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AN EQUITRIANGULAR INTEGRAL TRANSFORM AND ITS APPLICATIONS

by

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ABSTRACT

An integral transform is established to facilitate the solution of boundary value problems of the first kind in an equilateral triangular (equitriangular) region. The differential operator (of order 2n) of the problem contains only powers of the Laplacian ∇^2. The boundary values are zero in terms of $(\nabla^2)^i$, where i = 0, 1, 2,$\cdots$(n-1). The inhomogeneous terms and the initial conditions, if any, belong to a certain class of functions of which the constant is the most interesting member.

After some general discussions, this transform is applied to a number of problems in viscous flow and heat transfer all inside an equitriangular duct. Numerical work has been carried out for these problems to show the roles played by various parameters.

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I. INTRODUCTION

The method of transforms is well known and widely used in solving the partial differential equations of engineering and physics. In many cases, suitable transforms can be established especially for some finite regions. Finite Fourier and Hankel transforms are probably the most well-developed examples. In principle, the formalism of a finite integral transform has been developed by Kaplan and Sonnemann (ref. 1 and 2) for an arbitrary two- or three-dimensional region, using the eigenfunctions of a two- or three-dimensional counterpart of the Sturm-Liouville system. The explicit forms of the eigenfunctions are known only for a few simple regions, namely, the rectangle, the circle, the circular annulus (ref. 1), the circular sector and the circular-annulus sector (ref. 3) for boundary conditions of all three kinds. It is the purpose of this work to add to the above list the equilateral triangle with boundary condition of the first kind.

A set of the corresponding eigenfunctions has been discovered by Sen (ref. 4); he has also applied it to the problem of the deflection of thin plates (ref. 4 and 5). It has been recently applied to the transient viscous flow problems by Sen (ref. 6) and Crǎciun (ref. 7). All these applications are in the form of the classical method of separation of variables, and without any numerical results.

In the process of establishing this "equitriangular" integral transform, it is found that Sen's eigenfunctions do not form a complete set. However, by narrowing down the type of boundary value problems to be treated, it is seen that a transform can, after all, be established with Sen's incomplete set. Thus, in this work, not only is a new transform established, but a format is also discovered which shows how an incomplete set of eigenfunctions can be systematically employed to the greatest advantage. The problems that can be treated happen to be physically the most interesting. It is only required that the boundary condition (of the first kind) be the vanishing of the functional values, and that the inhomogeneous terms and the initial conditions belong to a certain class of functions of which the constant is the most interesting member.

The detailed properties of the transform will be presented after a short summary of the general theory developed by Kaplan and Sonnemann (ref. 1).

The transform is then used to solve a very general boundary value problem whose special cases are later identified with the following, heretofore unsolved, physical problems:

(1) Transient natural convection heat transfer.

Theoretical investigation of this problem has been made by Izumi (ref. 8) for a vertical circular tube. A similar analysis is made here for a vertical equitriangular duct.

(2) Steady combined natural and forced convection flow.

This problem with linearly varying wall temperatures has been solved for vertical ducts of various cross sections by Ostrach (ref. 9), Lu (ref. 3), Han (ref. 10) and Tao (ref. 11 and 12). Here, a solution in the vertical duct of equilateral triangular cross section is obtained by application of the equitriangular transform.

(3) Transient Viscous Flow with Suspended Particles.

The transient response of an incompressible fluid with suspended solid particles in a triangular duct, as a pressure gradient is applied, is investigated for the case of Stokes drag coupling. Although this is obviously the simplest example of the transient two-phase flow, its treatment, even in ducts of simple cross sections, seems to be absent from the existing literature.

(4) Transient Heat Conduction with Heat Generation.

This problem is well documented (ref. 13) for various kinds of regions. The conduction in an equitriangular column is, however, never attempted. It is solved in this work with uniform heat generation as an example.

Before the results of these problems are analyzed numerically, the following problems, whose solutions are known in closed forms, are treated extensively using the present method of transform:

(1) Viscous Flow of an Incompressible Fluid.

The steady solution of this problem exists in a very simple form (ref. 14). The method of transform yields the solution in an alternative form, namely, a series. As a by-product, the transient solutions (ref. 6 and 7) are also rederived by the present method.

(2) Steady Forced Convection of an Incompressible Fluid.

This problem was solved by Tao (ref. 15) in a closed form. Its series form is obtained here again by the method of equitriangular transform.

The above two problems are then used to investigate the rate of convergence and the Gibbs' phenomenon, if any, of the series. The comparison shows that the general behavior of the series is no different from an ordinary Fourier series. The details of the comparison are presented in the Appendix.

II. DEVELOPMENT OF AN EQUITRIANGULAR INTEGRAL TRANSFORM

A. General Theory of Finite Integral Transforms

A general development of the finite integral transform theory (ref. 1) based on the Helmholtz equation is briefly reviewed in this section for the boundary condition of the first kind.

Consider the Helmholtz equation

$$\nabla^2 \psi + \lambda^2 \psi = 0 \tag{1}$$

defined in a finite two-dimensional region R together with the boundary condition

$$\psi = 0 \qquad \text{on B} \tag{2}$$

where B is the boundary of R. This homogeneous problem has solutions only for a set of discrete values of λ, which are called the eigenvalues λ_n of the problem. The corresponding set of eigenfunctions associated with these eigenvalues is represented by ψ_n. Thus,

$$\nabla^2 \psi_n + \lambda^2{}_n \psi_n = 0 \tag{3}$$

$$\nabla^2 \psi_m + \lambda^2{}_m \psi_m = 0 \tag{4}$$

Multiplying (3) by ψ_m and (4) by ψ_n and subtracting, one obtains

$$(\lambda^2{}_m - \lambda^2{}_n)\psi_m \psi_n = \psi_m \nabla^2 \psi_n - \psi_n \nabla^2 \psi_m \tag{5}$$

Then, integrating over the region R, we have

$$(\lambda_m^2-\lambda_n^2)\int_R \psi_m \psi_n \, d\sigma = \int_R (\psi_m \nabla^2 \psi_n - \psi_n \nabla^2 \psi_m) d\sigma \tag{6}$$

By an application of the Green's theorem, this becomes

$$(\lambda_m^2-\lambda_n^2)\int_R \psi_m \psi_n \, d\sigma = \int_B (\psi_m \nabla \psi_n - \psi_n \nabla \psi_m)\cdot\hat{N} \, ds \tag{7}$$

But the boundary condition (2) demands that

$$\psi_m = 0 \quad \text{and} \quad \psi_n = 0 \quad \text{on } B \tag{8}$$

Then,

$$\int_R \psi_m \psi_n \, d\sigma = 0 \quad \text{if} \quad m \neq n \tag{9}$$

And for m = n, the integral

$$\int_R \psi_n^2 \, d\sigma = N_n \tag{10}$$

is called the norm N_n. Equation (9) demonstrates the orthogonality of the eigenfunctions. If the set ψ_n is complete, it is possible to express a large class of functions (ref. 16 and 17) as a series in ψ_n :

$$\phi = \sum_n C_n \psi_n \tag{11}$$

Multiplying (11) by ψ_m and integrating over the region R gives

$$\int_R \phi \, \psi_m \, d\sigma = \int_R \sum_n C_n \psi_n \psi_m \, d\sigma \tag{12}$$

But, by the previously established orthogonality of ψ_n, the right-hand side of (12) vanishes for $m \neq n$. For $m = n$, equation (12) gives C_n as

$$C_n = \frac{1}{N_n} \int_R \phi \, \psi_n \, d\sigma \tag{13}$$

The finite integral transform of the function ϕ, with respect to ψ_n, can now be defined as

$$\tilde{\phi} = \int_R \phi \, \psi_n \, d\sigma \tag{14}$$

And equation (11), which is the inversion formula for the transform takes the more convenient form

$$\phi = \sum_n \frac{\tilde{\phi} \, \psi_n}{N_n} \tag{15}$$

The transform of the Laplacian of ϕ may now be derived as follows. By the definition (14),

$$\widetilde{\nabla^2 \phi} = \int_R \psi_n \, \nabla^2 \phi \, d\sigma \tag{16}$$

The right-hand side of (16) can be written as

$$\int_R \nabla \cdot [\psi_n \nabla \phi - \phi \nabla \psi_n] \, d\sigma + \int_R \phi \, \nabla^2 \psi_n \, d\sigma$$

Applying Green's theorem to the first integral and using the relation (3) in the second integral, this becomes

$$\int_B [\psi_n \nabla \phi - \phi \nabla \psi_n] \cdot \hat{N} \, ds - \lambda^2{}_n \int_R \phi \, \psi_n \, d\sigma$$

The second integral here is clearly the transform of ϕ as given by (14).

Remembering that $\psi_n = 0$ on B, one finally obtains the relation

$$\overset{\frown}{\nabla^2 \phi} = - \lambda_n^2 \overset{\frown}{\phi} - \int_B \phi \, (\nabla \psi_n) \cdot \hat{N} \, ds \qquad (17)$$

The importance of this relationship is that it relates the transform of the Laplacian of ϕ to the transform of ϕ itself. The line integral in (17) involves only the boundary condition of the problem; i.e., the value of ϕ on B.

The task of establishing an integral transform reduces then to the search for the set of eigenvalues and eigenfunctions which satisfies the homogeneous differential equation (1) with boundary condition (2). The transform and inversion formulas are given by equations (14) and (15). The norm and transform of the Laplacian operator are found by direct calculations. The difficulty of course lies in the determination of the complete set of eigenfunctions. Just a set of eigenfunctions obtained by inspection, for example, is not enough. Beside the cases where the two-dimensional problem degenerates into two one-dimensional problems (e.g., the cases inside a rectangle, or a circular sector), there is no established method of getting a complete set of eigenfunctions. The only general method is that of an approximate nature based on the variational principle. However, it will be shown in the next section that even an incomplete set can be systematically employed for a large class of boundary-value problems.

B. An Equitriangular Integral Transform and Its Properties

An integral transform will be developed in this section for the region bounded by an equilateral triangle of side 2a, following the general procedure set out in the previous section. It is convenient to introduce here the trilinear coordinates (ref. 4). The region R under consideration and the coordinate system used are shown in Figure (1).

Sen (ref. 4) has shown that the system

$$\left\{ \begin{array}{ll} \nabla^2 \psi + \lambda^2 \psi = 0, \quad \text{in} \quad R & (18) \\ \psi = 0 \quad \text{on} \quad p_1, p_2, p_3 = 0 & (19) \end{array} \right.$$

has a set of eigenfunctions

$$\psi_n = \sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3 ,$$

$$\text{where} \quad p_3 = \sqrt{3}\, a - p_1 - p_2 \qquad (20)$$

together with the set of eigenvalues

$$\lambda_n = \frac{2n\pi}{\sqrt{3}a} \qquad n = 1,2,3,\ldots \qquad (21)$$

The norm of the set is

$$N_n = \int_R \psi_n^2 \, d\sigma$$

$$= \int_0^{\sqrt{3}a} \int_0^{\sqrt{3}a-p_1} \frac{2}{\sqrt{3}} \left[\sin \lambda_n p_1 + \sin \lambda_n p_2 - \sin \lambda_n (p_1+p_2)\right]^2 dp_1 dp_2$$

$$= \frac{3\sqrt{3}\, a^2}{2} \tag{22}$$

With respect to this set, one can define an equitriangular transform

$$\tilde{\phi}(n) = \int_0^{\sqrt{3}a} \int_0^{\sqrt{3}a-p_1} \frac{2}{\sqrt{3}} \phi(p_1,p_2) \Big[\sin \lambda_n p_1 + \sin \lambda_n p_2$$

$$- \sin \lambda_n (p_1 + p_2)\Big] dp_1 dp_2 \tag{23}$$

Although the eigenfunctions given by (20) are orthogonal with the eigenvalues (21), there is no assurance that ψ_n is a complete set. To have a complete set of orthogonal eigenfunctions, ψ_n as given by (20), with eigenvalues λ_n given by (21), is joined by the unknown complementary eigenfunctions Ω_m with associated eigenvalues ξ_m.

A function of a suitable class can now be expressed in terms of this complete set of eigenfunctions as

$$\phi(p_1,p_2) = \sum_n \frac{\tilde{\phi}(n)}{N_n} \psi_n(p_1,p_2) + \sum_m \frac{\bar{\phi}(m)}{M_m} \Omega_m(p_1,p_2) \tag{24}$$

where

$$\tilde{\phi}(n) = \int_R \phi(p_1,p_2)\ \psi_n(p_1,p_2)d\sigma \tag{25}$$

$$\bar{\phi}(m) = \int_R \phi(p_1,p_2)\ \Omega_m(p_1,p_2)d\,\sigma \tag{26}$$

$$N_n \quad = \int_R \psi^2{}_n(p_1,p_2)d\,\sigma \tag{27}$$

$$M_m \quad = \int_R \Omega^2{}_m(p_1,p_2)d\,\sigma \tag{28}$$

are two integral transforms and their norms. The transforms of the Laplacian operator in terms of the complete set of eigenvalues and eigenfunctions are

$$\widetilde{\nabla^2\phi}(n) = -\ \lambda^2{}_n\ \tilde{\phi}(n) - \int_B \phi(\ \nabla\psi_n)\cdot\hat{N}\ ds \tag{29}$$

$$\overline{\nabla^2\phi}(m) = -\ \xi^2{}_m\ \bar{\phi}(m) - \int_B \phi(\ \nabla\ \Omega_m)\cdot\hat{N}\ ds \tag{30}$$

Transforms of higher power of $\nabla^2\phi$, if needed, can be obtained by repeated use of (29) and (30).

In order to obtain a transform of functions having any prescribed value on the three sides, the eigenfunctions Ω_m must be known in detail. Then the integrations given by (26),(28), and (30) could be performed. However, if the functional value is zero on the boundary, the equations (29) and (30) become

$$\widetilde{\nabla^2}\ \phi(n) \quad = \quad -\ \lambda^2{}_n\ \tilde{\phi}(n) \tag{31}$$

$$\overline{\nabla^2 \phi(m)} = - \xi^2_m \bar{\phi}(m) \qquad (32)$$

Furthermore, Sen (ref. 6) has shown that a constant C can be expanded as

$$C = \sum_{n=1}^{\infty} \frac{2C}{n\pi} \left[\sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3 \right] = \sum_{n=1}^{\infty} \frac{2C}{n\pi} \psi_n(p_1, p_2)$$

$$(33)$$

Comparing this with (24) reveals that

$$\widetilde{C} = \frac{2C}{n\pi} N_n = \frac{3\sqrt{3}\, a^2}{n\pi} C \qquad (34)$$

$$\bar{C} = 0 \qquad (35)$$

(Note that C is a member of a class of functions whose "barred transforms" vanish. Another obvious member of this class is $\sin \lambda_5 p_1 + \sin \lambda_5 p_2 + \sin \lambda_5 p_3$. But in this work, we will only deal with the constant function).

Now consider the following problem:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \alpha \nabla^2 \phi + C & \text{, in the triangle R} \qquad (36) \\ \phi = 0 \quad \text{on B} & \qquad (37) \\ \phi = \phi_o \quad \text{at} \quad t = 0 & \qquad (38) \end{cases}$$

where ϕ_o is a constant.

Applying (31) and (34), one obtains

$$\begin{cases} \dfrac{d\tilde{\phi}}{dt} = -\alpha\lambda^2_n\tilde{\phi} + \dfrac{3\sqrt{3}a^2}{n\pi}C & (39) \\ \tilde{\phi} = \dfrac{3\sqrt{3}a^2}{n\pi}\phi_o \quad \text{at} \quad t = 0 & (40) \end{cases}$$

The solution for $\tilde{\phi}$ is then

$$\tilde{\phi} = \frac{3\sqrt{3}a^2}{n\pi}\left[\phi_o e^{-\alpha\lambda^2_n t} + \frac{C}{\alpha\lambda^2_n}(1 - e^{-\alpha\lambda^2_n t})\right] \qquad (41)$$

Now, applying (32) and (35) to the original system yields

$$\begin{cases} \dfrac{d\bar{\phi}}{dt} = -\alpha\xi^2_m\bar{\phi} & (42) \\ \bar{\phi} = 0 \quad \text{at} \quad t = 0 & (43) \end{cases}$$

The solution here is obviously

$$\bar{\phi} = 0 \qquad (44)$$

Applying the inversion (24), the solution to the original problem is seen to be

$$\phi = \sum_n \frac{2}{n\pi}(\sin\lambda_n p_1 + \sin\lambda_n p_2 + \sin\lambda_n p_3)\left[\phi_o e^{-\alpha\lambda^2_n t} + \frac{C}{\alpha\lambda^2_n}(1 - e^{-\alpha\lambda^2_n t})\right] \qquad (45)$$

Thus, solutions to this class of boundary value problems are obtained without a detailed knowledge of the eigenfunctions Ω_m at all.

This is then the systematic way of utilizing an incomplete set of eigenfunctions in solving a class of boundary-value problems. By employing the method of integral transforms, it is possible to state clearly what class of problems one can treat by using the "tilde transform" alone without the necessity of knowing the complementary set Ω_m. The problems must: (1) involve only powers of the Laplacian operator, say up to the nth order; (2) be of the first kind with zero boundary values of $(\nabla^2)^i$, where $i = 0, 1, 2 \ldots, (n-1)$; and (3) involve only constant inhomogeneous terms and initial conditions. The last requirement can be relaxed somewhat to admit function like $\sin \lambda_5 p_1 + \sin \lambda_5 p_2 + \sin \lambda_5 p_3$. But this is not of great practical interest.

The third general conclusion quoted above is obvious from the simple example. The first two can be seen easily by successive application of the transform on $(\nabla^2)^i$.

III. APPLICATIONS

A. A General Boundary Value Problem

A general boundary value problem is solved in this section by the equitriangular transform method. The viscous flow and heat transfer applications of the following sections are special cases of this general problem.

Consider the following problem:

$$\left\{\begin{array}{ll} \left.\begin{array}{l} a_1 \frac{\partial f}{\partial t} = a_2 \nabla^2 f + a_3 f + a_4 g + A(t), \qquad (46) \\ \\ b_1 \frac{\partial g}{\partial t} = b_2 \nabla^2 g + b_3 g + b_4 f + B(t), \qquad (47) \end{array}\right\} \text{in the triangle R} & \\ f = 0 \quad \text{on} \quad p_1, p_2, p_3 = 0 & (48) \\ g = 0 \quad \text{on} \quad p_1, p_2, p_3 = 0 & (49) \\ f = F(p_1, p_2) \quad \text{at} \quad t = 0 & (50) \\ g = G(p_1, p_2) \quad \text{at} \quad t = 0 & (51) \end{array}\right.$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are constants; $A(t)$, $B(t)$ are functions of time; $F(p_1, p_2)$, $G(p_1, p_2)$ are functions of the admissable class (e.g., constants or functions like $\sin \lambda_5 p_1 + \sin \lambda_5 p_2 + \sin \lambda_5 p_3$).

Applying the equitriangular transform, one obtains

$$\left\{ a_1 \frac{d\breve{f}}{dt} = -a_2 \lambda_n^2 \breve{f} + a_3 \breve{f} + a_4 \breve{g} + \frac{6a}{\lambda_n} A(t) \right. \tag{52}$$

$$\left\{\begin{array}{ll} b_1 \dfrac{d\breve{g}}{dt} = - b_2 \lambda_n^{\,2} \breve{g} + b_3 \breve{g} + b_4 \breve{f} + \dfrac{6a}{\lambda_n} B(t) & \qquad (53) \\[2ex] \breve{f} = \breve{F}(p_1, p_2) \quad \text{at} \quad t = 0 & \qquad (54) \\[2ex] \breve{g} = \breve{G}(p_1, p_2) \quad \text{at} \quad t = 0 & \qquad (55) \end{array}\right.$$

The transformed functions $\breve{f}$ and $\breve{g}$ can be determined by solving the ordinary differential equations (52) and (53) simultaneously with the conditions (54) and (55). The functions f and g are then given as

$$f = \frac{2}{3\sqrt{3}a^2} \sum_{n=1}^{\infty} \breve{f} \left[\sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3\right] \tag{56}$$

$$g = \frac{2}{3\sqrt{3}a^2} \sum_{n=1}^{\infty} \breve{g} \left[\sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3\right] \tag{57}$$

or, in Cartesian coordinates,

$$\begin{aligned} f = \frac{2}{3\sqrt{3}a^2} \sum_{n=1}^{\infty} \breve{f} \Bigg[& \sin\lambda_n \left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n \left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) \\ & + \sin\lambda_n \left(\frac{a}{\sqrt{3}} - X\right)\Bigg] \end{aligned} \tag{58}$$

$$\begin{aligned} g = \frac{2}{3\sqrt{3}a^2} \sum_{n=1}^{\infty} \breve{g} \Bigg[& \sin\lambda_n \left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n \left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) \\ & + \sin\lambda_n \left(\frac{a}{\sqrt{3}} - X\right)\Bigg] \end{aligned} \tag{59}$$

B. Viscous Flow of an Incompressible Fluid

The equation of motion for the fully developed flow of an incompressible viscous fluid in an equitriangular duct is (ref. 6 and 7)

$$\frac{\partial W}{\partial t} = \nu \nabla^2 W - \frac{1}{\rho} \frac{\partial P}{\partial Z}, \quad \text{in the triangle R} \tag{60}$$

where: W is the velocity; ν the kinematic viscosity; ρ the density; and $\frac{\partial P}{\partial Z}$ the pressure gradient which is a function of t only.

The no-slip viscous boundary condition at the walls is

$$W = 0 \quad \text{on} \quad p_1, p_2, p_3 = 0 \tag{61}$$

The initial condition is

$$W = W_o(p_1, p_2) \quad \text{at } t = 0 \tag{62}$$

This corresponds to the previous general boundary value problem with

$$\begin{aligned}
& f = W && g = 0 \\
& a_1 = 1 && b_1 = 0 \\
& a_2 = \nu && b_2 = 0 \\
& a_3 = 0 && b_3 = 0 \\
& a_4 = 0 && b_4 = 0 \\
& A(t) = -\frac{1}{\rho} \frac{\partial P}{\partial Z} && B(t) = 0 \\
& F(p_1, p_2) = W_o && G(p_1, p_2) = 0
\end{aligned}$$

Using these values in equations (52) through (55), one obtains,

$$\left\{ \begin{aligned}
& \frac{d\widetilde{W}}{dt} = -\nu \lambda^2{}_n \widetilde{W} + \frac{6a}{\lambda_n} \left(-\frac{1}{\rho} \frac{\partial P}{\partial Z} \right) && (63) \\
& \widetilde{W} = \widetilde{W}_o \quad \text{at} \quad t = 0 && (64)
\end{aligned} \right.$$

The following two initial conditions are considered:

1. Fluid at rest for $t < 0$. At $t = 0$ an exponential pressure gradient is applied.
2. Fluid in steady motion with constant pressure gradient for $t < 0$. At $t = 0$, the pressure gradient is exponentially dampened.

1. Fluid initially at rest.

For fluid initially at rest, the initial velocity condition is

$$W = W_o = 0 \quad \text{at} \quad t = 0 \tag{65}$$

The exponential pressure gradient is applied at $t = 0$ and is

$$-\frac{1}{\rho}\frac{\partial P}{\partial Z} = C\,e^{-\delta t} \quad \text{for } t \geq 0 \tag{66}$$

The solution to equations (63) and (64) with this initial condition and pressure gradient is

$$\tilde{W} = \frac{6\,a\,C}{\lambda_n(\nu\lambda_n^2-\delta)}\left(e^{-\delta t}-e^{-\nu\lambda_n^2 t}\right) \tag{67}$$

Substituting this into equation (56), we have

$$W = \frac{4C}{\sqrt{3}\,a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n(\nu\lambda_n^2-\delta)}\left(e^{-\delta t}-e^{-\nu\lambda_n^2 t}\right)\left[\sin\lambda_n p_1+\sin\lambda_n p_2+\sin\lambda_n p_3\right] \tag{68}$$

This is the solution obtained by Sen (ref. 6), here rederived by the method of transform.

2. Fluid initially in steady motion.

For fluid initially in steady motion with constant pressure gradient for $t < 0$ and exponential pressure damping applied at $t = 0$, the pressure gradient is

$$-\frac{1}{\rho}\frac{\partial P}{\partial Z} = \begin{cases} C & \text{for } t < 0 \\ Ce^{-\delta t} & \text{for } t \geq 0 \end{cases}$$

The initial velocity is the solution to the problem of steady viscous flow with a constant pressure gradient. This problem is

$$\begin{cases} \nu \nabla^2 W_o + C = 0 & (69) \\ W_o = 0 \quad \text{on } p_1, p_2, p_3 = 0 & (70) \end{cases}$$

This corresponds to the general boundary value problem with

$f = W_o$

$a_2 = \nu$

$A(t) = C$

as the only non-zero quantities.

Using these values in equations (52) through (55), one obtains

$$\tilde{W}_o = \frac{6aC}{\nu\lambda_n^{\;3}} \tag{71}$$

Note that W_o is an admissible initial condition as the inversion of (71) yields a series expansion in ϕ_n.

The solution to equations (63) and (64) with the initial condition (71) and the stated pressure gradient is

$$\tilde{W} = \frac{6aC}{\lambda_n(\nu\lambda_n^{\;2}-\delta)}\left(e^{-\delta t} - \frac{\delta}{\nu\,\lambda_n^{\;2}}\, e^{-\nu\lambda_n^{\;2} t}\right) \tag{72}$$

Substituting this into equation (56), we have

$$W = \frac{4C}{\sqrt{3}a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n(\nu\lambda_n^2 - \delta)} \left(e^{-\delta t} - \frac{\delta}{\nu\lambda_n^2} e^{-\nu\lambda_n^2 t} \right) \left[\sin\lambda_n p_1 + \sin\lambda_n p_2 + \sin\lambda_n p_3 \right] \tag{73}$$

This is the solution given by Crǎciun (ref. 7), again rederived by the method of transform.

The solution to the problem of steady viscous flow with a constant pressure gradient is obtained by substituting (71) into equation (58). Replacing W_o by W to emphasize that the solution is for the steady flow, we have

$$W = \frac{4K}{\sqrt{3}a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \left[\sin\lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} - X\right) \right] \tag{74}$$

where $K = C/\nu = -\frac{1}{\mu}\frac{\partial P}{\partial Z}$

Equation (74) can be nondimensionalized by letting

$$x = \frac{X}{2a/\sqrt{3}} \quad (-1 \leqslant x \leqslant +\tfrac{1}{2})$$

$$w = \frac{W}{K\,a^2}$$

$$y = \frac{Y}{a} \quad (-1 \leqslant y \leqslant +1)$$

Then, the nondimensional velocity distribution is

$$w = \frac{3}{2\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[\sin\frac{2n\pi}{3}(1 - \frac{3}{2}y + x) + \sin\frac{2n\pi}{3}(1 + \frac{3}{2}y + x) + \sin\frac{2n\pi}{3}(1 - 2x) \right] \tag{75}$$

The mean nondimensional velocity is

$$w_M = \frac{\int_R w \, d\sigma}{\int_R d\sigma} = \frac{9}{2\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{20} \qquad (76)$$

The nondimensional form of the closed form solution to this problem (ref. 14) is

$$w = \frac{1}{9} (1 - 2x)\left[(x + 1)^2 - \frac{9}{4} y^2 \right] \qquad (77)$$

The corresponding w_M is exactly 1/20.

Velocity profiles along the line $y = 0$ ($-1 \leqslant x \leqslant +.5$) were computed according to both equations (75) and (77), and plotted in Figure (2). These profiles, together with the rate of convergence of the series, are examined in detail in the Appendix.

C. Steady Forced Convection of an Incompressible Fluid

Consider the steady, full developed, laminar flow of an incompressible fluid with heat sources in an equitriangular duct. With constant axial pressure and temperature gradients, and with negligible energy dissipation, the momentum and energy equations are as follows:

$$\left.\begin{aligned} \nabla^2 W &= - K, \qquad (78) \\ \nabla^2 \theta &= K_1 W - K_2, \qquad (79) \end{aligned}\right\} \text{in the triangle } R$$

where W stands for the axial velocity; θ for the difference between T and T_w taken at the same axial position; and

$$K = - \frac{1}{\mu} \frac{\partial P}{\partial Z}$$

$$K_1 = \frac{\rho c}{k} \frac{\partial T}{\partial Z}$$

$$K_2 = \frac{Q}{k}$$

For other symbols, see Nomenclature.

The viscous and thermal boundary conditions at the wall are

$$\left. \begin{array}{l} W = 0 \\ \theta = 0 \end{array} \right\} \text{ on } B \tag{80}$$

This corresponds to the general boundary value problem with

$f = W$ $\qquad$ $g = \theta$

$a_2 = 1$ $\qquad$ $b_2 = 1$

$A(t) = K$ $\qquad$ $b_4 = - K_1$

$B(t) = K_2$

as the only non-zero quantities.

Using these in equations (52) through (55), one obtains

$$\tilde{W} = \frac{6a}{\lambda_n^{\,3}} K \tag{82}$$

$$\tilde{\theta} = - \frac{K_1}{\lambda_n^{\,2}} \tilde{W} + \frac{6a}{\lambda_n^{\,3}} K_2 \tag{83}$$

Then, the transformed temperature difference is

$$\tilde{\theta} = 6a \left(\frac{K_2}{\lambda_n^{\,3}} - \frac{KK_1}{\lambda_n^{\,5}} \right) \tag{84}$$

Substituting this into equation (58), we have

$$\theta = \frac{4}{\sqrt{3}a}\sum_{n=1}^{\infty}\left(\frac{K_2}{\lambda_n^3} - \frac{KK_1}{\lambda_n^5}\right)\left[\sin \lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} - X\right)\right] \qquad (85)$$

The velocity distribution given by the inversion of (82) was obtained in the previous section.

Equation (85) can be nondimensionalized by letting

$$\theta^* = \frac{\theta}{-KK_1 a^4} \qquad x = \frac{X}{2a/\sqrt{3}} \quad (-1 \leqslant x \leqslant + 1/2)$$

$$K_2^* = \frac{K_2}{-KK_1 a^2} \qquad y = \frac{Y}{a} \quad (-1 \leqslant y \leqslant + 1)$$

Then, the dimensionless temperature distribution is

$$\theta^* = \sum_{n=1}^{\infty}\left(\frac{9}{8n^5\pi^5} + \frac{3K_2^*}{2n^3\pi^3}\right)\left[\sin \frac{2n\pi}{3}(1 - \frac{3}{2}y + x) + \sin \frac{2n\pi}{3}(1 + \frac{3}{2}y + x) + \sin \frac{2n\pi}{3}(1-2x)\right] \qquad (86)$$

The closed form solution to this problem obtained by Tao (ref. 15) is

$$\theta^* = \frac{1}{108}(2x - 1)\left[(x + 1)^2 - \frac{9}{4}y^2\right]\left[(x^2 + \frac{3}{4}y^2 - 1) + 12 K_2^*\right] \qquad (87)$$

Dimensionless temperature profiles along the line $y = 0 (-1 \leqslant x \leqslant +.5)$ were computed according to both equations (86) and (87). The results are shown in Figure (3). These profiles are examined in detail in the Appendix.

Various heat transfer parameters are easily obtained using the velocity and temperature distributions. The mean temperature is given by

$$\theta^*_M = \int_R \theta^* \, d\sigma / \int_R d\sigma = \frac{1}{\sqrt{3}a^2} \int_{-R} \theta^* d\sigma \qquad (88)$$

The mean mixed temperature may also be calculated:

$$\theta^*_{MM} = \int_R \theta^* \, w \, d\sigma / \int_R w \, d\sigma = \frac{1}{\sqrt{3} \, a^2 w_M} \int_R \theta^* \, w \, d\sigma \qquad (89)$$

The average Nusselt number may be defined as

$$Nu = \frac{d}{k \, \theta^*_{MM}} \int_B q \, ds / \int_B ds = \frac{(K_2^* + w_M)}{3 \, \theta^*_{MM}} \qquad (90)$$

Using the series form of the velocity, equation (75), and temperature, equation (86), we have

$$\theta^*_M = \sum_{n=1}^{\infty} \left(\frac{27}{8n^6 \pi^6} + \frac{9K_2^*}{2n^4 \pi^4} \right) = \frac{1}{280} + \frac{1}{20} K_2^* \qquad (91)$$

$$\theta^*_{MM} = \frac{\sum\limits_{n=1}^{\infty} \left(\frac{9}{16n^8 \pi^4} + \frac{3K_2^*}{4n^6 \pi^2} \right)}{\sum\limits_{n=1}^{\infty} \frac{1}{n^4}} = \frac{3}{560} + \frac{1}{14} K_2^* \qquad (92)$$

$$Nu = \frac{\left(\frac{9}{2\pi^4} \sum_{n=1}^{\infty} 1/n^4\right) + K_2^*}{\left(\frac{27}{16\pi^4} \sum_{n=1}^{\infty} 1/n^8 \Big/ \sum_{n=1}^{\infty} 1/n^4\right) + \left(\frac{9}{4\pi^2} \sum_{n=1}^{\infty} K_2^*/n^6 \Big/ \sum_{n=1}^{\infty} 1/n^4\right)}$$

$$= \frac{\frac{1}{20} + K_2^*}{\frac{9}{560} + \frac{3}{14} K_2^*} \tag{93}$$

which are exactly the values given by Tao (ref. 15).

D. Transient Natural Convection

In this section, we treat the case of transient laminar natural heat convection in an equitriangular duct of infinite length. The axis of the duct is orientated parallel to the direction of the generating body force. Initially the fluid is at rest with constant temperature T_o. The duct wall is maintained at a constant temperature T_w for $t \geq 0$. The ends of the duct are open to fluid at constant temperature T_o and the flow is thermally and hydrodynamically fully developed. Assuming negligible energy dissipation, and quasi-incompressible fluid (i.e., the density changes only with the temperature variation), the momentum and energy equations can be written as:

$$\left.\begin{aligned} \frac{\partial W}{\partial t} &= \nu \nabla^2 W + \beta f_Z \theta - \beta f_Z \theta_o , & (94) \\ \frac{\partial \theta}{\partial t} &= \alpha \nabla^2 \theta , & (95) \end{aligned}\right\} \text{in the triangle R}$$

where

$$\theta = T - T_w$$

$$\theta_o = T_o - T_w$$

The boundary conditions at the wall are

$$\left.\begin{aligned} W &= 0 \\ \theta &= 0 \end{aligned}\right\} \quad \text{on } B \qquad \begin{aligned} &(96) \\ &(97) \end{aligned}$$

and the initial conditions are

$$W = 0 \qquad \text{at } t = 0 \qquad (98)$$

$$\theta = \theta_o \qquad \text{at } t = 0 \qquad (99)$$

This is a special case of the general boundary value problem with

$f = W$ $\qquad g = \theta$

$a_1 = 1$ $\qquad b_1 = 1$

$a_2 = \nu$ $\qquad b_2 = \alpha$

$a_3 = 0$ $\qquad b_3 = 0$

$a_4 = \beta f_Z$ $\qquad b_4 = 0$

$A(t) = -\beta f_Z \theta_o$ $\qquad A(t) = 0$

$F(p_1,p_2) = 0$ $\qquad G(p_1,p_2) = \theta_o$

Substituting these values into equations (52) through (55), and using the previously obtained equitriangular transform of a constant, one obtains

$$\left\{\begin{aligned} &\frac{d\breve{W}}{dt} = -\nu\lambda_n^2\breve{W} + \beta f_Z\breve{\theta} - \frac{6a}{\lambda_n}\beta f_Z\theta_o && (100)\\ &\frac{d\breve{\theta}}{dt} = -\alpha\lambda_n^2\breve{\theta} && (101)\\ &\breve{W} = 0 \quad \text{at } t = 0 && (102)\\ &\breve{\theta} = \frac{6a}{\lambda_n}\theta_o \quad \text{at } t = 0 && (103) \end{aligned}\right.$$

The solution to equation (101) with the initial condition (103) is

$$\breve{\theta} = \frac{6a}{\lambda_n}\theta_o e^{-\alpha\lambda_n^2 t} \tag{104}$$

Substituting this into equation (59), we have

$$\theta = \frac{4\theta_o}{\sqrt{3}\,a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}e^{-\alpha\lambda_n^2 t}\left[\sin\lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right)\right.$$

$$\left. + \sin\lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} - X\right)\right] \tag{105}$$

Using the transformed temperature solution (104) in equation (100) gives

$$\frac{d\breve{W}}{dt} + \nu\lambda_n^2\breve{W} = \frac{6a}{\lambda_n}\left(e^{-\alpha\lambda_n^2 t} - 1\right)\beta f_Z\theta_o \tag{106}$$

The solution to equation (106) with the initial condition (102) is

$$\tilde{W} = \frac{6a}{\lambda_n^3} \frac{\beta f_Z \theta_o}{\nu(Pr-1)} \left[(1-e^{-\nu \lambda_n^2 t}) - Pr\,(1-e^{-\alpha \lambda_n^2 t}) \right], \text{ if } Pr \neq 1 \tag{107}$$

and,

$$\tilde{W} = \frac{6a}{\lambda_n^3} \frac{\beta f_Z \theta}{\nu} \left[\lambda_n^2 \nu t - (1-e^{-\alpha \lambda_n^2 t}) \right] \text{ , if } Pr = 1 \tag{108}$$

where, $Pr = \nu/\alpha$ is the Prandtl number.

Substituting these into equation (58), we have

$$W = \frac{4\beta f_Z \theta_o}{\sqrt{3} a \nu (Pr-1)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \left[(1-e^{-\nu \lambda_n^2 t}) - Pr\,(1-e^{-\alpha \lambda_n^2 t}) \right] \tag{109}$$

$$\times \left[\sin \lambda_n \left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} - X \right) \right], \text{ if } Pr \neq 1$$

and,

$$W = \frac{4\beta f_Z \theta_o}{\sqrt{3}\, a \nu} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \left[\lambda_n^2 \nu t - (1-e^{-\alpha \lambda_n^2 t}) \right] \tag{110}$$

$$\times \left[\sin \lambda_n \left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} - X \right) \right], \text{if } Pr = 1$$

Equations (105), (109), and (110) can be nondimensionalized by letting

$$\theta^* = \frac{\theta}{\theta_o} = \frac{T-T_w}{T_o-T_w} \qquad x = \frac{X}{2a/\sqrt{3}} \ (-1 \leqslant x \leqslant + \tfrac{1}{2})$$

$$w = \frac{W}{\beta f_Z (T_w - T_o) a^2/\nu} \qquad y = \frac{Y}{a} \quad (-1 \leqslant y \leqslant + 1)$$

Then, the dimensionless temperature and velocity distributions are

$$\theta^* = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\omega_n Fo} \Big[\sin \frac{2n\pi}{3}(1-\frac{3}{2} y + x) + \sin \frac{2n\pi}{3}(1 + \frac{3}{2} y + x) + \sin \frac{2n\pi}{3} (1 - 2 x) \Big] \tag{111}$$

$$w = \frac{3}{2\pi^3(1-Pr)} \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[(1-e^{-\omega_n Fo\, Pr}) - Pr(1-e^{-\omega_n Fo}) \Big] \tag{112}$$

$$\times \Big[\sin \frac{2n\pi}{3}(1- \frac{3}{2}y + x) + \sin \frac{2n\pi}{3}(1+\frac{3}{2} y+x)+\sin\frac{2n\pi}{3}(1-2x) \Big],$$

if $Pr \neq 1$

$$w = \frac{3}{2\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[(1-e^{-\omega_n Fo}) - \omega_n Fo \ e^{-\omega_n Fo} \Big] \tag{113}$$

$$\Big[\sin \frac{2n\pi}{3}(1- \frac{3}{2} y + x)+ \sin \frac{2n\pi}{3}(1 +\frac{3}{2}y+x)+\sin \frac{2n\pi}{3}(1-2x) \Big], \text{if } Pr = 1$$

where $Fo = \alpha t/a^2$ is the Fourier number; and $\omega_n = 4n^2\pi^2/3$.

The mean temperature and velocity are

$$\theta^*_M = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\omega_n Fo} \tag{114}$$

$$w_M = \frac{9}{2\pi^4(1-Pr)} \sum_{n=1}^{\infty} \frac{1}{n^4} \left[(1-e^{-\omega_n FoPr})-Pr(1-e^{-\omega_n Fo})\right] \text{ if } Pr \neq 1 \tag{115}$$

$$w_M = \frac{9}{2\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left[(1-e^{-\omega_n Fo}) - \omega_n Fo\, e^{-\omega_n Fo}\right], \text{ if } Pr = 1 \tag{116}$$

Dimensionless temperature profiles along the line $y = 0$ $(-1 \leqslant x \leqslant +.5)$ were calculated from equation (111). The profiles are shown in Figure (4). The temperature gradient near the duct walls decreases with increasing time; and, at all times, the value near the base is larger than the value near the tip.

Nondimensional velocity distributions along the line $y = 0$ $(-1 \leqslant x \leqslant +.5)$ were computed according to equation (112) for $Pr = .7$ and $Pr = 3.$ with results shown in Figure (5). With increasing time, the flow evolves from an initial local fluid acceleration near the walls with the fluid near the duct center at rest, to the final acceleration of the fluid core to steady flow. The steady flow is due to the density difference of the fluid in the duct at T_w and the fluid at the open ends at T_o.

The dimensionless mean temperature and velocity were calculated from equations (114), (115), and (116); with results given in Figure (6). The mean temperature increases faster than the mean velocity. This time lag is reduced as the Prandtl number is increased.

E. Steady Combined Natural and Forced Convection

Consider the steady laminar fully-developed flow with heat sources, constant axial pressure and temperature gradients, in an equitriangular duct. The duct is orientated in a direction parallel to the generating body force. We again assume that the energy dissipation is negligible and the fluid is quasi-incompressible. Then, the momentum and energy equations are

$$\left.\begin{aligned} &\mu \nabla^2 W + \rho \beta f_Z \theta = \Pi , &(117)\\ &k \nabla^2 \theta - \rho c A W = -Q, &(118)\end{aligned}\right\} \text{ in the triangle R}$$

where θ is defined the same way as in section C. Π, the pressure gradient parameter, is introduced according to the established theory of semi- or quasi-incompressible fluids (ref. 9 and 3). The boundary conditions at the wall are again

$$\left.\begin{aligned} &W = 0 &(119)\\ &\theta = 0 &(120)\end{aligned}\right\} \text{ on B}$$

The general boundary value problem reduces to this set if we take

$$\begin{array}{ll}
f = W & g = \theta \\
a_1 = 0 & b_1 = 0 \\
a_2 = \mu & b_2 = k \\
a_3 = 0 & b_3 = 0 \\
a_4 = \rho\beta f_Z & b_4 = -\rho c\, A \\
A(t) = -\Pi & B(t) = Q \\
F(p_1, p_2) = 0 & G(p_1, p_2) = 0
\end{array}$$

Using these values in equations (52) through (55), we have

$$\mu\, \lambda_n^{\,2}\, \tilde{W} - \rho\, \beta\, f_Z\, \tilde{\theta} = -\frac{6a}{\lambda_n}\, \Pi \qquad (121)$$

$$\rho\, c\, A\tilde{W} + k\, \lambda_n^{\,2}\, \tilde{\theta} = \frac{6a}{\lambda_n}\, Q \qquad (122)$$

Solving these simultaneously, one obtains

$$\tilde{\theta} = \frac{6a}{\lambda_n} \left(\frac{\mu Q\, \lambda_n^{\,2} + \rho c\, A\, \Pi}{\mu\, k\, \lambda_n^{\,4} + \rho^2 c\, A\, \beta\, f_Z} \right) \qquad (123)$$

$$\tilde{W} = \frac{6a}{\lambda_n} \left(\frac{\beta\, \rho\, f_Z Q - k\, \Pi\, \lambda_n^{\,2}}{\mu\, k\, \lambda_n^{\,4} + \rho^2\, c\, A\, \beta\, f_Z} \right) \qquad (124)$$

Substituting these into equations (58) and (59), we have

$$\theta = \frac{4}{\sqrt{3}a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\frac{\mu \, Q \, \lambda_n^{2} + \rho \, c \, A \, \Pi}{\mu \, k \, \lambda_n^{4} + \rho^{2} c \, A \, \beta \, f_Z} \right) \tag{125}$$

$$\times \left[\sin \lambda_n \left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}\,Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} - X \right) \right]$$

and $$W = \frac{4}{\sqrt{3}a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\frac{\beta \, \rho \, f_Z \, Q - \; k \, \Pi \, \lambda_n^{2}}{\mu \, k \, \lambda_n^{4} + \rho^{2} c \, A \, \beta \, f_Z} \right)$$

$$\times \left[\sin \lambda_n \left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2} \right) + \sin \lambda_n \left(\frac{a}{\sqrt{3}} - X \right) \right] \tag{126}$$

In order to nondimensionalize the velocity and temperature distributions, the hydraulic diameter of the duct is used as the characteristic length. When $A \neq 0$, $(c\, E_A \, Ad/Pr)^{1/2}$ and Ad are used as the characteristic velocity and temperature, respectively, For $A = 0$, $(c\, ET_w/Pr)^{1/2}$ and T_w are used. Then, the dimensionless temperature and velocity distributions are

$$\theta^{*}_{A} = \frac{8}{3} \sum_{n=1}^{\infty} \frac{1}{\zeta_n} \left(\frac{\zeta_n^{2} F_A + L_A Ra_A}{\zeta_n^{4} + Ra_A} \right) \left[\sin \frac{\zeta_n}{2} (1 - \sqrt{3}\, y + x) \right.$$

$$\left. + \sin \frac{\zeta_n}{2} (1 + \sqrt{3} y + x) + \sin \frac{\zeta_n}{2} (1 - 2x) \right] \quad \text{if } A \neq 0 \tag{127}$$

$$w_A = \frac{8}{3} \sum_{n=1}^{\infty} \frac{1}{\zeta_n} \left(\frac{F_A - \zeta_n^{2} L_A}{\zeta_n^{4} + R_{a_A}} \right) \left[\sin \frac{\zeta_n}{2} (1 - \sqrt{3}y + x) \right.$$

if A ≠ 0

(128)

$$\left. + \sin \frac{\zeta_n}{2} (1 + \sqrt{3}y + x) + \sin \frac{\zeta_n}{2} (1 - 2x) \right]$$

and,

$$\theta^{*} = \frac{8}{3} \sum_{n=1}^{\infty} \frac{F}{\zeta_n^{3}} \left[\sin \frac{\zeta_n}{2} (1 - \sqrt{3}y + x) + \sin \frac{\zeta_n}{2} (1 + \sqrt{3}y + x) \right.$$

(129)

$$\left. + \sin \frac{\zeta_n}{2} (1 - 2x) \right]$$

for A = 0

$$w = \frac{8}{3} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^{3}} \left(\frac{F}{\zeta_n^{2}} - L \right) \left[\sin \frac{\zeta_n}{2} (1 - \sqrt{3}y + x) + \sin \frac{\zeta_n}{2} (1 + \sqrt{3}y + x) \right.$$

(130)

$$\left. + \sin \frac{\zeta_n}{2} (1 - 2x) \right]$$

for A = 0

where:

$$\theta^{*}_{A} = \frac{\theta}{Ad} \qquad \theta^{*} = \frac{\theta}{T_w}$$

$$w_A = \frac{W}{\left(\frac{c\ E_A\ Ad}{Pr} \right)^{1/2}} \qquad w = \frac{W}{\left(\frac{c\ E\ T_w}{Pr} \right)^{1/2}}$$

$$x = \frac{X}{d} = \frac{\sqrt{3}X}{2a} \quad (-1 \leqslant x \leqslant + \tfrac{1}{2}) \qquad y = \frac{Y}{d} = \frac{\sqrt{3}Y}{2a} \ (\tfrac{\sqrt{3}}{2} \leqslant y \leqslant \tfrac{\sqrt{3}}{2})$$

$$E_A = Ra_A \left(\frac{\beta\ f_Z d}{c} \right) \qquad E = Ra \left(\frac{\beta f_Z d}{c} \right)$$

$$F_A = \frac{Qd}{kA} \qquad F = \frac{Qd^2}{kT_w}$$

$$L_A = \left(\frac{Pr\ d^3}{c\ E_A\ A} \right)^{1/2} \frac{\Pi}{\mu} \qquad L = \left(\frac{Pr}{c\ E_A\ T_w} \right)^{1/2} \frac{\Pi d^2}{\mu}$$

$$Ra_A = Pr\ Gr_A \qquad Ra = Pr\ Gr$$

$$Pr = \frac{\mu c}{k} \qquad \zeta_n = \frac{4n\pi}{3}$$

$$Gr_A = \frac{\rho^2 \beta f_Z A d^4}{\mu^2} \qquad Gr = \frac{\rho^2\ \beta\ f_Z\ T_w\ d^3}{\mu^2}$$

The mass through-flow is

$$V = \rho \int_R W\ d\ \sigma$$

Then, the dimensionless mass through-flow is

$$v_A = 8\sqrt{3} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2} \left(\frac{F_A - \zeta_n^2\ L_A}{\zeta_n^4 + Ra_A} \right) \quad \text{if } A \neq 0 \qquad (131)$$

and,

$$v = 8\sqrt{3} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^4} \left(\frac{F}{\zeta_n^2} - L \right) \quad \text{for } A = 0 \qquad (132)$$

where

$$v_A = \frac{1}{E_A} \frac{\beta f_Z}{k} V \qquad v = \frac{1}{E} \frac{\beta f_Z}{k} V$$

The mean mixed temperature is

$$\theta_{MM} = \int_R \theta \, W \, d\sigma \Big/ \int_R W \, d\,\sigma$$

Then the dimensionless mean mixed temperature is

$$\theta^*_{AMM} = \frac{8\sqrt{3}}{v_A} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^{\,2}} \left[\frac{\left(\zeta_n^{\,2} F_A + L_A \; Ra_A\right)\left(F_A - \zeta_n^{\,2} L_A\right)}{\left(\zeta_n^{\,4} + Ra_A\right)^2} \right] \text{ if } A \neq 0 \qquad (133)$$

and,

$$\theta^*_{MM} = \frac{8\sqrt{3}}{v} \sum_{n=1}^{\infty} \frac{F}{\zeta_n^{\,6}} \left(\frac{F}{\zeta_n^{\,2}} - L \right) \qquad \text{for } A = 0 \qquad (134)$$

The average Nusselt number may be defined as

$$Nu_A = \frac{d}{k(Ad)} \int_B q \; ds \Big/ \int_B ds = \frac{Ra_A \; v_A}{3\sqrt{3}} - \frac{F_A}{4} \quad \text{if } A \neq 0 \qquad (135)$$

and,

$$Nu = \frac{d}{kT_w} \int_B q \; ds \Big/ \int_B ds = -\frac{1}{4} F \qquad \text{for } A = 0 \qquad (136)$$

The relationship between the dimensionless mass through-flow and the dimensionless heat source and pressure gradient parameters is, if $A \neq 0$,

$v_A = .002601 \, F_A - .04856 \, L_A$ for $Ra_A = 1$

$v_A = .002529 \, F_A - .04729 \, L_A$ for $Ra_A = 10$

$v_A = .001980 \, F_A - .03762 \, L_A$ for $Ra_A = 100$

$$v_A = .000614\ F_A - .01380\ L_A \qquad \text{for } Ra_A = 1000$$

$$v_A = .000093\ F_A - .00298\ L_A \qquad \text{for } Ra_A = 10000$$

Nusselt numbers were calculated according to equation (135) using these relationships. The results are given in Figure (7). When the pressure gradient parameter is positive and the Rayleigh number is high enough, large values of the heat source parameter will change the direction of the overall heat transfer. A comparison with the results (light curves in Figure 7) obtained by Lu for a duct with a cross section of a 60° circular sector (ref. 3), shows that the difference between the two shapes, although small at small values of the Rayleigh number, increases as the Rayleigh number increases. It is also seen that the Nusselt number approaches its final asymptotic value sooner in the case of the equitriangular cross section, $L_A = 0$.

F. Transient Viscous Flow of an Incompressible Fluid with Suspended Particles

The equation of motion for the transient laminar flow of an incompressible fluid with a uniform suspension of solid particles, for the case of Stokes drag coupling, is (ref. 18 and 19)

$$\frac{\partial W}{\partial t} = \nu \nabla^2 W + \frac{b}{\tau} (W_p - W) + K\nu, \text{ in the triangle } R \qquad (137)$$

where W is the fluid velocity; W_p, the particle velocity; ν, the kinematic viscosity; b, the ratio of the particle concentration density to the fluid density; τ, the momentum equilibrium time (ref. 18 and 19); and K, the axial pressure gradient parameter $- \frac{1}{\mu} \frac{\partial P}{\partial Z}$. The rate of change of particle velocity due to the Stokes drag force is

$$\frac{\partial W_P}{\partial t} = - \frac{1}{\tau} (W_P - W) \tag{138}$$

The no-slip boundary condition for the fluid is

$$W = 0 \qquad \text{on} \quad B \tag{139}$$

For the fluid originally at rest, with the particles suspended also at rest, the initial conditions are

$$\left.\begin{aligned} W &= 0 \qquad (140) \\ W_P &= 0 \qquad (141) \end{aligned}\right\} \text{ at } t = 0$$

This corresponds to the general boundary value problem with

$$\begin{aligned}
f &= W & g &= W_P \\
a_1 &= 1 & b_1 &= 1 \\
a_2 &= \nu & b_2 &= 0 \\
a_3 &= - b/\tau & b_3 &= -1/\tau \\
a_4 &= b/\tau & b_4 &= 1/\tau \\
A(t) &= K \nu & B(t) &= 0 \\
F(p_1, p_2) &= 0 & G(p_1, p_2) &= 0
\end{aligned}$$

Using these values in equations (52) through (55), one obtains,

$$\left\{ \frac{d\breve{W}}{dt} = - \nu \lambda_n^2 \breve{W} - \frac{b}{\tau} \breve{W} + \frac{b}{\tau} \breve{W}_P + \frac{6a}{\lambda_n} K\nu \right. \tag{142}$$

$$\begin{cases} \dfrac{d\tilde{W}_P}{dt} = \dfrac{1}{\tau}\tilde{W} - \dfrac{1}{\tau}\tilde{W}_P & (143) \\[2ex] \tilde{W} = 0 \quad \text{at } t = 0 & (144) \\[2ex] \tilde{W}_P = 0 \quad \text{at } t = 0 & (145) \end{cases}$$

Solving equations (142) and (143) simultaneously with the initial conditions (144) and (145), we have

$$\tilde{W} = \frac{6aK}{\lambda_n^3}\left[1 + \frac{D_1}{D_2 - D_1}(1+\tau D_2)e^{D_2 t} - \frac{D_2}{D_2 - D_1}(1+\tau D_1)e^{D_1 t}\right] \tag{146}$$

$$\tilde{W}_P = \frac{6aK}{\lambda_n^3}\left[1 + \frac{D_1}{D_2 - D_1}e^{D_2 t} - \frac{D_2}{D_2 - D_1}e^{D_1 t}\right] \tag{147}$$

where

$$D_1 = -\left(\frac{\nu\lambda_n^2}{2} + \frac{b+1}{2\tau}\right) + \frac{1}{2}\sqrt{\nu^2\lambda_n^4 + \frac{2\nu\lambda_n^2(b-1)}{\tau} + \frac{(b+1)^2}{\tau^2}} \tag{148}$$

$$D_2 = -\left(\frac{\nu\lambda_n^2}{2} + \frac{b+1}{2\tau}\right) - \frac{1}{2}\sqrt{\nu^2\lambda_n^4 + \frac{2\nu\lambda_n^2(b-1)}{\tau} + \frac{(b+1)^2}{\tau^2}} \tag{149}$$

Substituting equations (146) and (147) into the inversion formulas (58) and (59) gives

$$W = \frac{4K}{\sqrt{3}a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\left[1 + \frac{D_1}{D_2 - D_1}(1+\tau D_2)e^{D_2 t} - \frac{D_2}{D_2 - D_1}(1+\tau D_1)e^{D_1 t}\right] \tag{150}$$

$$\times\left[\sin\lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} - X\right)\right]$$

$$W_P = \frac{4K}{\sqrt{3}a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3} \left[1 + \frac{D_1}{D_2-D_1} e^{D_2 t} - \frac{D_2}{D_2-D_1} e^{D_1 t}\right]$$

$$\times \left[\sin \lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} - X\right)\right] \tag{151}$$

The fluid and particle velocity distributions can be nondimensionalized by letting

$$w = \frac{W}{Ka^2} \qquad x = \frac{X}{2a/\sqrt{3}} \qquad (-1 \leqslant x \leqslant + \tfrac{1}{2})$$

$$w_P = \frac{W_P}{Ka^2} \qquad y = \frac{Y}{a} \qquad (-1 \leqslant y \leqslant + 1)$$

Then, equations (150) and (151) become

$$w = \frac{3}{2\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}\left[1- \frac{D_2^*}{D_2^*-D_1^*}(1+\tau^* D_1^*)e^{D_1^* t^*} + \frac{D_1^*}{D_2^*-D_1^*}(1+\tau^* D_2^*)e^{D_2^* t^*}\right]$$

$$\times\left[\sin \frac{2n\pi}{3}(1- \frac{3}{2} y+x)+ \sin \frac{2n\pi}{3}(1 + \frac{3}{2} y + x)+ \sin \frac{2n\pi}{3} (1-2x)\right] \tag{152}$$

$$w_P = \frac{3}{2\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}\left[1 - \frac{D_2^*}{D_2^*-D_1^*} e^{D_1^* t^*} + \frac{D_1^*}{D_2^*-D_1^*} e^{D_2^* t^*}\right]$$

$$\times\left[\sin \frac{2n\pi}{3} (1- \frac{3}{2} y+x)+ \sin \frac{2n\pi}{3}(1+ \frac{3}{2} y+x)+ \sin \frac{2n\pi}{3}(1-2x)\right] \tag{153}$$

Here

$$t^* = \frac{t}{a^2/\nu} \qquad \tau^* = \frac{\tau}{a^2/\nu}$$

$$D_1^* = -\left(\frac{2n^2\pi^2}{3} + \frac{b+1}{2\tau^*}\right) + \sqrt{\frac{4n^4\pi^4}{9} + \frac{2n^2\pi^2(b-1)}{3\tau^*} + \frac{(b+1)^2}{4\tau^{*2}}}$$

$$D_2^* = -\left(\frac{2n^2\pi^2}{3} + \frac{b+1}{2\tau^*}\right) - \sqrt{\frac{4n^4\pi^4}{9} \quad \frac{2n^2\pi^2(b-1)}{3\tau^*} + \frac{(b+1)^2}{4\tau^{*2}}}$$

The dimensionless mean velocities are:

$$w_M = \frac{\int_R w \, d\sigma}{\int_R d\sigma} = \frac{9}{2\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}\left[1 - \frac{D_2^*}{D_2^* - D_1^*}(1+\tau^* D_1^*)e^{D_1^* t^*} + \frac{D_1^*}{D_2^* - D_1^*}(1+\tau D_2^*)e^{D_2^* t^*}\right] \tag{154}$$

$$w_{PM} = \frac{\int_R w_P \, d\sigma}{\int_R d\sigma} = \frac{9}{2\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}\left[1 - \frac{D_2^*}{D_2^* - D_1^*} e^{D_1^* t^*} + \frac{D_1^*}{D_2^* - D_1^*} e^{D_2^* t^*}\right] \tag{155}$$

These values were calculated and the results are given in Figures (8) through (12). For spherical particles of radius ϵ, the momentum equilibrium time according to Stokes drag law, is (ref. 14)

$$\tau = \frac{2}{9}\,\frac{\rho_P \epsilon^2}{\rho_L \nu}$$

or, in dimensionless form

$$\tau^* = \frac{2}{9} \frac{\rho_P}{\rho_L} \left(\frac{\epsilon}{a}\right)^2$$

Figure (8) shows that the fluid-particle velocity difference is small when the particles are small in size. For large particles, however, the fluid-particle velocity difference is large as shown by Figure (9). Increasing the number of particles increases the time needed to reach steady flow, although the time when the fluid and the particles start accelerating seems to be the same for all concentrations. At low particle concentrations, Figure (10), the fluid moves the particles with it at all times when the particle size is small. With large particles, the fluid flow is unaffected but large fluid-particle velocity differences result. When large particle concentrations are present, Figure (11), lower particle and fluid accelerations result. For large particle sizes, the fluid starts accelerating sooner but the particles start later. The time to reach steady flow is also increased as particle size is increased. The limiting case of very large b, with very small τ, is shown in Figure (12). Here, as the particles begin to accelerate, the fluid velocity approaches the intermediate value of a flow past stationary particles. Then the fluid and particles accelerate to the same final steady value of velocity.

G. Transient Conduction with Heat Generation

The governing equation for transient heat conduction with uniform heat generation in an equitriangular column is

$$\frac{\partial\theta}{\partial t} = \alpha \nabla^2\theta + \frac{Q}{\rho c} \text{ , in the triangle R} \tag{156}$$

where θ stands for $T-T_w$. The column is initially at temperature T_o and the wall surfaces are maintained at temperature T_w for $t \geqslant 0$. Then, the initial and boundary conditions are

$$\theta = 0 \qquad \text{on B} \tag{157}$$

$$\theta = T_o - T_w = \theta_o \qquad \text{at } t = 0 \tag{158}$$

This belongs to the general boundary value problem with

$f = \theta$

$a_1 = 1$

$a_2 = \alpha$

$A(t) = Q/\rho c$

$F(p_1,p_2) = \theta_o$

as the only non-zero quantities.

Substituting these into equations (51) through (55), and using the previously obtained equitriangular transform of a constant, one obtains,

$$\left\{\begin{array}{ll} \dfrac{d\breve{\theta}}{dt} = - \alpha\lambda_n{}^2\,\breve{\theta} + \dfrac{6a}{\lambda_n}\,\dfrac{Q}{\rho c} & (159) \\ \\ \breve{\theta} = \dfrac{6a}{\lambda_n}\,\theta_o \quad \text{at } t = 0 & (160) \end{array}\right.$$

The solution to this initial value problem is

$$\tilde{\theta} = \frac{6a}{\lambda_n}\left[\theta_o\, e^{-\alpha\lambda_n^2 t} + \frac{1}{\alpha\,\lambda_n^2}\;\frac{Q}{\rho c}\,(1-e^{-\alpha\lambda_n^2 t})\right] \tag{161}$$

Substituting this into equation (58), we have

$$\theta = \frac{4}{\sqrt{3}a}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left[\theta_o\, e^{-\alpha\lambda_n^2 t} + \frac{1}{\alpha\lambda_n^2}\;\frac{Q}{\rho c}\,(1-e^{-\alpha\lambda_n^2 t})\right]$$

$$\times\left[\sin\lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin\lambda_n\left(\frac{a}{\sqrt{3}} - X\right)\right] \tag{162}$$

Equation (162) can be nondimensicnalized by letting

$$\theta^* = \frac{\theta}{\theta_o} \qquad x = \frac{X}{2a/\sqrt{3}} \quad (-1 \leqslant x \leqslant +\tfrac{1}{2})$$

$$Q^* = \frac{Qa^2}{k\theta_o} \qquad y = \frac{Y}{a} \quad (-1 \leqslant y \leqslant +1)$$

Then, the dimensionless temperature distribution is

$$\theta^* = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[e^{-\omega_n Fo} + \frac{Q^*}{\omega_n}\left(1-e^{-\omega_n Fo}\right)\right]$$

$$\times\left[\sin\frac{2n\pi}{3}\left(1 - \frac{3}{2}y + x\right) + \sin\frac{2n\pi}{3}\left(1 + \frac{3}{2}y + x\right) + \sin\frac{2n\pi}{3}(1-2x)\right] \tag{163}$$

where Fo is the Fourier number:, and $\omega_n = \frac{4n^2\pi^2}{3}$.

The sustained temperature distribution (i.e., $F_o \to \infty$) corresponds to the steady viscous flow velocity distribution with w replaced by θ^*/Q^*. Thus, the profiles shown in Figure (2) also describe the distribution of θ^*/Q^*.

Equation (163) reduces to equation (111) for $Q^* = 0$. That is, the temperature profiles along the line $y = 0$ for transient heat conduction with no heat generation are also shown in Figure (4).

IV. CONCLUSIONS

An integral transform has been established for solving a class of boundary value problems of the first kind in an equitriangular region. The transform of the Laplacian of a function was shown to be related to the transform of the function itself. This fact also relates the transform of ∇^{2n} to that of $\nabla^{2(n-1)}$. Thus, by successive use of this property, partial differential equations involving powers of the Laplacian operator can be reduced to ordinary differential equations or algebraic equations.

The set of eigenfunctions used to establish this transform is not a complete set; however, by narrowing down the type of boundary value problems to be treated, the known set has been used to the greatest advantage. The procedure employed consisted of first joining the known set of eigenfunctions with an unknown complementary set of eigenfunctions to form the complete set. Then, the admissible type of boundary value problems becomes the class of problems which can be solved by the method of transform without any knowledge of the complementary eigenfunctions. This requires that the inhomogeneous terms and initial conditions be constants or functions like $\sin \lambda_5 p_1 + \sin \lambda_5 p_2 + \sin \lambda_5 p_3$. Also, the problem must have zero boundary values of $(\nabla^2)^i$; $i = 0, 1, 2, \ldots (n-1)$.

The above format could be followed in establishing integral transforms for other regions in which an incomplete set of eigenfunctions is known.

A general time dependent boundary value problem, in an equitriangular region, with position dependent initial conditions of the admissible class, was solved by the transform method. The case of a coupled system of two dependent variables, with differential operators $(\nabla^2)^1$ and $(\nabla^2)^0$, and zero functional value on the boundary, was considered. Boundary value problems corresponding to this general problem can be solved by direct substitution into the given transformed equations. It should be noted, however, that this is not the most general boundary value problem solvable by the method of equitriangular transform. The admissible class of problems includes the time dependent coupled system of m equations with m dependent variables, with higher powers of ∇^2 operating on each of the variables.

The velocity distribution for steady viscous incompressible fluid flow, and the temperature distribution for steady laminar forced convection, wereobtained as special cases of the general boundary value problem. These series solutions were numerically compared with previously obtained closed form solutions to investigate the rate of convergence and the Gibbs' phenomenon of the series. The analysis showed the general series behavior to be similar to that of an ordinary Fourier series.

The previously unsolved problems of transient natural convection, steady combined natural and forced convection, transient viscous flow with suspended particles, and transient conduction with heat generation were solved also as special cases of the general boundary value problem. Some numerical results were also worked out which should prove to be of practical value.

V. NOMENCLATURE

A = $\partial T/\partial Z$

$A(t)$ = function of time

a = half side of the equilateral triangle

a_1, a_2, a_3, a_4 = constants

B = boundary of R

$B(t)$ = function of time

b = ratio of particle concentration density to fluid density

b_1, b_2, b_3, b_4 = constants

C = constant

c = specific heat of an incompressible medium

D_1, D_2 = damping factors

D_1^*, D_2^* = dimensionless damping factors

d = $2a/\sqrt{3}$, hydraulic diameter

E = $Ra\ (\beta f_z d/c)$

E_A = $Ra_A\ (\beta f_z d/c)$

e = exponential base

F = Qd^2/kT_w, dimensionless heat source paramter

F_A = Qd/kA, dimensionless heat source parameter

$F(p_1, p_2)$ = function of position

Fo = $\alpha t/a^2$, Fourier number

f = a dependent variable

f_z = body force per unit mass, positive in -z direction

$G(p_1, p_2)$ = function of position

Gr = $(\rho^2 \beta f_z T_w d^3/\mu^2)$, Grashof number

Gr_A = $(\rho^2 \beta f_z A d^4/\mu^2)$, modified Grashof number

g = a dependent variable

K = $-\frac{1}{\mu} \frac{\partial P}{\partial Z}$, pressure gradient parameter

K_1 = $\rho c/k \quad \partial T/\partial Z$

K_2 = Q/k

$K_2{}^*$ = $K_2/-KK_1 a^2$

k = thermal conductivity

L = $(Pr/c\ E\ T_w)^{1/2}\ \Pi\, d^2/\mu$, dimensionless pressure gradient parameter

L_A = $(Prd^3/c\ EA)^{1/2}\ \Pi/\mu$, dimensionless pressure gradient parameter

M_m = norm of Ω_m

m = 1, 2, 3, etc.

$\hat{N}$ = outward unit normal vector (two-dimensional)of B

N_n = norm of ϕ_n

Nu = average Nusselt number

Nu_A = average Nusselt number

n = 1, 2, 3, etc.

P = pressure

Pr = ν/α, Prandtl number

p_1, p_2, p_3 = trilinear coordinates

Q = heat source term

Q^* = dimensionless heat source term

q = normal wall heat flux, positive inward

R = the two dimensional region under investigation

Ra = Pr Gr, Rayleigh number

Ra_A = Pr Gr_A, modified Rayleigh number

ds = line element along B

T = temperature

t = time

t^* = $\nu t/a^2$, dimensionles time

V = mass through-flow

v = $(1/E)\ (\beta f_z/k)V$, dimensionless mass-through flow

v_A = $(1/E_A)(\beta f_z/k)V$, dimensionless mass-through flow

W = axial velocity

W_P = particle axial velocity

w = dimensionless axial velocity

w_P = dimensionless particle axial velocity

X, Y, Z, = Cartesian coordinates

x = dimensionless x-coordinate

y = dimensionless y-coordinate

α = $k/\rho c$, thermal diffusivity

β = thermal coefficient of volumetric expansion

∇ = gradient

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} = \frac{\partial^2}{\partial p_1{}^2} + \frac{\partial^2}{\partial p_2{}^2} + \frac{\partial^2}{\partial p_3{}^2} - \frac{\partial^2}{\partial p_1 \partial p_2} - \frac{\partial^2}{\partial p_1 \partial p_3} - \frac{\partial^2}{\partial p_2 \partial p_3}$$

δ = damping factor

ϵ = particle radius

ζ_n = $4n\pi/3$

θ = $T - T_W$

θ^* = dimensionless temperature difference
λ_n = eigenvalues of ψ_n
μ = absolute viscosity coefficient
ν = μ/ρ, kinematic viscosity coefficient
ξ_m = eigenvalues of Ω_m
Π = $dP/dZ + \rho_W f_Z$, pressure gradient parameter
π = 3.14159
ρ = density
ρ_W = reference density evaluated at temperature T_W
$\sum_n$ = summation over n = 1, 2, 3, ...
$d\sigma$ = area element of R
τ = momentum equilibrium time
τ^* = $\nu\tau/a^2$, dimensionless momentum equilibrium time
ϕ = a dependent variable
ψ_n = eigenfunctions
Ω_m = eigenfunctions
ω_n = $4n^2\pi^2/3$

Subscripts

O = initial value
M = mean value
MM = mean mixed
m = 1,2,3, etc.
n = 1,2,3, etc.
W = wall value
Z = axial direction

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VII. APPENDIX

In this Appendix, numerical solutions to the steady viscous flow problem and the steady laminar forced convection problem are analyzed in detail. The solutions obtained by the equitriangular transform method are compared with the closed form solutions.

The dimensionless velocity distribution for the steady viscous flow of an incompressible fluid is given in series form by equation (75), and in closed form by equation (77). Along the line $y = 0$, these are

$$w = \frac{3}{2\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[2 \sin \frac{2n\pi}{3} (1+x) + \sin \frac{2n\pi}{3} (1-2x) \right]$$

and,

$$w = \frac{1}{9} (1-2x)(x+1)^2$$

where

$$-1 \leqslant x \leqslant + \frac{1}{2}$$

The percent changes in the value of w according to the series, due to the addition of each of the first 16 terms, are given for various values of x in Table II. These results show that a term contributing a small amount to the series value is often followed by a term making a substantial change. Thus, testing for convergence is better made on the change due to the addition of a group of terms. All numerical values calculated in this work are terminated when a three-term group changes the current series value by an amount smaller than 0.01 percent.

Velocity profiles given by the closed form and the series form are shown in Figure (2). No distinguishment can be made between the series and closed form profiles because of the minuteness of the difference between these values. The number of terms required for convergence within .01 percent, and the percent error as compared with the closed form solution are given in Table III for increments in x of 0.1. The table shows that the error is largest near the boundaries; and that, in general, large numbers of terms are required near the boundaries.

The number of terms required for convergence and the percent error between the series and closed form solutions are given in Table IV for values of x very close to the boundaries. The behavior of the series near the base is better than near the tip of the triangle. As the boundary is approached, the error increases. Thus, Gibbs' phenomenon seems to occur for $-1.000 < x < -.998$ or $.498 < x < .500$.

The dimensionless temperature distribution for steady laminar forced convection is given by equation (86) in the series form, and equation (87) in the closed form. Along the line $y = 0$, these are

$$\theta^* = \sum_{n=1}^{\infty} \left(\frac{9}{8n^5 \pi^5} + \frac{3K_2^*}{2n^3 \pi^3} \right) \left[2 \sin \frac{2n\pi}{3} (1+x) + \sin \frac{2n\pi}{3} (1-2x) \right]$$

and,

$$\theta^* = \frac{1}{108} (1-2x) (x+1)^2 (1-x^2 - 12K_2^*)$$

where

$$-1 \leqslant x \leqslant +\frac{1}{2}$$

Temperature profiles calculated according to these equations are shown in Figure (3). Again, no distinguishment can be made between the series and closed form profiles. The number of terms required for convergence within .01 percent, and the error referred to the closed form are given in Table V for the case of $K_2^* = 3.0$. Similar results were obtained in the range $-3.0 \leqslant K^* < +3.0$. The behavior of the series is poorest near the tip.

The above results seem to indicate a behavior similar to that of a Fourier series. The series seem to converge throughout the region; however, a large number of terms might be required near the boundary. The series would converge to the correct value except near the boundaries, where Gibbs' phenomenon could occur.

TABLE I

Description	Symbol	Form
Eigenfunctions (trilinear form)	ψ_n	$\sin \lambda_n p_1 + \sin \lambda_n p_2 + \sin \lambda_n p_3$
Eigenfunctions (Cartesian form)	ψ_n	$\sin \lambda_n\left(\frac{a}{\sqrt{3}} - \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right) + \sin \lambda_n\left(\frac{a}{\sqrt{3}} + \frac{\sqrt{3}Y}{2} + \frac{X}{2}\right)$ $+ \sin \lambda_n\left(\frac{a}{\sqrt{3}} - X\right)$
Eigenvalues	λ_n	$\frac{2n\pi}{\sqrt{3}a}$ $n = 1,2,3, \ldots$
Norm	N_n	$\frac{3\sqrt{3}a^2}{2}$
Transform of ϕ	$\breve{\phi}$	$\breve{\phi}$
Transform of the Laplacian of $\phi^{(1)}$	$\breve{\nabla^2 \phi}$	$- \lambda_n{}^2 \breve{\phi}$
Transform of a constant C	$\breve{C}$	$\frac{6a}{\lambda_n}$ C
Inversion of $\phi^{(2)}$	ϕ	$\frac{2}{3\sqrt{3}a^2}\sum_{n=1}^{\infty} \breve{\phi} \; \psi_n$

TABLE I CONTINUED

Description	Symbol	Form
Integration of ϕ_n over the equitriangular region	$\int_R \phi_n \, d\sigma$	$\frac{6a}{\lambda_n}$
Transform of $p_1\, p_2\, p_3$ (3)	$\overbrace{p_1\, p_2\, p_3}$	$\frac{6\sqrt{3}\, a^2}{\lambda_n^3}$
Transform of $p_1\, p_2\, p_3\left(p_1^2 + p_2^2 + p_3^2 - 3a^2\right)$ (4)	-	$\frac{144\sqrt{3}\, a^2}{\lambda_n^5}$

(1) ϕ here must vanish on the sides of the triangle

(2) ϕ here must be the solution of a boundary value problem satisfying the three requirements stated on page 14

(3) From equation (74) and ref. 14

(4) From equation (85) and ref. 15, or from ref. 4

TABLE II

Percent Change Due to the Addition of Term

Term	x = -.9	x = -.5	x = 0.0	x = .25	x = .45
1	100.00	100.00	100.00	100.00	100.00
2	49.18	27.27	-14.29	-6.16	-.27
3	31.75	<.01	<.01	4.04	9.70
4	22.69	-3.53	1.75	-2.27	-.24
5	17.98	-.61	-.91	.06	3.15
6	13.98	<.01	<.01	<.01	-.22
7	9.97	.22	.33	-.02	1.42
8	7.61	.44	.22	.28	-.20
9	5.71	<.01	<.01	-.15	.73
10	4.16	-.23	.11	.05	-.18
11	2.90	-.06	-.09	-.08	.40
12	1.88	<.01	<.01	<.01	-.16
13	1.07	.03	.05	.05	.22
14	.45	.08	-.04	-.02	-.13
15	<.01	<.01	<.01	.03	.12
16	-.30	-.06	.03	-.04	-.11

TABLE III

x	Number of Terms for Convergence within .01%	% Error Compared with Closed Form
-.95	54	-.347
-.90	27	-.324
-.8	42	-.017
-.7	18	-.050
-.6	36	<.001
-.5	30	+.005
-.4	18	<.001
-.3	18	-.005
-.2	15	-.006
-.1	15	+.012
0.0	15	-.006
.1	15	-.005
.2	12	-.014
.3	18	-.001
.40	27	-.009
.45	27	-.038

TABLE IV

(a) x near -1.0

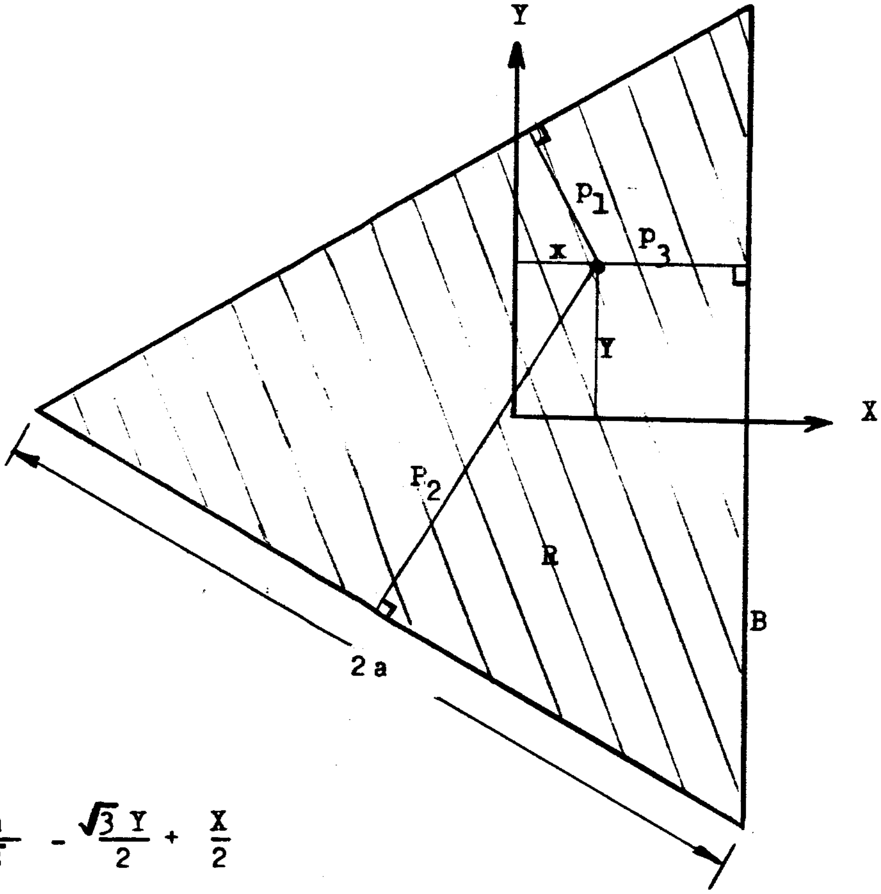
x	Number of Terms for Series Convergence within .01%	% Error from Closed Form
-.998	735	-28.80
-.996	369	- 4.78
-.994	246	- .29
-.990	261	- 1.42
-.986	105	+ 2.59
-.982	150	- .67
-.978	123	- .54
-.974	105	- .47
-.970	48	+ 3.08
-.966	81	- .40
-.962	72	- .38
-.958	66	- .37
-.954	60	- .36

(b) x near +.5

x	Number of Terms for Series Convergence within .01%	% Error from Closed Form
.454	18	+.10
.458	30	-.02
.462	36	-.02
.466	15	+.03
.470	30	+.06
.474	21	+.04
.478	36	+.07
.482	33	+.04
.486	39	+.01
.490	51	-.01
.494	63	-.01
.496	75	-.12
.498	87	-.20

TABLE V

x	Number of Terms for Convergence within .01%	% Error Compared with Closed Form
-.95	57	-.347
-.90	30	-.322
-.8	45	-.017
-.7	21	-.050
-.6	39	-.001
-.5	33	+.005
-.4	21	<.001
-.3	21	-.005
-.2	18	-.006
-.1	18	+.011
0.0	15	-.010
.1	18	-.050
.2	15	-.014
.3	21	-.001
.40	30	-.009
.45	30	-.027



$$p_1 = \frac{a}{\sqrt{3}} - \frac{\sqrt{3}\,Y}{2} + \frac{X}{2}$$

$$p_2 = \frac{a}{\sqrt{3}} + \frac{\sqrt{3}\,Y}{2} + \frac{X}{2}$$

$$p_3 = \frac{a}{\sqrt{3}} - X = \sqrt{3}\,a - (p_1 + p_2)$$

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} = \frac{\partial^2}{\partial p_1{}^2} + \frac{\partial^2}{\partial p_2{}^2} + \frac{\partial^2}{\partial p_3{}^2} - \frac{\partial^2}{\partial p_1 \partial p_2} - \frac{\partial^2}{\partial p_1 \partial p_3} - \frac{\partial^2}{\partial p_2 \partial p_3}$$

B: $p_1 = 0,\ p_2 = 0,\ p_3 = 0$

Figure (1) - The Equitriangular Region

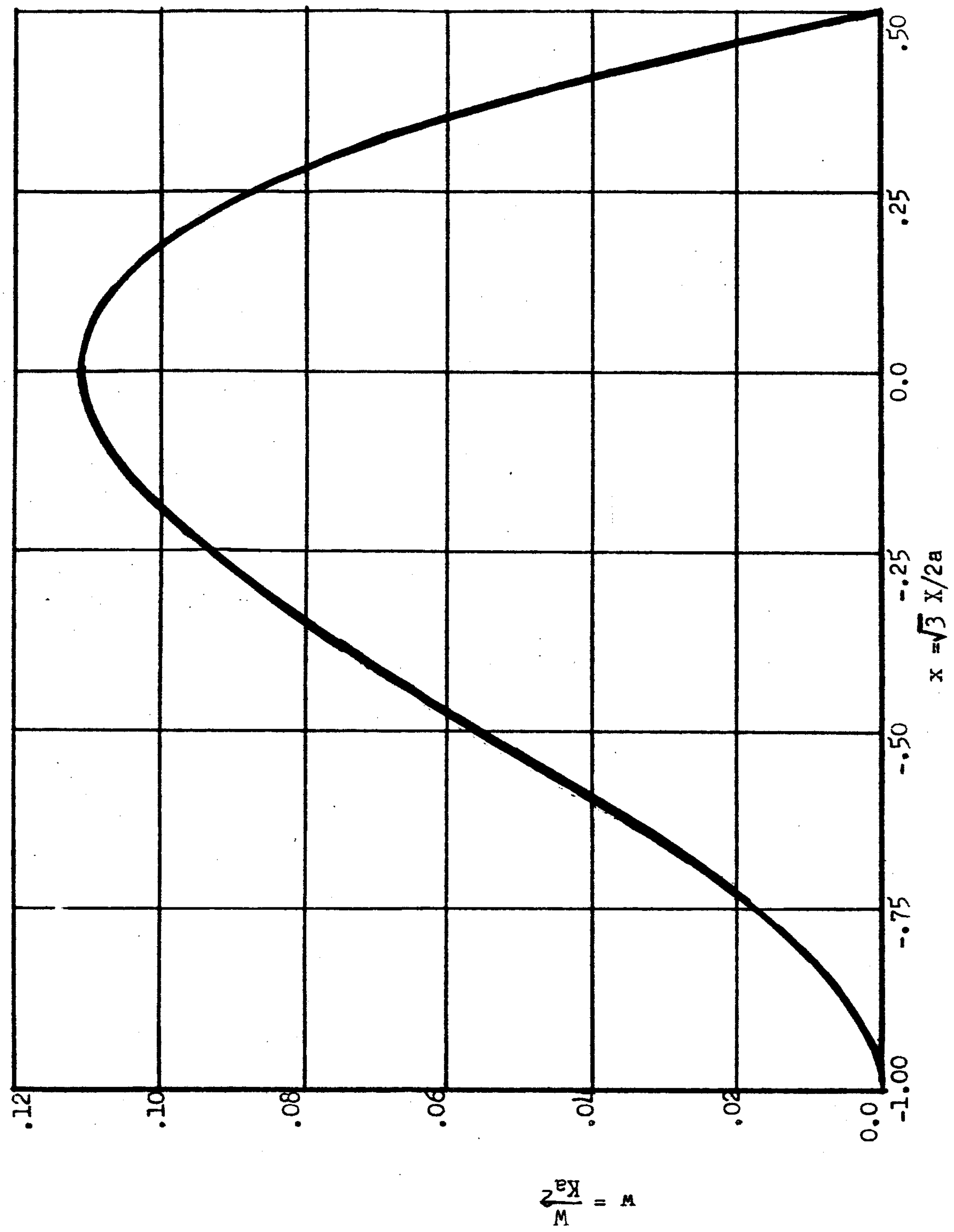


Figure (2) - Dimensionless velocity profile along y = 0 for steady viscous flow of an incompressible fluid.

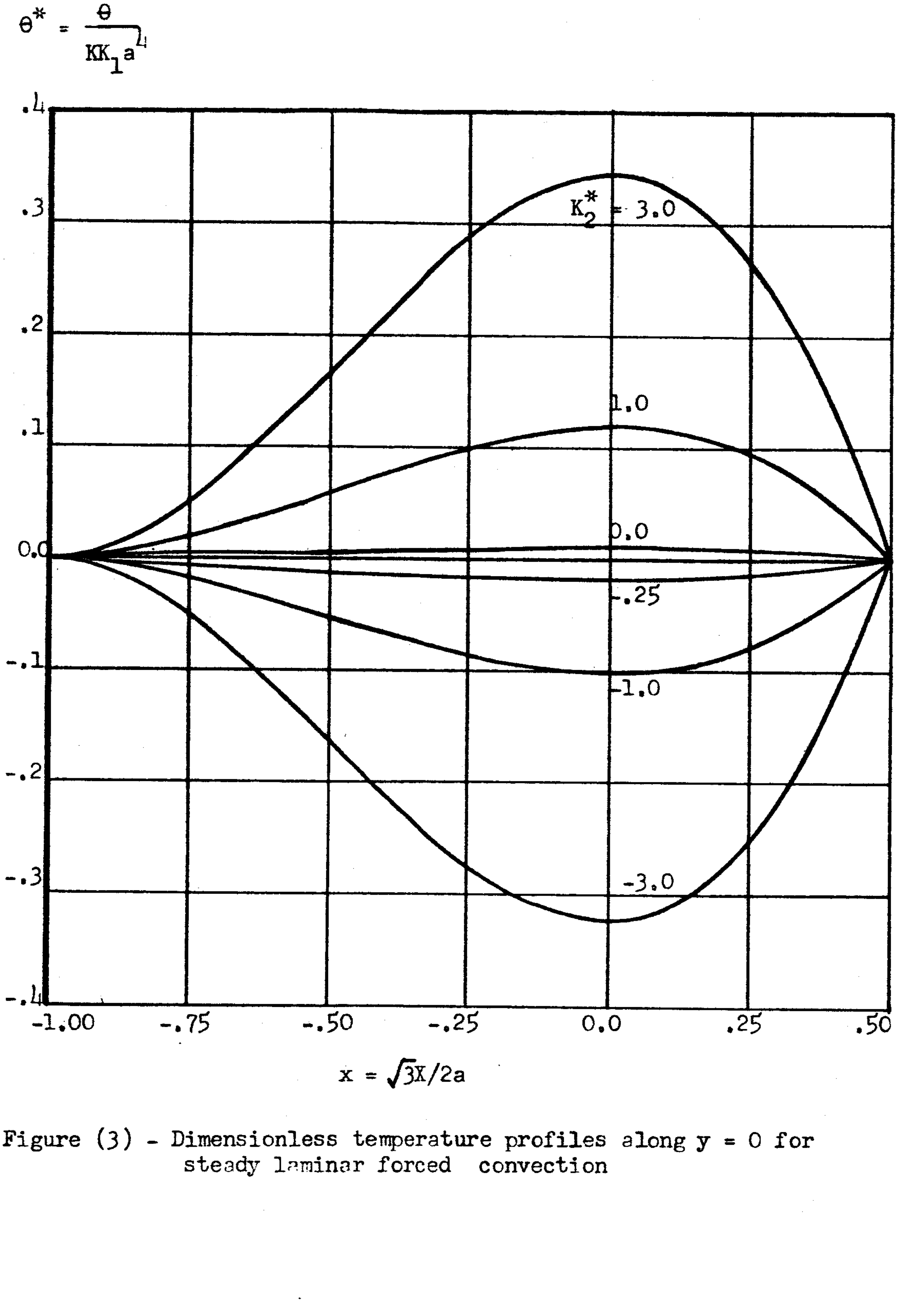


Figure (3) - Dimensionless temperature profiles along y = 0 for steady laminar forced convection

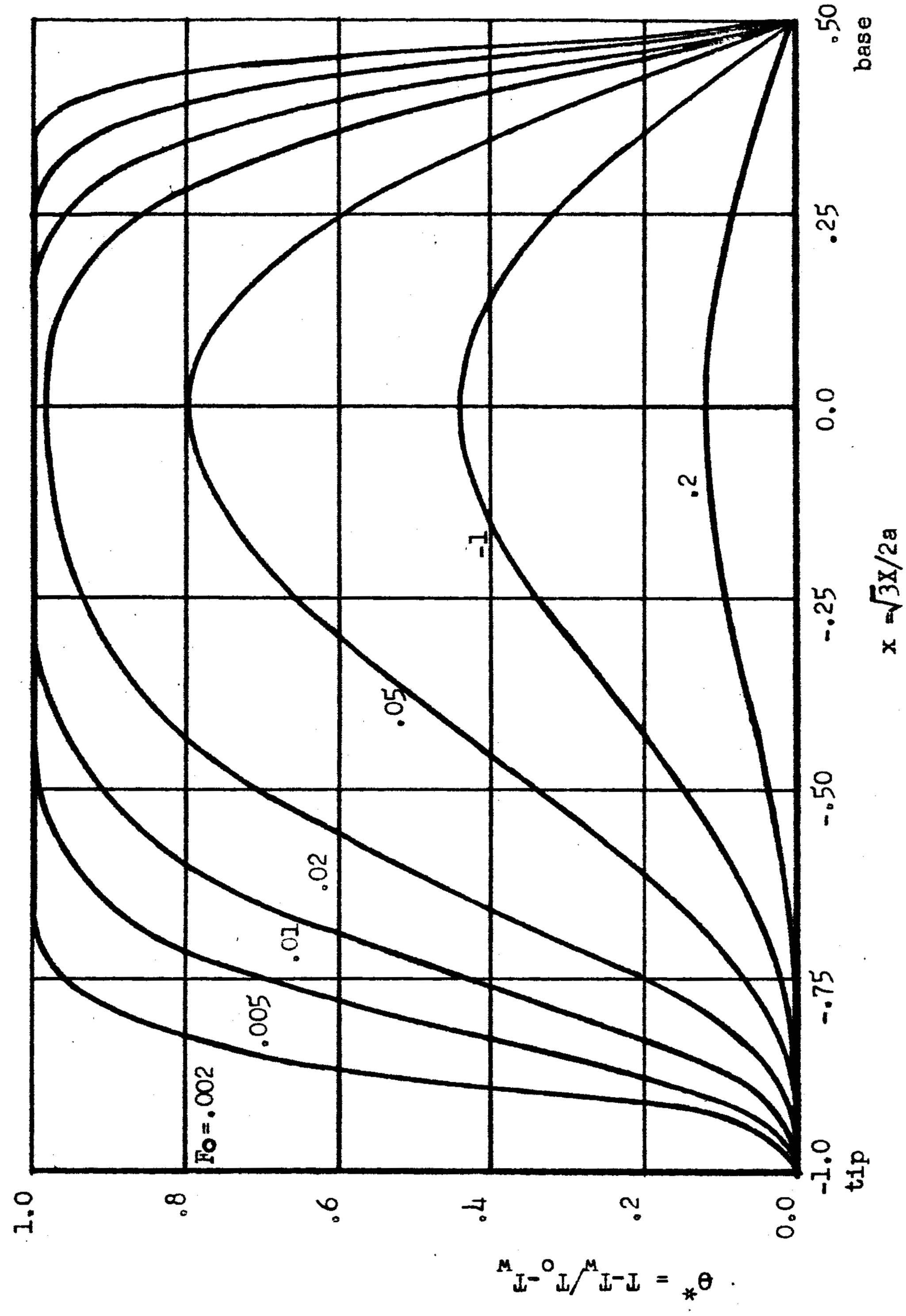


Figure (4) - Dimensionless temperature profiles along y = 0 for transient natural convection.

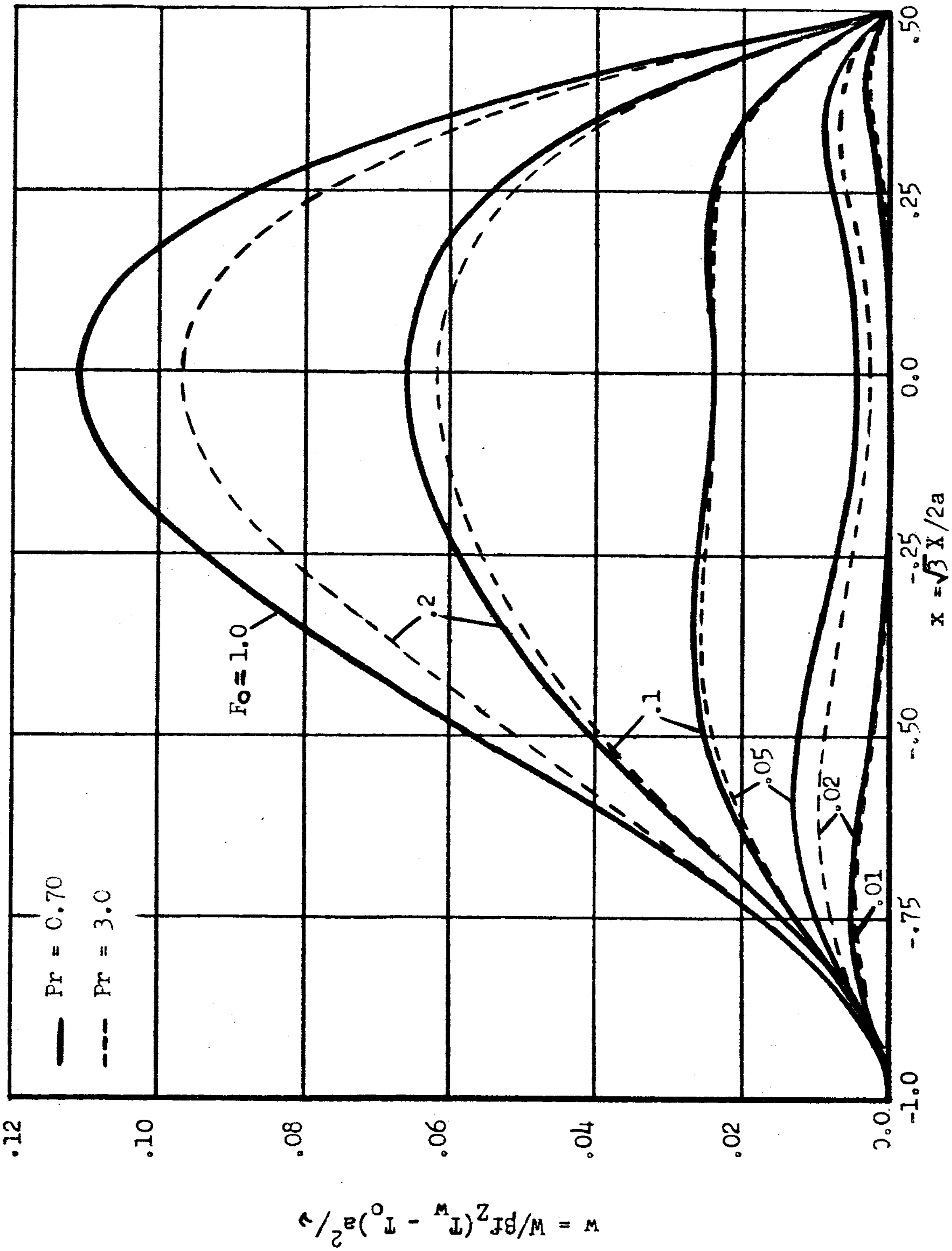


Figure (5) - Dimensionless velocity profiles along y = 0 for transient natural convection

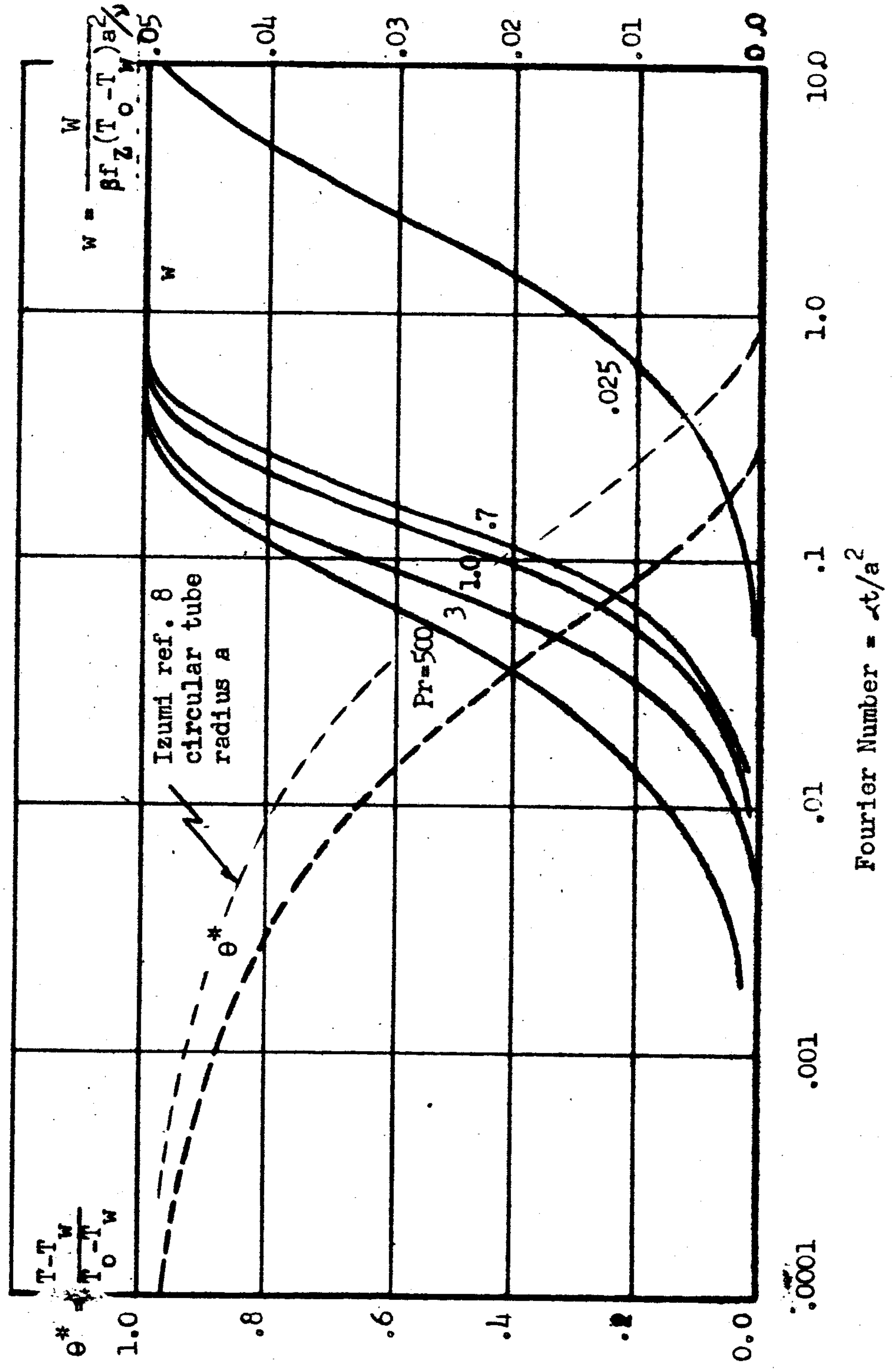


Figure (6) - Mean dimensionless temperature and velocity for transient natural convection.

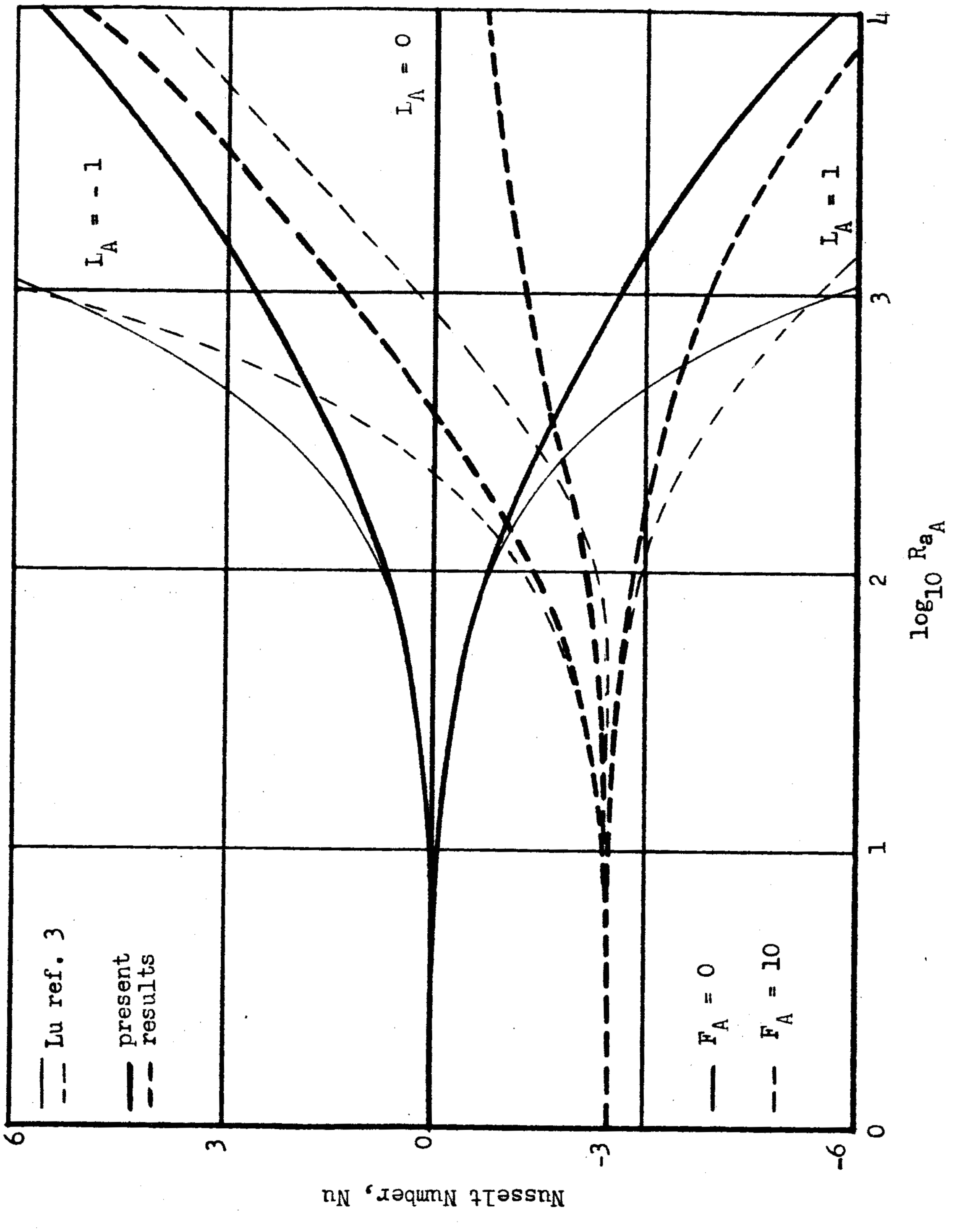


Figure (7) - Nusselt number for combined natural and forced convection

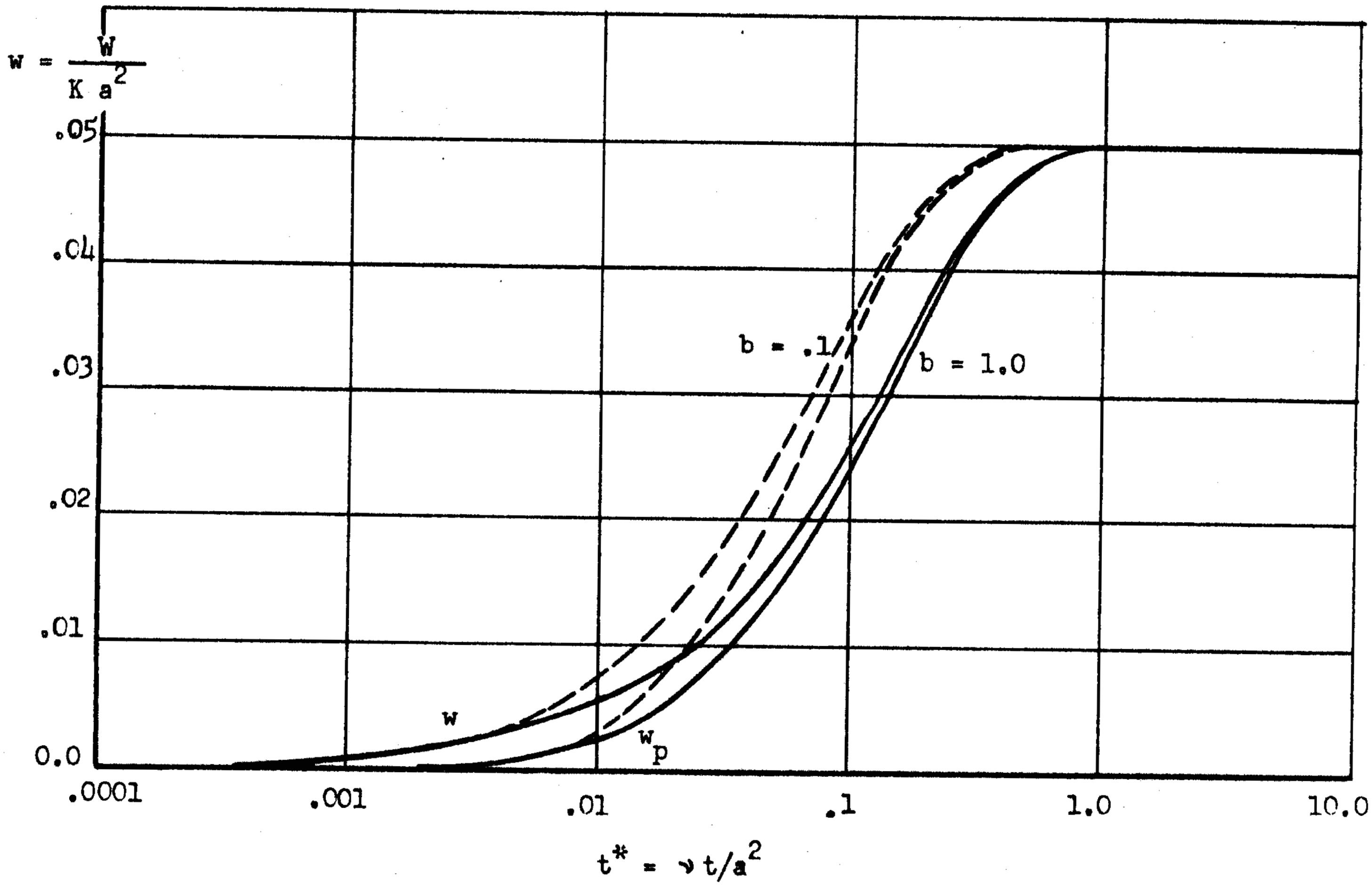


Figure (8) - Dimensionless mean fluid and particle velocities for transient flow with suspended particles with $\tau^* = .01$

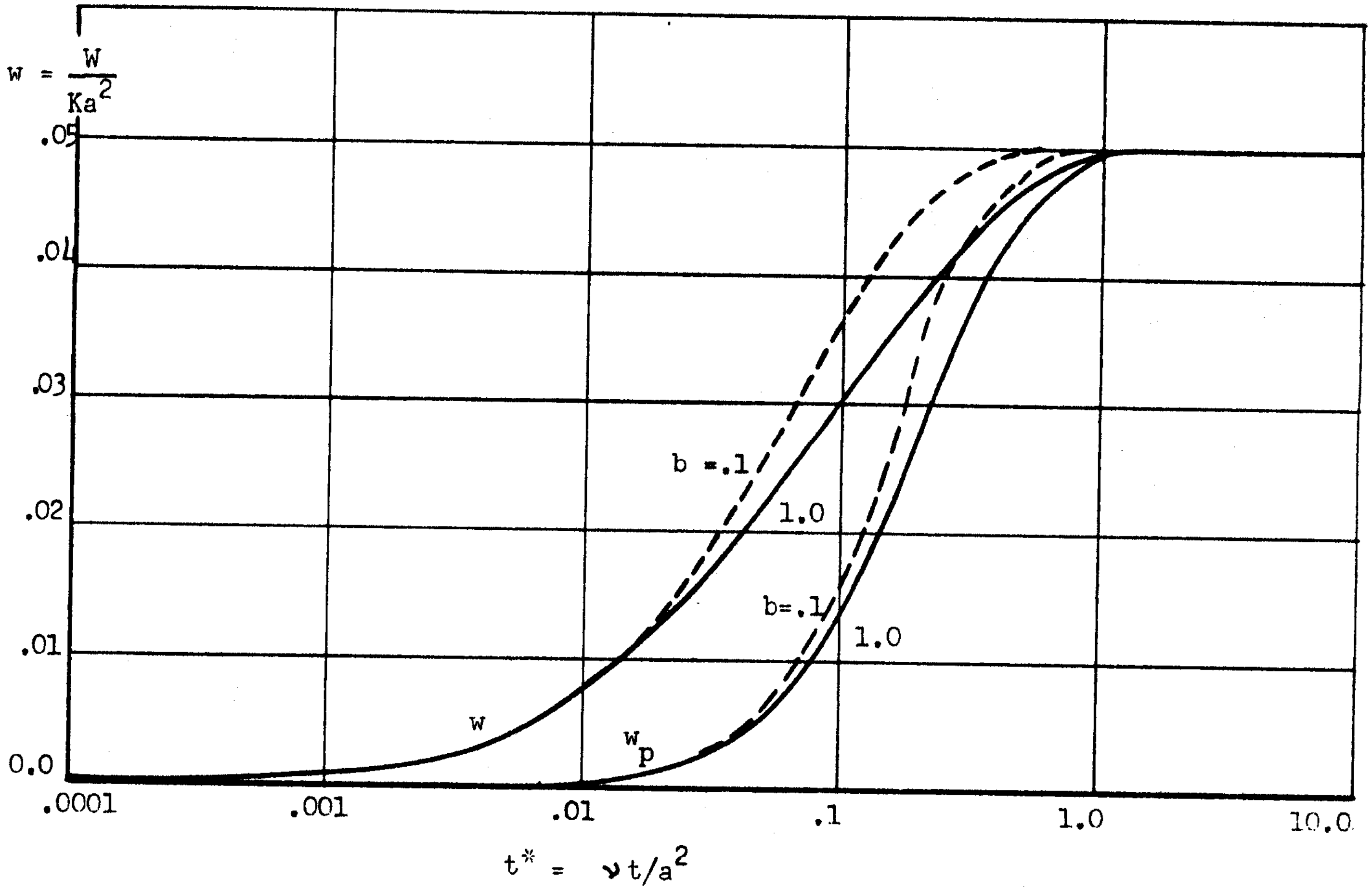


Figure (9) - Dimensionless mean fluid and particle velocities for transient flow with suspended particles with $\tau^* = .1$

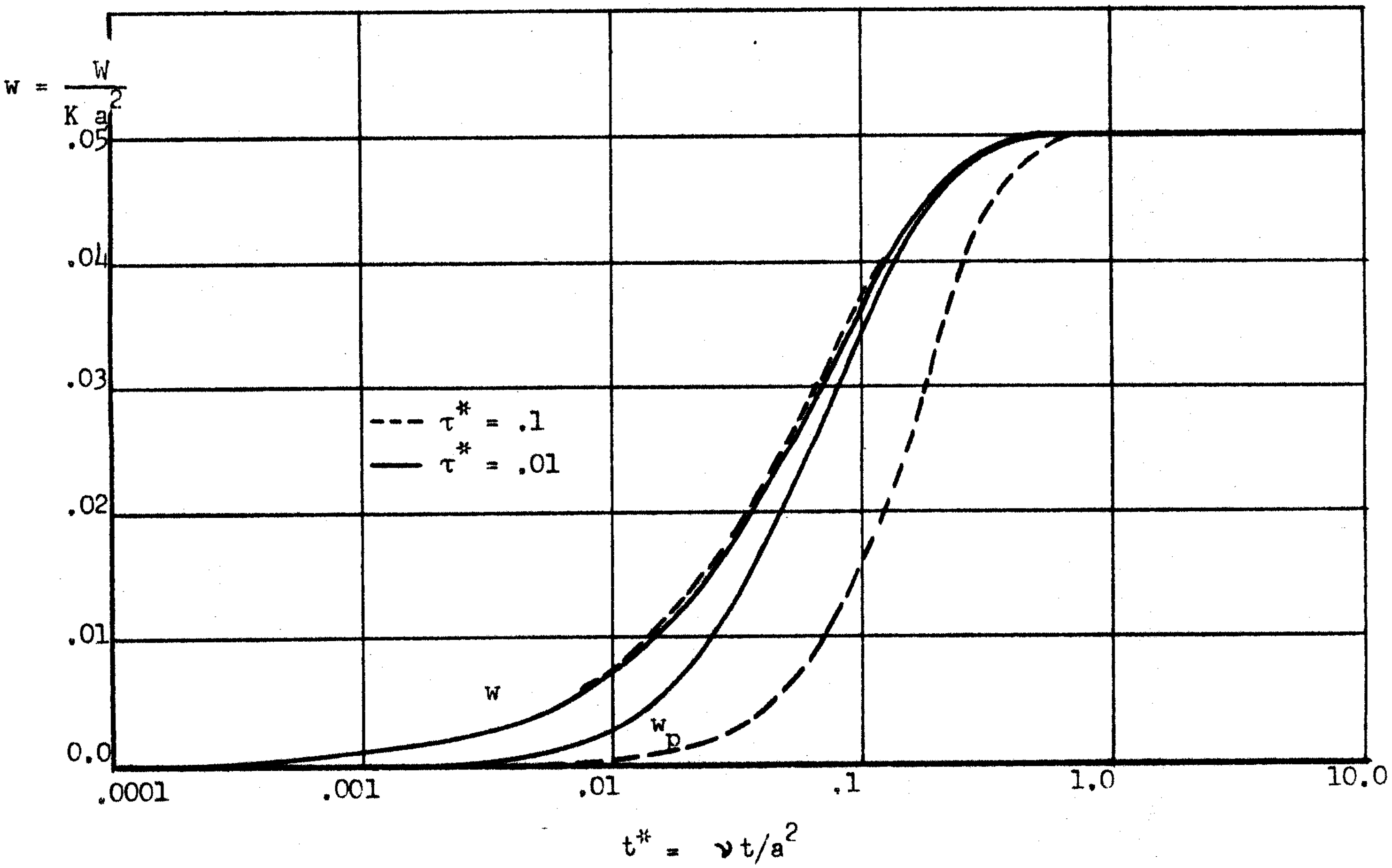


Figure (10) - Dimensionless mean fluid and particle velccities for transient flow with suspended particles with b = .1

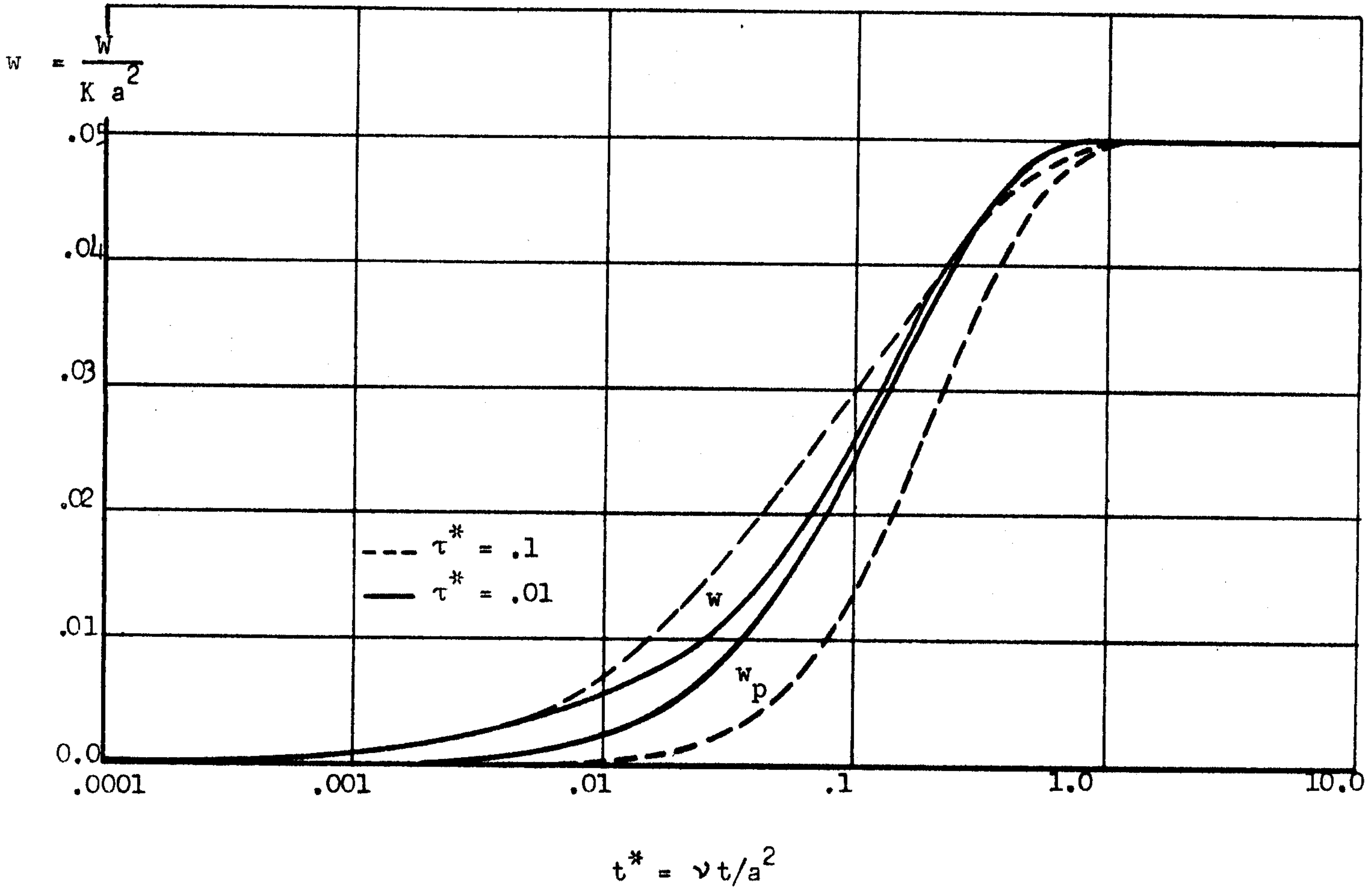


Figure (11) - Dimensionless mean fluid and particle velocities for transient flow with suspended particles with h = 1.0

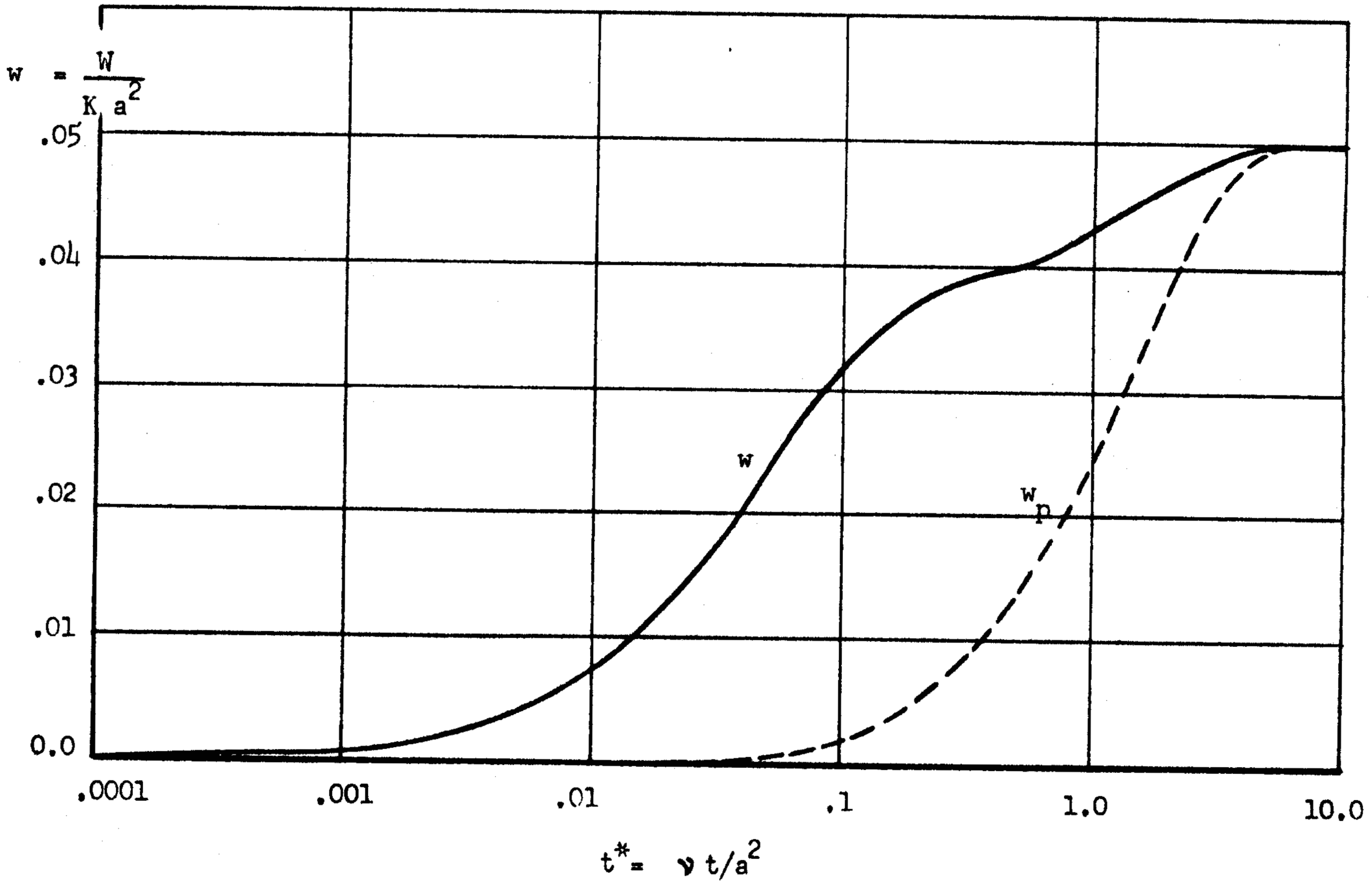


Figure (12) - Dimensionless mean fluid and particle velocities for transient flow with suspended particles with $b = 5.0$ and $\tau^* = .1$