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SECOND-ORDER SOLUTION OF THE POLAR OBLATENESS PROBLEM

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ABSTRACT

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This report employs the method of variation of parameters and the method of averaging to obtain a second-order solution for the motion of a satellite about an oblate earth, including the second, third, and fourth zonal harmonics. The parameters used are: two orthogonal unit vectors in the plane of the motion, one being in the direction of the initial-position vector; two quantities which are products of the eccentricity and the sine and cosine of the angle from the initial-position vector to the perigee vector, the magnitude of the angular momentum vector, and the epoch. The solution is well-behaved for negative energy, eccentricity less than one, and all inclinations with second-order secular terms occurring for certain inclinations. The second-order, short-period terms are not calculated.

CONTENTS

SECTION I

INTRODUCTION

This report presents the details of the derivation of a second-order, approximate solution for the motion of an artificial satellite about the earth. The earth is approximated by an oblate speroid with the oblateness specified by the second, third, and fourth zonal harmonics. The differential equations upon which the solution is based are obtained by the method of variation of parameters applied to the solution of the two-body problem. The parameters, in terms of which the two-body solution is expressed, are selected because of the relative simplicity of the differential equations to which their variations lead, and freedom from restrictions on initial conditions arising from circularity. The technique adopted for obtaining the solution is a modification of the method of averaging.

Section II contains an explanation of the method of averaging as it applies to the problem at hand and of the modification introduced. The modification consists of retaining some of the periodic terms in the averaged differential equations. This, obviously, is quite contrary to the intent of the method, rendering the use of the name, method of averaging, inappropriate. However, since the modification is slight, it would not be fitting to disguise the method by any other designation. Even though the incorporation of the periodic terms in the averaged, differential equations does add measurably to the complexity of their solutions and does not reduce significantly the labor required to compute the second-order averaged differential equations, it appears reasonable to expect an improvement in the basic, approximate solutions if at least all the linear terms are accounted for in the averaged equations.

The solutions contained in this report have one principal limitation which is a consequence of the set of parameters employed. The approximate solutions are not well-behaved in the region of rectilinear motion. It will be tacitly assumed, therefore, as is usually the case, that the energy is negative and the eccentricity less than one. Important qualitative properties of the solutions under these restrictions are treated by Kyner.[1] The same author has derived a technique for estimating the errors in the approximate solutions over a finite interval. The availability of such estimates, obviously, is an invaluable asset for any approximate procedure.

In Section III, the potential for the problem is given, and the perturbation equations resulting from the application of the method of variation of parameters to the two-body elements are detailed. Section IV contains the first-order solution. Section V describes the time-angle relation. In Section VI, the results of the calculations of the mean-values required for the second-order, differential equations are presented. In Section VII, the solutions of these equations are obtained.

The method of solution consists of solving the linearized equations. The non-linear terms are incorporated into the solutions by the variation of parameters of the solutions of the linearized, differential equations, substitution of the solutions, and integration.

It is of interest to note that the linearized, second-order, averaged, differential equations consist principally of two sets of four simultaneous equations. One of these sets takes into account the interdependence of certain parameters caused by the addition of the third zonal harmonic. Also, the contributions from the non-linear terms to the solutions introduce combinations of constants in the denominators which appear to be a source of difficulty. This characteristic, present in most solutions of the problem, is the small-divisor property and is commented on briefly in Sections IV and VIII. Finally, it should be noted that the second-order, short-period terms have not been computed.

SECTION II

THE METHOD OF AVERAGES

The method of averages,[2] as employed in this report, offers a technique for obtaining an approximate, closed form solution of differential equations under certain conditions. The differential equations must be such that a zero-order solution is already known, and the exact solution must be a small variation of the zero-order solution. In order to obtain higher order solutions, a second set of differential equations, derived by the application of the Method of variation of parameters to the parameters of the zero-order solution, must be developed. These equations determine the dependence of those parameters on that part of the original differential equations not satisfied by the zero-order solution. If the derived equations were amenable to exact solution, the total solution would then take the form of the zero-order solution with the parameters expressed as functions of the dependent variable. In general, however, exact expressions for the parameters cannot be found and approximate solutions must suffice. The Method of Averages is applied to the derived set of differential equations to obtain these approximate solutions.

Let the derived set of differential equations be denoted by

$$(d/dt)\,(\underline{x}) = \hat{\epsilon}\,\underline{X}\,(\underline{x}, t, \hat{\epsilon}) \qquad \text{(II-1)}$$

where

$\underline{x}$ is a vector whose components are the parameters of the zero-order solution;

$\hat{\epsilon}$ is a small quantity;

$\underline{X}$ is a vector whose components are trigonometric polynomials of the independent variable, t, with coefficients which are finite, rational functions of the x_i and $\hat{\epsilon}$.

Under these conditions, the existence of a solution for all time is assured, provided $\underline{x}$ is bounded. This latter condition will be met by the problem to be treated in this report.

A first-order solution is assumed to have the form

$$\underline{x}\,(\underline{x}_0, t, \hat{\epsilon}) = \underline{y}\,(\underline{y}_0, t, \hat{\epsilon}) + \epsilon\,\underline{H}\,(\underline{y}, t, 0)\,, \qquad \text{(II-2)}$$

where

$\underline{y}\,(\underline{y}_0, t, \hat{\epsilon})$ is the solution of

$$(d/dt)\left[\underline{y}\,(\underline{y}_0, t, \hat{\epsilon})\right] = \hat{\epsilon}\,\underline{Y}\,(\underline{y}, t, 0) = \hat{\epsilon}\left\{M_t\left[\underline{X}\,(\underline{y}, t, 0)\right] + \underline{Z}\,(\underline{y}, t, 0)\right\}, \qquad \text{(II-3)}$$

where

$\underline{y}_0 = \underline{y}\,(\underline{y}_0, 0, \hat{\epsilon})$, and

$\underline{x}_0 = \underline{x}\,(\underline{x}_0, 0, \hat{\epsilon})$.

$M_t\,(\ldots)$ is the averaging operator defined by

$$M_t\left[\underline{X}\,(\underline{y}, t, 0)\right] = \frac{1}{2\pi}\int_0^{2\pi}\left[\underline{X}\,(\underline{y}, t', 0)\right] dt'$$

with the y_i held constant.

The vector $\underline{H}\,(\underline{y}, t, 0)$ is defined by

$$\underline{H}(\underline{y}, t, 0) = \int\left[\underline{X}\,(\underline{y}, t', 0) - \underline{Y}\,(\underline{y}, t', 0)\right] dt'\,.$$

The addition of the vector $\underline{Z}\,(\underline{y}, t, 0)$ represents a modification of the usual method of averages. It is included to allow for the possibility of incorporating more than simply the mean value terms in Equation II-3 without adding significantly to the difficulty of solution. The choice of vector $\underline{Z}$ is left to the discretion of each investigator. However, it must be selected from the terms remaining in

$$\left\{\underline{X}(\underline{y}, t, 0) - M_t\,[\underline{X}(\underline{y}, t, 0)]\right\} .$$

Finally, that Equation II-2 satisfies Equation II-1 to first order may be verified by substitution.

A second-order solution is now assumed to have the form

$$\underline{x}(\underline{x}_0, t, \hat{\epsilon}) = \underline{y}(\underline{y}_0, t, \hat{\epsilon}) + \hat{\epsilon}\,\underline{H}(\underline{y}, t, 0) + \hat{\epsilon}^2\,\underline{J}(\underline{y}, t, 0) \qquad \text{(II-4)}$$

where

$$(d/dt)\left[\underline{y}(\underline{y}_0, t, \hat{\epsilon})\right] = \hat{\epsilon}\,\underline{Y}(\underline{y}, t, 0) + \hat{\epsilon}^2\,\underline{Y}'(\underline{y}, t, 0) \ . \qquad \text{(II-5)}$$

In order to determine $\underline{Y}'$ and $\underline{J}$, Equation II-4 is differentiated with respect to time and substituted in the left hand side of Equation II-1; Equation II-2 is substituted for vector $\underline{x}$ in the right hand side of Equation II-1, and vector $\underline{X}$ is expanded to second order about $\epsilon = 0$, and $\underline{x} = y$. The results are

$$\underline{Y}(\underline{y}, t, 0) + \hat{\epsilon}\,\underline{Y}'(\underline{y}, t, 0) + \underline{X}(\underline{y}, t, 0) - \underline{Y}(\underline{y}, t, 0) + \hat{\epsilon}\,(\partial/\partial \underline{y})\,[\underline{H}(\underline{y}, t, 0)]\,\underline{Y}(y, t, 0)$$

$$+ \hat{\epsilon}\,(\partial/\partial t)\,[\underline{J}(\underline{y}, t, 0)] = \underline{X}(\underline{y}, t, 0) + \hat{\epsilon}\,(\partial/\partial \underline{y})\,[\underline{X}(\underline{y}, t, 0)]\,\underline{H}(\underline{y}, t, 0)$$

$$+ \hat{\epsilon}\,(\partial/\partial \epsilon)\,[\underline{X}(\underline{y}, t, 0)\,] \ . \qquad \text{(II-6)}$$

Simplifying Equation II-6, we have

$$\underline{Y}'(\underline{y}, t, 0) + (\partial/\partial t)\,[\underline{J}(\underline{y}, t, 0) = (\partial/\partial \hat{\epsilon})\,[\underline{X}(\underline{y}, t, 0)] \qquad \text{(II-7)}$$

$$+ (\partial/\partial \underline{y})\,[\underline{X}(\underline{y}, t, 0)]\,\underline{H}(\underline{y}, t, 0) - (\partial/\partial \underline{y})\,[\underline{H}(\underline{y}, t, 0)]\,\underline{Y}(\underline{y}, t, 0) \ .$$

As in the first approximation, we take

$$\underline{Y}'(\underline{y},t,0) = M_t \Big\{ (\partial/\partial\hat{\epsilon}) [\underline{X}(\underline{y},t,0)] + (\partial/\partial\underline{y}) [\underline{X}(\underline{y},t,0)] \underline{H}(\underline{y},t,0)$$

$$- (\partial/\partial\underline{y}) [\underline{H}(\underline{y},t,0)] \underline{Y}(\underline{y},t,0) \Big\} + \underline{Z}'(\underline{y},t,0) \quad . \qquad \text{(II-8)}$$

The vector $\underline{Z}'$ is, again, some suitably chosen vector of functions selected from the right hand side of Equation II-7 after the mean value has been subtracted. The remaining periodic terms define the vector $\underline{J}$, i.e.,

$$\underline{J}(\underline{y},t,0) = \int \Big\{ (\partial/\partial\hat{\epsilon}) [\underline{X}(\underline{y},t',0)] + (\partial/\partial\underline{y}) [\underline{X}(\underline{y},t',0)] \underline{H}(\underline{y},t',0)$$

$$- (\partial/\partial\underline{y}) [\underline{H}(\underline{y},t',0)] \underline{Y}(\underline{y},t',0) - \underline{Y}'(\underline{y},t',0) \Big\} dt' . \qquad \text{(II-9)}$$

holding vector $\underline{y}$, constant.

To obtain a particular solution, use is made of a given set of initial conditions, $\underline{x}_0$, i.e.,

$$\underline{x}_0 = \underline{y}_0 + \hat{\epsilon}\, \underline{H}(\underline{y}_0, 0, 0) + \hat{\epsilon}^2\, \underline{J}(\underline{y}_0, 0, 0) \quad . \qquad \text{(II-10)}$$

A second-order solution for the vector $\underline{y}_0$ can be found by substituting the first-order solution of vector $\underline{y}_0$ from Equation II-2 in vector $\underline{H}$. This first-order solution for $\underline{y}_0$ is

$$\underline{y}_0 = \underline{x}_0 - \hat{\epsilon}\, \underline{H}(\underline{x}_0, 0, 0) \quad ,$$

so that the second-order expression for $\underline{y}_0$ is

$$\underline{y}_0 = \underline{x}_0 - \hat{\epsilon}\, \underline{H}(\underline{x}_0, 0, 0) - \hat{\epsilon}^2\, \underline{J}(\underline{x}_0, 0, 0) + \hat{\epsilon}^2 (\partial/\partial\underline{y}) \Big[\underline{H}(\underline{x}_0, 0, 0) \Big] \underline{H}(\underline{x}_0, 0, 0) \; .$$

SECTION III

EQUATIONS OF MOTION

The purpose of this report is to derive an approximate, closed form solution to the equations of motion of a body subject to the gravitational attraction of a mass centered at the origin of an inertial coordinate system, X, Y, Z, symmetric about the Z-axis, having the potential

$$V = -(\mu/r)\left[1 - \sum_{n=2}^{4} J_n/r^n \, P_n\,(z/r)\right]$$

with the radius of the mass in the X, Y-plane taken as unit length. Since it is intended that the solution to be derived shall hold for a satellite about the earth, the quantity μ is the universal gravitational constant times the mass of the earth, and the J_n, n = 2, 3, 4, are empirically determined, given quantities. (Appropriate replacements should be introduced for attracting centers other than the earth; e.g., the moon.) Taking into account the relations between the magnitudes of this J_n and substituting for the $P_n\,(z/r)$, the potential, V, may be rewritten, adopting Kyner's notation, as

$$V(\epsilon') = -(\mu/r)\left\{1 + (\epsilon'/3r^2)\left[1 - 3\,(z/r)^2\right] - (\epsilon'^2 B_3/2r^3)\,(z/r)\left[5\,(z/r)^2 - 3\right] + (\epsilon'^2 B_4\, 3/8r^4)\left[1 - 10\,(z/r)^2 + (35/3)\,(z/r)^4\right]\right\} \qquad \text{(III-1)}$$

where

$$\epsilon' = 3J_2/2$$

$$B_3 = (4/9)\,(J_3/J_2^2)$$

$$B_4 = -\,(4/9)\,(J_4/J_2^2)$$

The most recent values for the $J_{n's}$ should be obtained so that ϵ', B_3, and B_4 may be kept up to date.[3]

For the ensuing development, it is necessary to have available the gradient of the potential, V.

$$- \text{grad}\,(V) = -(\mu/r^3)\underline{R} + \underline{F}$$

$$\underline{F} = \underline{F}_2 + \underline{F}_3 + \underline{F}_4$$

$$\underline{F}_2 = -\epsilon'(\mu/r^4)\left\{(\underline{R}/r)\left[1 - 5\,(z/r)^2\right] + 2\underline{K}\,(z/r)\right\} \tag{III-2}$$

$$\underline{F}_3 = \epsilon'^2\, B_3(\mu/2r^5)\left\{(\underline{R}/r)5\,(z/r)\left[7\,(z/r)^2 - 3\right] - \underline{K}3\left[5(z/r)^2 - 1\right]\right\} \tag{III-3}$$

$$\underline{F}_4 = -\epsilon'^2\, B_4\,(15/8)\,(\mu/r^6)\left\{(\underline{R}/r)\left[1 - 14\,(z/r)^2 + 21\,(z/r)^4\right]\right.$$

$$\left. + 4\,\underline{K}\,(z/r)\left[1 - (7(z/r)^2/3)\right]\right\} \tag{III-4}$$

where $\underline{K}$ is a unit vector in the direction of the Z-coordinate.

The equations of motion for the problem as specified at the beginning of this section may then be expressed as

$$(d^2/dt^2)\,(\underline{R}) = -\,(\mu/r^3)\,\underline{R} + \underline{F} \tag{III-5}$$

since the vector $\underline{F}$ has the small quantity ϵ' as a common factor, the zero-order solution will be taken to be the solution of the two-body problem,

$$(d^2/dt^2)\,(\underline{R}) = -\,(\mu/r^3)\,\underline{R}$$

Of the numerous sets of parameters in terms of which this solution may be formulated, the following has been selected[4] g, $\underline{U}$, $\underline{V}$, e cos θ, and e sin θ. The parameter g is the magnitude of the angular momentum vector, $\underline{U}$ and $\underline{V}$ are unit orthogonal vectors which specify the plane of the motion. The parameters e and θ are the eccentricity and the angle measured from $\underline{U}$ to the perigee vector, respectively. The vector $G/g = \underline{U} \times \underline{V}$ is the unit angular momentum vector.

The state variables for the motion of the body may then be expressed as follows:

$$R = 4\,(\cos\varphi\,\underline{U} + \sin\varphi\,\underline{V}) \qquad \text{(III-6)}$$

$$(d/dt)\,(\underline{R}) = -(\mu/g)\left[(\sin\varphi + e\sin\theta)\,\underline{U} - (\cos\varphi + e\cos\theta)\,\underline{V}\right] \qquad \text{(III-7)}$$

$$t = \tau + \left[g^3/\mu^2\,(1-e^2)^{3/2}\right]\Big(2\Big\{\arctan\left[(1-e^2)^{1/2}\tan(f/2)/(1+e)\right]$$

$$+ \arctan\left[(1-e^2)^{1/2}\tan(\theta/2)/(1+e)\right]\Big\}$$

$$- (1-e^2)^{1/2}\Big\{\left[e\sin f/(1+e\cos f)\right] + e\sin\theta/(1+e\cos\theta)\Big\}\Big) \qquad \text{(III-8)}$$

$$r = g^2/\mu\,(1 + e\cos f) \qquad \text{(III-9)}$$

$$f = \varphi - \theta \qquad \text{(III-10)}$$

$$(d/dt)\,(\varphi) = g/r^2 \qquad \text{(III-11)}$$

In order to determine the perturbation equations for these parameters most conveniently, they should first be expressed as functions of the state variables, $\underline{R}$ and $(d/dt)\,(\underline{R})$. (The term τ is the time of $\underline{U}$ - passage. It will be treated in a separate section.)

$$g^2 = \left[\underline{R} \times (d/dt)\,(\underline{R})\right]^2 \qquad \text{(III-12)}$$

$$\underline{U} = (\cos\varphi)\,\underline{R}/r - \sin\varphi\left[\underline{G}/g \times \underline{R}/r\right] \qquad \text{(III-13)}$$

$$\underline{V} = (\sin\varphi)\,\underline{R}/r + \cos\varphi\left[\underline{G}/g \times \underline{R}/r\right] \qquad \text{(III-14)}$$

$$e\cos\theta = \cos\varphi\,(g^2/\mu r - 1) + \sin\varphi\,g/\mu r\,(\underline{R}\cdot(d/dt)\,(\underline{R})) \qquad \text{(III-15)}$$

$$e\sin\theta = \sin\varphi\,(g^2/\mu r - 1) - \cos\varphi\,g/\mu r\,(\underline{R}\cdot(d/dt)\,(\underline{R})) \qquad \text{(III-16)}$$

The perturbation equations are then obtained by taking the time-derivatives of Equations III-12 through III-16 and substituting for $(d^2/dt^2)\,(\underline{R})$ from Equation III-5 wherever it occurs. After cancelling like terms and simplifying, we have

$$(d/dt)\,(g) = (\underline{G}/g \times \underline{R}) \cdot \underline{F} \qquad \text{(III-17)}$$

$$(d/dt)\,(\underline{G}/g) = -\,(\underline{F} \cdot \underline{G}/g)\,(\underline{G}/g \times \underline{R}/g) \qquad \text{(III-18)}$$

$$(d/dt)\,(\underline{U}) = -\,(\underline{F} \cdot \underline{G}/g)\; r/g\,(\underline{G}/g) \sin\varphi \qquad \text{(III-19)}$$

$$(d/dt)\,(\underline{V}) = (\underline{F} \cdot \underline{G}/g)\,(r/g)\,(\underline{G}/g) \cos\varphi \qquad \text{(III-20)}$$

$$(d/dt)\,(e \cos\theta) = 2\Big[(d/dt)\,(g)/g\Big](1+e \cos f) \cos\varphi + \sin\varphi\Big\{\Big[(d/dt)\,(g)/g\Big]\, e \sin f + (g/\mu r)\,(\underline{R} \cdot \underline{F})\Big\} \qquad \text{(III-21)}$$

$$(d/dt)\,(e \sin\theta) = 2\Big[(d/dt)\,(g)/g\Big](1+e \cos f) \sin\varphi - \cos\varphi\Big\{\Big[(d/dt)\,(g)/g\Big]\, e \sin f + (g/\mu r)\,(\underline{R} \cdot \underline{F})\Big\} \qquad \text{(III-22)}$$

Equations III-17 through III-22 correspond to Equation II-1

$$(d/dt)\,(\underline{x}) = \epsilon'\, \underline{X}\,(\underline{x},\ t,\ \epsilon')$$

with the components of $\underline{x}$ being g, $\underline{U}$, $\underline{V}$, $e \cos\theta$, and $e \sin\theta$.

It is appropriate at this juncture to point out that there are two constants of the motion for the problem. The two constants are the energy, E, and the third component of the angular momentum, g_3. The equation for the energy is $E = v^2/2 + V$. This equation follows from the integration of Equation II-5 after dotting with the vector $(d/dt)\,(\underline{R})$,

$$(d/dt)\,(\underline{R}) \cdot (d^2/dt^2)\,(\underline{R}) = -\,\mu\Big[R \cdot (d/dt)\,(R)\Big]/r^3 + (d/dt)\,(\underline{R}) \cdot \underline{F}.$$

That the component g_3 is constant follows from

$$\underline{K} \cdot \Big[\underline{R} \times (d^2/dt^2)\,(\underline{R})\Big] = \underline{K} \cdot (\underline{R} \times \underline{F})$$

Since the vector $\underline{F}$ consists only of the vectors $\underline{R}$ and $\underline{K}$, the right hand side of the equation is clearly zero.

SECTION IV

FIRST-ORDER SOLUTION

As is evident from Equation II-2, it is necessary to distinguish between the vectors $\underline{x}$ and $\underline{y}$. The components of the vector $\underline{x}$ are the parameters for which a solution is sought. The components of the vector $\underline{y}$ are an auxiliary set of "smoothing" parameters. Since, in the following development, the same basic symbols will be used to denote the components of the vectors $\underline{x}$ and $\underline{y}$ - so that one does not lose sight of their significance - their distinction will be observed by affixing an asterisk to the components of the vector $\underline{x}$.

The basic set of equations to be solved in obtaining the first-order solution is

$$(d/dt)\ (\underline{x}) = \epsilon\ \underline{X}\ (\underline{x},\ t,\ 0) \qquad \text{(IV-1)}$$

This set of equations corresponds to the set III-17 through III-22, where the vector $\underline{F}$ is limited to the vector $\underline{F}_2$. Making a substitution for the vector $\underline{F}_2$ from Equation III-2 yields the following set:

$$(d/dt)\ (g) = -\ \epsilon'(\mu/r^3)\ (\underline{G}/g \times \underline{R}/r) \cdot \underline{K}\ 2\ (z/r) \qquad \text{(IV-2)}$$

$$(d/dt)\ (\underline{U}) = 2\ \epsilon'\ (\mu/r^4)\ (g_3/g)\ (r/g)\ (\underline{G}/g)\ (z/r)\ \sin\varphi \qquad \text{(IV-3)}$$

$$(d/dt)\ (\underline{V}) = -\ 2\ \epsilon'\ (\mu/r^4)\ (g_3/g)\ (r/g)\ (\underline{G}/g)\ (z/r)\ \cos\varphi \qquad \text{(IV-4)}$$

The parameters $e\cos\theta$ and $e\sin\theta$ will be treated in more detail shortly.

It now becomes evident that, since the equation

$$(d/dt)\ (\varphi) = g/r^2$$

holds, even for the perturbed problem, a change of variable may be effected. The general form of the set of differential equations is transformed from Equations IV-1 to

$$(d/d\varphi)\ (\underline{x}) = \epsilon\ \underline{X}\ (\underline{x},\ \varphi,\ 0) \qquad \text{(IV-5)}$$

Also, to simplify the equations somewhat and to make them, superficially at least, appear more meaningful, Equation II-2 will be written as

$$\underline{x}\,(\underline{x}_0,\ \varphi,\ \epsilon) = \underline{y}(\underline{y}_0,\ \varphi,\ \epsilon) + \underline{y}_s\,(\underline{y},\ \varphi,\ 0) \qquad \text{(IV-6)}$$

Along with these changes, the appropriate substitutions will be introduced into Equations IV-2 through IV-4 so that the equations are expressed in terms of the parameters and φ. In addition, let

$$\mu^2\,\epsilon'/g^4 = \epsilon$$

The equations then take the form

$$(d/d\varphi)\,(g^*) = \left\{-\,2\,\epsilon\,g\,(1+e\,\cos f)\,(v_3\,\cos\varphi - u_3\,\sin\varphi)\,(u_3\cos\varphi + v_3\sin\varphi)\right\}^* \qquad \text{(IV-7)}$$

$$(d/d\varphi)\,(\underline{U}^*) = \left\{2\,\epsilon\,(1+e\,\cos f)\,(u_3\cos\varphi + v_3\,\sin\varphi)(\underline{G}/g)\sin\varphi\right\}^* \qquad \text{(IV-8)}$$

$$(d/d\varphi)\,(\underline{V}^*) = \left\{-\,2\,\epsilon\,(1+e\cos f)\,(u_3\,\cos\varphi + v_3\,\sin\varphi)(\underline{G}/g)\cos\varphi\right\}^* \qquad \text{(IV-9)}$$

The equations are now in suitable form for applying the averaging technique. Equations IV-7 through IV-9 correspond to the φ- derivative of Equation IV-6, i.e.,

$$(d/d\varphi)\,(\underline{x}) = (d/d\varphi)\,(\underline{y}) + (d/d\varphi)\,(\underline{y}_s) \qquad \text{(IV-10)}$$

where

$$(d/d\varphi)\,(\underline{y}) = \epsilon\,M_\varphi\left[\underline{X}\,(\underline{y},\ \varphi,\ 0)\right] + \epsilon\,\underline{Z}\,(\underline{y},\ \varphi,\ 0) \qquad \text{(IV-11)}$$

and

$$(d/d\varphi)\,(\underline{y}_s) = \epsilon\left[\underline{X}\,(\underline{y},\ \varphi,\ 0) - \underline{Y}\,(\underline{y},\ \varphi,\ 0)\right] \qquad \text{(IV-12)}$$

The differential equation for the parameter g^*, Equation IV-7, has zero mean value. Although the terms $(v_3\,\cos\varphi - u_3\,\sin\varphi)\,(u_3\,\cos\varphi + v_3\,\sin\varphi)$ can be integrated exactly and appear to make an interesting choice for the $\underline{Z}$ expression, experience indicates that such a choice introduces complications in the second-order development. For this reason, it is preferred to take $\underline{Z} = 0$. The results are

$$(d/d\varphi)\ (g) = 0$$

or

$$g = g_0 \tag{IV-13}$$

It follows that $(d/d\varphi)(g_s)$ equals the right hand side of Equation IV-7, unstarred. Integrating Equation IV-7, holding the parameters constant, we have

$$g_s = -\epsilon g \Big(\{(z/r)^2 - [(d/d\varphi)\ (z/r)]^2\}/2 - (2/3)\Big\{(z/r)\,(d/d\varphi)\,(z/r)\,e \sin f$$

$$- [(z/r)^2 - \{(d/d\varphi)\,(z/r)\}^2]\, e \cos f \Big\}\Big) \tag{IV-14}$$

The first-order solution g^* is

$$g^* = g_0 + g_s \tag{IV-15}$$

The differential equations for the vectors $\underline{U}^*$ and $\underline{V}^*$ are Equations IV-8 and IV-9. The equations corresponding to Equation IV-11 are chosen to be

$$(d/d\varphi)\ (\underline{U}) = 2\,\epsilon\,(g_3/g)\ (z/r)\ (\underline{G}/g)\ \sin\varphi \tag{IV-16}$$

and

$$(d/d\varphi)\ (\underline{V}) = -\,2\,\epsilon\ (g_3/g)\ (z/r)\ (\underline{G}/g)\ \cos\varphi \tag{IV-17}$$

One may observe that in these equations the $\underline{Z}$-terms are not taken to be zero. These equations are divided into two sub-sets: those for u_3 and v_3, and those for u_1, u_2, v_1, and v_2.

The equations for u_3 and v_3 are

$$(d/d\varphi)\ (u_3) = 2\ \epsilon\,(g_3/g)^2\ (z/r)\ \sin\varphi \tag{IV-18}$$

and

$$(d/d\varphi)\ (v_3) = -\ 2\,\epsilon\ (g_3/g)^2\ (z/r)\ \cos\varphi \tag{IV-19}$$

These two equations are solved simultaneously, making use of the results of Equation IV-13, and setting

$$\epsilon\,(g_3/g)^2 = \epsilon_2$$

the results are

$$u_3 = \left[u_{30}\cos\varphi - v_{30}\sin\varphi\right]\cos\left[\sqrt{1+2\,\epsilon_{20}}\,\varphi\right]$$

$$+ \sin\left[\sqrt{1+2\,\epsilon_{20}}\,\varphi\right]\left\{u_{30}\sqrt{1+2\,\epsilon_{20}}\,\sin\varphi + (v_{30}/\sqrt{1+2\,\epsilon_{20}})\cos\varphi\right\} \quad \text{(IV-20)}$$

$$v_3 = \left[u_{30}\sin\varphi + v_{30}\cos\varphi\right]\cos\left[\sqrt{1+2\,\epsilon_{20}}\,\varphi\right]$$

$$- \sin\left[\sqrt{1+2\,\epsilon_{20}}\,\varphi\right]\left[u_{30}\sqrt{1+2\,\epsilon_{20}}\,\cos\varphi - (v_{30}/\sqrt{1+2\,\epsilon_{20}})\sin\varphi\right] \quad \text{(IV-21)}$$

The equations for u_1, u_2, v_1, and v_2 must be modified before the solution may be derived because the first two components of the vector $\underline{G}$ are not constants of the motion. These equations are re-expressed as

$$(d/d\varphi)(u_1) = 2\epsilon\,(g_3/g)\sin\varphi\,[u_2\sin\varphi - v_2\cos\varphi$$

$$+ (g_3/g)(u_1\cos\varphi + v_1\sin\varphi)] \quad \text{(IV-22)}$$

$$(d/d\varphi)(u_2) = 2\epsilon\,(g_3/g)\sin\varphi\,[-(u_1\sin\varphi - v_1\cos\varphi)$$

$$+ (g_3/g)(u_2\cos\varphi + v_2\sin\varphi)] \quad \text{(IV-23)}$$

$$(d/d\varphi)(v_1) = -2\,\epsilon\,(g_3/g)\cos\varphi\,[u_2\sin\varphi - v_2\cos\varphi$$

$$+ (g_3/g)(u_1\cos\varphi + v_1\sin\varphi)] \quad \text{(IV-24)}$$

$$(d/d\varphi)(v_2) = -2\,\epsilon\,(g_3/g)\cos\varphi\,[-(u_1\sin\varphi - v_1\cos\varphi)$$

$$+ (g_3/g)(u_2\cos\varphi + v_2\sin\varphi)] \quad \text{(IV-25)}$$

This set of equations is solved by standard methods. The solution is:

$$\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} = \frac{1}{4\sqrt{\alpha^2+2\beta+1}} \begin{bmatrix} a_c & a_s & b_s & -b_c \\ -a_s & +a_c & b_c & b_s \\ c_s & -c_c & d_c & d_s \\ c_c & c_s & -d_s & d_c \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ v_{10} \\ v_{20} \end{bmatrix} \qquad \text{(IV-26)}$$

where

$$a_c = -(|\lambda_2| + 2\beta)\cos|\lambda_1|\varphi + (|\lambda_1| + 2\beta)\cos|\lambda_2|\varphi - (|\lambda_4| - 2\beta)\cos|\lambda_3|\varphi + (|\lambda_3| - 2\beta)\cos|\lambda_4|\varphi$$

$$a_s = -(|\lambda_2| + 2\beta)\sin|\lambda_1|\varphi + (|\lambda_1| + 2\beta)\sin|\lambda_2|\varphi - (|\lambda_4| - 2\beta)\sin|\lambda_3|\varphi + (|\lambda_3| - 2\beta)\sin|\lambda_4|\varphi$$

$$b_s = (|\lambda_2| - 2\alpha)\sin|\lambda_1|\varphi - (|\lambda_1| - 2\alpha)\sin|\lambda_2|\varphi - (|\lambda_4| - 2\alpha)\sin|\lambda_3|\varphi + (|\lambda_3| - 2\alpha)\sin|\lambda_4|\varphi$$

$$b_c = (|\lambda_2| - 2\alpha)\cos|\lambda_1|\varphi - (|\lambda_1| - 2\alpha)\cos|\lambda_2|\varphi - (|\lambda_4| - 2\alpha)\cos|\lambda_3|\varphi + (|\lambda_3| - 2\alpha)\cos|\lambda_4|\varphi$$

$$c_c = -(|\lambda_2| + 2\beta)\cos|\lambda_1|\varphi + (|\lambda_1| + 2\beta)\cos|\lambda_2|\varphi + (|\lambda_4| - 2\beta)\cos|\lambda_3|\varphi - (|\lambda_3| - 2\beta)\cos|\lambda_4|\varphi$$

$$c_s = -(|\lambda_2| + 2\beta)\sin|\lambda_1|\varphi + (|\lambda_1| + 2\beta)\sin|\lambda_2|\varphi + (|\lambda_4| - 2\beta)\sin|\lambda_3|\varphi - (|\lambda_3| - 2\beta)\sin|\lambda_4|\varphi$$

$$d_c = -(|\lambda_2| - 2\alpha)\cos|\lambda_1|\varphi + (|\lambda_1| - 2\alpha)\cos|\lambda_2|\varphi - (|\lambda_4| - 2\alpha)\cos|\lambda_3|\varphi + (|\lambda_3| - 2\alpha)\cos|\lambda_4|\varphi$$

$$d_s = -(|\lambda_2| - 2\alpha)\sin|\lambda_1|\varphi + (|\lambda_1| - 2\alpha)\sin|\lambda_2|\varphi - (|\lambda_4| - 2\alpha)\sin|\lambda_3|\varphi$$
$$+ (|\lambda_3| - 2\alpha)\sin|\lambda_4|\varphi$$

and

$$\alpha = \epsilon \frac{g_3}{g_o}$$

$$\beta = \epsilon \left(\frac{g_3}{g_o}\right)^2$$

$$|\lambda_1| = \alpha + 1 + \sqrt{\alpha^2 + 2\beta + 1}$$

$$|\lambda_2| = \alpha + 1 - \sqrt{\alpha^2 + 2\beta + 1}$$

$$|\lambda_3| = (\alpha - 1) + \sqrt{\alpha^2 + 2\beta + 1}$$

$$|\lambda_4| = (\alpha - 1) - \sqrt{\alpha^2 + 2\beta + 1}$$

The absolute value signs used here indicate that a factor i is omitted from the λ's which are the characteristic roots for the system of Eqs. (IV-22) - (IV 25).

The vectors $\underline{U}_s$ and $\underline{V}_s$ are easily obtained.

$$\underline{U}_s = 2(\epsilon_1/3)(\underline{G}/g)\{e \sin f\,[2v_3 - (z/r)\sin\varphi] - e\cos f\,\{[(d/d\varphi)(z/r)]\sin\varphi$$
$$+ (z/r)\cos\varphi\}\} \qquad \text{(IV-27)}$$

$$\underline{V}_s = 2(\epsilon_1/3)(\underline{G}/g)\{e \sin f\,[(z/r)\cos\varphi - 2u_3] + e\cos f\,\{[(d/d\varphi)(z/r)\cos\varphi]$$
$$- (z/r)\sin\varphi\}\} \qquad \text{(IV-28)}$$

The complete first-order solutions for the vector $\underline{U}^*$ and $\underline{V}^*$ are obtained by making the appropriate substitutions from Equations IV-20, IV-26, IV-27 and IV-28 into

$$\underline{U}^* = \underline{U} + \underline{U}_s \qquad \text{(IV-29)}$$

and

$$\underline{V}^* = \underline{V} + \underline{V}_s \qquad \text{(IV-30)}$$

Since the parameters e cos θ and e sin θ will be treated in this report in a slightly different form from that in Reference 4, a more detailed development will be given here. Beginning with Equations III-21 and III-22, two substitutions may be introduced which, at least superficially, lead to some simplification. The first substitution is

$$(d/dt)\,(g)/g = [(d/dt)\,(\underline{R}) \cdot \underline{F}]\,(r^2/g^2) - (\underline{R} \cdot \underline{F})\,[R \cdot (d/dt)\,(R)]/g^2$$

Second, if we set

$$V = \sum_{n=2}^{4} V_n,$$

then

$$[(d/dt)\,(\underline{R}) \cdot \underline{F}] = \sum_{n=2}^{4} (d/dt)\,(V_n)$$

and

$$(\underline{R} \cdot \underline{F}) = -\sum_{n=2}^{4} (n+1)\,V_n$$

Making these substitutions and carrying out some reductions, we have

$$(d/d\varphi)\,(e \cos\theta) = \sum_{n=2}^{4} (d/d\varphi) \Big\{ V_n\,[2 \cos\varphi\,(r/\mu) + (r^2/g^2) \sin\varphi\ e \sin f]$$

$$- V_n\,\mu\,(r^3/g^4)\,[(2n-1)\,(1 + e \cos f)\,e \sin\theta + (n-1)\,(1-e^2) \sin\varphi] \Big\} \qquad \text{(IV-31)}$$

$$(d/d\varphi)\,(e \sin\theta) = \sum_{n=2}^{4} (d/d\varphi) \Big\{ V_n\,[2\,(r/\mu) \sin\varphi - (r^2/g^2)\,e \sin f \cos\varphi]$$

$$+ V_n\,\mu\,(r^3/g^4)\,[(2n-1)\,(1 + e \cos f)\,e \cos\theta + (1-e^2)\,(n-1) \cos\varphi\,] \Big\}^{*} \qquad \text{(IV-32)}$$

In those parts of the above equations subject to differentiation with respect to φ, the parameters are assumed to be constant. In this form it is an easy matter to pick out the mean values. For n = 2 and including selected non-zero Z-terms, the differential equations for e cos θ and e sin θ are

$$(d/d\varphi)\,(e\cos\theta) = \epsilon_3\,[e\sin\theta + (1+2e^2/3)\sin\varphi] \qquad \text{(IV-33)}$$

$$(d/d\varphi)\,(e\sin\theta) = -\,\epsilon_3\,[e\cos\theta + (1+2e^2/3)\cos\varphi] \qquad \text{(IV-34)}$$

where

$$\epsilon_3 = \epsilon\,[(3/2)\,(u_3^2 + v_3^2) - 1]$$

since

$$u_3^2 + v_3^2 = 1 - (g_3/g)^2$$

and since

$$g = g_0$$

to first order, one may set

$$\epsilon_3 = \epsilon_{30}$$

Also, it is well known that the eccentricity e contains no secular terms to first order. Consequently, we may put

$$e^2 = e_0^2$$

Equations IV-33 and IV-34 may then be integrated as a system by making use of the variation of parameters method. The solution is

$$e\cos\theta = (e\cos\theta)_0\cos(\epsilon_{30}\varphi) + (e\sin\theta)_0\sin(\epsilon_{30}\varphi)$$

$$+\,\epsilon_{30}\,(\cos\epsilon_{30}\varphi - \cos\varphi)\,(1+2e_o^2/3)\,/\,(1+\epsilon_{30}) \qquad \text{(IV-35)}$$

$$e\sin\theta = -\,[(e\cos\theta)_0\sin\epsilon_{30}\varphi - (e\sin\theta)_0\,\epsilon_{30}\varphi$$

$$+\,\epsilon_{30}\,(\sin\epsilon_{30}\varphi - \sin\varphi)\,(1+2e_0^2/3)\,/\,(1+\epsilon_{30})\,] \qquad \text{(IV-36)}$$

Incorporating the short-period terms in these solutions allows for a variation in $(e \cos \theta)^*$ and $(e \sin \theta)^*$ even when the initial conditions are for a circular orbit.

From Equations IV-31 and IV-32, it appears that at least a major part of $(e \cos \theta)_s$ and $(e \sin \theta)_s$ are immediately available. However, since straightforward integration gives rise to constants, and since second-order theory assumes that these terms have zero mean, the constants must be explicitly found and eliminated. The expressions for those terms are

$$(e \cos \theta)_s = (\epsilon_3/3)\left\{[(e \cos f)^2 \cos\varphi + 2 (e \cos f)(e \sin f) \sin\varphi - (3/2)[(e \cos f)(\cos\varphi) - (e \sin f) \sin\varphi]\right\} \quad (\epsilon/3)\left\{(z/r)^2 - [(d/d\varphi)(z/r)]^2\right\}\left\{-3 (e \cos\theta)(e \cos f) - (9/4) \cos\varphi (e \cos f) - (5/2) \cos\varphi + e^2 \cos\varphi - (21/8) e \cos\theta\right\} - (\epsilon/3)(z/r)(d/d\varphi)(z/r)\,[6\, e \sin\theta\;\, e \cos f + (9/2)(\sin\varphi(e \cos f) + (e \sin\theta)/2) + 2(1-e^2) \sin\varphi] \qquad \text{(IV-37)}$$

$$(e \sin \theta)_s = -\,\epsilon_3/3\;[2 \cos\varphi (e \cos f)(e \sin f) - \sin\varphi (e \cos f)^2 + (3/2)(e \sin f) \cos\varphi + (e \cos f) \sin\varphi] + (\epsilon/3)\left\{(z/r)^2 - [(d/d\varphi)(z/r)]^2\right\}\left\{-3\, e \cos f\,(e \sin\theta) - (9/4) \sin\varphi (e \cos f) - (5/2) \sin\varphi + (e^2) \sin\varphi - (21/8)\, e \sin\theta\right\} + \epsilon/3\;(z/r)(d/d\varphi)(z/r)\;[6\, e \cos\theta (e \cos f) + (9/2)(\cos\varphi (e \cos f) + (e \cos\theta)/2) + 2(1-e^2) \cos\varphi] \qquad \text{(IV-38)}$$

The complete first-order solutions for $(e \cos \theta)^*$ and $(e \sin \theta)^*$ are given by

$$(e \cos \theta)^* = e \cos \theta + (e \cos \theta)_s \qquad \text{(IV-39)}$$

and

$$(e \sin \theta)^* = e \sin \theta + (e \sin \theta)_s \qquad \text{(IV-40)}$$

with the appropriate substitutions from Equations IV-35, IV-36, IV-37 and IV-38.

It is useful for two reasons, at this point, to examine the first-order solutions of the auxiliary parameters,

$$p = u_3 \, e \cos\theta + v_3 \, e \sin\theta$$

$$q = - u_3 \, e \sin\theta + v_3 \, e \cos\theta$$

In more familiar terminology, these two quantities are, respectively, the product of the eccentricity and the third component of the unit, perigee vector, and the product of the eccentricity and the third component of the unit vector perpendicular to the perigee vector and lying in the plane of the motion. These quantities appear in the second-order, averaged, differential equations and must be integrated. Integration of these terms yields, in some developments, quantities which, at the critical angle of inclination, are undefined.

Neglecting the short-period terms and retaining only the first-order, long-period terms, the first-order expressions for the terms p and q are

$$p = p_0 \cos \epsilon_4 \varphi + q_0 \sin \epsilon_4 \theta$$

$$q = - (p_0 \sin \epsilon_4 \varphi - q_0 \cos \epsilon_4 \varphi)$$

It may be noted that, if the differential equations for the parameters u_3 and v_3 had not incorporated some short-period terms, namely, those denoted by the vector $\underline{Z}$, the argument of the trigonometric functions in the solutions for those parameters would have contained the factor ϵ_2. In that case, the quantity ϵ_4 would have had the form

$$\epsilon_4 = \epsilon_2 - \epsilon_3$$

Since

$$\epsilon_2 = \epsilon \, (g_3/g)^2$$

$$\epsilon_3 = \epsilon \left[(u_3^2 + v_3^2) \, 3/2 - 1 \right]$$

it follows that

$$\epsilon_2 - \epsilon_3 = \epsilon \left[5/2 \, (g_3/g)^2 - 1/2 \right]$$

The term (g_3/g) is the cosine of the inclination. Thus,

$$\epsilon_2 - \epsilon_3 = 0$$

when the inclination satisfies the equation

$$\cos i = \sqrt{5}/5$$

However, in the present treatment, the quantity ϵ_4 has a slightly different form.

$$\epsilon_4 = \sqrt{1+2\epsilon_2} - 1 - \epsilon_3$$

This quantity assumes the value zero when the inclination satisfies

$$\sin^2 i = (2/9\epsilon)\,(3\epsilon - 5 + 5\sqrt{1+(6\epsilon/25)}\,)$$

This solution gives a slightly higher value for the critical angle of inclination. It is possible that other values can be found for this critical angle. For instance, somewhat more complicated solutions for the parameters involved are given in a previous report,[4] and it appears that the argument of perigee would be constant even for a different angle of inclination. It is evident that, at the critical inclination both quantities, p and q, are constants. There are other similar combinations, such as

$$u_3\, p - v_3\, q \quad \text{and} \quad p\, e \cos\theta + q\, e \sin\theta$$

which embody differences of constants as a factor in the arguments of trigonometric functions, and which vanish for certain other inclinations. A rather large number - about twenty - occurs in the second-order, averaged differential equations and must be integrated. Therefore, it is pointed out that, at the inclination proper to each, the combination is a constant. Integration under such a condition leads to secular terms. While this is not a particularly desirable result, considering the nature of the parameters, at least these differences are not a source of initial singularities.

SECTION V

THE TIME-ANGLE RELATION

In order to complete the first-order solution, a relation must be derived which determines the interdependence of the time, t, and the angle variable, φ. From the zero-order solution, Kepler's equation is available. It is Equation III-8.

$$t = \tau + \left[g^3/\mu^2 (1-e^2)^{3/2}\right] \Big(2 \{\arctan [(1-e^2)^{\frac{1}{2}} \tan (f/2)/(1+e)$$

$$+ \arctan [(1-e^2)^{\frac{1}{2}} \tan (\theta/2)/(1+e)]\} \qquad \text{(V-1)}$$

$$- (1-e^2)^{\frac{1}{2}} \{[e \sin f/(1+e \cos f)] + e \sin \theta/(1+e \cos \theta)\}\Big)^*$$

where τ is the time of U- passage. This equation is valid only for elliptic motion. A similar equation is easily obtained for hyperbolic motion. The parabolic case, on the other hand, is somewhat more troublesome. Since most problems of practical interest regarding the perturbation force produced by the polar oblateness of the earth relate to near-earth-satellite orbits, and since for trajectories of this type all of the parameters are bounded, which guarantees a solution to Equation II-1**, consideration, from this point on, will be restricted to those sets of initial conditions which allow values for the quantities E and g, satisfying

$$E < 0$$

and

$$g > 0$$

---

* All parameters in this section are to be understood to be starred.

** See Section II

As a consequence, the equation for elliptic motion (Equation V-1) will be discussed here.

It is possible to derive a perturbation equation for the parameter τ having the form of Equation II-1; however, because of the very complicated structure of this equation - and especially the second-order perturbation equation - the following approximation is suggested as an alternative.

Let the osculating Kepler equation (Equation V-1) be written in the form

$$t = \tau + F\left[\underline{K}(\varphi), \varphi\right] \qquad \text{(V-2)}$$

where $\underline{K}$ has components, g, e cos θ, and e sin θ , and the variable φ is the differential true anomaly, $f + \theta$. For a given value of the variable φ , namely, φ_1, the parameters represented by the vector $\underline{K}$ and the parameter τ , along with the corresponding vectors $\underline{U}$ and $\underline{V}$, determine a unique osculating ellipse. If the motion were strictly elliptical, the time-history of the satellite would be given by

$$t = \tau_1 + F(\underline{K}_1, \varphi) \qquad \text{(V-3)}$$

In this equation, φ is the angle from the vector $\underline{U}_1$ to the position vector $\underline{R}$. Equation V-3 may now be evaluated at $t = \tau_0$, which is the time at which the satellite's position would have been in the direction $\underline{U}_0'$. The vector $\underline{U}_0'$ is the image of the vector $\underline{U}_0$ produced by the perturbation force, including not only the change of the osculating plane of the motion, but also the rotation about the osculating angular momentum vector $\underline{G}$. In other words, the vector $\underline{U}_0'$ lies in the plane determined by vectors $\underline{U}_1$ and $\underline{V}_1$, and differs from the vector $\underline{U}_1$ by the angle $\theta - \theta_0$. Equation V-3, evaluated at $t = \tau_0$, is then

$$t = \tau_0 + F\left[\underline{K}_1, (\theta - \theta_0)\right] \qquad \text{(V-4)}$$

Subtracting Equation V-4 from V-3 after evaluating at $t = t_1$ yields the desired time-angle relationship. The function F is replaced by its specific form given in Equation V-2. The results are

$$t = \tau_0 + \left[g^3/\mu^2 (1-e^2)^{3/2} \right] \Big(2 \arctan \{ (1-e^2)^{\frac{1}{2}} \sin(f+\theta_0) / \left[[1+\cos(f+\theta_0)] [1+e\cos(\varphi-\theta_0)] \right.$$

$$\left. + \sin(f+\theta_0) e \sin(\varphi-\theta_0) \right] \} - (1-e^2)^{\frac{1}{2}} [e \sin f/(1+e\cos f) + e \sin\theta_0/(1+e\cos\theta_0)] \Big) \qquad \text{(V-5)}$$

In this equation the parameters are replaced by their explicit solutions, first order, second order, etc. If the independent variable is the variable φ, the time t is easily calculated by evaluating the right hand side of Equation V-5 at specific values of the variable φ. On the other hand, if the time t is taken as the independent variable, an iterative procedure must be employed to obtain the corresponding value of the variable φ.

SECTION VI

THE SECOND-ORDER, AVERAGED DIFFERENTIAL EQUATIONS

The procedure to be followed in deriving the second-order, averaged, differential equations is contained in Section II. For purposes of convenience, the requisite equations are reproduced here.

According to Equation II-4, the complete second-order solution should have the form

$$\underline{x}(\underline{x}_0, t, \hat{\epsilon}) = \underline{y}(\underline{y}_0, t, \hat{\epsilon}) + \hat{\epsilon}\,\underline{H}(\underline{y}, t, 0) + \hat{\epsilon}^2\,\underline{J}(\underline{y}, t, 0) \qquad \text{(VI-1)}$$

The vector $\underline{y}$ is the solution of

$$(d/dt)\left[\underline{y}(\underline{y}_0, t, \hat{\epsilon})\right] = \hat{\epsilon}\,\underline{Y}(\underline{y}, t, 0) + \hat{\epsilon}^2\,\underline{Y}'(\underline{y}, t, 0) \qquad \text{(VI-2)}$$

Before Equation VI-2 can be solved, however, the explicit form of the vector $\underline{Y}'$ must be computed. The equation for this vector is given by

$$\begin{aligned}\underline{Y}'(\underline{y}, t, 0) = M_t\Big\{&(\partial/\partial\epsilon)[\underline{X}(\underline{y}, t, 0)] + (\partial/\partial\underline{y})[\underline{X}(\underline{y}, t, 0)]\,\underline{H}(\underline{y}, t, 0) \\ &- (\partial/\partial\underline{y})[\underline{H}(\underline{y}, t, 0)]\,\underline{Y}(\underline{y}, t, 0)\Big\} + \underline{Z}'(\underline{y}, t, 0)\end{aligned} \qquad \text{(VI-3)}$$

In these equations the vectors $\underline{Y}$ and $\underline{H}$ are identical in form with those derived in the first-order procedure. The vector $\underline{Z}'$ consists of short-period terms which are selected in a manner similar for determining the vector $\underline{Z}$, as described in the first-order theory. In this section, however, the components of that vector will, in most cases, be taken to be zero. Finally, the vector $\underline{J}$, which appears in Equation VI-1, is defined by Equation II-9. This vector has as a common factor the second-order small quantity ϵ^2, and its components consist of short-period terms. This vector will not be computed in this report.

In Equation VI-3, the components of the vector

$$M_t\left\{(\partial/\partial\epsilon)[\underline{X}(\underline{y},t,0)]\right\}$$

are obtained by substituting the vector sum $\underline{F}_3 + \underline{F}_4$ from Equations III-3 and III-4 into Equations III-17 through III-22 and expanding the results in a manner similar to that detailed in the derivation of the first-order, differential equations. The mean values are then extracted. The presence of the contributions of the third and fourth harmonics in the subsequent equations is evidenced by the respective factors, B_3 and B_4. The remaining vectors in Equations VI-3 are obtained by carrying out the indicated operations. The procedure is straightforward, requiring only care and patience. The results are

$$(d/d\phi)(u_3) = \left\{\epsilon(g_3/g)^2[\sin\phi\,(z/r) - \cos\phi\,(d/d\phi)(z/r)]\right\}_1 + \left\{\gamma_1 v_3 + \gamma_2 e\sin\theta\right\}_2 + \left\{\gamma_3\rho_3 - \gamma_7\rho_4 + \gamma_8\rho_5\right\}_3 \qquad \text{(VI-4)}$$

$$(d/d\phi)(v_3) = -\left\{\epsilon(g_3/g)^2[(z/r)\cos\phi + \sin\phi\,(d/d\phi)(z/r)]\right\}_1 - \left\{\gamma_1 u_3 + \gamma_2 e\cos\theta\right\}_2 - \left\{\gamma_3\tau_3 + \gamma_7\tau_4 + \gamma_8\tau_5\right\}_3 \qquad \text{(VI-5)}$$

$$(d/d\phi)(e\cos\theta) = \left\{\epsilon[(3/2)(u_3)^2 + (v_3)^2 - 1](1 + 2e^2/3)\sin\phi\right\}_1 + \left\{\delta_1 e\sin\theta + \delta_2 v_3\right\}_2 + \left\{\delta_3\rho_3 - \delta_4\rho_4 - \delta_5\rho_6\right\}_3 \qquad \text{(VI-6)}$$

$$(d/d\phi)(e\sin\theta) = -\left\{\epsilon[(3/2)(u_3)^2 + (v_3)^2 - 1](1 + 2e^2/3)\cos\phi\right\}_1 - \left\{\delta_1 e\cos\theta + \delta_2 u_3\right\}_2 - \left\{\delta_3\tau_3 + \delta_4\tau_4 + \delta_5\tau_6\right\} \qquad \text{(VI-7)}$$

$$(d/d\phi)(g) = \left\{ c_1\,pq + c_2 q \right\}_3 \tag{VI-8}$$

$$(d/d\phi)(u_1) = \left\{ 2[\,(z/r)\,(g_1/g)\,H_1 + (x/r)\,H_2\,]\,\sin\phi \right\}_2 + \left\{ \nu_1 (g_1/g)/(g_3/g) + \gamma_6 v_1 (p^2 - q^2) \right\}_3 \tag{VI-9}$$

$$(d/d\phi)(u_2) = \left\{ 2[\,(z/r)(g_2/g)\,H_1 + (y/r)\,H_2\,]\,\sin\phi \right\}_2 + \left\{ \nu_1 (g^2/g)/(g_3/g) + \gamma_6 v_2 (p^2 - q^2) \right\}_3 \tag{VI-10}$$

$$(d/d\phi)(v_1) = -\left\{ 2[(z/r)(g_1/g)\,H_1 + (x/r)\,H_2\,]\,\cos\phi \right\}_2 - \left\{ \nu_2 (g_1/g)/(g_3/g) + \gamma_6 u_1 (p^2 - q^2) \right\}_3 \tag{VI-11}$$

$$(d/d\phi)\,v_2 = -\left\{ 2[(z/r)(g_2/g)H_1 + (y/r)H_2\,]\,\cos\phi \right\}_2 - \left\{ \nu_2 (g_2/g)/(g_3/g) + \gamma_6 u_2 (p^2 - q^2) \right\}_3 \tag{VI-12}$$

where

$$c_1 = \epsilon^2 g \left\{ (7/6) - (5/4)(u_3{}^2 + v_3{}^2) - (45 B_4/8)\,[1 - (7/6)(u_3{}^2 + v_3{}^2)\,] \right\} \tag{VI-13}$$

$$C_2 = \epsilon^2 g^3 B_3 (3/2\mu)\,[\,(5/4)(u_3{}^2 + v_3{}^2) - 1\,] \tag{IV-14}$$

$$\gamma_1 = \epsilon (g_3/g)^2 \Big\{ 1 + \epsilon \Big[1 + (e^2/6) + (u_3{}^2 + v_3{}^2) [13e^2/12 - (11/12) + (15B_4/8)(2 + 3e^2) \{ 1 - (7/4)(u_3{}^2 + v_3{}^2) \}] \Big] \Big\} \quad \text{(VI-15)}$$

$$\gamma_2 = \epsilon^2 (g_3/g)^2 B_3 (g^2/\mu)(3/4) \Big[5(u_3{}^2 + v_3{}^2) - 2 \Big] \quad \text{(VI-16)}$$

$$\gamma_3 = \epsilon^2 (g_3/g)^2 B_3 \, 15g^2/8\mu \quad \text{(VI-17)}$$

$$\gamma_4 = \epsilon^2 (g_3/g)^2 \Big\{ (7/8)(u_3{}^2 + v_3{}^2) - (7/12) + (45 B_4/16) \Big[1 - (7/4)(u_3{}^2 + v_3{}^2) \Big] \Big\} \quad \text{(VI-18)}$$

$$\gamma_5 = \epsilon^2 (g_3/g)^2 \Big[(1/4) - 105 B_4/64 \Big] \quad \text{(VI-19)}$$

$$\gamma_6 = \epsilon^2 (g_3/g)^2/8 \quad \text{(VI-20)}$$

$$\gamma_7 = \epsilon^2 (g_3/g)^2 \Big\{ (u_3{}^2 + v_3{}^2) - (7/12) + (45B_4/16) \Big[1 - (7/4)(u_3{}^2 + v_3{}^2 \Big] \Big\} \quad \text{(VI-21)}$$

$$\gamma_8 = \epsilon^2 (g_3/g)^2 \Big[(3/8) - 105 B_4/64) \Big] \quad \text{(VI-22)}$$

$$\delta_1 = \epsilon \Big\{ (3/2)(u_3{}^2 + v_3{}^2) - 1 + \epsilon \Big[(2/3)(u_3{}^2 + v_3{}^2) - (5/6) - (55/48)(u_3{}^2 + v_3{}^2)^2 + (e^2/72) \{ 51(u_3{}^2 + v_3{}^2)^2 + 33(u_3{}^2 + v_3{}^2) - 37 \} + (15 B_4/16)(4 + 3e^2) \{ 1 - 5(u_3{}^2 + v_3{}^2) + (35/8)(u_3{}^2 + v_3{}^2)^2 \} \Big] \Big\} \quad \text{(VI-23)}$$

$$\delta_2 = \epsilon^2 B_3 (3g^2/4\mu)(2 + 3e^2)\left[(5/4)(u_3{}^2 + v_3{}^2) - 1\right] \quad \text{(VI-24)}$$

$$\delta_3 = \epsilon^2 \left\{2(u_3{}^2 + v_3{}^2) - (7/6) + (e^2/8)\left[5(u_3{}^2 + v_3{}^2) - (13/3)\right]\right\} \quad \text{(VI-25)}$$

$$\delta_4 = \epsilon^2 B_3 (15g^2/4\mu)\left[(5/4)(u_3{}^2 + v_3{}^2) - 1\right] \quad \text{(VI-26)}$$

$$\delta_5 = \epsilon^2 \left\{(7/16)(u_3{}^2 + v_3{}^2) - (23/24) + (315 B_4/64)\left[(7/6)(u_3^2 + v_3{}^2) - 1\right]\right\} \quad \text{(VI-27)}$$

$$H_1 = [\, 1/(g_3/g)]\,(\gamma_1 - H_2) \quad \text{(VI-28)}$$

$$H_2 = \epsilon^2 \,(g_3/g)^2\, (e^2/6)\, (u_3{}^2 + v_3{}^2) \quad \text{(VI-29)}$$

$$\rho_2 = e \sin \theta \quad \text{(VI-30)}$$

$$\rho_3 = u_3 q + v_3 p \quad \text{(IV-31)}$$

$$\rho_4 = - p e \sin \theta + q e \cos \theta \quad \text{(VI-32)}$$

$$\rho_5 = v_3(p^2 - q^2) + 2u_3 pq \quad \text{(VI-33)}$$

$$\rho_6 = - (p^2 - q^2) e \sin \theta + 2 pq e \cos \theta \quad \text{(VI-34)}$$

$$\tau_2 = e \cos \theta \qquad \text{(VI-35)}$$

$$\tau_3 = u_3 p - v_3 q \qquad \text{(VI-36)}$$

$$\tau_4 = pe \cos \theta + q e \sin \theta \qquad \text{(VI-37)}$$

$$\tau_5 = u_3(p^2 - q^2) - 2 v_3 pq \qquad \text{(VI-38)}$$

$$\tau_6 = (p^2 - q^2)e \cos \theta + 2pqe \sin \theta \qquad \text{(VI-39)}$$

$$\nu_1 = \gamma_2\rho_2 + \gamma_3\rho_3 - \gamma_4\rho_4 + \gamma_5\rho_5 \qquad \text{(VI-40)}$$

$$\nu_2 = \gamma_2\tau_2 + \gamma_3\tau_3 + \gamma_4\tau_4 + \gamma_5\tau_5 \qquad \text{(VI-41)}$$

SECTION VII

SOLUTION OF THE SECOND-ORDER, DIFFERENTIAL EQUATIONS

The group of equations, Equations VI-4 through VI-12, constitutes the basis for the second-order solution. Once a solution has been derived, it is substituted into Equation VI-1, yielding the complete, second-order solution. The next step, therefore, is to solve the set (Equations VI-4 through VI-12).

A cursory examination of these equations makes it evident that the best one can expect are approximate solutions. In general, the method of approximation adopted for deriving the solutions may be outlined as follows.

1) Those quantities which, in the first-order solution, were shown to be constants -- except for short-period terms -- are taken as constants on the right hand sides of Equations VI-4 through VI-12. They are, essentially, the quantities g, $u_3^2 + v_3^2$, and e^2. Thus, the terms defined by Equations VI-13 through VI-29 are constants.

2) The right hand sides of Equations VI-4 through VI-12 are divided into three parts: short-period, linear, and non-linear. The three types of terms are enclosed in subscripted brackets. The short-period terms are present in these equations through the vector $\underline{Y}$, which, it may be recalled, contains the vector $\underline{Z}$. The non-linear terms must be carefully examined to ensure that no linear quantities - such as $e^2 u_3$, or $(u_3^2 + v_3^2) e \cos \theta$ - are implicitly contained. The set of Equations VI-4 through VI-12 incorporates the first two steps.

3) The linearized equations are solved exactly, subject to the restrictions of the first step and are denoted by primes.

4) The non-linear parts of the equations are taken into account by the method of variation of parameters, substitution of the solutions of the linearized equations, followed by direct integration, retaining only terms of the second order.

A. SOLUTIONS FOR THE PARAMETERS u_3, v_3, e cos θ, AND e sin θ

Consideration of Equations VI-4 through VI-7 shows the interdependence of the parameters (u_3, v_3, e cos θ, e sin θ). The terms in the first set of brackets $\left[\ldots\ldots\ldots\right]_1$ are the short-period, first-order terms which were included in the first-order, averaged differential equations. However, because of the added complications in the second-order equations, they will be shifted, after integration, to the vector $\underline{H}$. The terms in the second brackets $\left[\ldots\ldots\ldots\right]_2$ are linear in the parameters u_3, v_3, e cos θ, and, e sin θ and constitute the linearized, differential equations which will be solved exactly. The equations are:

$$d/d\phi(u_3) = \gamma_1 v_3 + \gamma_2 e \sin\theta \quad \text{(VII-1)}$$

$$d/d\phi(v_3) = -\gamma_1 u_3 - \gamma_2 e \cos\theta \quad \text{(VII-2)}$$

$$d/d\phi(e \cos\theta) = \delta_2 v_3 + \delta_1 e \sin\theta \quad \text{(VII-3)}$$

$$d/d\phi(e \sin\theta) = -\delta_2 u_3 - \delta_1 e \cos\theta \quad \text{(VII-4)}$$

The solutions for the auxiliary parameters p and q are obtained relatively quickly and, since they are of some interest, are given here.

$$p' = p_o \cos(c_3\phi) + \left[q_o(\gamma_1 - \delta_1)/c_3\right] \sin(c_3\phi) + (1 - \cos c_3\phi) \left[4\delta_2\gamma_2 p_o - (\gamma_1 - \delta_1)\left\{\gamma_2 e_o^2 - \delta_2(u_{30}^2 + v_{30}^2)\right\}\right]/c_3^2 \quad \text{(VII-5)}$$

$$q' = -(\sin(c_3\phi)/c_3^2)\left[(\gamma_1 - \delta_1)p_o + \gamma_2 e_o^2 - \delta_2(u_{30}^2 + v_{30}^2)\right] + q_o \cos(c_3\phi)$$

where: (VII-6)

$$c_3 = \sqrt{(\gamma_1 - \delta_1)^2 + 4\gamma_2\delta_2}$$

The solutions for Equations VII-1 through VII-4 may be written as:

$$
\begin{bmatrix} u_3' \\ v_3' \\ e\cos\theta' \\ e\sin\theta' \end{bmatrix} = 1/c_3 \begin{bmatrix} c_3\cos(c_3\phi) & (\gamma_1-\delta_1)\sin(c_3\phi) & 0 & 2\gamma_2\sin(c_3\phi) \\ -(\gamma_1-\delta_1)\sin c_3\phi & c_3\cos(c_3\phi) & -2\gamma_2\sin(c_3\phi) & 0 \\ 0 & 2\delta_2\sin(c_3\phi) & c_3\cos(c_3\phi) & -(\gamma_1-\delta_1)\sin(c_3\phi) \\ -2\delta_2\sin(c_3\phi) & 0 & (\gamma_1-\delta_1)\sin(c_3\phi) & c_3\cos(c_3\phi) \end{bmatrix}
$$

$$
\cdot \begin{bmatrix} u_{30}\cos(\gamma_1-\delta_1)\phi + v_{30}\sin(\gamma_1-\delta_1)\phi \\ -u_{30}\sin(\gamma_1-\delta_1)\phi + v_{30}\cos(\gamma_1-\delta_1)\phi \\ (e\cos\theta)_o\cos(\gamma_1-\delta_1)\phi + (e\sin\theta)_o\sin(\gamma_1-\delta_1)\phi \\ -(e\cos\theta)_o\sin(\gamma_1-\delta_1)\phi + (e\sin\theta)_o\cos(\gamma_1-\delta_1)\phi \end{bmatrix} \qquad \text{(VII-7)}
$$

To implement the fourth step, let Equations VII-7 be represented by

$$\underline{W} = [M]\,\underline{W}_o$$

Then, by the method of variation of parameters, the vector $\underline{W}_o$ must satisfy

$$[M](d/d\phi\,\underline{W}_o) = \underline{T}$$

where the components of the vector $\underline{T}$ are the non-linear terms of Equations VI-4 through VI-7 or the third set of brackets $\left[\,\ldots\ldots\ldots\,\right]_3$. Since only terms of the second order are being retained, the matrix M is reduced to the identity matrix by setting the small quantity ϵ equal to zero. The vector $\underline{T}$ is then integrated. For this purpose, the first-order derivatives of the involved parameters are required. They are:

$$d/d\phi\,(u_3) = \gamma_1 v_1$$

$$d/d\phi\,(v_3) = -\gamma_1 u_3$$

$$d/d\phi\,(e\cos\theta) = \delta_1\, e\sin\theta$$

$$d/d\phi\,(e\sin\theta) = -\delta_1\, e\cos\theta$$

$$d/d\phi\,(p) = (\gamma_1-\delta_1)q$$

$$d/d\phi\,(q) = -(\gamma_1-\delta_1)p$$

With these approximations, the required integrals are easily obtained. They are listed below.

$$\int \rho_3 d\phi' = \tau_3/(2\gamma_1 - \delta_1) \qquad \text{(VII-8)}$$

$$\int \tau_3 d\phi' = \rho_3/(2\gamma_1 - \delta_1) \qquad \text{(VII-9)}$$

$$\int \rho_4 d\phi' = \tau_4/(\gamma_1 - 2\delta_1) \qquad \text{(VII-10)}$$

$$\int \tau_4 d\phi' = \rho_4/(\gamma_1 - 2\delta_1) \qquad \text{(VII-11)}$$

$$\int \rho_5 d\phi' = \tau_5/(3\gamma_1 - 2\delta_1) \qquad \text{(VII-12)}$$

$$\int \tau_5 d\phi' = -\rho_5/(3\gamma_1 - 2\delta_1) \qquad \text{(VII-13)}$$

$$\int \rho_6 d\phi' = \tau_6/(2\gamma_1 - 3\delta_1) \qquad \text{(VII-14)}$$

$$\int \tau_6 d\phi' = -\rho_6/(2\gamma_1 - 3\delta_1) \qquad \text{(VII-15)}$$

The solutions to Equations VI-4 through VI-7 may now be written as:

$$u_3 = (7.7)_1' + \gamma_3(7.8)' - \gamma_7(7.10)' + \gamma_8(7.12)' \qquad \text{(VII-16)}$$

$$v_3 = (7.7)_2' - \gamma_3(7.9)' - \gamma_7(7.11)' - \gamma_8(7.13)' \qquad \text{(VII-17)}$$

$$e \cos \theta = (7.7)_3' + \delta_3(7.8)' - \delta_4(7.10)' + \delta_5(7.14)' \qquad \text{(VII-18)}$$

$$e \sin \theta = (7.7)_4' - \delta_3(7.9)' - \delta_4(7.11)' - \delta_5(7.15)' \qquad \text{(VII-19)}$$

where subscripts refer to components of the vector equation (Equation VII-7), and equation numbers are to be replaced by the right hand sides of the equations to which they refer.

B. SOLUTION FOR THE PARAMETER g

Now that the solutions for linearized equations for the auxiliary parameters p and q and their first-order derivatives are available, Equation VI-8 is easily solved. The solution is:

$$g = g_o + \left[c_1 p'^2/2(\gamma_1 - \delta_1)\right] + c_2 p'/(\gamma_1 - \delta_1) \qquad \text{(VII-20)}$$

C. SOLUTIONS FOR THE PARAMETERS u_1, u_2, v_1, AND v_2

To obtain solutions for the parameters u_1, u_2, v_1, v_2, the terms in brackets subscripted 2 in Equations VI-9 through VI-12, i.e., the linearized equations, must first be rewritten in a manner similar to the procedure followed in Section IV for Equations IV-22 through IV-25. The equations then take the form

$$d/d\phi(u_1) = \left\{H_1(u_2 \sin\phi - v_2 \cos\phi) + \left[H_1(g_3/g) + H_2\right](u_1 \cos\phi + v_1 \sin\phi)\right\} 2 \sin\phi$$

$$d/d\phi(u_2) = \left\{H_1(-u_1 \sin\phi + v_1 \cos\phi) + \left[H_1(g_3/g) + H_2\right](u_2 \cos\phi + v_2 \sin\phi)\right\} 2 \sin\phi$$

$$d/d\phi(v_1) = -\left\{H_1(u_2 \sin\phi - v_2 \cos\phi) + \left[H_1(g_3/g) + H_2\right](u_1 \cos\phi + v_1 \sin\phi)\right\} 2 \cos\phi$$

$$d/d\phi(v_2) = -\left\{H_1(-u_1 \sin\phi + v_1 \cos\phi) + \left[H_1(g_3/g) + H_2\right](u_2 \cos\phi + v_2 \sin\phi)\right\} 2 \cos\phi$$

The solution to this set of equations is obviously identical in form with that given in Section IV (Equation IV-26) with the appropriate replacements for the constants. The substitutions are

$$\alpha = H_1$$
$$\beta = \gamma_1$$

To carry out the fourth step, the following first-order derivatives are required:

$$d/d\phi(u_1) = H_1 u_2 + \gamma_1 v_1$$

$$d/d\phi(u_2) = -H_1 u_1 + \gamma_1 v_2$$

$$d/d\phi(v_1) = H_1 v_2 - \gamma_1 u_1$$

$$d/d\phi(v_2) = -H_1 v_1 - \gamma_1 u_2$$

Pursuing the same reasoning as explained in subsection A, it is found that the appropriate integrals are:

$$\int (g_1/g)(\nu_1)d\phi' = -K_1(g_2/g)\Big\{\big[\gamma_2\rho_2/(K_1^2-K_2^2)\big] + \big[\gamma_3\rho_3/(K_1^2-K_3^2)\big] - \big[\gamma_4\rho_4/(K_1^2-K_4^2)\big] + \big[\gamma_5\rho_5/(K_1^2-K_5^2)\big]\Big\} - (g_1/g)\Big\{\big[K_2\gamma_2\tau_2/(K_1^2-K_2^2)\big] + \big[K_3\gamma_3\tau_3/(K_1^2-K_3^2)\big] + \big[K_4\gamma_4\tau_4/(K_1^2-K_4^2)\big] + \big[K_5\gamma_5\tau_5/(K_1^2-K_5^2)\big]\Big\} \tag{VII-21}$$

$$\int (g_2/g)\nu_1\, d\phi' = K_1(g_1/g)\Big\{\big[\gamma_2\rho_2/(K_1^2-K_2^2)\big] + \big[\gamma_3\rho_3/(K_1^2-K_3^2)\big] - \big[\gamma_4\rho_4/(K_1^2-K_4^2)\big] + \big[\gamma_5\rho_5/(K_1^2-K_5^2)\big]\Big\} - (g_2/g)\Big\{\big[K_2\gamma_2\tau_2/(K_1^2-K_2^2)\big] + \big[K_3\gamma_3\tau_3/(K_1^2-K_3^2)\big] + \big[K_4\gamma_4\tau_4/(K_1^2-K_4^2)\big] + \big[K_5\gamma_5\rho_5/(K_1^2-K_5^2)\big]\Big\} \tag{VII-22}$$

$$\int (g_1/g)\nu_2\, d\phi' = -K_1(g_2/g)\Big\{\big[\gamma_2\tau_2/(K_1^2-K_2^2)\big] + \big[\gamma_3\tau_3/(K_1^2-K_3^2)\big] + \big[\gamma_4\tau_4/(K_1^2-K_4^2)\big] + \big[\gamma_5\tau_5/(K_1^2-K_5^2)\big]\Big\} + (g_1/g)\Big\{\big[K_2\gamma_2\rho_2/(K_1^2-K_2^2)\big] + \big[K_3\gamma_3\rho_3/(K_1^2-K_3^2)\big] - \big[K_4\gamma_4\rho_4/(K_1^2-K_4^2)\big] + \big[K_5\gamma_5\rho_5/(K_1^2-K_5^2)\big]\Big\} \tag{VII-23}$$

$$\int (g_2/g)\nu_2\, d\phi' = K_1(g_1/g)\Big\{\big[\gamma_2\tau_2/(K_1^2-K_4^2)\big] + \big[\gamma_3\tau_3/(K_1^2-K_3^2)\big] + \big[\gamma_4\tau_4/(K_1^2-K_4^2)\big] + \big[\gamma_5\tau_5/(K_1^2-K_5^2)\big]\Big\} + (g_2/g)\Big\{\big[K_2\gamma_2\rho_2/(K_1^2-K_2^2)\big] + \big[K_3\gamma_3\rho_3/(K_1^2-K_3^2)\big] - \big[K_4\gamma_4\rho_4/(K_1^2-K_4^2)\big] + \big[K_5\gamma_5\rho_5/(K_1^2-K_5^2)\big]\Big\} \tag{VII-24}$$

where

$$K_1 = H_1$$

$$K_2 = \delta_2$$

$$K_3 = 2\gamma_1 - \delta_1$$

$$K_4 = \gamma_1 - 2\delta_1$$

$$K_5 = 3\gamma_1 - 2\delta_1$$

$$\int v_1(p^2-q^2)d\phi' = K_6\Big\{8(\gamma_1-\delta_1)H_1\Big[(\gamma_1-\delta_1)v_2(p^2-q^2) + pq(H_1v_1 - \gamma_1u_2)\Big] - \Big[4(\gamma_1-\delta_1)^2 - \gamma_1^2 + H_1^2\Big]\Big[(H_1v_2 + \gamma_1u_1)(p^2-q^2) + 4pq(v_1)(\gamma_1-\delta_1)\Big]\Big\} \quad \text{(VII-25)}$$

$$\int v_2(p^2-q^2)d\phi' = -K_6\Big\{8(\gamma_1-\delta_1)H_1\Big[(\gamma_1-\delta_1)v_1(p^2-q^2) - pq(H_1v_2 + \gamma_1u_1)\Big] - \Big[4(\gamma_1-\delta_1)^2 - \gamma_1^2 + H_1^2\Big]\Big[(H_1v_1 - \gamma_1u_2)(p^2-q^2) - 4pq(\gamma_1-\delta_1)v_2\Big]\Big\} \quad \text{(VII-26)}$$

$$\int u_1(p^2-q^2)d\phi' = K_6\Big\{8(\gamma_1-\delta_1)H_1\Big[(\gamma_1-\delta_1)u_2(p^2-q^2) + pq(H_1u_1 + \gamma_1v_2)\Big] - \Big[4(\gamma_1-\delta_1)^2 - \gamma_1^2 + H_1^2\Big]\Big[(H_1u_2 - \gamma_1v_1)(p^2-q^2) + pq(u_1)4(\gamma_1-\delta_1)\Big]\Big\} \quad \text{(VII-27)}$$

$$\int u_2(p^2-q^2)d\phi = -K_6\Big\{8(\gamma_1-\delta_1)H_1\Big[(\gamma_1-\delta_1)u_1(p^2-q^2) - pq(H_1u_2 - \gamma_1v_1)\Big] - \Big[4(\gamma_1-\delta_1)^2 - \gamma_1^2 - H_1^2\Big]\Big[(H_1u_1 + \gamma_1v_2)(p^2-q^2) - pq(u_2)4(\gamma_1-\delta_1)\Big]\Big\} \quad \text{(VII-28)}$$

where

$$K_6 = -1/16\, H_1^2(\gamma_1-\delta_1)^2 - \Big[4(\gamma_1-\delta_1)^2 - \gamma_1^2 + H_1^2\Big]^2$$

The solutions for the four parameters u_1, u_2, v_1, and v_2 are:

$$u_1 = (4.26)_1 + \left[1/(g_3/g)\right](7.21)' + \gamma_6(7.25)' \qquad \text{(VII-29)}$$

$$u_2 = (4.26)_2 + \left[1/(g_3/g)\right](7.22)' + \gamma_6(7.26)' \qquad \text{(VII-30)}$$

$$v_1 = (4.26)_3 - \left[1/(g_3/g)\right](7.23)' - \gamma_6(7.27)' \qquad \text{(VII-31)}$$

$$v_2 = (4.26)_4 - \left[1/(g_3/g)\right](7.24)' - \gamma_6(7.28)' \qquad \text{(VII-32)}$$

It is clear from the foregoing that the general structure of the solutions of Equations VI-4 through VI-12 is of the form

$$\underline{y} = \underline{y}' + \epsilon^2 \underline{L}(\underline{y}')$$

where the vector $\underline{y}'$ is the vector of solutions of the linearized equations, and vector $\underline{L}(\underline{y}')$ is the contribution to the solutions by the non-linear terms. The complete second-order solution has the form

$$\underline{x} = \underline{y} + \epsilon \underline{H}(\underline{y})$$

To determine the constants in these solutions, we make use of the initial conditions and the first-order approximations to the vector $\underline{y}_o$. The required constants are:

$$\underline{y}_o' = \underline{x}_o - \epsilon\, \underline{H}\left[\underline{x}_o - e\underline{H}(\underline{x}_o)\right] - \epsilon^2 \underline{L}(\underline{x}_o)$$

Once the constants have been obtained, the solutions for the parameters for a given value of the independent variable ϕ are computed by the following procedure.

1) The vector y' is calculated from Equation VII-7, the equation $g = g_o$, and Equation IV-26.

2) The vector $\underline{y}$ is derived by substituting vector $\underline{y}'$ in Equations VII-5, -6, -16, -17, -18, -19, -20, -29, -30, -31, and -32.

3) The final (starred) values for the parameters are obtained by substituting vector y into the general equation

$$\underline{x} = \underline{y} + \epsilon \, \underline{H} \, \underline{y}$$

where the components of vector $\underline{H}$ are the short-period terms derived in Section IV. As was noted in subsection A of Section VII, there are certain additions to be made to the components of vector $\underline{H}$ belonging to the parameters

$$u_3, \; v_3 \; e \cos \theta, \; e \sin \theta$$

These additions are, respectively,

$$- \epsilon (g_3/g)^2 \, (z/r) \cos \phi \, , \qquad - \epsilon \, (g_3/g)^2 \, (z/r) \sin \phi$$

$$- \epsilon \left[(u_3^2 + v_3^2)(3/2) - 1 \right] (1 + 2\epsilon^2/3) \cos \phi, \qquad - \epsilon \left[(u_3^2 + v_3^2)(3/2) - 1 \right] (1 + 2\epsilon^2/3) \sin \phi$$

4) The position and velocity vectors, $\underline{R}$ and $d/dt\,(\underline{R})$, are given by Equations III-6 and III-7. The equation for the time-angle relation is contained in Section V (Equation V-5).

SECTION VIII

CONCLUSION

The original purpose of the investigation which has culminated in this report was to derive a second-order solution to the polar oblateness problem that would be free of indeterminacies occasioned by particular initial angles of inclination and that would not introduce singularities of an equally constricting nature. This study has indicated that parameters too closely connected with the conventional elements should be avoided because of the difficulties associated with circular initial conditions. As a consequence, the initial vectors R_O and (d/dt) (R_O) were considered as candidates for parameters. A decision then had to be made whether to use differential, eccentric, or true anomaly as the independent variable. Unfortunately, the time-derivatives of both variables for the perturbed problem are quite complicated. This undesirable characteristic adds significantly to the calculations involved in solving the second-order differential equations. It was observed, however, that, if the vectors R_O and (d/dt) (R_O) are replaced by the equivalent set used in this report, the time-derivative of the differential true anomaly has the same mathematical structure as in the unperturbed case. The time-derivative of the differential eccentric anomaly, on the other hand, remains complex. The dissimilarity arises from the difference in the origin about which the two anomalies are measured. Finally, in order to avoid singularities in the perturbation equations themselves, the elements e cos θ and e sin θ were selected in preference to e and θ because the time-derivative of θ contains the eccentricity in its denominator.

A source of much concern has been the time-angle relationship, because not only is the equation (Equation V-1) relating the two complicated, but it also involves the parameter τ. Besides being complex, the perturbation equation for this parameter--and it does have one, because the osculating vectors $\underline{U}$ and $\underline{R}_O/r_D$, are not identical--has its own distinctive difficulties. An alternative approximation for deriving the relation between the time and the differential true anomaly has been suggested.

Regarding the method of solution of the perturbation equations, it is evident that the Von Zeipel technique was not applicable, since the parameters used in this study do not form a canonical set. The method of averaging was therefore adopted. It is interesting in this respect to note that recent studies[5] indicate the equivalence of the two methods, provided the constants of integration are properly chosen.

In solving the averaged, differential equations, two principal criteria dictated, insofar as possible, the nature of acceptable solutions. These guidelines were that no secular terms should occur in the solutions and that no terms appearing in the denominators should vanish for any inclinations. The reason for the first condition is evident: all the parameters except g are bounded between plus and minus one. This condition, at least in structure, has been met. The second condition is, apparently, far from being satisfied. The solutions contain numerous factors which, for an inclination proper to each, are undefined. Paradoxically, however, it is the first condition which is not fulfilled, while the second is.

As was mentioned at the end of Section IV, certain combinations of parameters occur in the second-order, averaged differential equations. These terms are trigonometric functions whose arguments involve factors which are differences of constants. When these differences vanish, the particular combination is a constant. Integration leads to a quantity which, instead of being undefined, is secular. Under such circumstances, it follows that the solution tends to deteriorate in time.

SECTION IX

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