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STUDY ON DETERMINING STABILITY DOMAINS FOR NONLINEAR DYNAMICAL SYSTEMS

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STUDY ON DETERMINING STABILITY DOMAINS FOR NONLINEAR DYNAMICAL SYSTEMS

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ABSTRACT

This memorandum is the first quarterly progress report prepared for the Astrodynamics and Guidance Theory Division, Aero-Astrodynamics Laboratory, NASA George C. Marshall Space Flight Center under Contract NAS 8-20306, "Study on Determining Stability Domains for Nonlinear Dynamical Systems." It reports the work performed during the period 1 May 1966 to 1 August 1966.

The procedure for formulating the problem of estimating the domain of attraction of an equilibrium solution of a nonlinear dynamical system as two extremal problems is briefly described. The numerical algorithm used to solve these extremal problems is outlined, and the qualitative aspects of the computational results are described. Some remaining problems are described and the plans for future work are stated.

TABLE OF CONTENTS

INTRODUCTION

During the period covered by this report, our effort has been devoted toward developing the procedure, described in Ref. 1, for estimating the domain of attraction of equilibrium motions of nonlinear dynamical systems.

REVIEW OF PROBLEM FORMULATION

Briefly, the procedure described in Ref. 1 is based upon choosing the quadratic form Liapunov function that yields the largest estimate of the domain of attraction for the given motion and system of equations. In particular, assume that the system is of the form

$$\dot{x} = Ax + f(x) \quad , \quad x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad , \quad f(0) = 0 \quad , \quad A \text{ stable} ; \tag{1}$$

i.e., it is n-dimensional, autonomous, quasi-linear, and stable. As a result of these assumptions the Liapunov function V,

$$V(x) = x^T Px \quad , \quad P > 0 \, , \tag{2}$$

will have as its time derivative

$$\dot{V}(x) = - x^T Qx + 2x^T Pf(x) \, , \tag{3}$$

where

$$- Q = A^T P + PA \, . \tag{4}$$

If Q is chosen to be positive definite, then P will be positive definite and $\dot{V}$ will be negative in the region

$$D: \quad \left(x \,\middle|\, \frac{\|f(x)\|}{\|x\|} < \frac{\lambda^{min}(Q)}{2\lambda^{max}(P)} \right) \, , \tag{5}$$

where $\lambda^{min}(Q)$ and $\lambda^{max}(P)$ are, respectively, the minimum eigenvalue of Q and the maximum eigenvalue of P.

According to LaSalle and Lefschetz (Ref. 2) an estimate of the domain of attraction of the equilibrium solution $x(t) = 0$ of Eq. (1)* is given by

$$\Omega_\ell: \quad \left(x \,|\, V(x) < \ell \;,\; \dot{V}(x) < 0\right) \tag{6}$$

if Ω_ℓ is bounded. Thus, relative to this choice of $V(x)$, i.e., the choice of Q, the best estimate is obtained by defining the set E as

$$E: \quad \left(x \,|\, \dot{V}(x) = 0 \;,\; x \neq 0\right) \tag{7}$$

and then choosing ℓ to be

$$\ell = \min_{x \,\epsilon\, E} V(x) \;. \tag{8}$$

Then, the optimal choice of Q from the set of all positive definite $n \times n$ matrices, denoted Q^0, is defined by

$$J(Q^0) = \max_{Q > 0} J(Q) \;, \tag{9}$$

where

$$J(Q) = \ell^{n/2} \left(\prod_{i=1}^{n} \lambda_i(P)\right)^{-\frac{1}{2}} = \left(\frac{\ell^n}{\det P}\right)^{\frac{1}{2}} \;. \tag{10}$$

This definition of Q^0 [Eq. (9)] will yield the best estimate in terms of enclosed volume, of the domain of attraction under the constraint that $V(x)$ be a positive definite quadratic form.

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*N.B. Hereafter it will be understood that we are concerned with the equilibrium solution $x(t) = 0$ of Eq. (1).

Thus an optimal estimate of the domain of attraction with respect to quadratic form Liapunov functions can be obtained via a numerical algorithm that solves Eqs. (8) and (9). Our efforts during the past quarter have been concentrated on the development of such an algorithm.

DEVELOPMENT OF THE NUMERICAL ALGORITHM

To solve the constrained minimum problem we have formulated an unconstrained minimum problem by using the penalty function technique; i.e., replace Eq. (8) by

$$\ell = \min_{x} \left\{ V(x) + k\dot{V}^2(x) \|x\|^{-m} \right\} , \tag{11}$$

where $k > 0$ is chosen to assure satisfaction of the constraint $\dot{V}(x) = 0$ to some prescribed accuracy, and m is chosen to be the least positive even integer such that

$$\lim_{x = 0} \dot{V}^2(x) \|x\|^{-m} = \infty . \tag{12}$$

The term $\|x\|^{-m}$ was originally introduced to exclude the trivial solution $x = 0$, which is the global minimum of the problem for $m = 0$; however, this term also modifies the function to be minimized for large $\|x\|$ in such a way as to de-emphasize the penalty term. In order to avoid this undesirable effect we formulated another unconstrained problem, viz,

$$\ell = \min_{x} \left\{ V(x) + k\dot{V}^2(x) g(x) \right\} , \tag{13}$$

where $g(x)$ was chosen to be

$$g(x) = (1 + c\|x\|^{-m}) . \tag{14}$$

Again, m is chosen as above, and $c > 0$ is chosen to appropriately limit the region within which $g(x)$ materially affects the penalty term of Eq. (13).

Equation (9) can also be reformulated as a more tractable constrained problem as follows. As is well known, the canonical form for the set of positive definite symmetric matrices is the diagonal matrix with positive eigenvalues. Thus, we form the arbitrary positive definite matrix Q via

$$Q = R^T \Lambda R \ , \tag{15}$$

where R is an arbitrary unitary transformation; i.e.,

$$R^T R = I \ , \tag{16}$$

$$\Lambda = \begin{pmatrix} \lambda_1(Q) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(Q) \end{pmatrix} , \tag{17}$$

and $\lambda_i(Q) > 0$ for $i = 1, 2, \ldots, n$. (Usually $\lambda_1(Q)$ will be normalized to unity.) This formulation obviates the necessity to apply the Sylvester criterion to Q at every iteration. The problem of generating R is readily resolved for $n = 2$, viz,

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{18}$$

We conjecture that for $n > 2$, R can be represented as a product of rotation matrices, i.e.,

$$R = \prod_{i=1}^{n} R_i \ , \tag{19}$$

where

$$R_i = \begin{array}{cc} & i \\ & \vdots \\ i\cdots & \left(\begin{array}{ccc|cc|ccc} 1 & & 0 & \vdots & 0 & & 0 & \\ & \ddots & & \vdots & & & & \\ 0 & & 1 & \vdots & & & & \\ \hline \cdots & \cdots & \cdots & \cos\theta_i & \sin\theta_i & & & \\ & 0 & & -\sin\theta_i & \cos\theta_i & & 0 & \\ \hline & 0 & & \multicolumn{2}{c|}{0} & 1 & & 0 \\ & & & & & & \ddots & \\ & & & & & 0 & & 1 \end{array}\right) \end{array} . \quad (20)$$

This conjecture is based upon the consideration that in essence we are attempting to scale and rotate an n-dimensional ellipsoid to fit the domain of attraction in some sense, such that all initial points within the ellipsoid produce trajectories that never leave it. That is, we are attempting to tailor the ellipsoid to the domain of attraction and the "flow" in the state space.

NUMERICAL ALGORITHM FOR MINIMUM PROBLEMS

The numerical algorithm being used to compute solutions to Eq. (11) or Eq. (13), and Eq. (9), is being developed at Grumman by McGill and Taylor and is based upon the work of Davidon (Ref. 3) and Fletcher and Powell (Ref. 4). The algorithm is based upon a modified gradient search concept and proceeds as follows.

To find the minimum over all x of $f(x)$, where $x^T = (x_1, \ldots, x_n)$ and $f(x)$ is a scalar function, choose an initial point x_0, and an arbitrary $n \times n$ positive definite symmetric matrix H_0 (e.g., the identity matrix). Then, let

$$s_k = - H_k f_k' \; , \tag{21}$$

where

$$f_k' = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)^T , \tag{22}$$

and find $\alpha_k > 0$ such that $f(x_k + \alpha_k s_k)$ is minimum with respect to α_k. Now, let

$$x_{k+1} = x_k + \alpha_k s_k \; , \tag{23}$$

and compute $f(x_{k+1})$ and $f'(x_{k+1})$. Define

$$y_k = f_{k+1}' - f_k' \; , \tag{24}$$

and then compute H_{k+1} as follows

$$H_{k+1} = H_k + A_k + B_k \; , \tag{25}$$

where

$$A_k = (\alpha_k s_k^T y_k)^{-1} \alpha_k^2 s_k s_k^T \tag{26}$$

and

$$B_k = -(y_k^T H_k y_k)^{-1} H_k y_k y_k^T H_k . \tag{27}$$

This procedure is repeated until

$$\|x_{k+1} - x_k\| < \epsilon , \; 0 < \epsilon \ll 1 . \tag{28}$$

RESULTS OF NUMERICAL EXPERIMENTS

The numerical experiments to date have dealt primarily with the solution of Eqs. (11) and (13), via the algorithm just described, for the damped Duffing equation, viz,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= - x_2 - x_1 + 0.04x_1^3 \ . \end{aligned} \tag{29}$$

Although this system of equations is not directly related to the booster guidance stability problem, it is an example for which ℓ can be obtained analytically, and moreover, Q^0 can be calculated analytically (see Ref. 1). Thus it is a good example for determining the accuracy of the numerical results. Because this example is not directly related to the goal of this study, the qualitative results are of more importance than the quantitative results, and the latter will be omitted from this discussion.

The functions that are to be minimized in Eqs. (11) and (13) have, for this example, four local minima; two are introduced by the modification that removes the global minimum, and two are the sought solutions to the estimation problem (the problem has point symmetry about the origin). As a result, the value computed by the minimization algorithm depends on the initial search point. Thus, although the known global minimum has been removed, two unknown local minima have been introduced, and the sensitivity of the solution to the initial point has not really been reduced. This problem is even more acute for the Van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= - x_2(1 - x_1^2) - x_1 \end{aligned} \tag{30}$$

because the function to be minimized has one global minimum and four local minima for $m = 0$. Thus a problem requiring further consideration is that of the existence of many local minima and the resulting dependence of the solution upon the initial search point.

As indicated in Ref. 1, the algorithm did not converge satisfactorily in the neighborhood of the optimal value of Q. This is attributed to the fact that

$$Q^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{31}$$

and is thus semidefinite; i.e., the optimal value is on the boundary of the allowable set of Q matrices. We have made changes in the logic in our program, which has enabled us to compute ℓ successfully for Q matrices close to Q^0; however, this problem may arise again when we try to compute Q^0 numerically.

We have programmed and are debugging a program which will compute Q^0 and ℓ for the Duffing equation; i.e., it determines ℓ for an initial Q and then iterates on the λ and θ of Eqs. (17) and (18) until $\left(J(Q)\right)^{-1}$ is minimized. The program has successfully computed to the boundary of the allowed set of Q's, but has as yet not been able to follow this boundary to Q^0. Enabling modifications are now in process.

PLANS FOR FUTURE WORK

In view of the results obtained to date together with discussions with Mr. C. C. Dearman, Jr. of NASA Huntsville, we plan to focus our efforts on the following areas:

1. Devising a procedure for computing an initial search point on the $\dot{V}(x) = 0$ constraint in order to eliminate the sensitivity to the initial search point.

2. Reformulating the problem for determining ℓ such that the desired value occurs as the global minimum of the function to be minimized.

3. Formulating examples that are representative of the booster guidance stability problem.

4. Developing techniques for estimating the temporal behavior of the function $V(x)$ so that relations between the initial state error $\|x(t_0)\|$ and the error at some later time, $\|x(t)\|$ $t_0 < t < \infty$, can be drawn. (Note that our present results are for infinite operating periods, while booster problems always concern finite operating periods.)

5. Proving our conjecture concerning the representation of the unitary transformation, Eqs. (19) and (20), and developing an efficient procedure for solving the Liapunov equation, Eq. (4).