AERO-ASTRONAUTICS REPORT NO. 21

LIFTING BODIES OF MINIMUM DRAG IN HYPERSONIC FLOW

by

ARTHUR H. LUSTY, JR.

RICE UNIVERSITY

1966

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N66 34677

SUMMARY

The problem of minimizing the drag of a slender, flat-top, homothetic body in hypersonic flow is considered under the assumptions that the pressure coefficient is Newtonian and the skin-friction coefficient is constant. The indirect methods of the calculus of variations are employed, and the necessary conditions to be satisfied by an optimum body are derived for arbitrary conditions imposed on the lift, the wetted area, the volume, the length, and the thickness. The particular cases treated are the following: (a) given lift, (b) given lift and thickness, (c) given lift and wetted area, (d) given lift

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(*) This research was supported by the Langley Research Center of the National Aeronautics and Space Administration under Grant No. NGR-44-006-034.

(**) Research Associate in Astronautics, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

1. INTRODUCTION

Currently, it is of interest to develop a vehicle which can cruise at moderate hypersonic speeds. Since the lift must be equal to the weight at every point of the cruise trajectory, it is of interest to find the vehicle shape which minimizes the drag for a given lift, for example, the average lift between the end points of the cruise. In addition to this aerodynamic constraint, several constraints of a geometric nature may be of interest. For instance, one may choose to specify the wetted area, the volume, the length, or the thickness of the configuration.

In this paper, the above problem is treated for a particular class of bodies, that of slender, flat-top, homothetic bodies (Ref. 1). For these bodies, the optimum longitudinal contour is determined under the assumption that the transversal contour is semicircular. While a semicircular contour is not necessarily the best, the extension of the results to the case of a body of arbitrary transversal contour is immediate if one applies the similarity law derived in Ref. 2.

The complete list of hypotheses is as follows: (a) a plane of symmetry exists between the left-hand and right-hand sides of the body; (b) the upper surface of the body is a plane

2. AERODYNAMIC AND GEOMETRIC QUANTITIES

In order to relate the aerodynamic and geometric quantities of a flat-top, homothetic body to its geometry, it is necessary to define two coordinate systems: a Cartesian system Oxyz and a cylindrical system $Oxr\theta$. For the Cartesian coordinate system, the origin O is the apex of the body; the x-axis is the intersection of the plane of symmetry with the flat top and is positive toward the base; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed. For the cylindrical coordinate system, r is the distance of any point from the x-axis, and θ measures the angular position of the vector $\vec{r}$ with respect to the xy-plane.

If hypotheses (a) through (h) are considered and if the lower surface is represented by the relationship $r = r(x)$, the drag D, the lift L, the wetted area S, and the volume V are given by (Ref. 2)

3. MINIMAL PROBLEM

The problem of minimizing the drag for arbitrary values imposed on the lift, the wetted area, the volume, the length, and the thickness is now formulated as follows: "In the class of functions r(x) which satisfy the integral constraints (1-2) through (1-4) and the prescribed boundary conditions, find that particular function which minimizes the integral (1-1)." According to standard variational procedures (see, for instance, Chapter 1 of Ref. 3), this problem is equivalent to that of minimizing the functional

$$I = \int_{x_i}^{x_f} F(r, \dot{r}, \lambda_1, \lambda_2, \lambda_3)\, dx \tag{4}$$

subject to the constraints (1-2) through (1-4) and the prescribed boundary conditions with the understanding that the fundamental function F is defined as

$$F = r\dot{r}^3 + \lambda_1 r\dot{r}^2 + (\lambda_2 + nC_f)\, r + \lambda_3 r^2 \tag{5}$$

where λ_1, λ_2, λ_3 denote constant Lagrange multipliers.

if the length is free and

$$t\dot{r}_f(3\dot{r}_f + 2\lambda_1) = 0 \tag{10}$$

if the thickness is free.

Once the solution of the Euler equation is obtained, it is necessary to verify that it actually minimizes the functional (4). In this connection, the Legendre necessary condition

$$3\dot{r} + \lambda_1 \geq 0 \tag{11}$$

must be satisfied at every point of the optimum shape.

6. GIVEN LIFT

For this case, the first integral (7) in conjunction with the natural boundary condition (9) and the conditions $\lambda_2 = \lambda_3 = 0$ leads to the following differential equation to be satisfied by the optimum shape:

$$2\dot{r}^3 + \lambda_1 \dot{r}^2 = nC_f \tag{15}$$

Since the multiplier λ_1 and the friction coefficient C_f are constant, this equation has the solution

$$\dot{r} = \text{Const} \tag{16}$$

Hence, the optimum flat-top body is the semicone

$$r = \dot{r}x \tag{17}$$

which, in the nondimensional system (12), becomes

$$\rho = \xi \tag{18}$$

The evaluation of the integrals (1) yields the relationships

$$\begin{aligned} D &= \pi q \ell^2 \dot{r}(\dot{r}^3 + nC_f) \\ L &= 2q\ell^2 \dot{r}^3 \\ S &= (1 + \pi/2)\ell^2 \dot{r} \\ V &= (\pi/6)\ell^3 \dot{r}^2 \end{aligned} \tag{19}$$

$$
\begin{aligned}
E_* &= (4/3\pi)(2n)^{-1/3} \cong 0.360 \\
S_* &= (\pi/4)(2n)^{1/3} \cong 0.926 \\
V_* &= (\pi/12\sqrt{2})(2n)^{-5/6} \cong 0.123 \\
\ell_* &= (1/2)\, n^{-1/2} \cong 0.553 \\
t_* &= (1/\sqrt{2})(2n)^{-1/6} \cong 0.651
\end{aligned} \tag{23}
$$

Equation (23-1) represents the highest lift-to-drag ratio which can be obtained with a flat-top body of semicircular cross section subjected to a flow parallel to the flat top (Ref. 1). Should the body be required to satisfy a certain number of geometric constraints, a decrease in the lift-to-drag ratio would occur with respect to that predicted by Eq. (23-1).

8. GIVEN LIFT AND WETTED AREA

For this case, the relationships $C = \lambda_3 = 0$ are valid and, as a consequence, the first integral (7) reduces to

$$2\dot{r}^3 + \lambda_1 \dot{r}^2 = \lambda_2 + nC_f \tag{26}$$

Since λ_1, λ_2, and C_f are constant, this equation has the solution (16) so that Eqs. (17) through (20) are valid. Since the lift and the wetted area are prescribed, the parameter S_* is known a priori. Hence, because of Eq. (20-2), the optimum thickness ratio is given by (Fig. 4)

$$\tau_* = (1/2)(2+\pi)^{1/2} S_*^{-1/2} \tag{27}$$

The corresponding length can be written as (Fig. 5)

$$\ell_* = 2(2+\pi)^{-3/4} S_*^{3/4} \tag{28}$$

Finally, the lift-to-drag ratio becomes (Fig. 6)

$$E_* = \frac{4S_*^{1/2}}{\pi(2+\pi)^{1/2} + 4S_*^{3/2}} \tag{29}$$

and achieves the maximum value (23-1) for the value of S_* defined by Eq. (23-2).

After the nondimensional coordinates (12) are employed in combination with the definitions

$$\alpha = (1/\dot{r})C_f^{1/3} \quad , \quad \lambda = \lambda_1 C_f^{-1/3} \tag{34}$$

Eqs. (33) can be rewritten as

$$\xi = A(\alpha, \lambda)/A(\alpha_f, \lambda) \qquad , \qquad \rho = B(\alpha, \lambda)/B(\alpha_f, \lambda) \tag{35}$$

where

$$A(\alpha, \lambda) = \int_0^{\alpha} u^3 (3 + \lambda u)(2 + \lambda u - nu^3)^{-2} du$$

$$B(\alpha, \lambda) = \alpha^3 (2 + \lambda\alpha - n\alpha^3)^{-1} \tag{36}$$

and where, because of Eq. (32),

$$\alpha_f = -3/2\lambda \tag{37}$$

The next step consists of relating the quantity λ to the prescribed values of the lift and the length as well as calculating the maximum lift-to-drag ratio. By combining Eqs. (35) with the integrals (1) and the definitions (13), (14), and (34), we obtain the relationships

Solutions of Class II. For these solutions, the first integral (30) is solved by

$$r = 0 \qquad \text{or} \qquad 2\dot{r}^3 + \lambda_1 \dot{r}^2 = nC_f \tag{43}$$

If the corner conditions (see Chapter 1 of Ref. 3) and the natural boundary condition (10) are combined with Eqs. (43), we see that the optimum body is a spike of zero thickness followed by a semicone of apex angle identical with that of Section 6. If the nondimensional coordinates (12) are used, the shape is given by (Fig. 7)

$$\begin{aligned} \rho &= 0 , & 0 \le \xi \le \xi_o \\ \rho &= (\xi - \xi_o)/(1 - \xi_o) , & \xi_o \le \xi \le 1 \end{aligned} \tag{44}$$

where the dimensionless abcissa of the transition point satisfies the relationship

$$\xi_o = 1 - (1/2)n^{-1/2} \ell_*^{-1} \tag{45}$$

The thickness ratio and the lift-to-drag ratio of this body are given by (Figs. 8 and 9)

$$\tau_* = (1/\sqrt{2})(2n)^{-1/6} \ell_*^{-1} , \quad E_* = (4/3\pi)(2n)^{-1/3} \tag{46}$$

These solutions occur in the range

$$(1/2)n^{-1/2} \le \ell_* \le \infty \tag{47}$$

and are characterized by a lift-to-drag ratio identical with that obtained in the case where

10. GIVEN LIFT AND VOLUME

For this case, the first integral (7) in conjunction with the conditions $C = \lambda_2 = 0$ leads to the following differential equation of the optimum shape:

$$- 2\dot{r}^3 - \lambda_1 \dot{r}^2 + nC_f + \lambda_3 r = 0 \tag{48}$$

which, at the initial point, becomes

$$2\dot{r}_i^3 + \lambda_1 \dot{r}_i^2 - nC_f = 0 \tag{49}$$

Furthermore, by applying the natural boundary condition (10), we see that two classes of solutions are possible

$$\begin{aligned} &\underline{\text{Class I}} \qquad && \dot{r}_f = -(2/3)\lambda_1 \\ &\underline{\text{Class II}} \qquad && \dot{r}_f = 0 \end{aligned} \tag{50}$$

As a first step, we solve Eq. (48) in parametric form as follows:

$$\begin{aligned} x &= \int_0^r (1/\dot{r})\,dr = (2/\lambda_3) \int_{\dot{r}_i}^{\dot{r}} (3\dot{r} + \lambda_1)\,d\dot{r} \\ r &= (1/\lambda_3)(2\dot{r}^3 + \lambda_1 \dot{r}^2 - nC_f) \end{aligned} \tag{51}$$

The next step consists of relating the quantity β_i to the prescribed values of the lift and the volume as well as calculating the unknown values of the length, the thickness ratio, and the lift-to-drag ratio. By combining Eqs. (53) with the integrals (1) and the definitions (13), (14), and (52), we obtain the relationships

$$
\begin{aligned}
\ell_* &= (1/2\sqrt{2})\, A(\beta_f, \beta_i)\,[G(\beta_f, \beta_i)]^{-1/2} \\
\tau_* &= B(\beta_f, \beta_i)/A(\beta_f, \beta_i) \\
E_* &= (2/\pi)\,[G(\beta_f, \beta_i)/H(\beta_f, \beta_i)] \\
V_* &= (\pi/16\sqrt{2})\, K(\beta_f, \beta_i)\,[G(\beta_f, \beta_i)]^{-3/2}
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
G(\beta, \beta_i) &= \left[\frac{6}{7}u^7 + \frac{5}{6}\lambda u^6 + \frac{1}{5}\lambda^2 u^5 - \frac{3}{4}nu^4 - \frac{1}{3}\lambda nu^3\right]_{\beta_i}^{\beta} \\
H(\beta, \beta_i) &= \left[\frac{3}{4}u^8 + \frac{5}{7}\lambda u^7 + \frac{1}{6}\lambda^2 u^6 + \frac{3}{5}nu^5 + \lambda nu^4 + \frac{1}{3}\lambda^2 nu^3 - \frac{3}{2}n^2u^2 - \lambda n^2 u\right]_{\beta_i}^{\beta} \\
K(\beta, \beta_i) &= \pm\left[\frac{3}{2}u^8 + \frac{16}{7}\lambda u^7 + \frac{7}{6}\lambda^2 u^6 + \frac{1}{5}(\lambda^3 - 12n)u^5 - \frac{5}{2}\lambda nu^4 - \frac{2}{3}\lambda^2 nu^3 + \frac{3}{2}n^2u^2 + \lambda n^2 u\right]_{\beta_i}^{\beta}
\end{aligned}
\tag{58}
$$

The final step consists of eliminating the quantities β_i and β_f from Eqs. (53) through (57). If this is done, one obtains the functional relationships

$$
\rho = f_1(\xi, V_*) \tag{59}
$$

11. CONCLUSIONS

In the previous sections, the problem of minimizing the drag of a slender, flat-top, homothetic body of semicircular cross section in hypersonic flow is investigated under the assumptions that the pressure coefficient is Newtonian and the skin-friction coefficient is constant. The indirect methods of the calculus of variations are employed, and the necessary conditions to be satisfied by an optimum body are derived for arbitrary conditions imposed on the lift, the wetted area, the volume, the length, and the thickness. The particular problems treated are the following: (a) given lift, (b) given lift and thickness, (c) given lift and wetted area, (d) given lift and length, and (e) given lift and volume.

For case (a), the optimum body is a semicone of thickness ratio $\tau = 1.18\, C_f^{1/3}$ and lift-to-drag ratio $E = 0.360\, C_f^{-1/3}$. For each of the cases (b) and (c), the optimum body is a semicone but the thickness ratio and lift-to-drag ratio are generally different from those pertaining to case (a). For case (d), two solutions are possible depending on whether the length is smaller of larger than that associated with the optimum semicone of case (a). If the length is smaller, a blunt-nosed body is obtained while, if the length is

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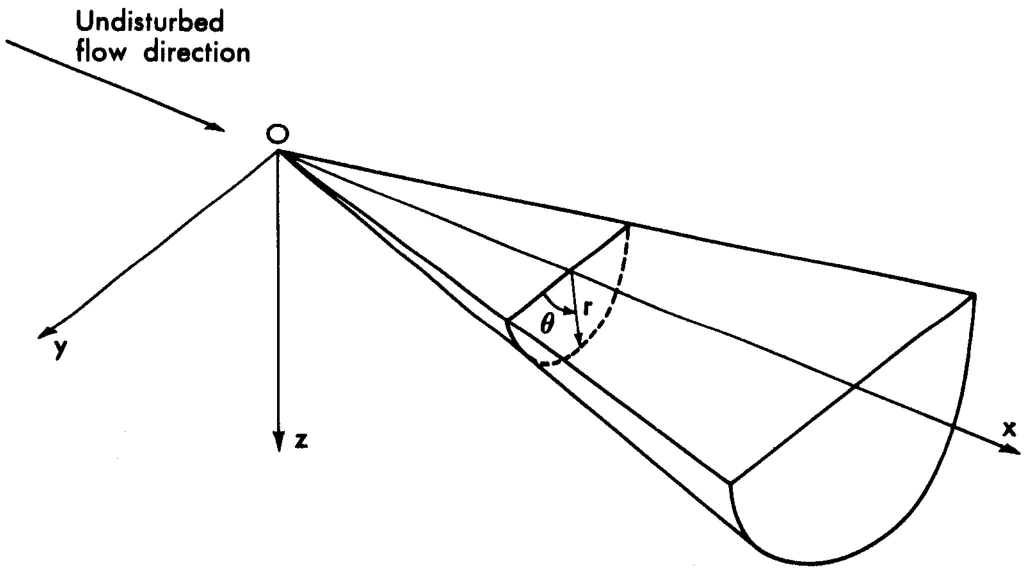

Fig. 1 Coordinate system.

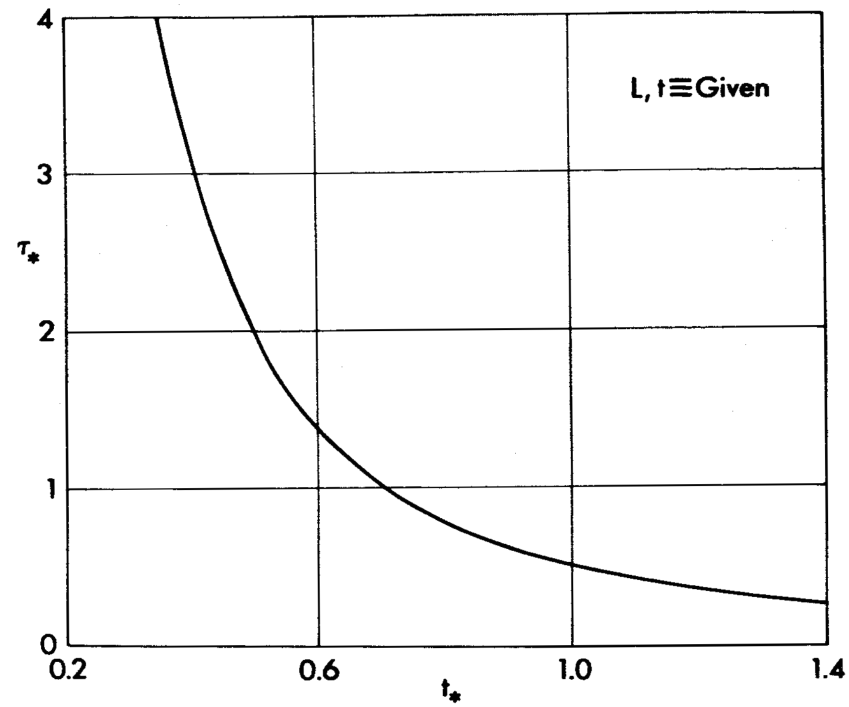

Fig. 2 Optimum thickness ratio.

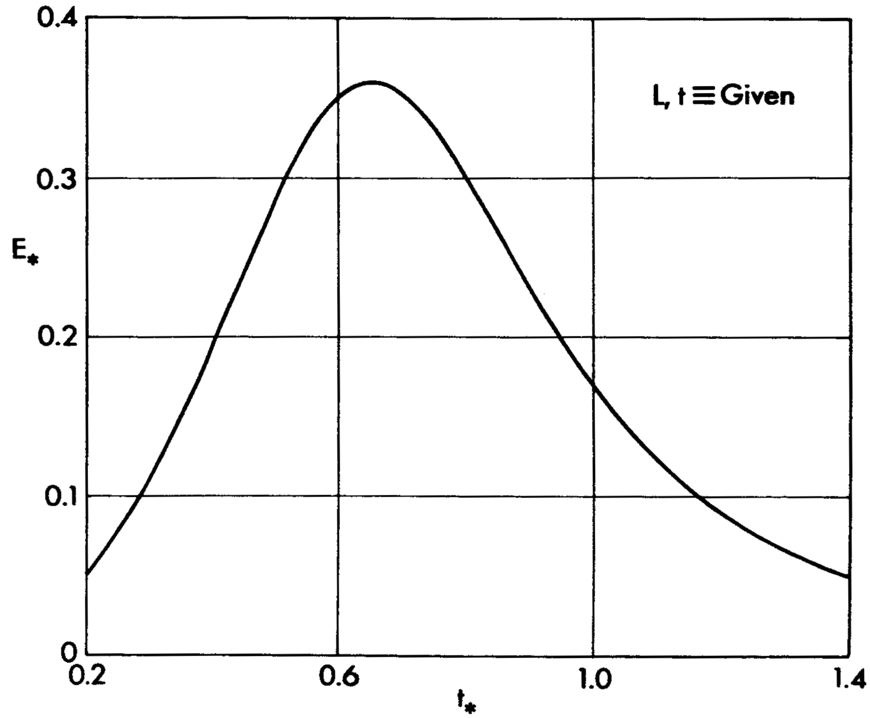

Fig. 3 Maximum lift-to-drag ratio.

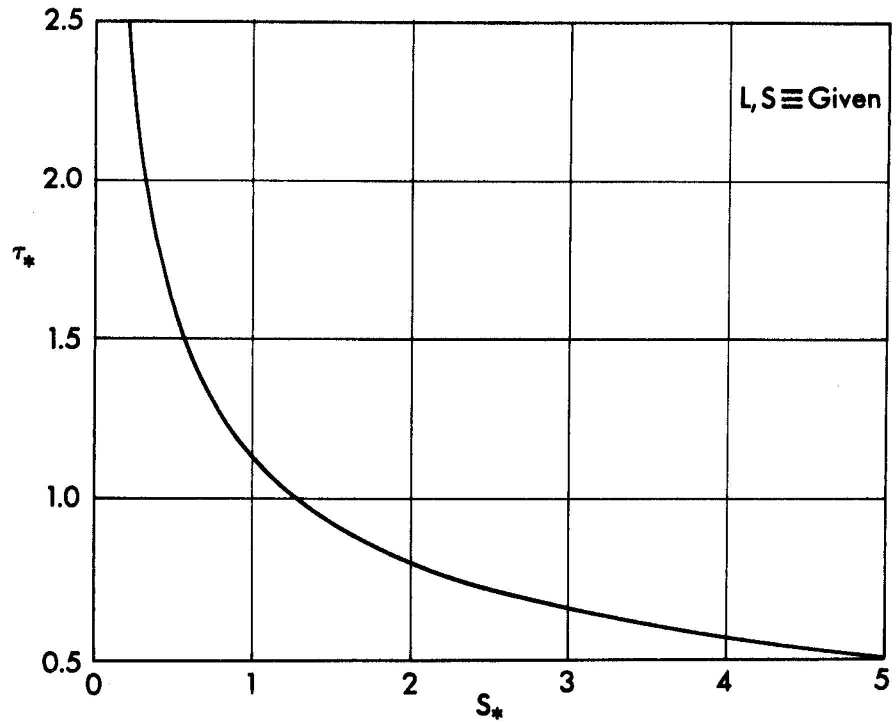

Fig. 4 Optimum thickness ratio.

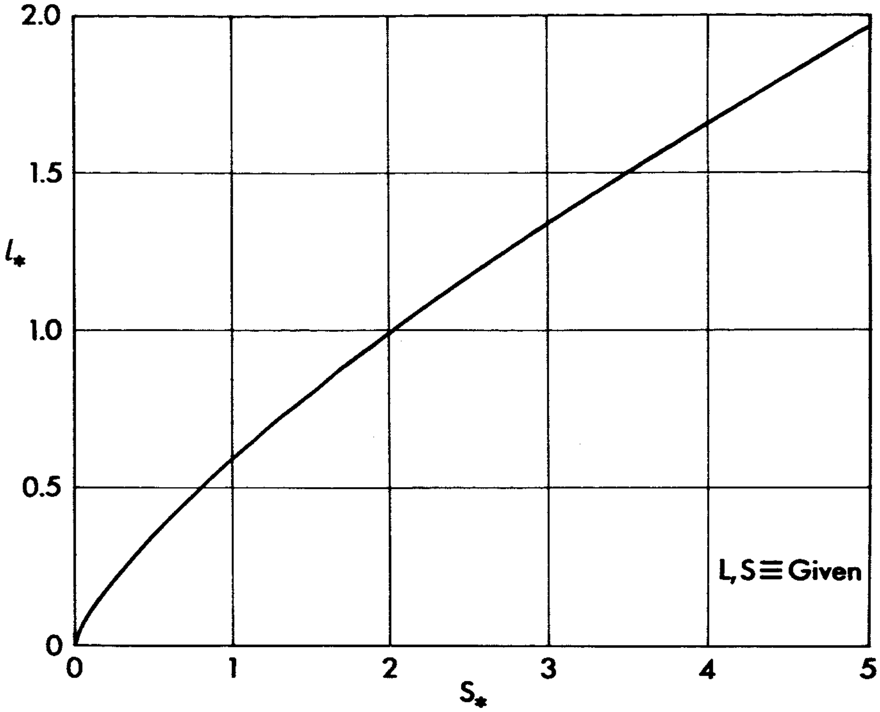

Fig. 5 Optimum length.

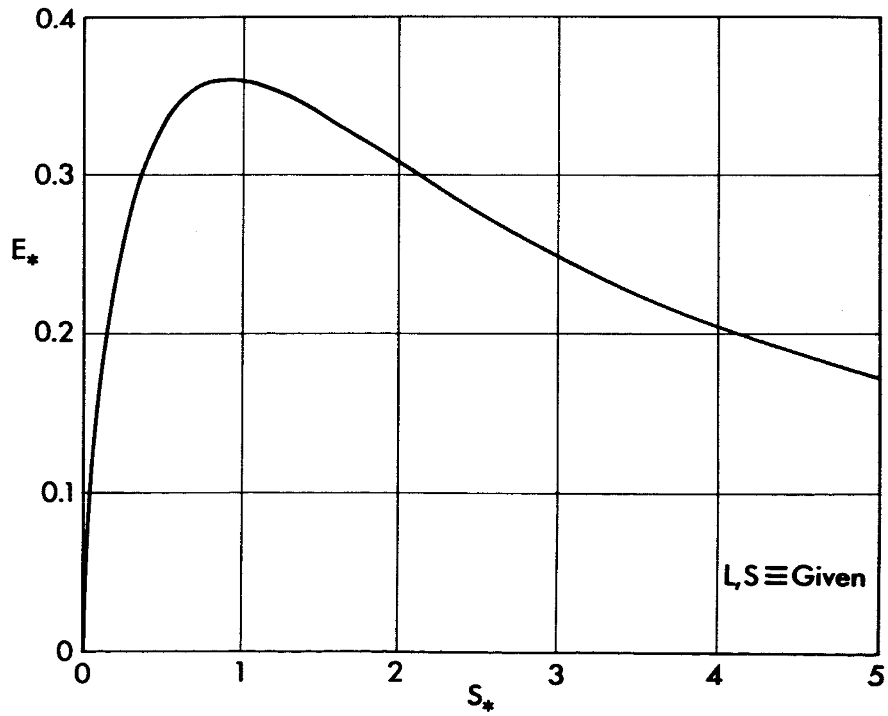

Fig. 6 Maximum lift-to-drag ratio.

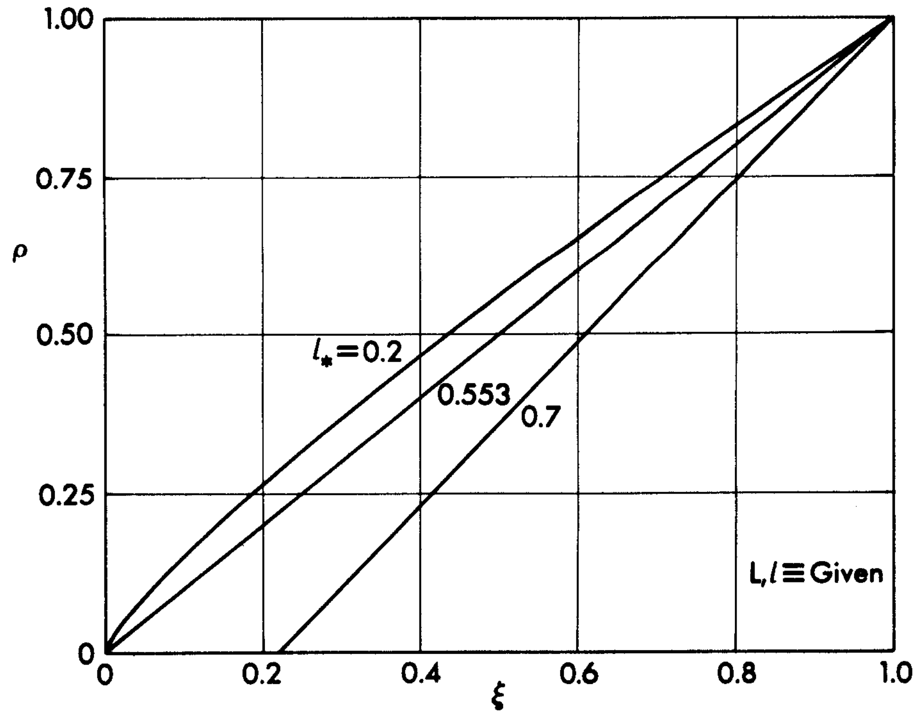

Fig. 7 Optimum shape.

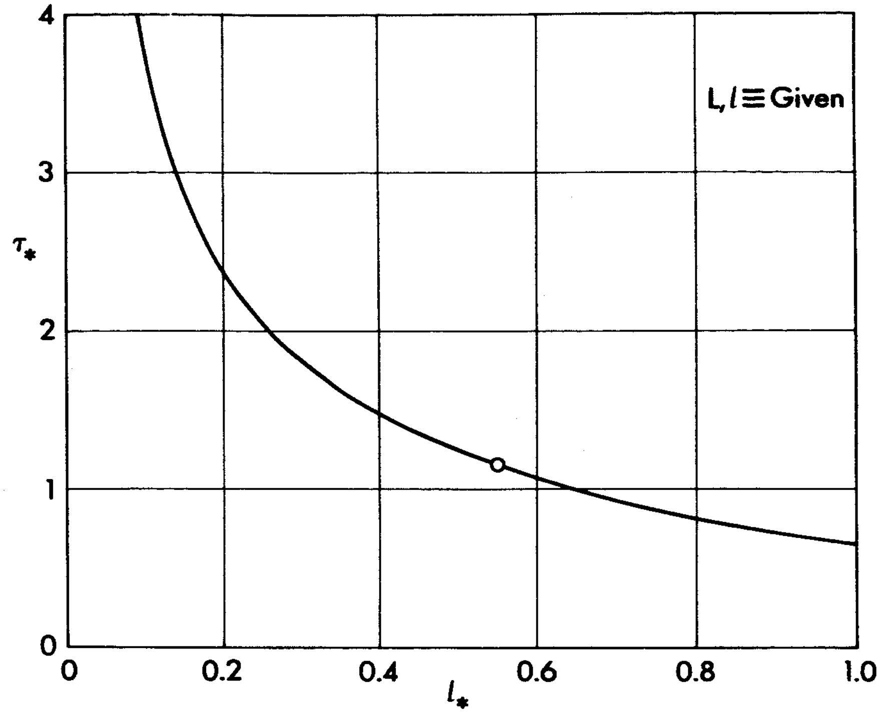

Fig. 8 Optimum thickness ratio.

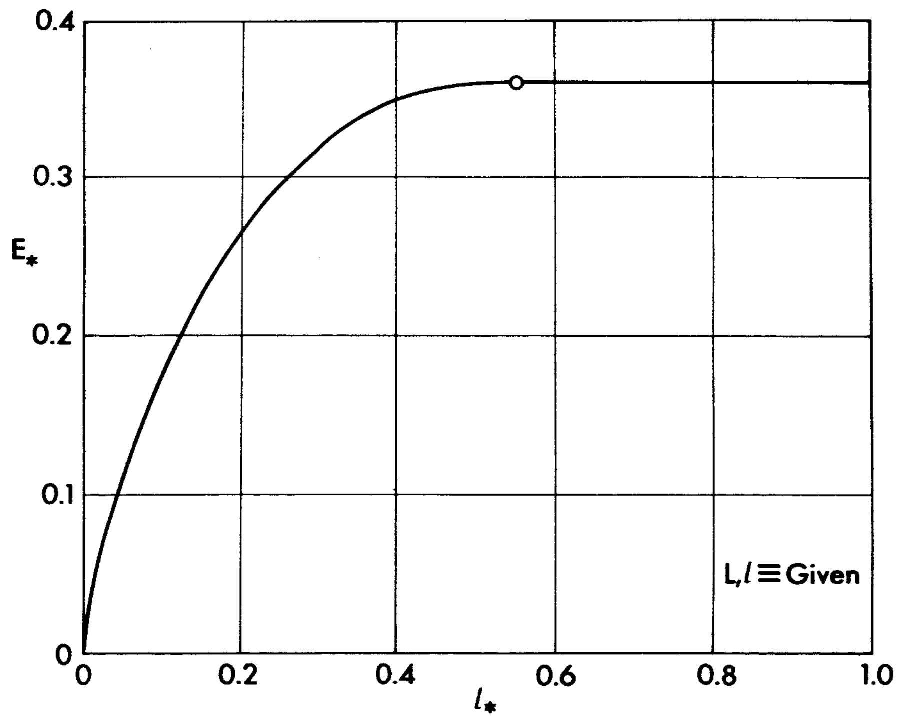

Fig. 9 Maximum lift-to-drag ratio.

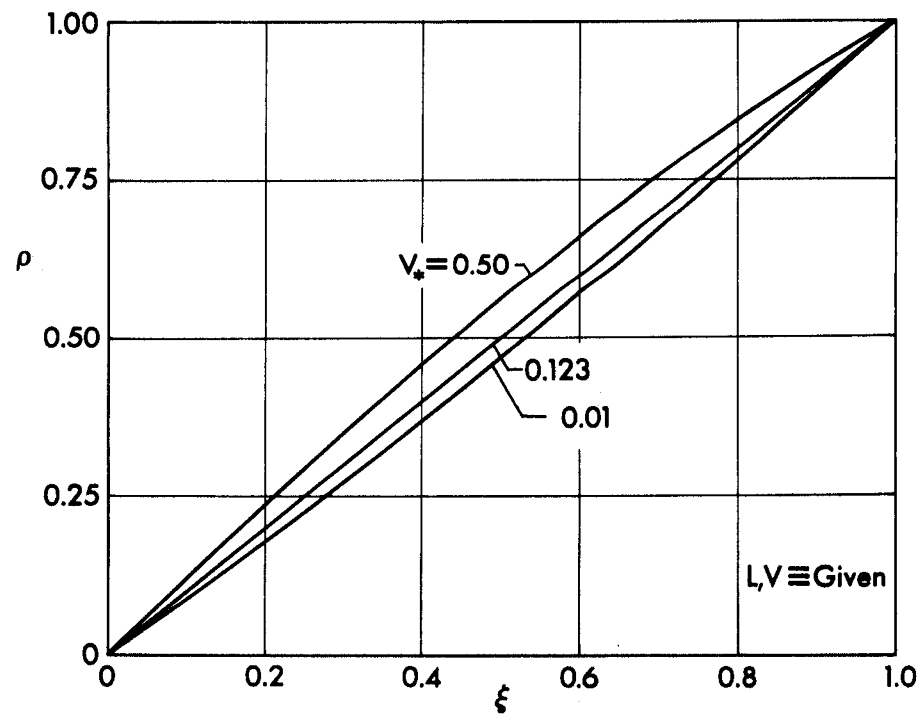

Fig. 10 Optimum shape.

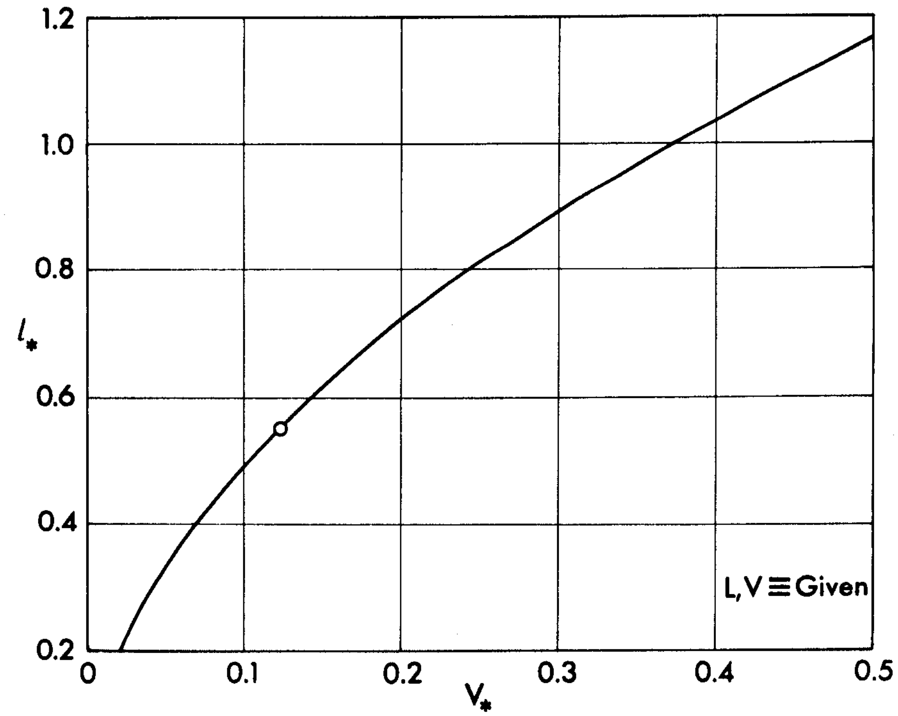

Fig. 11 Optimum length.

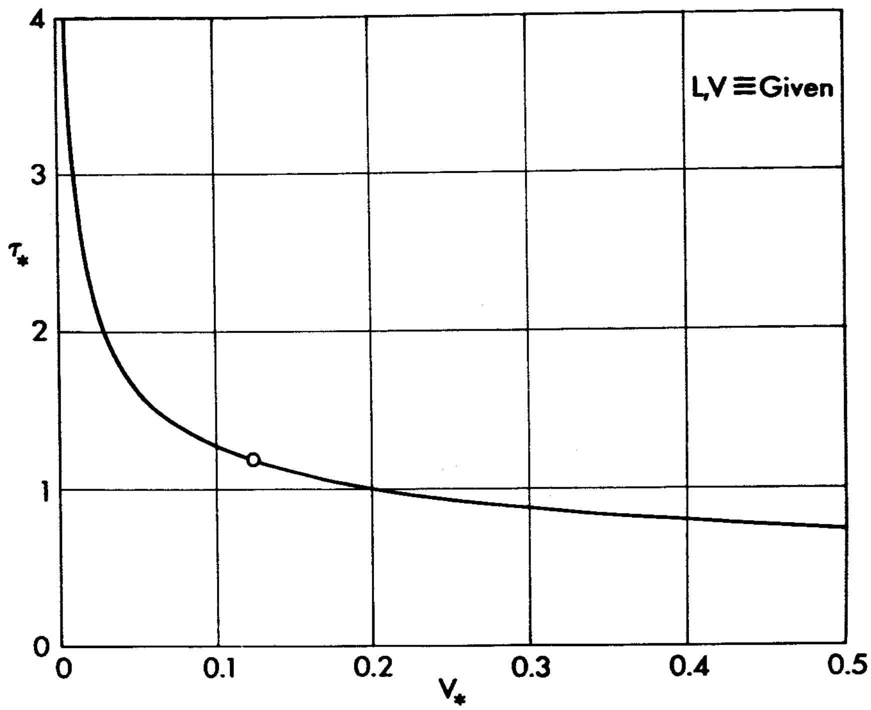

Fig. 12 Optimum thickness ratio.

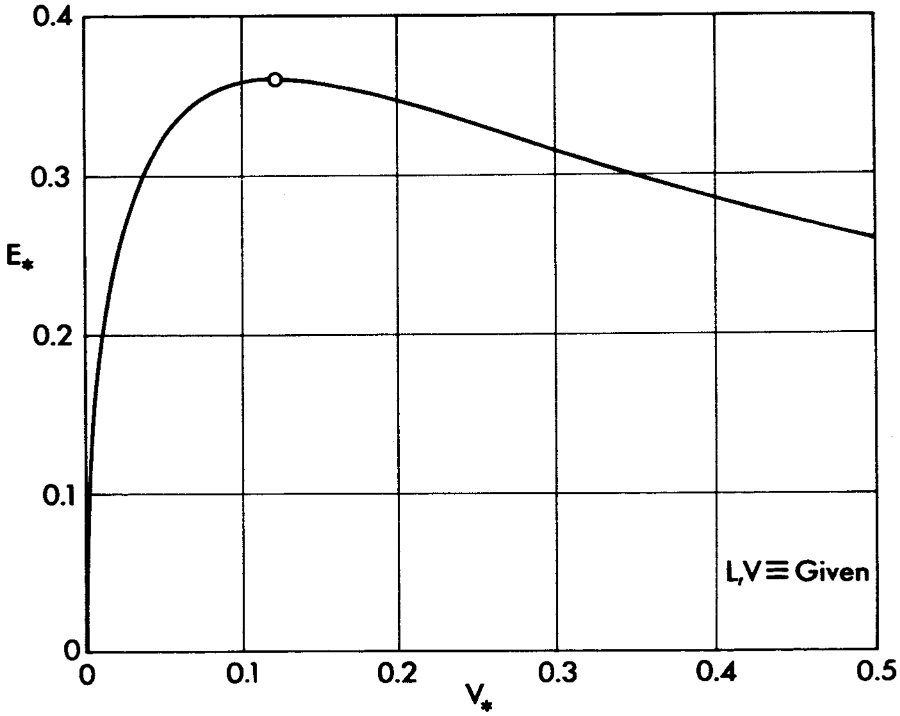

Fig. 13 Maximum lift-to-drag ratio.