^2 \frac{\omega_0^2}{\omega^2}\}} \qquad (4.15)$$

$$\Delta \frac{A_{3y}}{b_1} = \frac{1}{\omega}\kappa(\beta+\gamma)^2 v_2 v_{1z}[v_3^2(1-\frac{2\omega_0^2}{\omega^2}) - c_4^2] - i\omega(\beta+\gamma)\frac{v_2^2}{v_4^2}\frac{v_{1z}}{\omega_0^2}$$

$$\times [\alpha + (\alpha+\beta+\gamma)\tan^2\theta_1][v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]\{(\mu+\kappa)[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2] + \kappa v_4^2 \frac{\omega_0^2}{\omega^2}$$

(4.16)

$$\Delta \frac{A_{4y}}{b_1} = -\frac{i}{\omega}\kappa(\beta+\gamma)^2 v_2 \, v_{1z}[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2] - i\omega(\beta+\gamma)\frac{v_2^2}{v_3^2}\frac{v_{1z}}{\omega_0^2}$$

$$x[\alpha + (\alpha+\beta+\gamma)\tan^2\theta_1][v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]\{(\mu+\kappa)[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]$$

$$+ \kappa v_3^2 \frac{\omega_0^2}{\omega^2}\} \quad (4.17)$$

where

$$\Delta \equiv \kappa(\beta+\gamma)\, v_2^2 \frac{\omega_0^2}{\omega^2}(\gamma \tan\theta_3 - \gamma \tan\theta_4 - \beta \cot\theta_3 + \beta \cot\theta_4)$$

$$+ \frac{v_2^2}{v_3^2}\{(\beta+\gamma)^2\tan\theta_1 - [\alpha + (\alpha+\beta+\gamma)\tan^2\theta_1][\gamma \tan\theta_4 - \beta \cot\theta_4]\}\{(\mu+\kappa)[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]$$

$$+ \kappa v_3^2 \frac{\omega_0^2}{\omega^2}\} - \frac{v_2^2}{v_4^2}\{(\beta+\gamma)^2\tan\theta_1 - [\alpha + (\alpha+\beta+\gamma)\tan^2\theta_1][\gamma \tan\theta_3 - \beta \cot\theta_3]\}$$

$$x\ \{(\mu+\kappa)[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2] + \kappa\, v_4^2 \frac{\omega_0^2}{\omega^2}\} \quad (4.18)$$

It can be observed that in the general case when all the angles are real and the waves are reflected into the medium as in Figure 4.1 the ratio b_2/b_1 is real and the two ratios A_{3y}/b_1 and A_{4y}/b_1 are pure imaginary. Physically, this means that the reflected longitudinal wave (b_2) will be in phase with the incident wave (b_1) or out of phase by 180° depending upon whether the ratio is positive or negative. Similarly, the reflected transverse waves (A_{3y} and A_{4y}) will lead or lag the incident wave (b_1) by 90° depending upon the ratios being either positive or negative.

Special Cases

From (4.11) we can see that θ_4 is never zero since $v_4 < v_2$. If the inequality (1.61) is satisfied (this should be satisfied for the majority of materials as j is small) then $v_3 < v_2$ and the angle θ_3 is never zero either. We consider the possibilities of the amplitude ratios being zero and end the section by considering the case of θ_1 tending to zero.

i. $\theta_1 = 90^\circ$. Then $\nu_{1y} = \cos\theta_1 = 0$ and $\nu_{1z} = -1$, i.e., we have an incident wave normal to the boundary. From (4.11) we see that $\nu_{2y} = \nu_{3y} = \nu_{4y} = 0$, and hence $\nu_{2z} = \nu_{3z} = \nu_{4z} = 1$. Substitution of these values into equations (4.12), (4.13) and (4.14) will yield $b_2 = -b_1$ and two linear simultaneous homogeneous equations for A_{3y} and A_{4y}. If ω is not infinite, then $v_3 \neq v_4$ and the determinant of the coefficients of these equations is nonzero. Thus we conclude that

$$A_{3y} = A_{4y} = 0 \qquad b_2 = -b_1 \tag{4.19}$$

and an incident longitudinal microrotation wave normal to the surface reflects a similar type wave with a phase shift of 180° also normal to the surface.

ii. Amplitude ratio $A_{3y}/b_1 = 0$. Here we wish to determine the angle θ_1 such that the amplitude ratio A_{3y}/b_1 vanishes. As usual, setting the numerator (of (4.16)) equal to zero we find that θ_1 must satisfy

$$\tan^2\theta_1 = -\alpha(\alpha+\beta+\gamma)^{-1} - \kappa(\beta+\gamma)(\alpha+\beta+\gamma)^{-1} v_4^2 \frac{\omega_0^2}{\omega^2} \{\mu[c_4^2 - v_4^2 + v_4^2 \frac{2\omega_0^2}{\omega^2}] + \kappa[c_4^2 - v_4^2 + v_4^2 \frac{\omega_0^2}{\omega^2}]\}^{-1} \tag{4.20}$$

Now since (see Fig. 1.3) $c_4^2 \geq v_4^2$ we see that $\tan^2\theta_1$ is always negative and θ_1 is complex. Physically, this shows that there is no value of θ_1 between 0^o and 90^o that will make the amplitude ratio A_{3y}/b_1 vanish.

iii. Amplitude ratio $A_{4y}/b_1 = 0$. In a manner similar to that above, we consider the numerator of equation (4.17) to obtain

$$\tan^2\theta_1 = -\alpha(\alpha+\beta+\gamma)^{-1}$$

$$-\kappa(\beta+\gamma)(\alpha+\beta+\gamma)^{-1} v_3^2 \frac{\omega_0^2}{\omega^2}\{(\mu+\kappa)[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]+\kappa v_3^2\frac{\omega_0^2}{\omega^2}\}^{-1} \quad (4.21)$$

Thus we again conclude that there is no real value for the angle between 0^o and 90^o such that the amplitude ratio A_{4y}/b_1 vanishes.

iv. $\theta_1 = 0$. We now consider the case of a grazing incident longitudinal microrotation wave. Letting $\theta_1 = 0$ in the general solution we see that the angles θ_2, θ_3 and θ_4 are given by (4.11) as

$$\theta_2 = 0 \quad , \quad \cos\theta_3 = \frac{v_3}{v_2} \quad , \quad \cos\theta_4 = \frac{v_4}{v_2} \qquad (4.22)$$

In general $v_2 > v_3 > v_4$ and the angles are all real. Equations (4.15), (4.16) and (4.17) reduce to

$$b_2 = -b_1 \qquad A_{3y} = A_{4y} = 0 \qquad (4.23)$$

which again reduces to zero motion of the medium as can be seen by substitution of (4.23) into the potentials (4.2) to (4.7). Thus we realize the need of a limiting analysis once more as the incident angle θ_1 tends to zero.

In order to use a limiting procedure we go back to the three equations (4.12), (4.13) and (4.14) to get the amplitude ratios $A_{3y}/(b_1 + b_2)$ and $A_{4y}/(b_1 + b_2)$

$$\Delta \frac{A_{3y}}{b_1+b_2} = i\{\kappa(\beta+\gamma)k_4 \frac{v_{4y}}{v_2} \omega_0^2 [v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

$$+ \alpha k_2^2(\mu+\kappa) \frac{\omega}{v_4^2} + \alpha\kappa k_2^2 \frac{\omega_0^2}{\omega} [v_4^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\} \qquad (4.24)$$

$$\Delta \frac{A_{4y}}{b_1+b_2} = i\{\kappa(\beta+\gamma)k_3 \frac{v_{3y}}{v_2} \omega_0^2[v_3^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}$$

$$- \alpha(\mu+\kappa)k_2^2 \frac{\omega}{v_3^2} - \alpha\kappa k_2^2 \frac{\omega_0^2}{\omega} [v_3^2(1 - \frac{2\omega_0^2}{\omega^2}) - c_4^2]^{-1}\} \qquad (4.25)$$

where

$$\Delta \equiv (\beta+\gamma)v_{4y}k_4\omega_0^2[v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}\{(\mu+\kappa) \frac{\omega}{v_3^2} + \kappa \frac{\omega_0^2}{\omega}[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}\}$$

$$- (\beta+\gamma)v_{3y}k_3\omega_0^2[v_3^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}\{(\mu+\kappa)\frac{\omega}{v_4^2} + \kappa \frac{\omega_0^2}{\omega} [v_4^2(1-\frac{2\omega_0^2}{\omega^2})-c_4^2]^{-1}\}$$

$$(4.26)$$

where we have set $\theta_1 = 0$.

Thus at $\theta_1 = 0$, $b_1 + b_2$ finite but $b_2 - b_1 = \infty$, we predict that there exists two sets of coupled transverse waves along with the longitudinal wave. One set of coupled waves is at speed v_3 and angle θ_3 and the other is at speed v_4 and angle θ_4 where the angles are given by (4.22). The amplitude of the grazing longitudinal wave is $b_1 + b_2$ and the corresponding amplitudes of the two sets of coupled waves reflected into the medium are given by (4.24) and (4.25). From these the potentials, displacements, and finally the stresses can be obtained.

Summary

In summarizing the reflection of a longitudinal microrotational wave we first note that the problem exists only if $\omega > \omega_c$. If $\omega < \omega_c$ the incident wave degenerates into a vibratory motion of the medium and we have no reflection problem.

When the incident wave strikes normal to the boundary there is reflected a wave of the same type also normal to the boundary but with a 180° phase shift from the incident wave. As the incident angle θ_1 is decreased from 90° (normal incidence) to 0° (grazing incidence) we see that the general solution prevails and there are reflected two sets of coupled transverse waves at speeds v_3 and v_4, respectively, along with a wave of the incident type. The two sets of coupled waves suffer phase shifts with respect to the incident wave. The reflected longitudinal wave may or may not have a 180° phase shift depending on whether the amplitude ratio b_2/b_1 is negative or positive, respectively.

We concluded the analysis of this chapter by considering the reflection of a grazing longitudinal microrotation wave. By using a limit analysis similar to that of Roesler we predicted two sets of coupled transverse waves at angles θ_3 and θ_4 given by (4.22). In addition to these we have a resultant longitudinal wave which is the sum of the incident and reflected longitudinal waves.

CONCLUDING REMARKS

This article has presented the basic theory of wave propagation and three typical reflection problems for a micropolar elastic solid. These results are all steady-state solutions of plane waves reflecting at an infinite plane stress-free boundary. The extent of the solutions is such that for a given incident wave we know which waves are reflected, the laws of reflection, and the amplitude ratios of the potentials representing them.

A knowledge of the magnitudes of the constitutive coefficients would make this analysis more meaningful. There is much experimental work to be done by future workers to determine these material constants. The experimental work will be difficult as these effects are small for the classical problems and only become observable for waves with frequencies around one megacycle and above. To date there has been no experimental work done with elastic waves of such frequency, except possibly in crystals. An effort to correlate this theory with waves in crystals appears to be worthwhile.

The determination of how these waves or vibrations are established in the medium, the transient problem, has been completely neglected. Again, there is much work yet to be done before full understanding of these problems is achieved.

Other problems along these lines can now be completed. Of particular interest is the case of reflection and refraction at an interface between two different media. Another problem that is amenable to solution is that of the reflection from a fixed $(u = v = w = 0$, etc.) boundary. Work in this area is expected to start in the near future.

Clearly the present solutions in the micropolar wave theory only scratch the surface and much remains to be done on other theoretical problems, e.g., propagation of waves in finite bodies, initial value problems, diffraction theory and vibration problems.

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