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SUMMARY REPORT

Azimuth Axis Optical Alignment System

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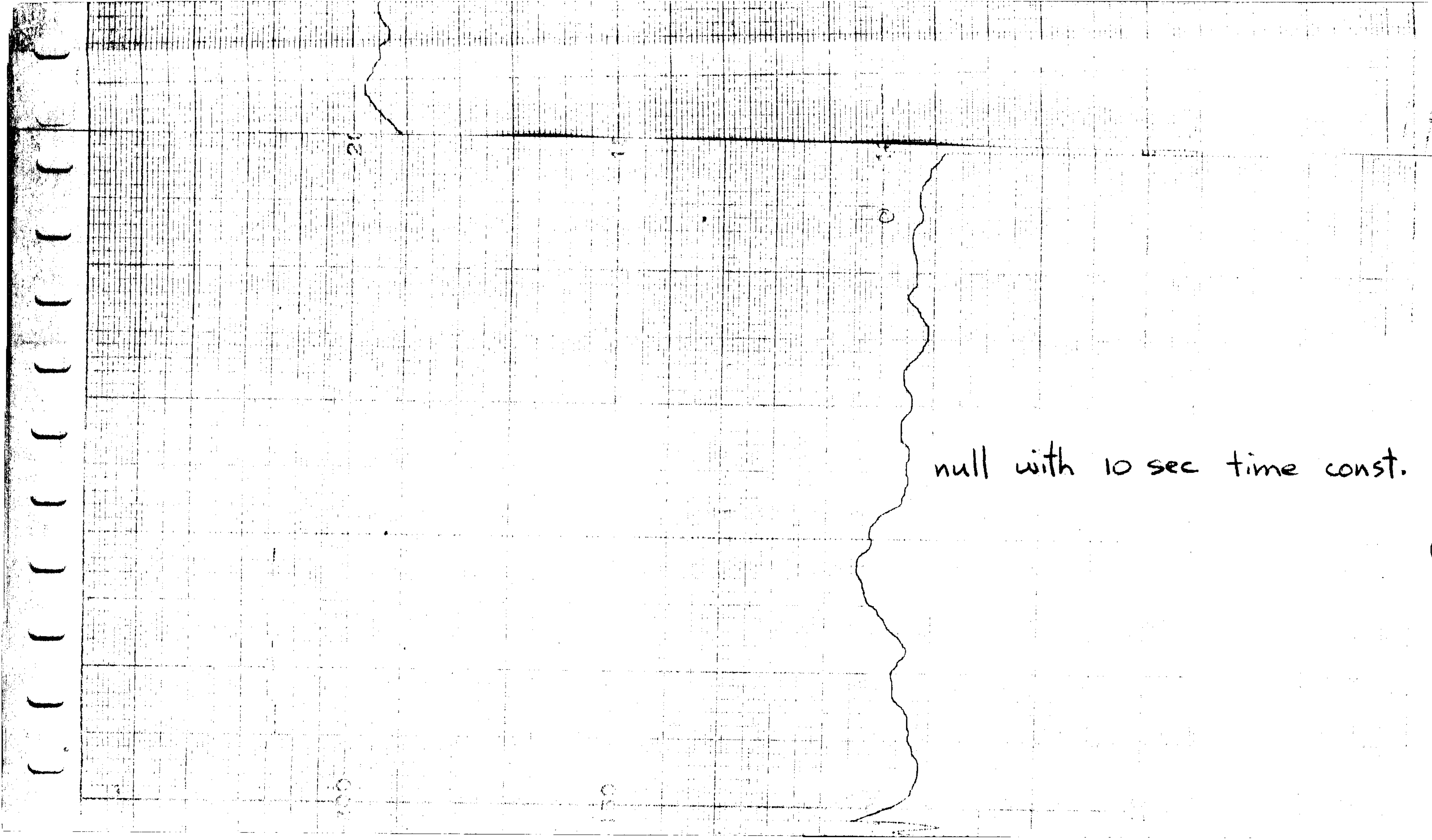
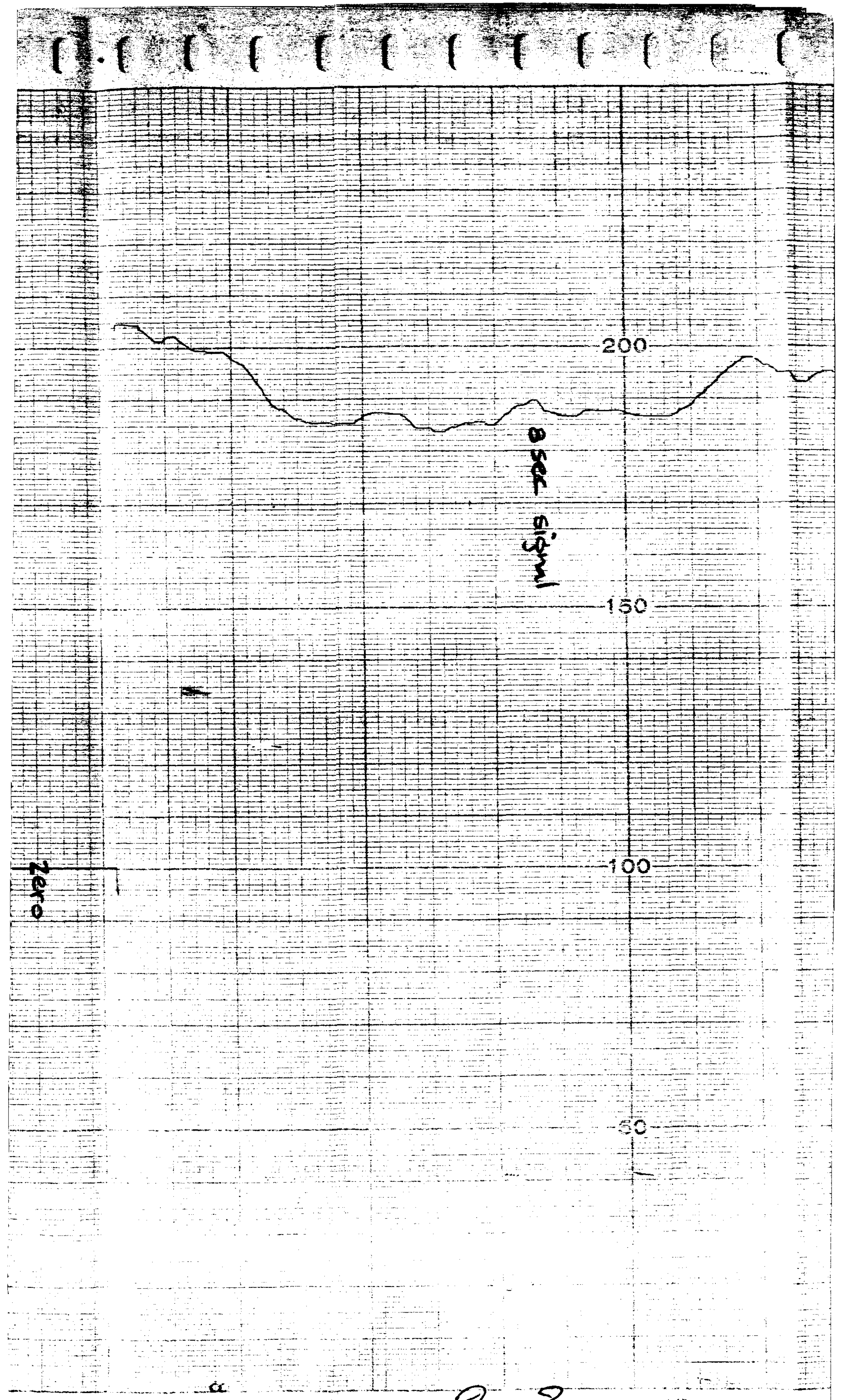
I. TECHNICAL FINDINGS

The Azimuth Axis Optical Alignment System was subjected to intensive testing at GM Defense Research Laboratories prior to delivery to NASA. The system was tested under a variety of weather conditions and at different ranges, up to 1000 feet. The signal to noise ratio at 1000 feet is illustrated by the recording shown in Figure 1. The repeatability of null, or accuracy, was found to be within plus or minus one second of arc. The ability of the system to accept translations of the vehicle unit was excellent. Translations of plus or minus 2-1/2 feet did not cause loss of acquisition or errors greater than plus or minus one second of arc. The instantaneous acquisition angle was found to be greater than plus and minus one full degree, as shown by Figure 2. The error signal linearity about null can also be seen in Figure 2. The maximum scale factor available with the electronics supplied is greater than 150 millivolts per arc second at the 1000 feet range.

II. CONCLUSIONS AND RECOMMENDATIONS

The Azimuth Axis Optical Alignment System operates on an entirely different principle than autocollimators. It does not use a geometrical image reflected from a plane mirror, but phase information imposed on the returning beam of light. A retroreflector, rather than a plane mirror, is used to return the transmitted beam. Neither angular nor translational movements of the target over a great range affect acquisition. Because of this, the Azimuth Axis Optical Alignment System can do things heretofore impossible for an optical system. For instance, a plus or minus one degree acquisition angle at 1000 feet with a 1-1/2 inch target mirror would require a 35 foot aperture standard autocollimator. The Azimuth Axis Optical Alignment System has a mere 1.4 inch receiver aperture, and yet it has greater than a plus or minus one degree acquisition angle.


null with 10 sec time const.



2-2

figure 1

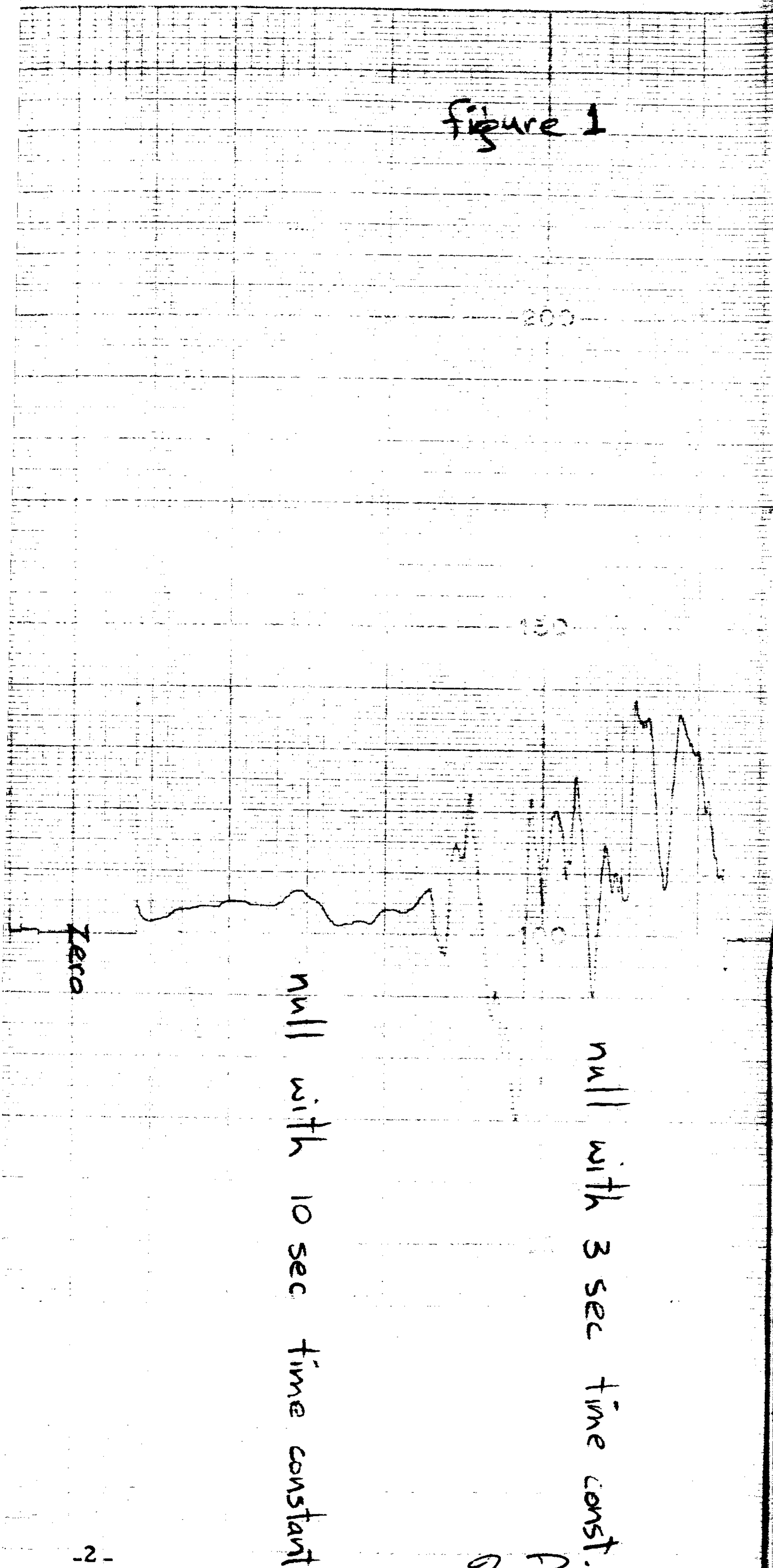
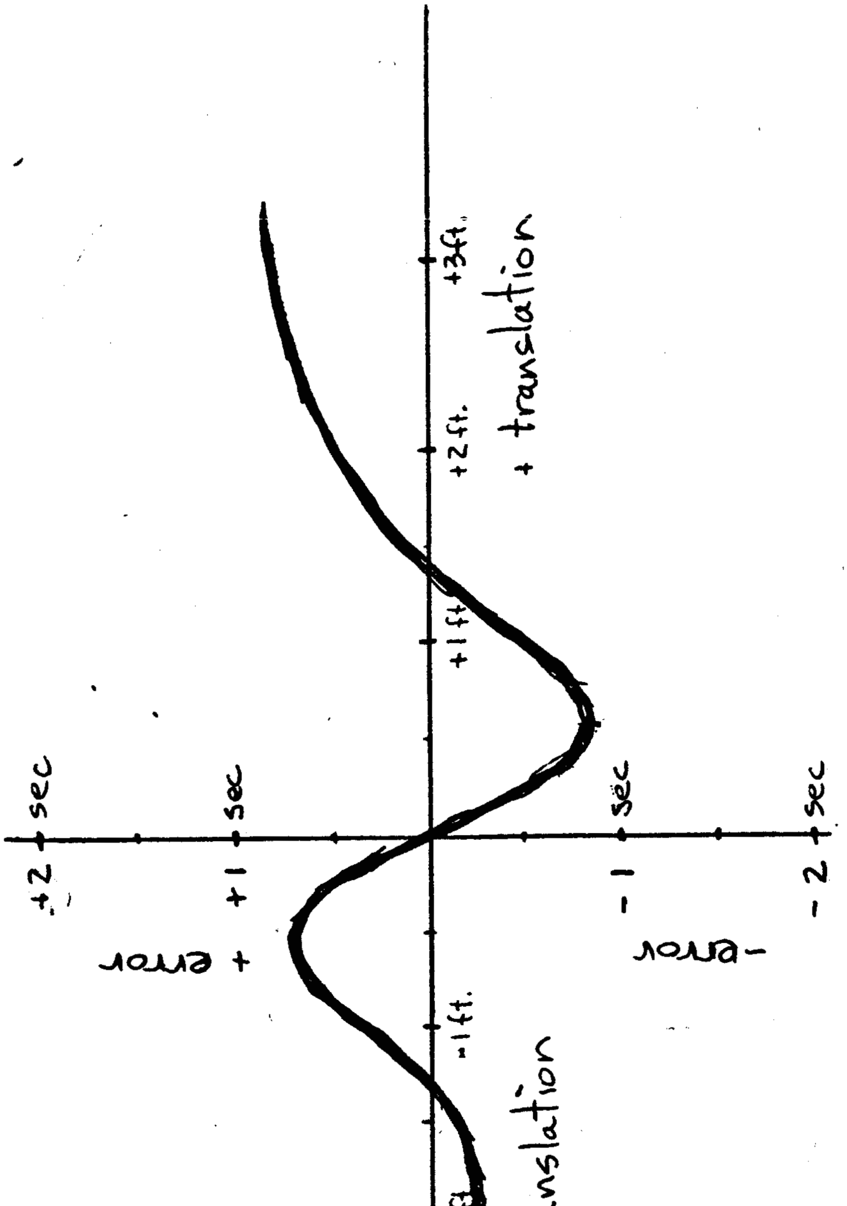

200
150
100
Zero
null with 10 sec time constant.
null with 3 sec time const.

figure 2



Error Signal VS Translation

There are, however, difficulties remaining. The signal to noise ratio experimentally determined for the 1000 feet range was somewhat lower than anticipated. The losses have been more or less accounted for as being due to high reflection losses, slight misalignment of the receiver elements, and an imperfect retroreflector. The initial alignment and adjustment of the Savart plate elements were very difficult and time consuming, and leaves something to be desired. The half-wave plates of mica are less than ideal retarders. Finally, the Pockels cell is difficult to use in practice.

The above mentioned problems are not without solutions. The signal level could be increased by anti-reflection coatings, more precise alignment, a better retroreflector, and a larger aperture on transmitter and receiver. The experience and information gained during assembly of the Savart plates could be used to design and build much better Savart plate assemblies. The half-wave plates could be made of crystal quartz, rather than mica. The Pockels cell could be made of a non-birefringent crystal like cuprous chloride instead of ADP or KDP. These steps could easily result in a better, less costly Azimuth Axis Optical Alignment System.

III. DESIGN PRINCIPLES

The Azimuth Axis Optical Alignment System monitors and measures the attitude or angular position of a remote object about the azimuth axis. The separation between the remote unit and the measuring unit can be as great as 1000 feet. The accuracy of angular information transfer is of the order of plus and minus one second of arc. This device has an instantaneous acquisition angle in excess of plus and minus one full degree. Lateral translation of the remote unit can be as great as five feet without affecting the output or sensitivity of the system.

Optical Principles

The angular attitude of the remote object is sensed by mounting upon it a polarization filter and birefringent element or plate. The phase information imposed on a beam of polarized light by passage through the birefringent element is then analyzed to determine the angle of passage of the beam through the plate. Consequently, the angular position of the plate about those axes which are normal to the line of sight can be determined.

A second and identical birefringent element is used to perform the analysis of this phase information. The two birefringent plates are oriented in such a way that when they are parallel (and only then) will there be a total cancellation of the phase difference imposed by the first element. If, after passage through both plates, there remains some measure of ellipticity, or retardation, it is known that the plates are not parallel.

Plane polarized light passes through a birefringent element and exits as two orthogonal components with a relative retardation proportional to the angle of passage. This light then passes through a second and identical element whose fast axis is parallel to the slow axis of the first. After passage through the second element, the previously unretarded component is retarded an amount just sufficient for the two components to exit in phase. If the second birefringent element is not parallel to the first, they will not emerge in phase. This fact is discerned as an error signal.

To have a system which is totally insensitive to changes in elevation angle while sensing azimuth, it is necessary to have birefringent plates which cause retardation as a function of angle of passage about one axis only. The retardation characteristics as a function of angle of passage of any given birefringent element can be directly viewed by placing the element between crossed polarizers. The lined patterns, or interference fringes, which are seen represent families of passage angles which provide equal retardation.

The fringe pattern which will provide sensitivity about only one axis consists of a series of parallel and straight lines. Such a pattern is closely approximated by the Savart plate.

The first order approximate analysis for Savart plate retardation does indeed suggest a straight and parallel isochromatic pattern. The exact analysis, including second-order terms for Savart plate retardation, however, shows that the fringes are not exactly straight and parallel. A modified Savart plate has been designed and is used in the Azimuth Axis Optical Alignment System to give straight and parallel fringes and to eliminate elevation sensitivity.

Optical Sub-System

The Azimuth Axis Optical Alignment System optical sub-system consists of two basic parts: the ground unit, which is a transmitter-receiver; and the vehicle unit, which is a small, passive retroreflector unit. Processing of the electrical signals will be covered in the Electronic Sub-System section.

Ground Unit

The ground unit consists of a projector unit, a receiver unit, and a sensing unit.

The projector unit sends a beam of tungsten light with a divergence angle of about one degree out towards the vehicle unit. A mechanical chopper amplitude modulates this light at 2000 cycles per second to tag it for background discrimination purposes. The projector is essentially a Newtonian telescope with a large diagonal mirror. A hole in the diagonal permits entry of the returned beam into the receiver unit. The light projector consists of a six-inch diameter concave mirror of 16 inch focal length, a 45-degree elliptical mirror with a 1.3 inch diameter hole and a 150-watt Quartz-Iodine lamp. This projector is mounted concentric to the receiver unit. The projector is mounted in such a way as to be mechanically isolated from the Savart plate mount. Dimensional instabilities caused by thermal gradients, etc. do not, therefore, affect the azimuth setting.

The receiver unit receives and analyzes the phase information imposed on the light by the two sensing units, the one in the vehicle unit, and the other in the ground unit. The receiver unit consists of: an electro-optic light modulator, driven at 400 cycles per second, which modulates the polarization form of the returned signal; an analyzing polarizer; an objective lens which concentrates the returned beam to a small spot; a solid state silicon photodetector; and a reflex viewer for aiming the projector-receiver.

The sensing unit consists of a modified Savart plate mounted in a holder, which can be aligned in azimuth. The Savart plate and its holder has an azimuth adjustment range of ± 1.5 degrees by means of opposed tangent screws. An elevation adjustment of $\pm 5^{o}$ is also possible, although use of this adjustment requires requalification of the porro prism to the Savart plates' sensitive axis. The sensitive axis of the ground unit is rotated in the azimuth plane by means of two opposed micrometer movements which bear against a stainless steel ball on the end of a radius arm. This adjustment is self-locking when the micrometer movements tighten against one another. The azimuth direction of the sensitive axis is qualified to a porro prism which is attached directly to the azimuth pivoting axis. The angular position of this porro prism can be sensed with an autocollimator or a theodolite by means of auto-reflection or auto-collimation.

Vehicle Unit

The vehicle unit consists of: a corner reflector which redirects the incident beam back towards the ground unit; a polarizer which plane polarizes the light; and a modified Savart plate which impresses phase angular information on the polarized beam as described previously. The vehicle unit is completely passive.

Electronic Sub-System

The electronic sub-system for the Azimuth Axis Optical Alignment System consists of three basic groups of electronic hardware. A signal processor synchronously detects both the tagging frequency signal (2KHz) and the information frequency signal (400 Hz) and displays the error on a readout device. Modulation of the Pockel cell is provided by a high voltage modulator. Power for portions of the signal processor is furnished by two stable power supplies.

Shown in Figure 4 is a block diagram of the Azimuth Axis Optical Alignment electronic sub-system. Each of the three groups will be described in more detail in the following paragraphs, and critical design principles will be included.

Signal Processor

Amplitude modulated light incident on a high sensitivity silicon photodiode yields an analogue signal consisting of the 2 KHz tagging frequency component amplitude modulated by the 400 Hz error information signal.

The electrical equivalent circuit of the detector photodiode is shown schematically in Figure 3

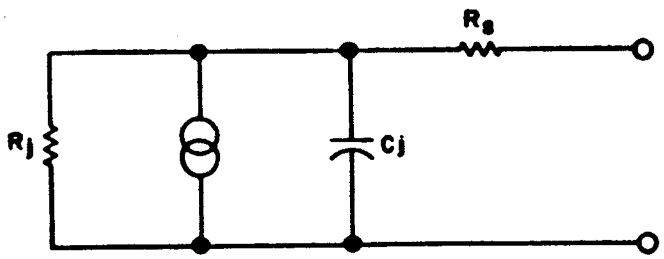


Fig. 3 Electrical equivalent circuit for detector diode

The barrier resistance (R_j) can be determined from the slope of the V-I curve in the non-saturated region. The barrier capacitance (C_j) is voltage dependent and decreases with increasing bias. R_S is the dynamic internal series resistance which is made up of contact, film, and bulk resistances, with the bulk resistance being the dominant factor.

Some additional typical parameter values for the photodiode are listed below:

Rise Time:	t_r =	4×10^{-9} sec at -90v
Fall Time:	t_f =	15×10^{-9} sec at -90v
Sensitivity:	S =	0.25 $\mu a \mu w^{-1}$ at 0.9 micron
Luminous Response: (See Note (a))	=	3,500 μa per lumen or 0.27 μa per footcandle of 2870° K color temp. 1,000 μa per lumen or 0.078 μa per footcandle of pulsed xenon illumination
Area:	A =	0.073 cm^2
Dark Current: (See Note (b))	I_o =	0.2 μa at 10 volts bias at 25° C
Junction Capacitance:	C_j =	30 pf at 10 volts bias (Refer to Fig. 5)
Series Resistance:	R_S =	200 ohms
Figure of Merit:	D^* =	2.7×10^{11} cm $cps^{1/2}$ $watts^{-1}$ at Load Resistance R_L = 1 meg ohm

The diode is operated in the photoconductive mode with a back bias of 1.2 v at 51 nanoamperes, provided by a battery and 1 megohm bias resistor.

At frequencies beyond the 1/f noise, the two primary sources of noise arise from leakage currents and the thermal noise in the internal device resistance, R_S, and the load resistance, R_L.

The noise currents are random in nature and their mean square values may be defined as in (a) and (b) below.

(a) **Shot Noise**

$$\overline{i^2_{N_o}} = 2qI_o\Delta f \quad (1)$$

(b) **Johnson Noise**

$$\overline{i^2_{NR_S}} + \overline{i^2_{NR_L}} = \frac{4KT\Delta f}{R_T} \quad \text{where } R_T = R_S + R_L \quad (2)$$

Substituting the values of the bias current (I_o = 51 nanoamperes) and the bias resistor ($R_T \approx R_L$ = 1 megohm) into the above equations yields the following mean square noise currents.

Shot Noise

$$\overline{i^2_{SN}} = 1.63 \times 10^{-26} \quad (3)$$

Johnson Noise

$$\overline{i^2_{JN}} = 1.7 \times 10^{-26} \quad (4)$$

From the equations (3) and (4) it is seen that the photodiode and its biasing resistor produce approximately an equal amount of Shot and Johnson noise. The N.E.P (noise equivalent power) of a detector is defined as the power necessary to generate a signal current equal to the noise current. Therefore, the sensitivity, S, and the total noise current, i_N, may be combined as shown in equation (5) to yield the N.E.P. of the detector.

$$\text{N.E.P} = i_N/S \text{ watts} \quad (5)$$

By minimizing the N.E.P. of the detector, the detection capabilities of the diode are enhanced. From equations

(2) and (5) it is seen that the N. E. P. can be minimized by increasing the value of the load (bias) resistor, thereby reducing the Johnson noise component and allowing only the Shot noise of the diode itself (equation 1) to predominate. In this system, however, the frequency response required (2.4 KHz) dictates that the load resistance be not greater than 1 megohm lest the low pass structure formed by shunt capacities $\left[C_j + \text{cable}\right]$ and R_L attenuate the 2 KHz tagging frequency signal.

The noise contributed by the detector-load resistor combination may now be calculated as follows:

$$\overline{i^2_{ND}} = \overline{i^2_{SN}} + \overline{i^2_{JN}} \tag{6}$$

$$= 1.63 \times 10^{-26} + 1.7 \times 10^{-26}$$

$$= 3.3 \times 10^{-26} \text{ amps}/\Delta f$$

Therefore, the r. m. s. noise voltage contribution of the diode resistor combination is:

$$E_{ND} = \left[i^2_{ND} \right]^{1/2} R_L \tag{7}$$

$$= 1.8 \times 10^{-7} \text{ volts r.m.s.} /\sqrt{\Delta f}$$

Following the photodiode is a low-noise preamplifier, which utilizes an input field effect transistor stage in a negative feedback configuration which yeilds an input impedance of 1000 megohms and 40 db of stable gain. The preamplifier possesses an equivalent noise resistance of less than 100 kilohms which yields a noise figure of less than 0.42 db.

The noise contribution of the amplifier is given by:

$$E_{NP} = \left[4KT\ fR_{eq} \right]^{1/2} \tag{8}$$
$$= 4.4 \times 10^{-8} \text{ volts rms}/\sqrt{\Delta f}$$

The total rms noise of the input system, consisting of detector diode, load resistor, and equivalent preamplifier Johnson noise, is given by:

$$E_{N_{total}} = \left[\overline{E_{ND}}^{2} + \overline{E_{P}}^{2} \right]^{1/2} \tag{9}$$
$$= 1.85 \times 10^{-7} \text{ volts rms}/\sqrt{\Delta f}$$

For purposes of N. E. P. calculations, this total noise voltage existing at the input can be converted to the equivalent noise current referenced to the 1 megohm load resistor. If this operation is done, an equivalent input rms noise current of 1.85×10^{-13} amps / $\sqrt{\Delta f}$ results.

Since the signal processor is bandlimited to 0.1 Hz at its output, the noise contribution at the input of the detector, load resistor and preamp is 0.6×10^{-7} volts rms. After a gain of 120 decibels in the signal processor, a peak to peak noise voltage of 0.48 volts is seen at the output of the Princeton lock-in amplifier.

After amplification in the preamp, the 400 Hz information signal is recovered by synchronous detection of the 2KHz tagging signal. This phase coherent detection process is realized by using a simple diode ring demodulator as shown in the block diagram of Figure 4 and in the circuit diagram of Figure 5. In order to obtain a reference drive for the ring demodulator, a photovoltaic reference diode mounted in the projector housing is utilized. The 2KHz reference waveform derived from this diode is first amplified and then filtered in a band pass structure. The resulting sine wave is shifted in phase by an externally controlled circuit so that optimum demodulation of the 2KHz signal may be accomplished in the previously mentioned synchronous detector. A screwdriver reference drive gain adjustment is available on the front panel of the signal processor.

Referring to Figure 4, the output of the synchronous detector is split into two channels, one of which contains the azimuth error signal (400Hz) information, and the other a direct current component, which after integration and amplification, is displayed on a front panel meter as an indication of the incident light intensity. In the signal channel, the 400Hz error information is filtered to reject higher order demodulation products and fed to a Princeton lock-in amplifier for further processing.

The Princeton lock-in amplifier is a correlation receiver which utilizes a synchronous detector with a continuously variable predetection bandpass amplifier. Additional capabilities include control of gain and phase of both signal and reference channels, and provision for an adjustable output filter time instant.

Figure 4. RAM ELECTRONIC SUBSYSTEM

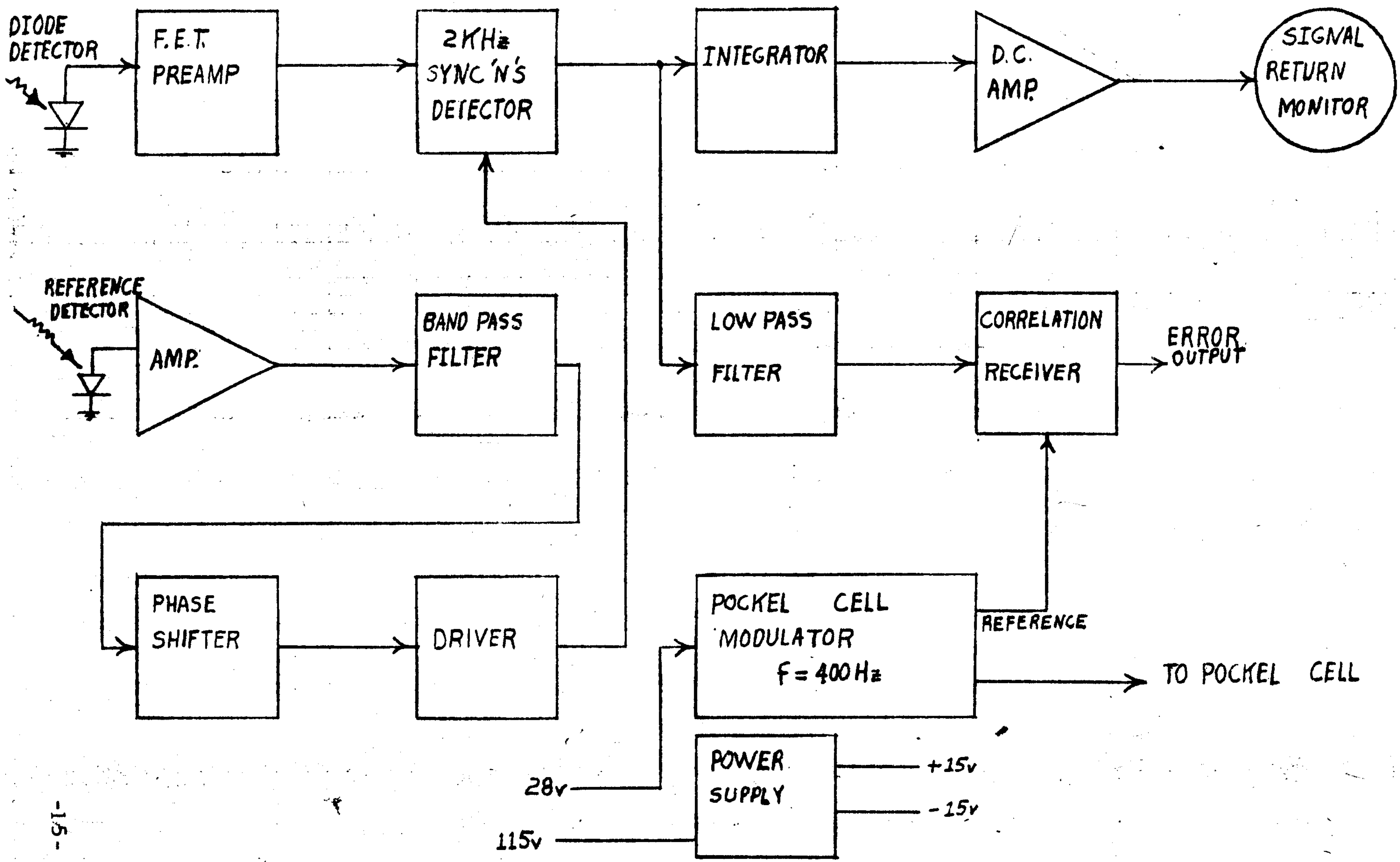
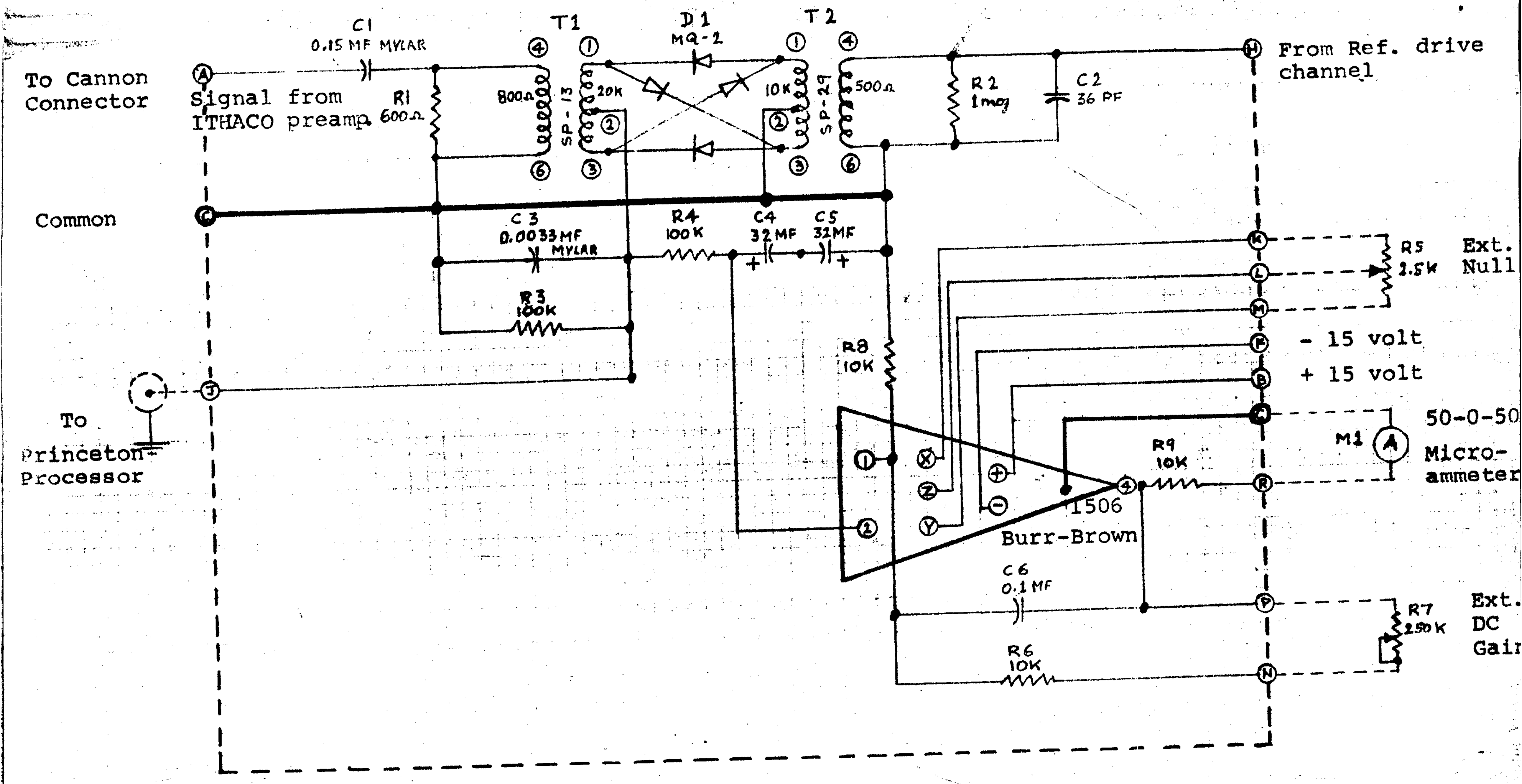


fig 4



A.O. Jan. 12, 1966

An external reference drive to the Princeton receiver is supplied by the Pockel cell modulator, which has a separate buffered output for this purpose.

Pockel Cell Modulator

The electro-optic light modulator used in the RAM system requires very high drive voltages in order to provide a high percentage autio modulation of the light returned from the remotely mounted Savart plate. The voltage required in the unit used is approximately 7200 volts peak-to-peak. The electronics provide a square wave function of this magnitude at approximately a 400 cycle audio rate. The circuit used to provice such a voltage is in the form of a D-C to A-C converter. This simple and efficient converter contains two power transistors and a saturating transformer, so connected that a regenerative switching action exists between the transistors. The transformer has a core material with a hysteresis curve approaching a square loop, and the output at the secondary of the transformer is an almost perfect square wave. A low level output is taken from the emitter of one of the transistors to supply the reference signal to the correlation receiver discussed previously.

Power Supply

The signal processor unit described above requires stable voltages of ± 15 volts dc for high performance. These voltages are supplied by two silicon modules which are precision regulated dc power supplies operating from 115 volts at 60 cps.

These supplies provide a dc output with an adjustable range of ± 2 volts dc (screwdriver adjustments at rear of units). The supplies are regulated against changes in line or load conditions and emply an automatic short-circuit current protection circuit. The Pockel cell modulator operates from the standard 28 volt dc line voltage which is a required input to the system.

IV. METHODS OF CALCULATION

Mueller Matrices

The Savart plate is comprised of two equal-thickness, plane-parallel pieces of crystalline quartz with their optic axes at 45 degrees to the principal plane-parallel surfaces. The relative retardation, d_a, produced by the first plate as a function of angle of incidence can be written as:[1, 2]

$$d_a = \frac{2\pi e}{\lambda}\left\{\frac{1}{c} - \frac{1}{b} - \left(\frac{b^2 - a^2}{2c^2}\right) E + \frac{1}{2}\left[\left(b - \frac{a^2}{c}\right)N^2 + \left(b - \frac{a^2 b^2}{c^3}\right)E^2\right]\right\},$$

where: λ = wavelength of the light

e = plate thickness

$a = 1/n_e$, the reciprocal of the extraordinary refractive index

$b = 1/n_o$, the reciprocal of the ordinary refractive index

$c^2 = (a^2 + b^2)/2$

$E = \sin i \cos \Theta$

$N = \sin i \sin \Theta$

i = angle of incidence of the ray under consideration,

and Θ = angle between the plane containing the optic axis and the normal to the plate, and the plane containing the incident ray and the plate normal.

Since the second plate is rotated 90 degrees with respect to the first plate its retardation, d_b, can be expressed by changing the sign of its terms:

$$d_b = \frac{2\pi e}{\lambda}\left\{\frac{1}{b} - \frac{1}{c} - \left(\frac{b^2 - a^2}{2c^2}\right) E_b - \frac{1}{2}\left[\left(b - \frac{a^2}{c}\right)N_b^2 + \left(b - \frac{a^2 b^2}{c^3}\right)E_b^2\right]\right\}.$$

but E_b and N_b can be written in terms of E and N; thus,

$E_b = -N$, and $N_b = E$

The expression for d_b is now:

$$d_b = \frac{2\pi e}{\lambda}\left\{\frac{1}{b} - \frac{1}{c} - \left(\frac{b^2 - a^2}{2c^2}\right) N + \frac{1}{2}\left[\left(\frac{a^2}{c} - b\right)E^2 + \left(\frac{a^2 b^2}{c^3} - b\right)N^2\right]\right\}.$$

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(1) Francon, Optica Acta, Vol. 1, pg. 50 (1954)

(2) Francon, J. O. S. A., Vol. 47, pg. 528 (1957)

The total retardation of the Savart plate, d_1, becomes the sum of the retardations of the two plates, that is:

$$d_a + d_b = d_1 = \frac{2\pi e}{\lambda}\left\{\left(-\frac{b^2-a^2}{2c^2}\right)\left(E + N\right) + \frac{1}{2}\left[\left(\frac{a^2b^2}{c^3} - \frac{a^2}{c}\right)\left(N^2 - E^2\right)\right]\right\}$$

It can be seen that this expression only approximates a system of straight and parallel fringes when viewed between crossed polarizers. A second Savart plate, of equal thickness, but rotated to 180 degrees from the first, causes $E_2 = -E$, and $N_2 = -N$ and provides relative retardation, d_2, as expressed by:

$$d_2 = \frac{2\pi e}{\lambda}\left\{\left(\frac{b^2-a^2}{2c^2}\right)\left(E + N\right) + \frac{1}{2}\left[\left(\frac{a^2b^2}{c^3} - \frac{a^2}{c}\right)\left(N^2 - E^2\right)\right]\right\}$$

Combining the retardation of these two plates (when they are parallel):

$$d_1 + d_2 = \frac{2\pi e}{\lambda}\left(\frac{a^2b^2}{c^3} - \frac{a^2}{c}\right)\left(N^2 - E^2\right)$$

This means that the combination will not cause total cancellation when viewed between crossed polarizers, but will produce a double system of equilateral hyperbolas.

The limited useful angular field of the crossed Savart plates can be overcome by a modification of their construction(3) as described below.

If the second component of the Savart plate is rotated 180 degrees, instead of 90 degrees, the expression for d_b becomes:

$$d_b = \frac{2\pi e}{\lambda}\left\{\frac{1}{c} - \frac{1}{b} + \left(\frac{b^2 - a^2}{2c^2}\right) E + \frac{1}{2}\left[\left(b - \frac{a^2}{c}\right) N^2 + \left(b - \frac{a^2b^2}{c^3}\right)E^2\right]\right\}$$

since $E_b = -E$, and $N_b = -N$, for 180-degree rotation. If either a half wave plate, or a 90 degree polarization rotator, (4, 5) which effectively causes a sign reversal of the retardation produced by

(3) See ref. 2, Francon, pg, 534

(4) C. J. Koester, J. O. S. A., Vol. 49, No. 4, pp. 405 – 409 (1959)

(5) S. Pancharatnam, Proc. Indian Acad. Sci. A41, pp 130 – 144 (1955)

the first component, is inserted between the two components, the expression for d_a now becomes:

$$d_a = \frac{2\pi e}{\lambda}\left\{\frac{1}{b} - \frac{1}{c} - \left(\frac{b^2-a^2}{2c^2}\right) E - \frac{1}{2}\left[\left(b-\frac{a^2}{c}\right) N^2 + \left(b - \frac{a^2 b^2}{c^3}\right) E^2\right]\right\},$$

and now $d_a + d_b$ is equal to:

$$d_1 = \frac{4\pi e}{\lambda}\left(\frac{b^2 - a^2}{2c^2}\right)\left(E\right),$$

which would be seen as a system of truly straight and parallel fringes between crossed polarizers, and with the additional difference of being oriented at 45 degrees to the fringe pattern of a normal Savart plate. If an identical modified Savart plate is rotated with respect to the first 180 degrees about their normal, the net retardation produced is:

$$d_1 + d_2 = \frac{4\pi e}{\lambda}\left(\frac{b^2-a^2}{2c^2}\right)(E) + \frac{4\pi e}{\lambda}\left(\frac{b^2-a^2}{2c^2}\right)(E_2);$$

but since $E_2 = -E$ for 180-degree rotation,

$$d_1 + d_2 = 0$$

This combination will appear as a totally dark field when viewed between crossed polarizers.

A matched pair of these modified Savart plates can now be considered as the heart of a system which transfers attitude information about two orthogonal axes via a beam of polarized light.

The system can be treated, for analysis purposes, as though the light source were located behind the missile unit polarizer.

Proceeding from this apparent position of the light source, the system consists of:

Missile Unit

1. A plane polarizer
2. A modified Savart plate (fast axis at 45 degrees to polarizer axis).

Ground Unit

1. A matched modified Savart plate
2. A Pockel cell (electrically induced axis at 45 degrees to Savart plate fast axis)
3. A plane polarizer (axis at 0 degree to Savart plate fast axis)
4. A photodetector

The performance of this system can be efficiently analyzed by a useful matrix-algebraic tool, the Mueller calculus[6]. With this calculus, an otherwise complicated system of polarizers, retarders, and variable retarders can be reduced to a single 4 x 4 matrix. Light can then be introduced into the system in the form of a four-parameter Stokes vector, and the magnitude of the exiting light computed is proportional to the signal received by the photodetector in the system under analysis.

In the first part of this analysis it will be assumed that there is only roll and elevation error (the missile unit fixed with respect to the ground unit about the azimuth axis). The second part of this

(6) W. A. Shurcliff, "Polarized Light", Harvard Univ. Press, (Cambridge, 1962) pp 109 – 117.

analysis will consider the general case, where there can be any orientation between the missile unit and the ground unit.

The first step in this analysis is to determine the Mueller matrix for each component of the system. Beginning with the last active element, for convenience in calculation, one may designate $[M_a]$ the matrix of the analyzing polarizer.

This is a plane polarizer oriented at 45 degrees to the reference direction, and the matrix which describes it is simply:

$$[M_a] = 1/2 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The next element in the system (still proceeding backwards) is the Pockel cell. This device can be represented by a more complex matrix:

$$[M_e] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos d_e & \sin d_e \\ 0 & 0 & -\sin d_e & \cos d_e \end{bmatrix} ,$$

which is the matrix of a linear retarder, of retardation d_e, whose fast (or slow, since d_e is a variable and can go negative) axis is oriented at 0 degree to the reference direction, and $d_e = \pi/2 \sin \omega t$, the relative phase retardation electrically induced in the Pockel cell.

The last element in the ground unit is the modified Savart plate (birefrigent element 2). The matrix describing birefrigent element 2 is:

$$\left[M_{b2}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d_2 & 0 & \sin d_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin d_2 & 0 & \cos d_2 \end{bmatrix},$$

which again is the matrix of a linear retarder, of retardation d_2, but in this case having its fast axis at 45 degrees to the reference direction. The expression for d_2 is:

$$d_2 = \frac{4\pi e}{\lambda} \left(\frac{b^2 - a^2}{2c^2} \right) (E),$$

as previously derived (except for the term E, which is here given positive sign, because the matrix accounts for its orientation).

The first element (still proceeding backwards) in the missile unit is its birefringent element, described by:

$$\left[M_{b_1}\right] \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d_1 & 0 & -\sin d_1 \\ 0 & 0 & 1 & 0 \\ 0 & \sin d_1 & 0 & \cos d_1 \end{bmatrix},$$

which represents a linear retarder, of retardation d_1, and whose fast axis is oriented at -45 degrees from the reference direction. The expression for d_1 is the same as previously derived.

The last element in the missile unit is the plane polarizer oriented at 0 degree to the reference direction, which is described by:

$$\left[M_p\right] = 1/2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this analysis, the ground, or analyzing, unit will be treated as though the individual elements are fixed, both with respect to each other and with respect to an arbitrary reference direction. The missile unit will also be treated as though its individual elements were fixed with respect to each other – first as though it were also fixed with respect to the reference direction, and then for the general case where it can have any orientation with respect to the main unit.

The matrix, $\left[M_g\right]$, which describes the ground unit, is actually the product of the matrices of its individual components, $\left[M_g\right] = \left[M_a\right]\left[M_e\right]\left[M_{b2}\right]$, where $\left[M_a\right]$ is the matrix of the analyzing polarizer, $\left[M_e\right]$ is the matrix of the Pockel cell, and $\left[M_{b2}\right]$ is the matrix of the main birefringent element:

$$\left[M_g\right] = 1/2 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos d_e & \sin d_e \\ 0 & 0 & -\sin d_e & \cos d_e \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d_2 & 0 & \sin d_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin d_2 & 0 & \cos d_2 \end{bmatrix},$$

$$\left[M_g\right] = 1/2 \begin{bmatrix} 1 & (-\sin d_e \quad \sin d_2) & \cos d_e & (\sin d_e \quad \cos d_2) \\ 0 & 0 & 0 & 0 \\ 1 & (-\sin d_e \quad \sin d_2) & \cos d_e & (\sin d_e \quad \cos d_2) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The missile unit, considered fixed with respect to the reference direction, can be described by the matrix, $\left[M_m\right]$, which is again the product of the individual components, $\left[M_m\right] = \left[M_{b1}\right]\left[M_p\right]$, where $\left[M_{b1}\right]$ is the matrix of the missile birefringent element, and $\left[M_p\right]$ is the matrix of the missile polarizer:

$$\left[M_m\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos d_1 & 0 & -\sin d_1 \\ 0 & 0 & 1 & 0 \\ 0 & \sin d_1 & 0 & \cos d_1 \end{bmatrix} 1/2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left[M_m\right] = 1/2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ \cos d_1 & \cos d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin d_1 & \sin d_1 & 0 & 0 \end{bmatrix},$$

where $d_1 = \left(\frac{2\pi e}{\lambda} \frac{n_e^2 - n_o^2}{n_e^2 + n_o^2}\right) (\sin i_1 \cos \theta_1)$, the relative phase retardation introduced by the missile birefringent element to light passing at an angle (i_1, θ_1).

The complete system can now be expressed as a single 4 x 4 matrix, $\left[M_s\right]$, which is the product of the missile unit matrix and the ground unit matrix, $\left[M_s\right] = \left[M_g\right]\left[M_m\right]$:

$$\left[M_s\right] = 1/2 \begin{bmatrix} 1 - \sin d_e \sin d_2 & \cos d_e & \sin d_e \cos d_2 \\ 0 & 0 & 0 & 0 \\ 1 - \sin d_e \sin d_2 & \cos d_e & \sin d_e \cos d_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} 1/2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ \cos d_1 & \cos d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin d_1 & \sin d_1 & 0 & 0 \end{bmatrix},$$

$$\left[Ms\right] = 1/4 \begin{bmatrix} 1 + \sin d_e \sin (d_1 - d_2) & 1 + \sin d_e \sin (d_1 - d_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 + \sin d_e \sin (d_1 - d_2) & 1 + \sin d_e \sin (d_1 - d_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The effect of sending a beam of unpolarized light through the system can now be computed by multiplying the stokes vector, $\left[V_i\right]$, of unit intensity incident light by the matrix of the system, $\left[M_s\right]$, which determines the Stokes vector of the emergent beam, $\left[V_s\right]$. That is, $\left[V_s\right] = \left[M_s\right]\left[V_i\right]$:

$$\left[V_s \right] = 1/4 \begin{bmatrix} 1 + \sin d_e \sin(d_1 - d_2) & 1 + \sin d_e \sin(d_1 - d_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 + \sin d_e \sin(d_1 - d_2) & 1 + \sin d_e \sin(d_1 - d_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\left[V_s \right] = 1/4 \begin{bmatrix} 1 + \sin d_e \sin (d_1 - d_2) \\ 0 \\ 1 + \sin d_e \sin (d_1 - d_2) \\ 0 \end{bmatrix},$$

which is a vector describing 100% plane polarized light, polarized at an angle of 45 degrees from the horizontal, with an intensity equal to:

$$I_s = 1/4 \left[1 + \sin d_e \sin (d_1 - d_2) \right],$$

which is; $I_s = 1/4 \; (1 + \sin \; (\pi/2 \sin \omega t) \sin [(4 \frac{\pi e}{\lambda})(\frac{n_e^2 - n_o^2}{n_e^2 + n_o^2})$

$(\sin i_1 \; \cos \theta_1 \; - \sin i_2 \cos \theta_2)]$

It can be seen that the first derivative with respect to $(d_1 - d_2)$ reaches a maximum when $(d_1 - d_2)$ is equal to zero, indicating that the sensitivity of the system is greatest at the "null" or no error condition. This "null" however, is ambiguous, and occurs whenever d_1 is equal to zero or an integer times π, (assuming d_2 is equal to zero), that is, whenever the expression $(\frac{4\pi e}{\lambda})(\frac{n_e^2 - n_o^2}{n_e^2 + n_o^2})$ $(\sin i_1 \cos \theta_1)$ has an integer value (0, 1, 2, ---). This ambiguity can be eliminated over any desired range of values of (i_1, θ_1) (again assuming d_2 equal to zero), that is, for any desired angular field, either by choosing "e" very small, or by choosing two or more values of "λ". Smaller values of "e", the crystal thickness, give greater angular fields, but at the cost of reduced sensitivity to small changes in the difference between $(\sin i_1 \cos \theta_1)$ and $(\sin i_2 \cos \theta_2)$. The observation, separately in time, of two colors can eliminate any ambiguity until both of the two separate signals are equal to zero. This technique permits a wide angular field to be covered without any loss of sensitivity, since a null occurs for both of the two wavelengths (λa and λb) only when the expression $\left(\frac{4\pi e}{\lambda}\right)\left(\frac{n_e^2 - n_o^2}{n_e^2 + n_o^2}\right)(\sin i_1 \cos \theta_1)$ is equal either to zero or to N_b, where

$$N_b = N_a\left(\frac{\lambda a}{\lambda b}\right),$$

and both N_a and N_b are integers. It can be seen that by choosing appropriate values of λ_a and λ_b, $(\sin i_1 \cos \theta_1)$ can be caused to approach $(\cos \theta_1)$, indicating a horizontal angular field of view limited only by such practical things as the system signal-to-noise ratio, bandwidth of optical filters, vignetting, etc.

The general case, where the missile unit can have any orientation with respect to a reference and to the ground unit, will now be considered. The matrix describing any optical device such as a polarizer or retarder, or the matrix for any combination of such individual devices, oriented at angle φ_o, can be converted to the matrix describing the device at any angle φ (rotation about the axis of the light direction). The means of converting such a matrix, $\left[M\varphi_o\right]$ to a general matrix, $\left[M\varphi\right]$, is simply a matter of computing the product $\left[M\varphi\right] = \left[T(-2\varphi)\right]\left[M\varphi_o\right]\left[T(2\varphi)\right]$, where $\left[T(2\varphi)\right]$ is the rotator matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $T(-2\varphi)$ is the counter-rotator matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & \sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing this operation, one can determine that:

$$M_{m\varphi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & \cos 2\varphi & -\sin 2\varphi & 0 \\ 0 & \sin 2\varphi & \cos 2\varphi & 0 \\ & 0 & 0 & 1 \end{bmatrix} 1/2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ \cos d_1 & \cos d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin d_1 & \sin d_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & -\sin 2\varphi & 0 \\ 0 & \sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$M_{m\varphi} = 1/2 \begin{bmatrix} 1 & \cos 2\varphi & \sin 2\varphi & 0 \\ \cos 2\varphi \cos d_1 & \cos^2 2\varphi \cos d_1 & \cos 2\varphi \sin 2\varphi \cos d_1 & 0 \\ \sin 2\varphi \cos d_1 & \cos 2\varphi \sin 2\varphi \cos d_1 & \sin^2 2\varphi \cos d_1 & 0 \\ \sin d_1 & \cos 2\varphi \sin d_1 & \sin 2\varphi \sin d_1 & 0 \end{bmatrix}$$

Now the complete system, with any orientation of the missile unit with respect to the ground unit, can be expressed as before, the product of the individual unit matrices; $\left[M_{g\varphi}\right] = \left[M_m\right]\left[M_{m\varphi}\right]$,

$$M_{s\varphi} = 1/2 \begin{bmatrix} 1 & -\sin d_e \sin d_2 & \cos d_e & \sin d_e \cos d_2 \\ 0 & 0 & 0 & 0 \\ 1 & -\sin d_e \sin d_2 & \cos d_e & \sin d_e \cos d_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$1/2 \begin{bmatrix} 1 & \cos 2\varphi & \sin 2\varphi & 0 \\ \cos 2\varphi \cos d_1 & \cos^2 2\varphi \cos d_1 & \cos 2\varphi \sin 2\varphi \cos d_1 & 0 \\ \sin 2\varphi \cos d_1 & \cos 2\varphi \sin 2\varphi \cos d_1 & \sin^2 2\varphi \cos d_1 & 0 \\ \sin d_1 & \cos 2\varphi \sin d_1 & \sin 2\varphi \sin d_1 & 0 \end{bmatrix},$$

$$M_{s_\varphi} = 1/4 \begin{bmatrix} 1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1) & \cos 2\varphi\,[1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1)] & \sin 2\varphi\,[1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1)] & 0 \\ 0 & 0 & 0 & 0 \\ 1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1) & \cos 2\varphi\,[1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1)] & \sin 2\varphi\,[1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1)] & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again multiplying the system matrix by the Stokes vector of the incident beam, one finds the Stokes vector of the emergent beam, $\left[V_{s_\varphi}\right] = \left[M_{s_\varphi}\right]\left[V_i\right]$,

$$V_{s_\varphi} = 1/4 \begin{bmatrix} 1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1) \\ 0 \\ 1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1) \\ 0 \end{bmatrix},$$

which is again 100% plane polarized light, polarized at an angle of 45 degrees from the horizontal, with an intensity equal to:

$$I_{s_\varphi} = 1/4 \left[1+\sin d_e (\cos d_2 \sin d_1 - \cos 2\varphi \sin d_2 \cos d_1) + \cos d_e (\sin 2\varphi \cos d_1)\right].$$

This primarily differs from I_s by containing a cosine function of "d_e", the electrically induced retardation, which can be sensed electronically

and used to drive the orientation of the missile unit to coincide with the orientation of the main unit. It can be seen that the sensitivity to misalignment of this kind is at a maximum when "φ" is equal to zero, but that the magnitude of the "cos d_e" term depends on the "cos d_1" term, which, however, can easily be driven to a maximum value corresponding to the null condition about the other axis of sensitivity. This means that this is a system which can independently detect misalignments about two axes of rotation — one axis which is parallel to the line of sight, and another axis which is normal to the line of sight.

Signal to Noise Calculations

The total power, P, available to the detector is equal to:

$$P = O_f L_f N$$

where: O_f is the optical factor which takes into account the size and configuration of the projector and receiver as well as the operating range

L_f is the loss factor due to absorption and reflection, and

N is the radiance of the source

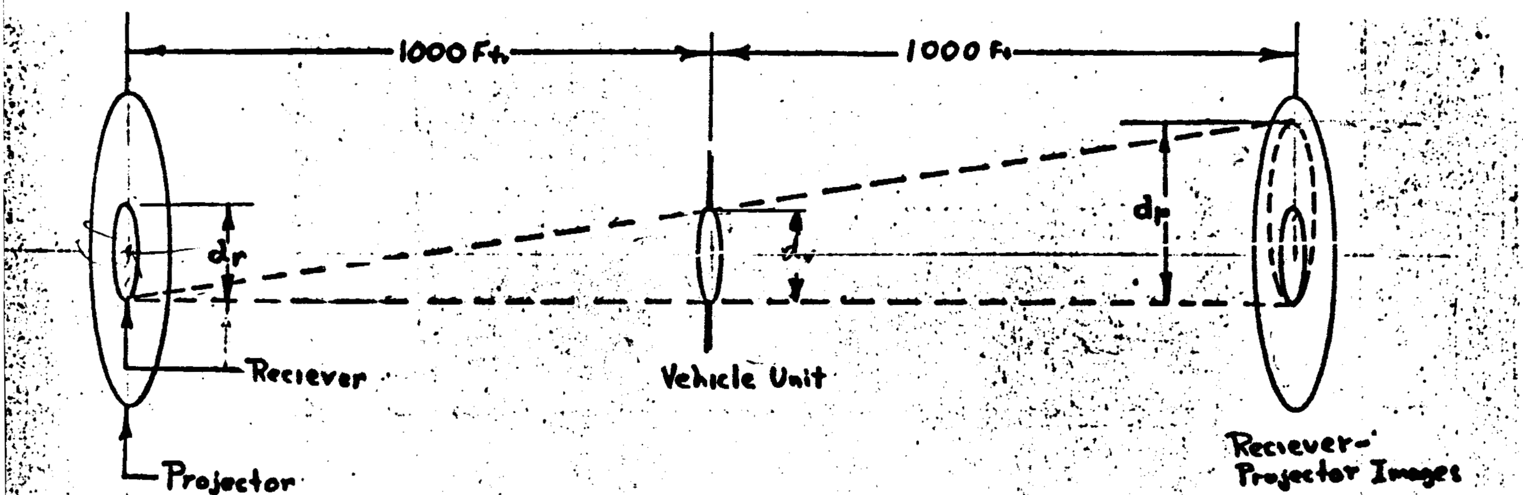


Fig. 6 Optical Factor Geometry

The optical factor is derived as follows: The projector consists of a simple spherical mirror with an annular plane mirror.

The source is placed near the focal point of the spherical mirror; the light reflecting from it is collimated. The plane mirror deflects the beam by 45° so that the beam is coaxial with the receiver. The projected light is returned by a corner reflector in the vehicle unit and passes thru the hole in the plane mirror to the receiver. To a first approximation, the corner reflector acts like a perfectly aligned plane mirror, forming an image of the projector at a distance R behind it i.e. a distance 2R from the receiver. Figure 6 illustrates the geometry. The receiver in effect is viewing the projector image thru the clear aperture of the vehicle unit. Notice from the figure that the effective diameter of the projector is limited by the vehicle unit to just twice the clear aperture of the corner. And further, part of the area included is the receiver image area. Thus, the total effective collecting area of the projector is:

$$A_p = \pi/4 \left(d_p^2 - d_r^2\right)$$

$$= \pi/4 \left(4 d_v^2 - d_r^2\right)$$

Approximately, the vehicle unit clear aperture is equal to that of the receiver; thus,

$$A_p = 3/4\, \pi\, d_r^2$$

The power collected by the receiver is a function of the solid angle, Ω, subtended by the receiver; where

$$\Omega = \frac{\pi}{4} \frac{d_r^2}{(2R)^2}$$

The total optical factor is

$$O_f = \frac{3\pi^2}{64} \frac{d_r^4}{R^2}$$

substituting $d_r = 1.4''$ $(3.55\ cm)$; $R = 1000\,ft$ $(3.05 \times 10^4\ cm)$:

$$O_f = 7.9 \times 10^{-8}\ cm^2\ steradian$$

The approximation made on the corner reflector image must now be reexamined. Actually, a corner returns the beam after three reflections and therefore returns 3! or 6 separate images. In a perfect corner, the 6 images are superimposed; in any practical device, however, the angles between faces differ from 90^o by some amount θ and the 6 images are divergent by an angle of up to 6θ (2θ for each reflection). Thus, to avoid losing power, the projector diameter must be increased to compensate for the divergence by an amount equal to $6\theta R$,

$$d_p' = 3d_r + 6\theta R$$

In the Azimuth Axis Optical Alignment System, θ = 8 arcseconds or 4×10^{-5} radian and dr = 1.5 inches. Thus, dp = 7.4 inches. The actual projector diameter is 6", indicating that the expression derived above for the optical factor may be too high; some of the rays which entered into that calculation do not in fact hit the detector. How much error is involved depends upon the specific corner and is not likely to be great, perhaps 20%. (Notice that there could actually be an increase in energy due to the divergence since the receiver in Fig. 6 may now be looking at a part of the projector image which does not include the center hole.

A corner with θ = 4 arcseconds would give dp = 6 inches. Replica corners are available on the market with θ = 2 arcseconds. It is therefore a possibility that the S/N ratio of the Azimuth Axis Optical Alignment system could be improved by substituting a more expensive corner.

The loss factor is approximated by the following considerations:

Projector		.33
Lamp envelope	.92	
Two mirrors	.80	
Chopper	.45	
Atmosphere (2 ways)		.88

Vehicle Unit			.061
	Polarizer (in)	.38	
	Polarizer (out)	.76	
	1st Savart Plate (in and out)	.25	
	Corner	.85	
Receiver			.057
	2nd Savart Plate	.50	
	Pockel Cell	.40	
	Two lenses	.85	
	Filter	.88	
	Analyzer	.38	
Total L_f			.001

Many of the above items were verified by laboratory and field measurements. Obviously, the atmospheric transmission is a variable. It is interesting to note that the original L_f estimate was higher by an order of magnitude. This is due partly to the introduction of the chopper. Almost the entire remaining difference is due to the losses of the Savart Plate. Fresnel reflection losses resulting from the use of warped mica retarders in making the modified Savart plates turned out to be very great; the light effectively makes three passes through Savart plates.

The effective radiance or brightness of the source is

$$N' = \frac{\int N_\lambda D^*_\lambda T_\lambda \, d\lambda}{D^*_p \, T_p}$$

where N_λ, D_λ, and T_λ are the spectral radiance, spectral D* of the detector, and spectral transmission of the optics, respectively. The peak D* and peak optical transmission, $D_p{}^*$ and T_p, appear in order to normalize the corresponding spectral quantities. Since the spectral band of interest is fairly narrow, average values can be used; thus (excluding average optical transmission which was in the loss factor, L_f) the effective radiance is:

$$N = \int_{.75\mu}^{.9\mu} N_{\lambda(B.B.)}\, d\lambda \; \frac{D^*_{ave}}{D_p^*} \; \epsilon_{ave}$$

where the effective limits on wavelength are placed by the optical filter.

The spectral radiance from .75 to .9 microns wavelength of a blackbody is quite sensitive to source temperature. Figure 7 , showing the integrated radiance vs. temperature, illustrates the gain to be made by using hotter quartz line bulbs instead of normal 2800°K tungsten filaments. At 3000°K the blackbody spectral radiance is 44 watts cm^{-2} $steradian^{-1}$. For a Silicon photodiode, $\frac{D^*_{ave}}{D_p^*} = .7$. The average emissivity, ϵ_{ave}, is about .3. The effective radiance is thus calculated to be:

$$N = 9.2 \text{ watts } cm^{-2} \text{ steradian}^{-1}$$

The total power on the detector is therefore:

$$P = (7.9 \times 10^{-8})(10^{-3})(9.2) = 7.3 \times 10^{-10} \text{ watts}$$

Since the magnitude of the error signal (S_e) is expressed by:

$$S_e = M P \sin\left[\frac{4\pi t}{\lambda}\left(\frac{n_o^2 - n_e^2}{n_o^2 + n_e^2}\right)(\sin e \cdot \cos\theta)\right]$$

M = modulation efficiency (.75)

where e is the error angle (1 arcsecond)

t is the single plate thickness (.38 cm)

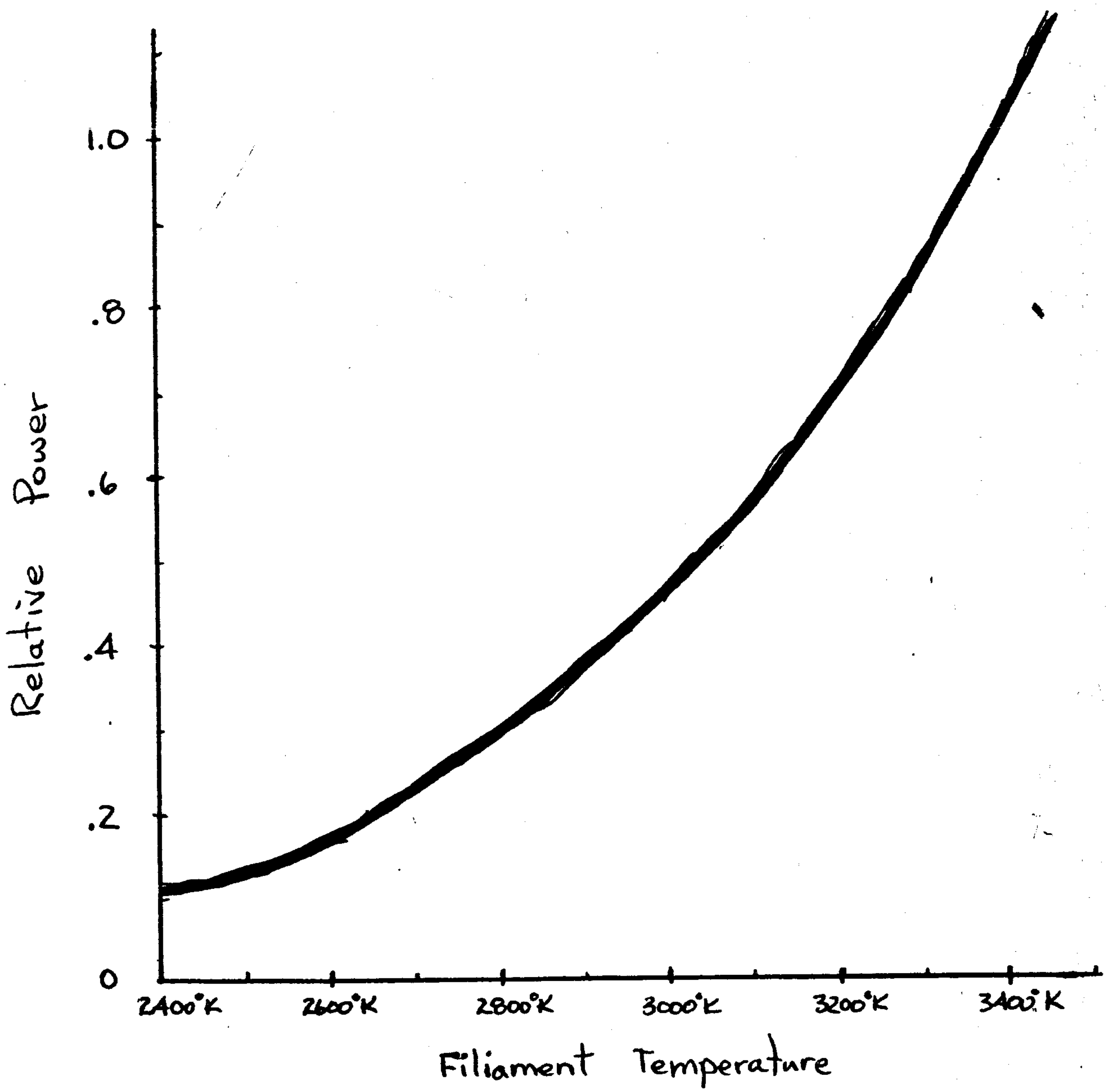
n_o is the ordinary index of refraction (1.544)

n_e is the extraordinary index of refraction (1.553)

λ is the wavelength (.8)

and θ is the angle between the aximuth axis and the Savart plate fast axis (45°).

figure 7



Filiament Temperature VS Relative Power
(as seen by detector - Si diode + polarizer included)

Substituting these numerical values in the signal equation, we have:

$$S_e = (.75)\ (7.3 \times 10^{-10})\ (1.2 \times 10^{-3}) = 6.5 \times 10^{-13} \text{ watts}$$

This is roughly equal to the detector NEP, agreeing with measured S/N ratio.