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THE TIME-HISTORY OF A SATELLITE

AROUND AN OBLATE PLANET

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ABSTRACT

In the previous work on the motion of a close satellite around an oblate planet, the orbital elements and perturbations have been described as functions of the central angle between the instantaneous node and the satellite. A complete theory, however, requires the elements and perturbations as functions of the time. Therefore the relationship between time and the central angle between node and satellite is presented here. Although the problem is mathematically nothing but a quadrature, it is practically quite complicated because the evaluation of the occurring integral requires some rather lengthy algebraic manipulations. Part of the calculations are avoided by using a new technique to numerically evaluate certain coefficients that depend only on the initial conditions. Furthermore, the energy integral is used to evaluate certain terms that otherwise would require consideration of higher order terms in the differential equations.

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1. INTRODUCTION

Eckstein, Shi, and Kevorkian (1964) discussed the geometry of the orbit of a satellite around an oblate planet. Particular emphasis was given to the nature of the orbit near "the critical inclination" and to the question of defining the motion for all inclinations by means of uniform asymptotic expansions. This paper is a continuation of the mentioned reference and discusses the time-history of the motion. The notation and definitions of the cited reference will be used throughout, and equations shown there will be referred to by the corresponding number followed by an asterisk.

The solution of the time-history of the motion is formally very simple, since (cf. (3.5*)) it reduces to a quadrature once the orbit has been defined. However, the practical problem of exhibiting an explicit analytic formula for the time, correct to some order, is quite involved for the following reasons.

Inclusion of short-period terms in the solution introduces considerable algebraic complexity in this problem. Struble (1962) neglected these terms by using an averaged differential equation as a starting point for calculating the first approximation to the long-period and secular terms in the time-history. In this paper all terms in the solution are retained in order to define the time-history completely to order ε. The presence of perturbations of course eliminates the advantage of using the eccentric anomaly as an independent variable. Therefore, short-period terms are expressed by their convergent Fourier series expansions in terms of ϕ, the central angle between the instantaneous node and the radius vector.

In addition to the algebraic complexity of the problem where short-period terms are included, the inherent difficulty associated with the quadrature of the equation for $dt/d\phi$ is the presence of long-period terms on the right hand side. (These are trigonometric terms whose arguments are integer multiples of $\epsilon\phi$.) Since such terms drop in order by one power of ϵ after integration, it is necessary to consider terms of order ϵ^{n+1} in order to include all long-period terms of order ϵ^n in the solution. Thus, the major difficulty in deriving the solution for t correct to order ϵ stems from the presence of certain unknown long-period terms of order ϵ^2 in $dt/d\phi$.

To evaluate these terms would in general require consideration of perturbations of order ϵ^3 for the inclination and reciprocal radius.* However, in this particular case it is shown that the existence of the energy integral is sufficient to define all long-period terms that arise to order ϵ^2.

In order to avoid excessive algebraic manipulations, some of the coefficients in the present solution are not evaluated literally. These coefficients occur in expressions where the general form is known and they may be expressed either as definite integrals or as the solutions of systems of linear algebraic equations. As these coefficients are constant for any given motion, one need only evaluate them numerically once for each set of initial conditions.

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* Briefly, this is due to the fact that the long-period behavior of a term of order ϵ^n is governed by boundedness criteria on terms of order ϵ^{n+1}.

2. DISCUSSION

2.1 The Differential Equation for the Time

Struble (1960) gives the following general result for the rate of change of ϕ for a problem with an arbitrary cylindrically symmetric potential.

$$\frac{d\phi}{dt} = \frac{pu^2}{\cos i} + \frac{\cos^3 i \cos\theta}{p \sin^2 i \sin\theta} \frac{\partial U}{\partial \theta} \tag{2.1}$$

After substituting for θ and $\frac{\partial U}{\partial \theta}$ from (3.5h*) and (3.2*) and some manipulation one obtains

$$\frac{dt}{d\phi} = \frac{\cos i}{pu^2} - \varepsilon[2 \frac{\cos^5 i}{p^3 u} \sin^2\phi] + \varepsilon^2[4 \frac{\cos^9 i}{p^5} \sin^4\phi + 28 \frac{cu}{p^3} \cos^5 i \sin^2 i \sin^4\phi$$

$$- 12 \frac{cu}{p^3} \cos^5 i \sin^2\phi] + 0(\varepsilon^3) \tag{2.2}$$

Upon substitution of the expansions for i and u into the right hand side, (2.2) reduces to

$$\frac{dt}{d\phi} = \frac{\cos i_o}{pu_o^2} - \varepsilon[\frac{\sin i_o}{pu_o^2} i_1 + 2 \frac{\cos i_o}{pu_o^3} u_1 + 2 \frac{\cos^5 i_o}{p^3 u_o} \sin^2\phi]$$

$$+ \varepsilon^2[- \frac{1}{2} \frac{\cos i_o}{pu_o^2} i_1^2 - \frac{\sin i_o}{pu_o^2} i_2 + 2 \frac{\sin i_o}{pu_o^3} i_1 u_1 + 3 \frac{\cos i_o}{pu_o^4} u_1^2$$

$$- 2 \frac{\cos i_o}{pu_o^3} u_2 + 10 i_1 \frac{\cos^4 i_o \sin i_o}{p^3 u_o} \sin^2\phi + 2 \frac{\cos^5 i_o}{p^3 u_o^2} u_1 \sin^2\phi$$

$$+ 4 \frac{\cos^9 i_o}{p^5} \sin^4\phi + 28 \frac{cu_o}{p^3} \cos^5 i_o \sin^2 i_o \sin^4\phi$$

$$- 12 \frac{cu_o}{p^3} \cos^5 i_o \sin^2\phi] + 0(\varepsilon^3) \qquad (2.3)$$

All terms of the order unity and ε on the right hand side of (2.3) are given in the original reference of the authors. However, as mentioned previously, a solution for t to order ε must also include consideration of the long-period terms proportional to ε^2 in (2.3). This necessitates the evaluation of i_2 and u_2 occurring to order ε^2 in (2.3). The lengthy expressions for these two quantities which had not been computed earlier are shown in Appendix 1. In addition, the leading term of (2.3), viz. $\cos i_o / pu_o^2$, contains the unknown terms i_{o2} and e_2 to order ε^2. As was pointed out in the original reference the determination of these terms requires the knowledge of the differential equations for $\dot{i}$ and e correct to order ε^3. It will be shown in the next section that for the present case, one can deduce the form in which these terms occur by using the energy integral.

2.2 Use of the Energy Integral to Compute Long-Period Terms

The energy integral

$$\frac{1}{2} (\frac{dr}{dt})^2 + \frac{1}{2} r^2 (\frac{d\psi}{dt})^2 \sin^2\theta + \frac{1}{2} r^2 (\frac{d\theta}{dt})^2 - U = E \qquad (2.4)$$

can be brought to the following exact form in terms of the present variables

$$u^4 \left(\frac{dt}{d\phi}\right)^2 = \left(\frac{du}{d\phi}\right)^2 \left[2E + u\left(2 - \frac{p^2}{\cos^2 i} u\right) + \frac{2}{3} \varepsilon u^3 (1 - 3 \sin^2 i \sin^2 \phi) \right.$$

$$\left. + \varepsilon^2 c u^5 \left(14 \sin^4 i \sin^4 \phi - 12 \sin^2 i \sin^2 \phi + \frac{6}{5}\right]^{-1} \qquad (2.5)$$

Here, E is the energy constant expressible in terms of the initial conditions, and the potential U, given by (3.2*), has been used. It is clear that knowledge of the orbit (i.e., $u(\phi,\varepsilon)$, $i(\phi,\varepsilon)$) together with $dt/d\phi$, as given by (2.3) to any order ε^n should lead to an identity to $O(\varepsilon^n)$ when these values are substituted into (2.5). Thus, one could use (2.5) to check the solution to any order. In the present case however, a converse use of the energy integral will be invoked. The orbit defined by u and i is known to $O(\varepsilon^2)$ and it will be assumed that these results are free of algebraic errors. The energy integral will then be used to define those terms of $dt/d\phi$ which are unknown to the order the calculations have been carried out. A partial check of the accuracy of the results used for u and i will be the cancellation of all known terms after substitution into (2.5) leaving only a definition of the unknown long-period terms one seeks.

As was pointed out earlier, the long period terms of order ε^2 in the leading term of (2.3) are unknown. To evaluate these, $\cos i_o / p u_o^2$ is expanded in its Fourier series as shown below

$$\frac{\cos i_o}{p u_o^2} = \frac{p^3}{\cos^3 i_o} \left[\frac{1}{(1 - e^2)^{3/2}} + \frac{2}{(1 + e^2)^{3/2}} \sum_{k=1}^{\infty} \left(\frac{\sqrt{1 - e^2} - 1}{e}\right)^k \cdot \right.$$

$$\left. (1 + k\sqrt{1 - e^2}) \cos k (\phi - \omega)\right] \qquad (2.6)$$

The terms under the summation sign in the above have a short period and hence need not be evaluated to $O(\varepsilon^2)$. Conversely, it is necessary to compute the form of the first term, $p^3/\cos^3 i_o\ (1 - e^2)^{3/2}$ correct to $O(\varepsilon^2)$ since this term depends only on $\tilde{\phi}$ (or $\bar{\phi}$). It will be shown in what follows that substitution of all known terms into the energy integral leads to precisely the expression for the above term.

When $dt/d\phi$ is eliminated from (2.5) by using (2.2) one obtains

$$\left(\frac{du}{d\phi}\right)^2 = \left\{\frac{\cos^2 i}{p^2} - \varepsilon\left[4u \frac{\cos^6 i}{p^4} \sin^2\phi\right] + \varepsilon^2\left[12u^2 \frac{\cos^{10} i}{p^6} \sin^4\phi + 56c \frac{u^3}{p^4} \cos^6 i \sin^2 i \sin^4\phi - 24c \frac{u^3}{p^4} \cos^6 i \sin^2\phi\right] + O(\varepsilon^3)\right\}\cdot\left\{2E + u\left(2 - \frac{p^2}{\cos^2 i} u\right) + \varepsilon \frac{2}{3} u^3 (1 - 3 \sin^2 i \sin^2\phi) + \varepsilon^2 c u^5 \left(14 \sin^4 i \sin^4\phi - 12 \sin^2 i \sin^2\phi + \frac{6}{5}\right)\right\} \tag{2.7}$$

After substituting the expansions for the variables on the right hand side of (2.7) in the form assumed in the original reference one obtains after some manipulation and ordering

$$2 \frac{\cos^2 i_o}{p^2} E + u_o \left(2 \frac{\cos^2 i_o}{p^2} - u_o\right) - \left(\frac{\partial u_o}{\partial \phi}\right)^2 + \varepsilon\left\{- 2 \frac{\partial u_1}{\partial \phi} \frac{\partial u_o}{\partial \phi} - 2 \frac{\partial u_o}{\partial \phi} \frac{\partial u_o}{\partial \tilde{\phi}} - 2 \frac{\sin 2i_o}{p^2} i_1 (E + u_o) + 2u_1\left(\frac{\cos^2 i_o}{p^2} - u_o\right) - 4 \frac{\cos^6 i_o}{p^4} u_o\left[2E + u_o\left(2 - \frac{p^2}{\cos^2 i_o} u_o\right)\right]\sin^2\phi + \frac{2}{3} \frac{\cos^2 i_o}{p^2} u_o^3 (1 - 3 \sin^2 i_o \sin^2\phi)\right\}$$

$$+ \varepsilon^2\{-(\frac{\partial u_1}{\partial\phi})^2 - 2\frac{\partial u_1}{\partial\phi}\frac{\partial u_o}{\partial\tilde{\phi}} - (\frac{\partial u_o}{\partial\tilde{\phi}})^2 - 2\frac{\partial u_2}{\partial\phi}\frac{\partial u_o}{\partial\phi} - 2\frac{\partial u_1}{\partial\tilde{\phi}}\frac{\partial u_o}{\partial\phi}$$

$$- 2\frac{\sin 2i_o}{p^2}(E + u_o)i_2 - 2\frac{\cos 2i_o}{p^2}(E + u_o)i_1^2 - 2\frac{\sin 2i_o}{p^2}u_1 i_1$$

$$+ 2(\frac{\cos^2 i_o}{p^2} - u_o)u_2 - u_1^2 + 48\frac{\cos^5 i_o \sin i_o}{p^4} i_1 u_o(E + u_o)\sin^2\phi$$

$$- 8\frac{\cos^6 i_o}{p^4}(E + 2u_o)u_1 \sin^2\phi - 16\frac{\cos^3 i_o \sin i_o}{p^2} i_1 u_o^3 \sin^2\phi$$

$$+ 12\frac{\cos^4 i_o}{p^2}u_o^2 u_1 \sin^2\phi - \frac{2}{3}\frac{\sin 2i_o}{p^2} i_1 u_o^3 - \frac{\sin 4i_o}{p^2} i_1 u_o^3 \sin^2\phi$$

$$+ 2\frac{\cos^2 i_o}{p^2}u_o^2 u_1 (1 - 3\sin^2 i_o \sin^2\phi)$$

$$+ [2E + u_o(2 - \frac{p^2}{\cos^2 i_o}u_o)][12\frac{\cos^{10} i_o}{p^6}u_o^2 \sin^4\phi$$

$$+ 56c\frac{\cos^6 i_o}{p^4}u_o^3 \sin^2 i_o \sin^4\phi - 24c\frac{\cos^6 i_o}{p^4}u_o^3 \sin^2\phi]$$

$$- \frac{8}{3}\frac{\cos^6 i_o}{p^4}u_o^4 \sin^2\phi (1 - 3\sin^2 i_o \sin^2\phi)$$

$$+ cu_o^5\frac{\cos^2 i_o}{p^2}(\frac{6}{5} - 12\sin^2 i_o \sin^2\phi + 14\sin^4 i_o \sin^4\phi)\} = 0(\varepsilon^3) \qquad (2.8)$$

In the above the known expressions for u_o, u_1, u_2, i_1, and i_2 have not yet been used. If these results are now substituted into (2.8), all short-period terms cancel identically as expected, and what remains is the following expression for the long-period term under consideration.

$$\frac{\cos^2 i_o}{p^2}(1 - e^2) = -2E + \varepsilon\frac{\cos^6 i_o}{p^6}\left[\left(\frac{1}{3} + e^2\right)(1 - 3\cos^2 i_o)\right.$$

$$\left. + \frac{e^2}{4}(1 + \cos^2 i_o)\cos 2\omega\right]$$

$$+ \varepsilon^2[\eta_o + \eta_2 \cos 2\omega + \eta_4 \cos 4\omega] + O(\varepsilon^3) \qquad (2.9)$$

Actually, what is needed is the negative three-halves power of (2.9) correct to $O(\varepsilon^2)$. This can be easily computed once the quantities denoted by η_o, η_2, and η_4 have been defined.

Before discussing the evaluation of η_o, η_2, and η_4, it is pointed out that in the term of order ε in (2.9), e and i_o have not been expanded because the higher order terms in these expansions are no longer negligible. In contrast, for the evaluation of (3.15*) and (3.20*) e and i_o or (e* and i_o*) were replaced by e_o* and i_{oo}* since the solution was valid only to $O(\varepsilon)$ there. However, one need only use e_o and i_{oo} for the values of e and i_o occurring in the term proportional to ε^2 in (2.9).

Thus η_o, η_1, and η_4 formally depend on the initial values e_o and i_{oo}. The form of the term of $O(\varepsilon^2)$ in (2.9) can be deduced by inspection of the terms of $O(\varepsilon^2)$ in (2.8). Each of these may be expressed as the following double sum

$$\sum_{n=0}^{8} \sum_{k=-2}^{2} a_{nk} \cos[n(\phi - \omega) + 2k\omega]$$

where the a_{nk} are functions of e_o and i_{oo}.

Now the identity (2.8) must be independent of ϕ to any order in ε. In particular, to $0(\varepsilon^2)$ all the contributions $a_{nk}\cos[n(\phi - \omega) \pm 2k\omega]$ must cancel when $n \neq 0$ leaving only terms which can be brought to the form shown in (2.9).

Unfortunately, the explicit evaluation of η_o, η_2, and η_4 in terms of i_{oo} and e_o requires extremely involved algebraic calculations. Furthermore, since the energy integral has already been used in arriving to this stage there is no independent procedure for checking the results if one were to actually derive explicit formulae for η_o, η_2, and η_4.

Therefore, the following direct scheme for computing the numerical value of these functions for any set of initial conditions is proposed. Let $E_2(\omega)$, defined in Appendix 2, denote the lengthy expression for the terms of order ε^2 in (2.8). Even though the form shown in the Appendix contains ϕ explicitly, it was pointed out that upon simplification the terms depending upon ϕ must cancel identically. Thus, strictly speaking, E_2 is a function of ω and the two initial values e_o and i_{oo}. Furthermore, it is relatively straightforward to show that upon simplification E_2 will reduce to the following form

$$\eta_o(e_o, i_{oo}) + \eta_2(e_o, i_{oo}) \cos 2\omega + \eta_4(e_o, i_{oo}) \cos 4\omega = E_2(\omega, e_o, i_{oo}) \tag{2.10}$$

The next step consisting of exhibiting the explicit dependence of the η_i upon the initial values e_o, and i_{oo}, is very arduous. The problem of predicting the numerical value of the η_i for any given pair e_o, and i_{oo}, is however quite simple. For any value of ω and a given pair e_o, and i_{oo}, (2.10) is a linear algebraic equation for the three η_i. The numerical value of the right hand side can be computed from the definition of E_2 in Appendix 2 while the left hand side contains two coefficients which depend on ω. Thus, if three distinct values of ω are chosen, (2.10) produces three linear equations for the η_i.*

One may, for instance, choose to evaluate E_2 at $\omega = 0, \frac{\pi}{4}$, and $\frac{\pi}{2}$, to simplify the left-hand side of (2.10). Moreover, since E_2 does not depend on ϕ in its final form, a choice of ϕ which results in considerable simplification of the numerical work of evaluating the right hand side of (2.10) is $\phi = 0$.

Thus, a possible set of equations that results are

$$\eta_o + \eta_2 + \eta_4 = E_2(0) \quad \text{(a)}$$

$$\eta_o - \eta_4 = E_2(\tfrac{\pi}{4}) \quad (2.11) \quad \text{(b)}$$

$$\eta_o - \eta_2 + \eta_4 = E_2(\tfrac{\pi}{2}) \quad \text{(c)}$$

The solution of the above system gives

$$\eta_o = \frac{1}{4}\left[E_2(0) + 2E_2(\tfrac{\pi}{4}) + E_2(\tfrac{\pi}{2})\right] \quad \text{(a)}$$

* It is allowable to choose arbitrary values for ω since (2.10) is an algebraic identity even though the range of ω, as predicted by the solution, is restricted for the case of critical inclination.

$$\eta_2 = \frac{1}{2}\,[E_2(0) - E_2(\tfrac{\pi}{2})] \qquad (2.12) \qquad (b)$$

$$\eta_4 = \frac{1}{4}\,[E_2(0) - 2E_2(\tfrac{\pi}{4}) + E_2(\tfrac{\pi}{2})] \qquad (c)$$

Equation (2.9) can now be used to express the leading term of the Fourier series for $\cos i_o \,/\, pu_o^{\;2}$ in terms of known quantities in the form:

$$\frac{p^3}{\cos^3 i_o (1 - e^2)^{3/2}} = (-\,2E)^{-3/2} + \frac{3}{2}\;(-\,2E)^{-5/2}\,\frac{\cos^6 i_o}{p^6}\,[(\tfrac{1}{3} + e^2)(1 - 3\cos^2 i_o)$$

$$+ \frac{e^2}{4}\,(1 + \cos^2 i_o)\cos 2\omega]$$

$$+ \varepsilon^2\,\{+ \frac{3}{2}\,(-\,2E)^{-5/2}[\eta_o + \eta_2\cos 2\omega + \eta_4\cos 4\omega]$$

$$+ \frac{3}{8}\,(-\,2E)^{-7/2}\,\frac{\cos^{12} i_o}{p^{12}}\,[(\tfrac{1}{3} + e^2)^2(1 - 3\cos^2 i_o)^2$$

$$+ \frac{e^2}{2}\,(\tfrac{1}{3} + e^2)(1 - 2\cos^2 i_o - 3\cos^4 i_o)\cos 2\omega$$

$$+ \frac{e^4}{32}\,(1 + \cos^2 i_o)^2(1 + \cos 4\omega)]\} \qquad (2.13)$$

2.3 The First Order Term

With the results of the previous section and use of the known solutions for u_o, u_1, and i_1, the term of order ε in the differential equation for the time becomes

$$
\begin{aligned}
T_s^{(1)} + T_\ell^{(1)} = \frac{\varepsilon \cos i_o}{p[1 + e\cos(\phi - \omega)]^3} \{1 &+ \frac{e^2}{2} - (4 + 3e^2)\cos^2 i_o \\
&+ (-\frac{1}{2} + \frac{5}{4}\cos^2 i_o)e^2 \cos 2\omega - 2e\cos^2 i_o \cos(\phi - \omega) \\
&- \frac{e}{4}(3 - 7\cos^2 i_o)\cos(\phi + \omega) + [-\frac{1}{6}(1 - e^2) \\
&+ \frac{7}{6}\cos^2 i_o]\cos 2\phi - \frac{e^2}{6}(1 - 6\cos^2 i_o)\cos(2\phi - 2\omega) \\
&+ \frac{1}{2} e\cos^2 i_o \cos(3\phi - \omega) + \frac{e^2}{12}\cos^2 i_o \cos(4\phi - 2\omega)\} \\
&+ \frac{3}{2}\varepsilon(-2E)^{-5/2}\frac{\cos^6 i_o}{p^6}[(\frac{1}{3} + e^2)(1 - 3\cos^2 i_o) \\
&+ \frac{e^2}{4}(1 + \cos^2 i_o)\cos 2\omega] \qquad (2.14)
\end{aligned}
$$

The notation $T_s^{(1)}$ and $T_\ell^{(1)}$ has been introduced to denote the fact that the right hand side of (2.14) consists of two parts. One is purely periodic in ϕ and is denoted by $T_s^{(1)}$ while the remainder, expressed as $T_\ell^{(1)}$, depends only on $\tilde{\phi}$ (or $\bar{\phi}$). The terms comprising $T_\ell^{(1)}$ are functions of e, i_o, and ω, and will be studied separately because they give rise to either secular or long-period terms (dropping in order by one power of ε, after integration in the time solution). In order to effect the separation of the short-period terms, the Fourier expansion of $[1 + e\cos(\phi - \omega)]^{-3}$, given in Appendix 3, is used. It can then be shown that $T_\ell^{(1)}$ becomes

$$T_\ell^{(1)}(e, i_o, \omega) = \frac{\cos i_o}{2p} \Big\{ b_{30}[1 + \frac{e^2}{2} - (4 + 3e^2) \cos^2 i_o$$

$$- \frac{e^2}{4} (2 - 5 \cos^2 i_o) \cos 2\omega]$$

$$- b_{31}[2e \cos^2 i_o + \frac{e}{4} (3 - 7 \cos^2 i_o) \cos 2\omega]$$

$$+ b_{32}[- \frac{e^2}{6} (1 - 6 \cos^2 i_o) - \frac{1}{6} (1 - e^2 - 7 \cos^2 i_o) \cos 2\omega]$$

$$+ \frac{1}{2} b_{33}\, e \cos^2 i_o \cos 2\omega + \frac{1}{12} b_{34}\, e^2 \cos^2 i_o \cos 2\omega \Big\}$$

$$+ \frac{3}{2} (- 2E)^{-5/2} \frac{\cos^6 i_o}{p^6} [(\frac{1}{3} + e^2)(1 - 3\cos^2 i_o)$$

$$+ \frac{e^2}{4} (1 + \cos^2 i_o) \cos 2\omega] \tag{2.15}$$

Note that since $T_\ell^{(1)}$ comprises the long-period terms and is multiplied by ε in (2.3) one must evaluate it correct to $O(\varepsilon)$ in order to insure the validity of the differential equation for the time correct to $O(\varepsilon^2)$. This can be achieved for the case of critical inclination by substituting the known expansions for e and i_o (cf. (3.28b*), (3.35*), (3.37*) and (3.28a*), (3.33*), (3.36*)) giving the following formal representation of $T_\ell^{(1)}$

$$T_\ell^{(1)}(e, i_o, \omega) = T_\ell^{(1)*}(e_o^*, i_{oo}^*, \omega_c, \varepsilon)$$

$$= T_{\ell o}^{(1)*}(e_o^*, i_{oo}^*, \varepsilon) + T_{\ell 1}^{(1)*}(e_o^*, i_{oo}^*, \varepsilon) \sqrt{\bar{\kappa}_o - \kappa_1 \cos 2\omega_c}$$

$$+ T_{\ell 2}^{(1)*}(e_o^*, i_{oo}^*, \varepsilon)(\sqrt{\bar{\kappa}_o - \kappa_1 \cos 2\omega_c})^2$$

$$+ T_{\ell 3}^{(1)*}(e_o^*, i_{oo}^*, \varepsilon)(\sqrt{\bar{\kappa}_o - \kappa_1 \cos 2\omega_c})^3$$

$$+ T_{\ell 4}^{(1)*}(e_o^*, i_{oo}^*, \varepsilon)(\sqrt{\bar{\kappa}_o - \kappa_1 \cos 2\omega_c})^4 \qquad (2.16)$$

Again, the explicit evaluation of the coefficients $T_{\ell i}^{(1)*}$, $i = 0, 1, \ldots 4$ is a formidable task. However, a numerical procedure identical to the one proposed in section 2.2 can be used.

This consists of numerically evaluating $T_{\ell}^{(1)}$ for any given pair of initial values e_o and i_{oo} for five different and consistent (cf. what follows) values of $\cos 2\omega_c$, say $\cos 2\omega_c^{(\nu)}$, $\nu = 0, 1, \ldots 4$.

Clearly, the values of $\omega_c^{(\nu)}$ must be consistent with the permissible range given by the solution near the critical inclination, since now the explicit form of (2.19) depends on whether or not i_{oo} is near the critical value. It was shown in (3.51*) and in (3.52*) that near the critical inclination $\omega_c^{(\nu)}$ must obey the inequality

$$\sin^2 \omega_c^{(\nu)} > \frac{\kappa_1 - \bar{\kappa}_o}{2\kappa_1} \qquad (2.17)$$

and possible choices of $\omega_c^{(\nu)}$ are

$$\omega_c^{(\nu)} = \frac{1}{2}\cos^{-1}\left[1 - \frac{\nu \cdot \sqrt{\bar{\kappa}_o - \kappa_1}}{\sqrt{\bar{\kappa}_o + \kappa_1} + \sqrt{\bar{\kappa}_o - \kappa_1}} - \frac{\nu^2}{4}\frac{\kappa_1}{(\sqrt{\bar{\kappa}_o + \kappa_1} + \sqrt{\bar{\kappa}_o - \kappa_1})^2}\right]$$

$$\text{if } \bar{\kappa}_o > \kappa_1 \qquad (2.18)$$

or

$$\omega_c^{(\nu)} = \frac{1}{2}\cos^{-1}\left[\frac{\bar{\kappa}_o}{\kappa_1} - \nu^2\left(\frac{\bar{\kappa}_o}{\kappa_1} + 1\right)\right] \qquad \text{if } -\kappa_1 < \bar{\kappa}_o \leq \kappa_1 \qquad (2.19)$$

When each of the above values of $\omega_c^{(\nu)}$ is substituted into the right hand side of (2.15) one obtains a value for the corresponding $T_\ell^{(1)*}$ which can then be used in conjunction with (2.16) to derive a set of five linear algebraic equations for the five unknown $T_{\ell_i}^{(1)*}$. The results of such a calculation are summarized in Appendix 4.

If i_{oo} is not critical the form of $T_\ell^{(1)*}$ is simplified to

$$T_\ell^{(1)}(e, i_o, \omega) = T_\ell^{(1)}(e_o, i_{oo}, \omega, \varepsilon)$$

$$= T_{\ell o}^{(1)}(e_o, i_{oo}, \varepsilon) + T_{\ell 2}^{(1)}(e_o, i_{oo}, \varepsilon)\cos 2\omega$$

$$+ T_{\ell 4}^{(1)}(e_o, i_{oo}, \varepsilon)\cos 4\omega \qquad (2.20.$$

In this case there are only three unknown functions: $T_{\ell o}^{(1)}$, $T_{\ell 2}^{(1)}$, and $T_{\ell 4}^{(1)}$. Since ω undergoes a secular variation in this case one may use the following three equally spaced values of $\omega = 0, \frac{\pi}{4}, \frac{\pi}{2}$ to obtain the following system of three linear algebraic equations.

$$T_{\ell o}^{(1)} + T_{\ell 2}^{(1)} + T_{\ell 4}^{(1)} = T_{\ell}^{(1)} (e_o, i_{oo}, 0, \varepsilon)$$

$$T_{\ell o}^{(1)} - T_{\ell 4}^{(1)} = T_{\ell}^{(1)} (e_o, i_{oo}, \tfrac{\pi}{4}, \varepsilon)$$

$$T_{\ell o}^{(1)} - T_{\ell 2}^{(1)} + T_{\ell 4}^{(1)} = T_{\ell}^{(1)} (e_o, i_{oo}, \tfrac{\pi}{2}, \varepsilon) \qquad (2.21)$$

The solution of (2.21) gives

$$T_{\ell o}^{(1)} = \tfrac{1}{4} [T_{\ell}^{(1)} (e_o, i_{oo}, 0, \varepsilon) + 2T_{\ell}^{(1)}(e_o, i_{oo}, \tfrac{\pi}{4}, \varepsilon) + T_{\ell}^{(1)} (e_o, i_{oo}, \tfrac{\pi}{2}, \varepsilon)]$$

$$T_{\ell 2}^{(1)} = \tfrac{1}{2} [T_{\ell}^{(1)} (e_o, i_{oo}, 0, \varepsilon) - T_{\ell}^{(1)} (e_o, i_{oo}, \tfrac{\pi}{2}, \varepsilon)]$$

$$T_{\ell 4}^{(1)} = \tfrac{1}{4} [T_{\ell}^{(1)} (e_o, i_{oo}, 0, \varepsilon) - 2T_{\ell}^{(1)} (e_o, i_{oo}, \tfrac{\pi}{4}, \varepsilon) + T_{\ell}^{(1)}(e_o, i_{oo}, \tfrac{\pi}{2}, \varepsilon)]$$

(2.22)

With the definition given in Appendix 6 for the short-period terms to $O(\varepsilon)$, the results are in a form suitable for quadrature. This will be discussed in Section 2.5 after the terms of $O(\varepsilon^2)$ have also been calculated.

2.4 The Second Order Term

For abbreviation, the whole expression to order ε^2 of (2.3) plus the terms of order ε^2 of the right side of (2.12), are denoted by $T^{(2)}(i_{oo}, e_o, \omega, \phi)$, (cf. Appendix 5). When the solutions for u_o, u_1, u_2, i_o, i_1, i_2 are substituted and only the leading terms e_o, i_{oo} are kept, one may express $T^{(2)}$ as a Fourier series in the form

$$T^{(2)}(i_{oo}, e_o, \omega, \phi) = \frac{1}{2} T_o^{(2)}(i_{oo}, e_o, \omega) + \sum_{m=1}^{\infty} T_m^{(2)}(i_{oo}, e_o, \omega)\cos m(\phi - \omega) \tag{2.23}$$

Only the first term, $\frac{1}{2} T^{(2)}(i_{oo}, e_o, \omega)$, is of interest because this is the only term that contributes to order ε in the solution for the time t. From inspection of the expression for $T^{(2)}$, it can be shown that $T_o^{(2)}(i_{oo}, e_o, \omega)$ is of the form

$$T_o^{(2)}(i_{oo}, e_o, \omega) = T_{oo}^{(2)}(i_{oo}, e_o) + T_{o2}^{(2)}(i_{oo}, e_o) \cos 2\omega + T_{o4}^{(2)}(i_{oo}, e_o) \cos 4\omega \tag{2.24}$$

The coefficients $T_{oo}^{(2)}$, $T_{o2}^{(2)}$, and $T_{o4}^{(2)}$ can be evaluated by substitution of three different values for ω in $T_o^{(2)}(\omega)$. The procedure is analogous to the one used for equations (2.10) and (2.11) and the solution is

$$T_{oo}^{(2)}(i_{oo}, e_o) = \frac{1}{4} [T_o^{(2)}(i_{oo}, e_o, 0) + 2 T_o^{(2)}(i_{oo}, e_o, \frac{\pi}{4}) + T_o^{(2)}(i_{oo}, e_o, \frac{\pi}{2})]$$

$$T_{o2}^{(2)}(i_{oo}, e_o) = \frac{1}{2} [T_o^{(2)}(i_{oo}, e_o, 0) - T_o^{(2)}(i_{oo}, e_o, \frac{\pi}{2})]$$

$$T_{o4}^{(2)}(i_{oo}, e_o) = \frac{1}{4} [T_o^{(2)}(i_{oo}, e_o, 0) - 2 T_o^{(2)}(i_{oo}, e_o, \frac{\pi}{4}) + T_o^{(2)}(i_{oo}, e_o, \frac{\pi}{2})] \tag{2.25}$$

where

$$T_o^{(2)}(i_{oo}, e_o, \omega) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} T^{(2)}(i_{oo}, e_o, \omega, \phi) \, d\phi \tag{2.26}$$

The definite integral of (2.26) has to be evaluated numerically for $\omega = 0$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$, with the e_o and i_{oo} given by the initial conditions.

2.5 Integration of the Time Equation

Using the results of the previous sections equation (2.3) may now be written in the following form:

$$\begin{aligned} \frac{dt}{d\phi} &= (-2E)^{-3/2} + \frac{p^3}{\cos^3 i_o^*} \sum_{k=1}^{\infty} b_{2k} \cos k(\phi - \omega) \\ &+ \varepsilon [T_{\ell o}^{(1)*} + T_{\ell 1}^{(1)*} [\bar{\kappa}_o - \kappa_1 \cos 2\omega_c]^{1/2} + T_{\ell 2}^{(1)*} (\bar{\kappa}_o - \kappa_1 \cos 2\omega_c) \\ &+ T_{\ell 3}^{(1)*} [\bar{\kappa}_o - \kappa_1 \cos 2\omega_c]^{3/2} + T_{\ell 4}^{(1)*} (\bar{\kappa}_o - \kappa_1 \cos 2\omega_c)^2 + \frac{d\sigma}{d\phi}] \\ &+ \varepsilon^2 [T_{oo}^{(2)} + T_{o2}^{(2)} \cos 2\omega_c + T_{o4}^{(2)} \cos 4\omega_c] + O(\varepsilon^{5/2}) \end{aligned} \tag{2.27}$$

where $\frac{d\sigma}{d\phi}$ stands for all the short period terms arising from the expression in (2.3) to order ε. This equation holds for all inclinations. For inclinations away from the critical, the following somewhat simpler form results

$$\frac{dt}{d\phi} = (-2E)^{-3/2} + \frac{p^3}{\cos^3 i_o}\sum_{k=1}^{\infty} b_{2k}\cos k(\phi - \omega) + \varepsilon[T_{\ell o}^{(1)} + T_{\ell 2}^{(1)} \cos 2\omega + T_{\ell 4}^{(1)} \cos 4\omega]$$

$$+ \frac{d\sigma}{d\phi} + \varepsilon^2[T_{oo}^{(2)} + T_{oo}^{(2)} \cos 2\omega + T_{o4}^{(2)} \cos 4\omega] + O(\varepsilon^3) \qquad (2.28)$$

and is applicable to most cases.

Integration of (2.28) yields

$$t = (-2E)^{-3/2}\phi + \frac{1}{2}\frac{T_{\ell 2}^{(1)}}{S_o} \sin 2\omega + \frac{1}{4}\frac{T_{\ell 4}^{(1)}}{S_o} \sin 4\omega + \frac{p^3}{\cos^3 i_o}\sum_{k=1}^{\infty}\frac{b_{2k}}{k} \sin k(\phi - \omega)$$

$$+ \varepsilon[T_{\ell o}^{(1)}\phi + \frac{1}{2}\frac{T_{o2}^{(2)}}{S_o} \sin 2\omega + \frac{1}{4}\frac{T_{o4}^{(2)}}{S_o} \sin 4\omega + \frac{S_o p^3}{\cos^3 i_o}\sum_{k=1}^{\infty}\frac{b_{2k}}{k} \sin k(\phi - \omega)$$

$$+ \sigma(\phi, \omega)] + \varepsilon^2 T_{oo}^{(2)} \cdot \phi + t_o \qquad (2.29)$$

where σ is given in Appendix 6 and t_o is a constant to be determined by the initial conditions. Integration of the general equation yields the more complicated expression

$$t = (-2E)^{-3/2}\phi + \varepsilon^{-1/2}\, T_{\ell 1}^{(1)}\, (\omega_o^* - w) + \varepsilon^{-1/2}\int_{w}^{\omega_c} (\bar{\kappa}_o - \kappa_1 \cos 2\xi)^{1/2}[T_{\ell 2}^{(1)*}$$

$$+ T_{\ell 3}^{(1)*}\sqrt{\bar{\kappa}_o - \kappa_1 \cos 2\xi} + T_{\ell 4}^{(1)*}(\bar{\kappa}_o - \kappa_1 \cos 2\xi)]d\xi$$

$$+ \frac{p^3}{\cos^3 i_o} \sum_{k=1}^{\infty} \frac{b_{2k}}{k} \sin k(\phi - \omega_c) + \varepsilon^{1/2} \int_w^{\omega_c} \frac{T_{02}^{(2)} \cos 2\xi + T_{04}^{(2)} \cos 4\xi}{(\bar{\kappa}_o - \kappa_1 \cos 2\xi)^{1/2}} d\xi$$

$$+ \varepsilon[T_{\ell o}^{(1)^*} \phi + \sigma] + \varepsilon^2 T_{oo}^{(2)} \cdot \phi + t_o^* \qquad (2.30)$$

where t_o^* is the integration constant, to be determined by the initial conditions.

The extremely slow motion of the apse in case of critical inclination is responsible for the occurrence of the large terms of order $\varepsilon^{-1/2}$ appearing in (2.30). However, it should be noted that these terms, although multiplied by $\varepsilon^{-1/2}$, are of order ε for $\phi = 0(1)$ and grow to order $\varepsilon^{-1/2}$ only when $(\omega_c - w) = 0(1)$ which is equivalent to $\phi = 0(\varepsilon^{-3/2})$. The evaluation of the integrals occurring in (2.30) leads to elliptic functions and will not be exhibited here.

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APPENDIX 1

$$i_2 = \frac{\cos^9 i_{oo} \sin i_{oo}}{p^8} \{ - \frac{e_o}{48} (19 - 21 \cos^2 i_{oo}) \cos(\phi - \omega)$$

$$- \frac{e_o}{4} [- 1 + 9 \cos^2 i_{oo} + 6c (1 + \frac{e_o^2}{4})(1 - 7 \cos^2 i_{oo})] \cos(\phi + \omega)$$

$$+ \frac{e_o^3}{8} c (1 - 7 \cos^2 i_{oo}) \cos(\phi - 3\omega) + \frac{7}{16} e_o^3 c \sin^2 i_{oo} \cos(\phi + 3\omega)$$

$$- \frac{1}{4} [1 + \frac{e_o^2}{2} + 2c (1 + \frac{3}{2} e_o^2) - (1 + \frac{3}{2} e_o^2)(1 + 14c)\cos^2 i_{oo}]\cos 2\phi$$

$$- \frac{e_o^2}{24} (2 - \cos^2 i_{oo}) \cos(2\phi - 2\omega) - \frac{e_o^2}{32} [5-7\cos^2 i_{oo} - 42c \sin^2 i_{oo}]\cos(2\phi+2\omega)$$

$$- \frac{e_o}{36} [1 + 7 \cos^2 i_{oo} + 18c (1 + \frac{e_o^2}{4})(1 - 7 \cos^2 i_{oo})] \cos(3\phi - \omega)$$

$$- \frac{e_o}{24} [9 - 16 \cos^2 i_{oo} - 42 c (1 + \frac{e_o^2}{4}) \sin^2 i_{oo}] \cos (3\phi + \omega)$$

$$- \frac{1}{48} [10 (1 + \frac{3}{4} e_o^2) - (19 + \frac{27}{2} e_o^2) \cos^2 i_{oo} - 42c(1 + \frac{3}{2} e_o^2)\sin^2 i_{oo}]\cos 4\phi$$

$$+ \frac{e_o^2}{96} [1 - 15 \cos^2 i_{oo} - 18 c (1 - 7 \cos^2 i_{oo})] \cos (4\phi - 2\omega)$$

$$- \frac{e_o}{48} [7 - 15 \cos^2 i_o - \frac{252}{5} c (1 + \frac{e_o^2}{4})\sin^2 i_{oo}]\cos(5\phi - \omega)$$

$$- \frac{e_o^3}{40} c (1 - 7 \cos^2 i_o)\cos(5\phi - 3\omega)$$

$$- \frac{e_o^{\ 2}}{288} [7 - 17 \cos^2 i_{oo} - 126\ c \sin^2 i_{oo}] \cos(6\phi - 2\omega)$$

$$+ \frac{e_o^{\ 3}}{16}\ c \sin^2 i_{oo} \cos(7\phi - 3\omega)\}$$

$$u_2 = \frac{\cos^{10} i_{oo}}{p^{10}} \{ - \frac{5}{24} - \frac{287}{288} e_o^{\ 2} + \frac{9}{8} c (1 + 3\ e_o^{\ 2} + \frac{3}{8} e_o^{\ 4}) + (- \frac{5}{8} + \frac{211}{48} e_o^{\ 2}$$

$$- \frac{45}{4} c - \frac{81}{2} c\ e_o^{\ 2} - \frac{189}{32} c\ e_o^{\ 4}) \cos^2 i_{oo} + (\frac{17}{6} - \frac{1039}{288} e_o^{\ 2} + \frac{105}{8} c$$

$$+ \frac{441}{8} c\ e_o^{\ 2} + \frac{567}{64} c\ e_o^{\ 4}) \cos^4 i_{oo}$$

$$+ [(\frac{5}{12} e_o^{\ 2} - 3\ c\ e_o^{\ 2} - \frac{1}{2} c\ e_o^{\ 4}) + (- \frac{143}{24} e_o^{\ 2} + 30\ c\ e_o^{\ 2} + 6\ c\ e_o^{\ 4}) \cos^2 i_{oo}$$

$$+ (\frac{63}{8} e_o^{\ 2} - \frac{63}{2} c\ e_o^{\ 2} - 7\ c\ e_o^{\ 4}) \cos^4 i_{oo}] \cos 2\omega$$

$$+ \frac{7}{128} c\ e_o^{\ 4} \sin^2 i_{oo} (1 - 5 \cos^2 i_{oo}) \cos 4\omega$$

$$+ [+ \frac{1}{9} + \frac{1}{2} c - \frac{37}{144} e_o^{\ 2} + 2\ c\ e_o^{\ 2} + \frac{5}{16} c\ e_o^{\ 4} + (\frac{1}{6} - 2c + \frac{49}{12} e_o^{\ 2}$$

$$- 20\ c\ e_o^{\ 2} - \frac{7}{2} c\ e_o^{\ 4}) \cos^2 i_{oo} + (- \frac{5}{18} + \frac{7}{2} c - \frac{775}{144} e_o^{\ 2} + 21\ c\ e_o^{\ 2}$$

$$+ \frac{63}{16} c\ e_o^{\ 4}) \cos^4 i_{oo})] \cos 2\phi$$

$$+ [\frac{25}{96} e_o^{\ 2} - \frac{9}{8} c\ e_o^{\ 2} - \frac{3}{16} c\ e_o^{\ 4} + (- \frac{35}{16} e_o^{\ 2} + 18\ c\ e_o^{\ 2} + \frac{27}{8} c\ e_o^{\ 4}) \cos^2 i_{oo}$$

$$+ \left(\frac{245}{96} e_o^2 - \frac{231}{8} c\, e_o^2 - \frac{91}{16} c\, e_o^4\right) \cos^4 i_{oo}\Big] \cos(2\phi - 2\omega)$$

$$+ \Big[- \frac{1}{48} e_o^2 - \frac{7}{16} c\, e_o^2 - \frac{7}{32} c\, e_o^4 + \left(\frac{11}{48} e_o^2 + \frac{7}{16} c\, e_o^4\right) \cos^2 i_{oo}$$

$$+ \left(- \frac{1}{6} e_o^2 + \frac{7}{16} c\, e_o^2 - \frac{7}{32} c\, e_o^4\right) \cos^4 i_{oo}\Big] \cos(2\phi + 2\omega)$$

$$+ \frac{1}{32} c\, e_o^4 [1 - 24 \cos^2 i_o + 35 \cos^4 i_{oo}] \cos(2\phi + 4\omega)$$

$$+ \Big[\frac{13}{288} e_o + \frac{9}{16} c\, e_o + \frac{e_o^3}{128} + \frac{31}{64} e_o^3 c + \left(\frac{23}{24} e_o - \frac{13}{2} c\, e_o + \frac{25}{48} e_o^3\right.$$

$$\left.- \frac{49}{8} c\, e_o^3\right) \cos^2 i_{oo} + \left(- \frac{433}{288} e_o + \frac{119}{16} c\, e_o - \frac{85}{128} e_o^3\right.$$

$$\left.+ \frac{469}{64} c\, e_o^3\right) \cos^4 i_{oo}\Big] \cos(3\phi - \omega) + \Big[\frac{5}{96} e_o - \frac{7}{16} c\, e_o + \frac{25}{384} e_o^3$$

$$- \frac{35}{64} c\, e_o^3 + \left(- \frac{5}{48} e_o + \frac{7}{16} c\, e_o - \frac{51}{192} e_o^3 + \frac{35}{32} c\, e_o^3\right) \cos^2 i_{oo}$$

$$+ \left(+ \frac{e_o}{12} + \frac{27}{128} e_o^3 - \frac{35}{64} c\, e_o^3\right) \cos^4 i_{oo}\Big] \cos(3\phi + \omega)$$

$$+ \Big[\frac{11}{384} e_o^3 - \frac{9}{64} c\, e_o^3 + \left(- \frac{73}{192} e_o^3 + \frac{99}{32} c\, e_o^3\right) \cos^2 i_{oo}$$

$$+ \left(\frac{125}{128} e_o^3 - \frac{357}{64} c\, e_o^3\right) \cos^4 i_{oo}\Big] \cos(3\phi - 3\omega) + \Big[\frac{1}{24} - \frac{7}{40} c + \frac{13}{96} e_o^2$$

$$- \frac{7}{10} c\, e_o^2 - \frac{7}{64} c\, e_o^4 + \left(- \frac{1}{6} + \frac{7}{20} c - \frac{7}{16} e_o^2 + \frac{7}{4} c\, e_o^2 - \frac{49}{160} c\, e_o^4\right) \cos^2 i_{oo}$$

$$+ \left(\frac{1}{8} - \frac{7}{40} c + \frac{23}{32} e_o^2 - \frac{21}{20} c\, e_o^2 - \frac{63}{320} c\, e_o^4\right) \cos^4 i_{oo}\Big] \cos 4\phi$$

$$+ [- \frac{e_o^2}{36} + \frac{11}{40} c\, e_o^2 + \frac{1}{20} c\, e_o^4 + (+ \frac{19}{24} e_o^2 - \frac{21}{5} c\, e_o^2 - \frac{4}{5} c\, e_o^4) \cos^2 i_{oo}$$

$$+ (- \frac{11}{8} e_o^2 + \frac{217}{40} c\, e_o^2 + \frac{21}{20} c\, e_o^4) \cos^4 i_{oo}] \cos (4\phi - 2\omega)$$

$$+ \frac{1}{320} c\, e_o^4 [- 3 + 90 \cos^2 i_{oo} + 175 \cos^4 i_{oo}] \cos (4\phi - 4\omega)$$

$$+ [\frac{7}{96} e_o - \frac{7}{20} ce_o + \frac{7}{192} e_o^3 - \frac{49}{160} ce_o^3 + (- \frac{67}{144} e_o + \frac{91}{80} ce_o - \frac{23}{96} e_o^3$$

$$- \frac{21}{20} ce_o^3) \cos^2 i_{oo} + (\frac{35}{72} e_o - \frac{63}{80} ce_o + \frac{49}{192} e_o^3 - \frac{119}{160} ce_o^3) \cos^4 i_{oo}] \cos (5\phi - \omega)$$

$$+ [- \frac{e_o^3}{128} + \frac{21}{320} c\, e_o^3 + (\frac{23}{144} e_o^3 - \frac{51}{40} c\, e_o^3) \cos^2 i_{oo}$$

$$+ (- \frac{43}{128} e_o^3 + \frac{567}{320} c\, e_o^3) \cos^4 i_{oo}] \cos (5\phi - 3\omega) + [\frac{e_o^2}{32} - \frac{19}{80} c\, e_o^2$$

$$+ (- \frac{e_o^2}{4} + c\, e_o^2) \cos^2 i_{oo} + (\frac{29}{96} e_o^2 - \frac{61}{80} c\, e_o^2$$

$$- \frac{23}{160} c\, e_o^4) \cos^4 i_{oo}] \cos (6\phi - 2\omega) + \frac{1}{160} c\, e_o^4 [1 - 24 \cos^2 i_{oo}$$

$$+ 35 \cos^4 i_{oo}] \cos (6\phi - 4\omega) + [+ \frac{e_o^3}{256} - \frac{9}{128} c\, e_o^3 + (- \frac{43}{1152} e_o^3$$

$$+ \frac{21}{64} c\, e_o^3) \cos^2 i_{oo} + (\frac{13}{256} e_o^3 - \frac{37}{128} c\, e_o^3) \cos^4 i_{oo}] \cos (7\phi - 3\omega)$$

$$- \frac{1}{128} c\, e_o^4 \sin^2 i_{oo} (1 - 5 \cos^2 i_{oo}) \cos (8\phi - 4\omega)\}$$

APPENDIX 2

$$E_2(\phi, \omega) = \frac{p^2}{\cos^2 i_{oo}} \{ - (\frac{\partial u_1}{\partial \phi})^2 - 2 \frac{\partial u_1}{\partial \phi} \frac{\partial u_o}{\partial \tilde{\phi}} - (\frac{\partial u_o}{\partial \tilde{\phi}})^2 - 2 \frac{\partial u_2}{\partial \phi} \frac{\partial u_o}{\partial \phi} - 2 \frac{\partial u_1}{\partial \tilde{\phi}} \frac{\partial u_o}{\partial \phi}$$

$$- 2 \frac{\sin 2i_{oo}}{p^2} (E + u_o) i_2 - 2 \frac{\cos 2i_{oo}}{p^2} (E + u_o) i_1^2 - 2 \frac{\sin 2i_{oo}}{p^2} u_1 i_1$$

$$+ 2 (\frac{\cos^2 i_{oo}}{p^2} - u_o) u_2 - u_1^2 + 48 \frac{\cos^5 i_{oo}}{p^4} \sin i_{oo}\, i_1 u_o (E + u_o) \sin^2\phi$$

$$- 8 \frac{\cos^6 i_{oo}}{p^2} (E + 2u_o) u_1 \sin^2\phi - 16 \frac{\cos^3 i_{oo}}{p^2} \sin i_{oo}\, i_1 u_o^3 \sin^2\phi$$

$$+ 12 \frac{\cos^4 i_{oo}}{p^2} u_o^2 u_1 \sin^2\phi - \frac{2}{3} \frac{\sin 2i_{oo}}{p^2} i_1 u_o^3 - \frac{\sin 4i_o}{p^2} i_1 u_o^3 \sin^2\phi$$

$$+ 2 \frac{\cos^2 i_{oo}}{p^2} u_o^2 u_1 (1 - 3 \sin^2 i_{oo} \sin^2\phi)$$

$$+ 12 \frac{\cos^{12} i_{oo}}{p^8} e_o^2 u_o^2 \sin^2(\phi - \omega) \sin^4\phi$$

$$+ 56\, c \frac{\cos^8 i_{oo}}{p^6} \sin^2 i_{oo}\, e_o^2 u_o^3 \sin^2(\phi - \omega) \sin^4\phi$$

$$- 24\, c \frac{\cos^8 i_{oo}}{p^6} e_o^2 u_o^3 \sin^2(\phi - \omega) \sin^2\phi$$

$$- \frac{8}{3} \frac{\cos^6 i_{oo}}{p^4} u_o^4 (\sin^2\phi - 3 \sin^2 i_{oo} \sin^4\phi)$$

$$+ 14\, c \frac{\cos^2 i_{oo}}{p^2} u_o^5 \sin^4 i_{oo} \sin^4\phi - 12\, c \frac{\cos^2 i_{oo}}{p^2} u_o^5 \sin^2 i_{oo} \sin^2\phi$$

$$+ \frac{6}{5}\, c \frac{\cos^2 i_{oo}}{p^2} u_o^5$$

$$- \frac{\cos^6 i_{oo}}{p^8} i_1 \sin 2i_{oo} [(\frac{1}{3} + e_o^2)(1 - 3 \cos^2 i_{oo})$$

$$+ \frac{e_o^2}{4} (1 + \cos^2 i_{oo}) \cos 2\omega]$$

$$- 4 \frac{\cos^{12} i_{oo}}{p^{10}} u_o \sin^2\phi [(\frac{1}{3} + e_o^2)(1 - 3 \cos^2 i_{oo})$$

$$+ \frac{e_o^2}{4} (1 + \cos^2 i_{oo}) \cos 2\omega]$$

$$+ 2 \frac{\partial u_o}{\partial \phi} \frac{\sin 2i_{oo}}{p^2} C_2 \sin 2\omega[1 + e_o \cos (\phi - \omega)]$$

$$- 2 \frac{\partial u_o}{\partial \phi} \frac{\cos^2 i_{oo}}{p^2} B_2 \sin 2\omega \cos(\phi - \omega)\}$$

APPENDIX 3

Fourier Expansion of the Function

$$\frac{1}{[1 + e\cos(\phi - \omega)]^n}$$

$$\frac{1}{[1 + e\cos(\phi - \omega)]^n} = \frac{b_{no}}{2} + \sum_{k=1}^{\infty} b_{nk} \cos k(\phi - \omega)$$

where, for k = 0, 1, 2 ...

$$b_{1k} = \frac{2}{\sqrt{1 - e^2}} \left(\frac{\sqrt{1 - e^2} - 1}{e}\right)^k$$

$$b_{2k} = \frac{2}{(1 - e^2)^{3/2}} \left(\frac{\sqrt{1 - e^2} - 1}{e}\right)^k (1 + k\sqrt{1 - e^2})$$

$$b_{3k} = \frac{2}{(1 - e^2)^{5/2}} \left(\frac{\sqrt{1 - e^2} - 1}{e}\right)^k \left(1 + \frac{e^2}{2} + \frac{3}{2} k \sqrt{1 - e^2} + \frac{k^2}{2} (1 - e^2)\right)$$

$$b_{4k} = \frac{2}{(1 - e^2)^{7/2}} \left(\frac{\sqrt{1 - e^2} - 1}{e}\right)^k \Big[1 + \frac{3}{2} e^2 + \frac{11 + 4e^2}{6} k\sqrt{1 - e^2}$$

$$+ k^2(1 - e^2) + \frac{k^3}{6} (1 - e^2)^{3/2}\Big]$$

APPENDIX 4

$$T_{10}^*(e_o^*,i_{oo}^*,\varepsilon) = [(1 + \frac{25}{12}y + \frac{35}{24}y^2 + \frac{5}{12}y^3 + \frac{1}{24}y^4)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(o)},\varepsilon)$$

$$- (4y + \frac{13}{3}y^2 + \frac{3}{2}y^3 + \frac{1}{6}y^4)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(1)},\varepsilon)$$

$$+ (3y + \frac{19}{4}y^2 + 2y^3 + \frac{1}{4}y^4)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(2)},\varepsilon)$$

$$- (\frac{4}{3}y + \frac{7}{3}y^2 + \frac{7}{6}y^3 + \frac{1}{6}y^4)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(3)},\varepsilon)$$

$$+ (\frac{1}{4}y + \frac{11}{24}y^2 + \frac{1}{4}y^3 + \frac{1}{24}y^4)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(4)},\varepsilon)]$$

$$T_{11}^*(e_o^*,i_{oo}^*,\varepsilon) = x[-(\frac{25}{12} + \frac{35}{12}y + \frac{5}{4}y^2 + \frac{1}{6}y^3)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(o)},\varepsilon)$$

$$+ (4 + \frac{26}{3}y + \frac{9}{2}y^2 + \frac{2}{3}y^3)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(1)},\varepsilon)$$

$$- (3 + \frac{19}{2}y + 6y^2 + y^3)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(2)},\varepsilon)$$

$$+ (\frac{4}{3} + \frac{14}{3}y + \frac{7}{2}y^2 + \frac{2}{3}y^3)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(3)},\varepsilon)$$

$$- (\frac{1}{4} + \frac{11}{12}y + \frac{3}{4}y^2 + \frac{1}{6}y^3)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(4)},\varepsilon)$$

$$T_{12}^*(e_o^*,i_{oo}^*,\varepsilon) = x^2[(\frac{35}{24} + \frac{5}{4} y + \frac{1}{4} y^2)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(o)},\varepsilon)$$

$$- (\frac{13}{3} + \frac{9}{2} y + y^2)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(1)},\varepsilon)$$

$$+ (\frac{19}{4} + 6y + \frac{3}{2} y^2)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(2)},\varepsilon)$$

$$- (\frac{7}{3} + \frac{7}{2} y + y^2)\ T_1^*\ (e_o^*,i_{oo}^*,\omega_c^{(3)},\varepsilon)$$

$$+ (\frac{11}{24} + \frac{3}{4} y + \frac{1}{4} y^2)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(4)},\varepsilon)]$$

$$T_{13}^*(e_o^*,i_{oo}^*,\varepsilon) = x^3[- (\frac{5}{12} + \frac{1}{6} y)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(o)},\varepsilon) + (\frac{3}{2} + \frac{2}{3} y)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(1)},\varepsilon)$$

$$- (2 + y)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(2)},\varepsilon) + (\frac{7}{6} + \frac{2}{3} y)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(3)},\varepsilon)$$

$$- (\frac{1}{4} + \frac{1}{6} y)\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(4)},\varepsilon)]$$

$$T_{14}^*(e_o^*,i_{oo}^*,\varepsilon) = x^4[\frac{1}{24}\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(o)},\varepsilon) - \frac{1}{6}\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(1)},\varepsilon)$$

$$+ \frac{1}{4}\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(2)},\varepsilon) - \frac{1}{6}\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(3)},\varepsilon)$$

$$+ \frac{1}{24}\ T_1^*(e_o^*,i_{oo}^*,\omega_c^{(4)},\varepsilon)]$$

where

$$x = \begin{cases} \dfrac{4}{\sqrt{\bar{\kappa}_o - \kappa_1}}, & \text{if } -\kappa_1 < \bar{\kappa}_o < +\kappa_1 \\[2ex] \dfrac{4}{\sqrt{\bar{\kappa}_o + \kappa_1} - \sqrt{\bar{\kappa}_o - \kappa_1}}, & \text{if } \bar{\kappa}_o \geqq \kappa_1 \end{cases}$$

$$y = \begin{cases} 0 & , \text{if } -\kappa_1 < \bar{\kappa}_o < +\kappa_1 \\[2ex] 4\,\dfrac{\sqrt{\bar{\kappa}_o - \kappa_1}}{\sqrt{\bar{\kappa}_o + \kappa_1} - \sqrt{\bar{\kappa}_o - \kappa_1}}, & \text{if } \bar{\kappa}_o \geqq \kappa_1 \end{cases}$$

APPENDIX 5

$$T_2(i_{oo},e_o,\omega,\phi) = -\frac{1}{2}\frac{\cos i_{oo}}{pu_o^2} i_1^2 - \frac{\sin i_{oo}}{pu_o^2} i_2 + 2\frac{\sin i_{oo}}{pu_o^3} i_1 u_1$$

$$+ 3\frac{\cos i_{oo}}{pu_o^4} u_1^2 - 2\frac{\cos i_{oo}}{pu_o^3} u_2 + 10\frac{\cos^4 i_o \sin i_{oo}}{p^3 u_o} i_1 \sin^2\phi$$

$$+ 2\frac{\cos^5 i_{oo}}{p^3 u_o^2} u_1 \sin^2\phi + 4\frac{\cos^9 i_{oo}}{p^5} \sin^4\phi$$

$$+ 28\frac{cu_o}{p^3} \cos^5 i_{oo} \sin^2 i_{oo} \sin^4\phi$$

$$- 12\frac{cu_o}{p^3} \cos^5 i_{oo} \sin^2\phi$$

$$+ \frac{3}{2}(-2E)^{-5/2}[n_o + n_2 \cos 2\omega + n_4 \cos 4\omega]$$

$$+ \frac{3}{8}(-2E)^{-7/2}\frac{\cos^{12} i_{oo}}{p^{12}}[(\frac{1}{3} + e_o)^2(1 - 3\cos^2 i_{oo})^2$$

$$+ \frac{e_o^2}{2}(\frac{1}{3} + e_o^2)(1 - 2\cos^2 i_{oo} - 3\cos^4 i_{oo}) \cos 2\omega$$

$$+ \frac{e_o^4}{32}(1 + \cos^2 i_{oo})^2(1 + \cos 4\omega)]$$

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APPENDIX 6

$$\sigma = \frac{\cos i_o}{p} \sum_{k=1}^{\infty} b_{3k}\left[\frac{1}{k}(1 + \frac{e^2}{2} - (4 + 3e^2) \cos^2 i_o) \sin k(\phi - \omega)\right.$$

$$- \frac{e^2}{8k} (2 - 5 \cos^2 i_o) \sin\{k\phi - (k - 2)\omega\} - \frac{e}{k + 1} \cos^2 i_o \sin\{(k + 1)(\phi - \omega)\}$$

$$- \frac{e^2}{8k} (2 - 5 \cos^2 i_o) \sin\{k\phi - (k + 2)\omega\} - \frac{e}{8} \frac{(3 - 7 \cos^2 i_o)}{k + 1} \sin\{(k+1)\phi-(k-1)\omega\}$$

$$- \frac{1}{12(k + 2)} (1 - e^2 - 7 \cos^2 i_o) \sin\{(k + 2)\phi- k\omega\}$$

$$- \frac{e^2}{12} \frac{(1 - 6 \cos^2 i_o)}{(k + 2)} \sin\{(k + 2)(\phi - \omega)\}$$

$$\left. + \frac{e}{4} \frac{\cos^2 i_o}{k + 3} \sin\{(k + 3)\phi - (k + 1)\omega\} + \frac{e^2}{24} \frac{\cos^2 i_o}{(k + 4)} \sin\{(k + 4)\phi - (k + 2)\omega\}\right]$$

$$- \frac{\cos i_o}{p} \sum_{k=2}^{\infty} b_{3k}\left[\frac{e \cos^2 i_o}{k - 1} \sin(k - 1)(\phi - \omega)\right.$$

$$\left. + \frac{e}{8} \frac{(3 - 7 \cos^2 i_o)}{(k - 1)} \sin\{(k - 1)\phi - (k + 1)\omega\}\right]$$

$$- \frac{\cos i_o}{p} \sum_{\substack{k=1 \\ k\neq 2}}^{\infty} \frac{1}{12} \frac{b_{3k}}{(k - 2)} \left[- (1 - e^2 - 7 \cos^2 i_o) \sin\{(k - 2)\phi - k\omega\}\right.$$

$$\left. + e^2(1 - 6 \cos^2 i_o) \sin\{(k - 2)(\phi - \omega)\}\right]$$

$$+ \frac{\cos i_o}{p} \sum_{\substack{k=1 \\ k\neq 3}}^{\infty} \frac{1}{4} \frac{b_{3k}}{(k - 3)} e \cos^2 i_o \sin\{(k - 3)\phi - (k - 1)\omega\}$$

$$+ \frac{\cos i_o}{p} \sum_{\substack{k=1 \\ k\neq 4}}^{\infty} \frac{1}{24} \frac{b_{3k}}{(k - 4)} e^2 \cos^2 i_o \sin\{(k - 4)\phi - (k - 2)\omega\}$$

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