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SUMMARY

The problem of maximizing the lift-to-drag ratio of a slender, three-dimensional, flat-top wing of given planform in hypersonic flow is considered under the assumptions that the pressure coefficient is modified Newtonian and the skin-friction coefficient is constant. The indirect methods of the calculus of variations in two independent variables are employed, and the maximum lift-to-drag ratio problem is solved for (a) unconstrained volume and (b) given volume.

If the volume is unconstrained, the optimum wing surface is unique; it has a constant chordwise slope and a trailing edge thickness distribution similar to the chord distribution.

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If the volume is given, the chordwise slope of the optimum wing is constant in the spanwise sense but not in the chordwise sense. A one-parameter family of extremal solutions exists, depending on the value of the volume parameter: this parameter is directly proportional to the volume and inversely proportional to the span and the root chord squared. If the volume parameter exceeds a certain critical value, the optimum wing is convex. If the volume parameter is equal to the critical value, the optimum wing is identical with that of case (a). Finally, if the volume parameter is smaller than the critical value, the optimum wing is slightly concave.

1. INTRODUCTION

For three-dimensional, hypersonic wings, two extremal problems are of interest, that is, to minimize the drag for a given lift and to maximize the lift-to-drag ratio for unconstrained lift. Since problems of the former type were considered in Ref. 1, the object of this paper is to consider problems of the latter type in connection with wings whose planform shape is given, whose thickness distribution on the trailing edge is free, and whose volume is either free or arbitrarily given. The following hypotheses are employed: (a) a plane of symmetry exists between the left-hand and right-hand parts of the wing; (b) the upper surface is flat; (c) the free-stream velocity is parallel to the line of intersection of the plane of symmetry and the flat top; (d) the wing is slender in both the chordwise and spanwise senses, that is, the squares of both the chordwise and spanwise slopes are small with respect to one; (e) the pressure coefficient is proportional to the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element;(f) the skin-friction coefficient is constant and equal to some suitably chosen average value; (g) the base drag is neglected; and (h) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces.

2. FORMULATION OF THE VARIATIONAL PROBLEM

In order to relate the lift-to-drag ratio and the volume of a three-dimensional, flat-top wing to its geometry, the following reference system xyz is used: x is a chordwise coordinate measured from the leading edge and positive toward the base; y is a spanwise coordinate measured from the plane of symmetry and positive toward the wing tip; and z is a coordinate perpendicular to the flat-top plane, measured from the flat top, and positive downward. In this coordinate system, the boundary of the wing is described by the equations (Ref. 1)

$$\begin{aligned} &\underline{\text{Leading edge}} \qquad x = 0 \ , \quad z = 0 \\ &\underline{\text{Trailing edge}} \qquad x = c(y) \ , \quad z = t(y) \end{aligned} \tag{1}$$

where the chord distribution c(y) is given and the trailing edge thickness distribution t(y) is free. Furthermore, if z(x, y) denotes the lower surface, $p \equiv \partial z/\partial x$ the chordwise slope, and $q \equiv \partial z/\partial y$ the spanwise slope, the lift-to-drag ratio E = L/D is given by (Ref. 1)

$$E = \frac{\int_0^{b/2} \int_0^{c(y)} p^2 dxdy}{\int_0^{b/2} \int_0^{c(y)} (p^3 + K) dxdy} \tag{2}$$

where

$$K = C_f/n \tag{3}$$

and where b is the prescribed span, C_f the skin-friction coefficient, and n a factor modifying the Newtonian pressure distribution(*). This lift-to-drag ratio is to be

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(*) The pressure coefficient employed in Eq. (2) is $C_p = 2np^2$.

maximized subjected to the isoperimetric constraint of given volume

$$V/2 = \int_0^{b/2} \int_0^{c(y)} z\,dxdy \tag{4}$$

and the inequality constraint

$$p \geq 0 \tag{5}$$

which expresses the limit of validity of the Newtonian pressure law.

The previous problem consists of extremizing a ratio of integrals and is governed by the theory of Ref. 2. Therefore, we study the maximization of the functional (see Chapter 3 of Ref. 3)

$$I = \int_0^{b/2} \int_0^{c(y)} F(z, p, E, \lambda)\,dxdy \tag{6}$$

subject to the boundary conditions (1) and the isoperimetric constraint (4) with the understanding that the fundamental function is defined as(*)

$$F = p^2 - E(p^3 + K) + \lambda z \tag{7}$$

The unknown maximum lift-to-drag ratio E and the undetermined Lagrange multiplier λ are held constant during the extremization process. The fundamental function (7) is characterized by the first partial derivatives

$$F_p = 2p - 3Ep^2 \quad , \quad F_q = 0 \quad , \quad F_z = \lambda \tag{8}$$

(*) For simplicity, Ineq. (5) has not been accounted for in the definition (7). However, it can be verified a posteriori that it is satisfied by all of the extremal surfaces calculated here.

and the second partial derivatives

$$F_{pp} = 2(1 - 3Ep) \ , \quad F_{qq} = 0 \ , \quad F_{pq} = 0$$

$$F_{zz} = 0 \ , \quad F_{zp} = 0 \ , \quad F_{zq} = 0 \qquad (9)$$

3. NECESSARY CONDITIONS

The function z(x, y) extremizing the functional (6) must be a solution of the Euler equation

$$\partial F_p/\partial x - F_z = 0 \tag{10}$$

which, in the light of Eqs. (8), can be rewritten as

$$\partial(2p - 3Ep^2)/\partial x - \lambda = 0 \tag{11}$$

Therefore, upon integrating Eq. (11) in the x-direction, we see that the following first integral is valid:

$$3Ep^2 - 2p + \lambda x = f(y) \tag{12}$$

where f(y) is an arbitrary function of the spanwise coordinate.

The boundary conditions for the Euler equation are partly of the prescribed type and partly of the natural type. The latter are to be determined from the transversality condition

$$(F - pF_p)(\delta x - \dot{x}\delta y) + F_p(\delta z - \dot{z}\delta y) = 0 \tag{13}$$

($\dot{x} = dx/dy$ and $\dot{z} = dz/dy$ denote total derivatives evaluated on the boundary) which must be satisfied for every set of variations consistent with the conditions imposed on the planform shape and the thickness distribution on the periphery of the planform. For the leading edge, the following relations hold:

$$\delta x - \dot{x}\delta y = 0 \quad , \quad \delta z - \dot{z}\delta y = 0 \tag{14}$$

so that Eq. (13) is satisfied. For the trailing edge, the variations δx and δy satisfy the relationship

$$\delta x - \dot{x}\delta y = 0 \tag{15}$$

and Eq. (13) is satisfied providing

$$F_p = 0 \tag{16}$$

that is, because of Eq. (8-1), providing

<u>Trailing edge</u>

$$3Ep^2 - 2p = 0 \tag{17}$$

Once the solution of the Euler equation is obtained, it is necessary to verify that it maximizes the functional (6). In this connection, the Legendre condition

$$F_{pp} \leq 0 \tag{18}$$

must be satisfied and ensures a relative maximum with respect to weak variations. Because of Eq. (9-1), its explicit form is

$$p \geq 1/3E \tag{19}$$

If strong variations of the slope are considered, the Legendre condition is to be replaced by the Weierstrass condition

$$F(z, p_c, E, \lambda) - F(z, p, E, \lambda) - F_p(z, p, E, \lambda)(p_c - p) \leq 0 \tag{20}$$

where z and p are the ordinate and the slope of the extremal surface and p_c is the slope of the comparison surface. The explicit form of this inequality

$$(p_c - p)^2(1 - 2Ep - Ep_c) \leq 0 \tag{21}$$

holds for every comparison slope consistent with the constraint (5) providing

$$p \geq 1/2E \tag{22}$$

Clearly, this inequality is more restrictive than Ineq. (19).

4. NONDIMENSIONAL QUANTITIES

In order to simplify the representation of the results for the particular cases, it is convenient to introduce the nondimensional coordinates

$$\xi = x/c(y) \quad , \quad \eta = 2y/b \quad , \quad \zeta = z/t(y) \tag{23}$$

the nondimensional chord distribution

$$\gamma = c(y)/c(0) \tag{24}$$

and the thickness ratio of the generic airfoil

$$\tau = t(y)/c(y) \tag{25}$$

Also, we define the thickness ratio, lift-to-drag ratio, and volume parameters

$$\begin{aligned} \tau_* &= \tau/\sqrt[3]{K} \\ E_* &= E\,\sqrt[3]{K} \\ V_* &= V/bc^2(0)\,\sqrt[3]{K} \end{aligned} \tag{26}$$

5. VOLUME FREE

If the volume is free, the relationship $\lambda = 0$ holds so that the extremal surface is described by the first integral

$$3Ep^2 - 2p = f(y) \tag{27}$$

and the boundary conditions (1) in which the thickness distribution t(y) is to be determined so that the natural boundary condition

Trailing edge $$3Ep^2 - 2p = 0 \tag{28}$$

is satisfied. Because of the Weierstrass condition (22), this equation admits the solution

Trailing edge $$p = 2/3E \tag{29}$$

indicating that the chordwise slope is constant over the trailing edge. By combining Eqs. (27) and (28), we see that

$$f(y) = 0 \tag{30}$$

meaning that the chordwise slope is also constant over the entire extremal surface. As a consequence, the optimum wing is described by the partial differential equation

$$p = 2/3E \tag{31}$$

which, in the light of the conditions (1-1), admits the particular integral

$$z = (2/3E)x \tag{32}$$

Next, the conditions (1-2) are applied to obtain the relationship

$$t(y) = (2/3E)\,c(y) \tag{33}$$

implying that the trailing edge thickness distribution is similar to the chord distribution. Then, by forming the ratio of the above equations and introducing the dimensionless coordinates (23), we conclude that the optimum wing surface is given by

$$\zeta = \xi \tag{34}$$

The final step consists of combining Eqs. (2), (4), (26), (31),and (32) to obtain the following values of the thickness ratio , lift-to-drag ratio, and volume parameters:

$$\tau_* = \sqrt[3]{2} \quad , \quad E_* = \sqrt[3]{4}\,/3 \quad , \quad V_* = (1/\sqrt[3]{4})\int_0^1 \gamma^2 d\eta \tag{35}$$

Equation (35-2) represents the highest lift-to-drag ratio which can be obtained with a three-dimensional, flat-top wing subjected to a flow parallel to the flat top. Should a wing of given planform be required to have a volume other than that given by Eq. (35-3), a loss in the lift-to-drag ratio would occur with respect to that predicted by Eq. (35-2).

6. VOLUME GIVEN

If the volume is given, the extremal surface is described by the first integral

$$3Ep^2 - 2p + \lambda x = f(y) \tag{36}$$

and the boundary conditions (1) in which the thickness distribution t(y) is to be determined so that the natural boundary condition

Trailing edge $$3Ep^2 - 2p = 0 \tag{37}$$

is satisfied. Because of the Weierstrass condition (22), this equation admits the solution

Trailing edge $$p = 2/3E \tag{38}$$

indicating that the chordwise slope is constant over the trailing edge.

If the first integral (36) is applied at the trailing edge and is combined with Eq. (37), it is seen that

$$f(y) = \lambda c(y) \tag{39}$$

and that

$$3Ep^2 - 2p - \lambda(c - x) = 0 \tag{40}$$

This is an algebraic equation of the second degree in p which--in the light of the Legendre condition (19)--admits the solution

$$3Ep = 1 + [1 + 3E\lambda(c - x)]^{1/2} \tag{41}$$

If this partial differential equation is integrated in x-direction and the conditions (1-1) are imposed, we obtain the relationship

$$3Ez = x + (2/9E\lambda)\left\{[1 + 3E\lambda c]^{3/2} - [1 + 3E\lambda(c - x)]^{3/2}\right\} \tag{42}$$

which, at the trailing edge, becomes

$$3Et = c + (2/9E\lambda)[(1 + 3E\lambda c)^{3/2} - 1] \tag{43}$$

Therefore, if the following definitions are introduced:

$$\alpha = 3E\lambda\, c(0) \tag{44}$$

and

$$G(\xi, \eta, \alpha) = 3\alpha\xi\gamma + 2\left\{[1 + \alpha\gamma]^{3/2} - [1 + \alpha\gamma(1 - \xi)]^{3/2}\right\} \tag{45}$$

Eqs. (41) and (42) can be rewritten as

$$\begin{aligned} (27E^2\lambda)z &= G(\xi, \eta, \alpha) \\ (27E^2\lambda)t &= G(1, \eta, \alpha) \end{aligned} \tag{46}$$

Consequently, the optimum shape is described in nondimensional form by

$$\zeta = \frac{G(\xi, \eta, \alpha)}{G(1, \eta, \alpha)} \tag{47}$$

The next step is to relate the quantity α to the prescribed value of the volume as well as to determine the optimum dimensions and the lift-to-drag ratio. To do this, we combine Eqs. (2), (4), (43), and (47) to obtain the relations

$$\tau_* = A(\eta, \alpha)/B(\alpha)$$

$$E_* = B(\alpha)/3 \tag{48}$$

$$V_* = C(\alpha)/B(\alpha)$$

where

$$A(\eta, \alpha) = 1 + (2/3\alpha\gamma)[(1 + \alpha\gamma)^{3/2} - 1]$$

$$B^3(\alpha) = 2 - (2/5\alpha)\int_0^1 [4 - (4 - \alpha\gamma)(1 + \alpha\gamma)^{3/2}]\, d\eta \left[\int_0^1 \gamma\, d\eta\right]^{-1} \tag{49}$$

$$C(\alpha) = \int_0^1 \left\{\gamma^2/2 + (2\gamma/3\alpha)(1 + \alpha\gamma)^{3/2} + (4/15\alpha^2)[1 - (1 + \alpha\gamma)^{5/2}]\right\} d\eta$$

The final step consists of eliminating the quantity α between Eqs. (47) and (48) to obtain the functional relationships

$$\zeta = f_1(\xi, \eta, V_*)$$

$$\tau_* = f_2(\eta, V_*) \tag{50}$$

$$E_* = f_3(V_*)$$

The corresponding extremal surface is convex for

$$C(0)/B(0) \leq V_* \leq C(\alpha_1)/B(\alpha_1) = \infty \tag{51}$$

and concave for

$$C(\alpha_2)/B(\alpha_2) \leq V_* \leq C(0)/B(0) \tag{52}$$

where α_1 and α_2 are defined by the relations

$$B(\alpha_1) = 0 \quad , \quad \alpha_2 + 3/4 = 0 \tag{53}$$

For values of V_* smaller than the lower bound in Ineq. (52), the optimum wing is composed of two subsurfaces, one of which is a flat-plate starting at the leading edge. However, these solutions are not analyzed here in view of their limited technical interest.

With the aid of Eqs. (47) through (53), the proposed problem can be solved numerically or analytically for each volume parameter V_* and chord distribution $\gamma(\eta)$. In this connection, the results for a constant chord (Ref. 4), that is, for

$$\gamma(\eta) = 1 \tag{54}$$

are plotted in Figs. 1 through 3. Since the limiting values of α are

$$\alpha_1 = 5.42 \quad , \quad \alpha_2 = -3/4 \tag{55}$$

the extremal solution is convex for

$$0.630 \leq V_* \leq \infty \tag{56}$$

and concave for

$$0.543 \leq V_* \leq 0.630 \tag{57}$$

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LIST OF CAPTIONS

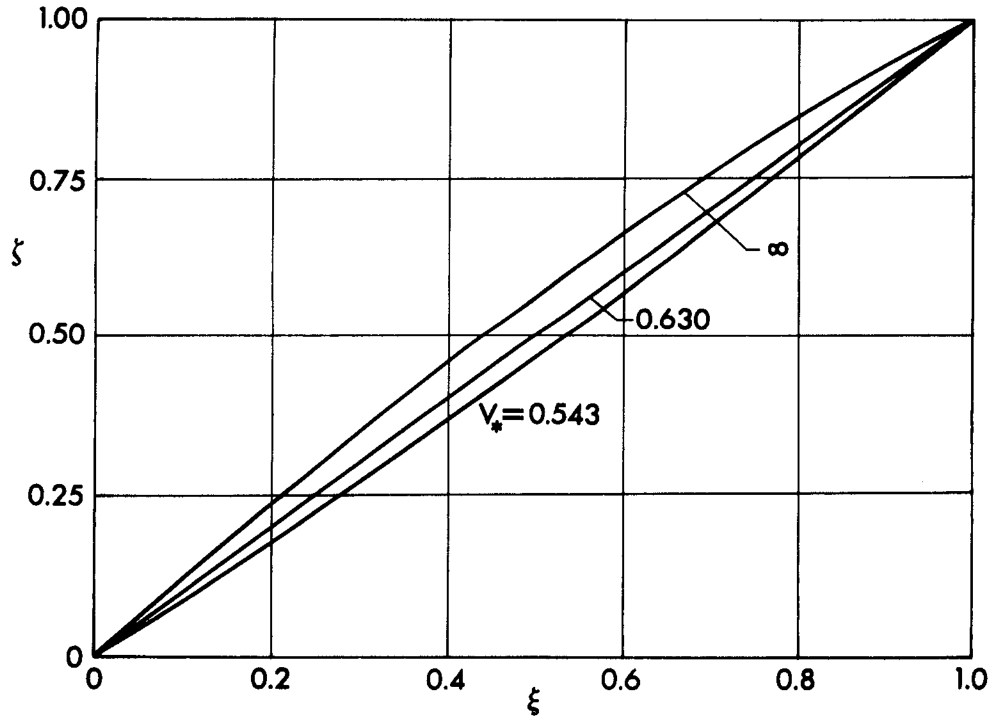

Fig. 1 Optimum shape.

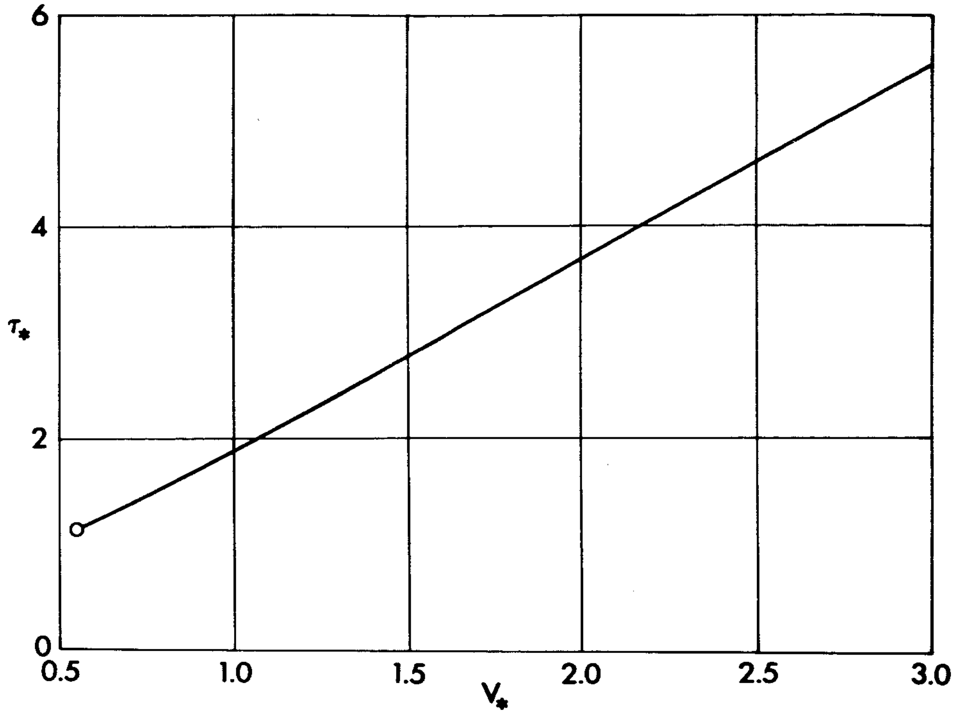

Fig. 2 Optimum thickness ratio.

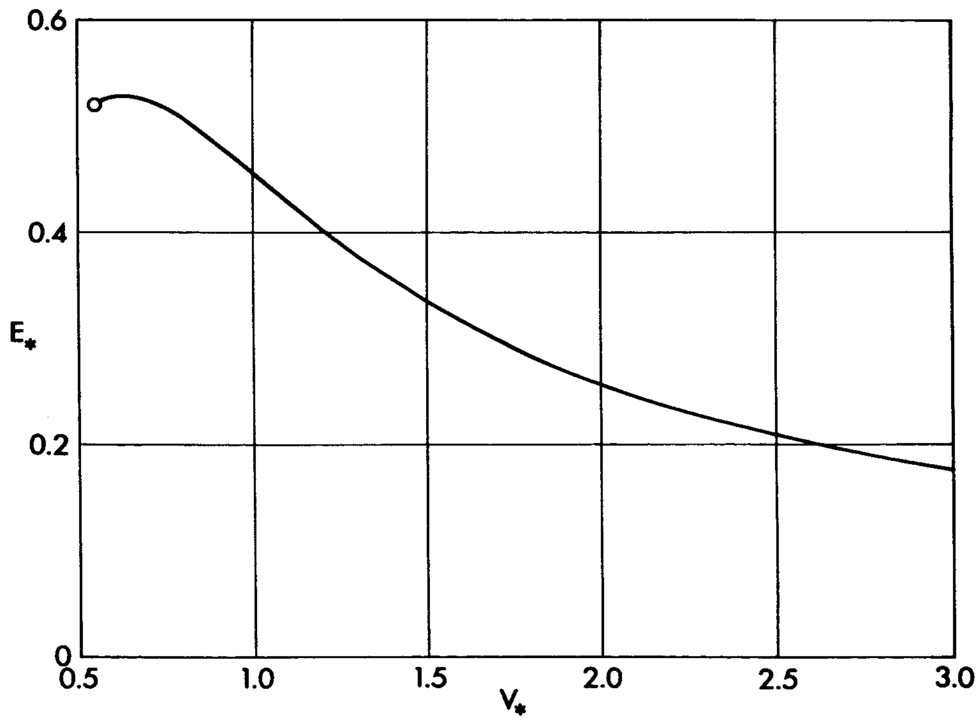

Fig. 3 Maximum lift-to-drag ratio.