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RADIO CERENKOV RADIATION FROM A PRIMARY COSMIC RAY *

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ABSTRACT

The radiation field of a primary cosmic ray moving in straight line motion in the Earth's atmosphere is discussed. The particle is taken to have a velocity in excess of the velocity of light in the medium, and also to have a Larmor radius greater than a scale height of the atmosphere. It is shown that the polarization of the electric vector is nearly radial in the plane perpendicular to the track of the cosmic ray.

It is also shown that the radiation is detectable at centimeter wavelengths provided a radio telescope of size in excess of about 80 meters is used, that the integration time of the receiver is about 20 microseconds and that the noise temperature is less than about 20° K.

I. Introduction

In a recent paper (Lerche, 1965) the near field synchrotron radiation was discussed for a primary cosmic ray moving in the Ecrth's atmosphere with a velocity exceeding that of light in the medium. It was found that while the resultant energy field was strong enough to be observed by radio methods there existed two mutually incompatible conditions which made it seem unlikely that this radiation field could be used as a means of detecting high energy cosmic rays. The first condition required the velocity, u, of the particle to be greater than c/n where c is the velocity of light in vacuo and n is the refractive index of air. The second condition demanded that the Larmor radius of the primary be less than a scale height of the atmosphere in order that the calculation be applicable. Using a scale height of about 8 km. it was found that these two conditions were not compatible and hence the calculation could not be applied to particles producing synchrotron radiation unless the variation of refractive index with height was allowed for. Since the refractive index becomes very close to unity above one atmospheric scale height it seems unlikely that particles will be able to produce enough Cerenkov radiation for their detection by radio means to be assured.

It is then of interest to consider fast primary cosmic rays which have a large Larmor radius compared with the scale height of the atmosphere. In such a case we can neglect the effect of the Earth's magnetic field on the particle's motion and consider the particle to be moving in a straight line. We expect intuitively that the radiation field from such a particle will be weaker than that produced by the synchrotron effect since the particle is not being accelerated so much as seen by an observer at rest. However provided the radiation is not too weak we should be able to detect it by radio means, especially since we no longer have the restrictive condition on the Larmor radius which prevented the synchrotron radiation from being observed. In this treatment we assumed for simplicity that the refractive index of the medium shows neither time nor spatial variation.

2. The Wave Equation

We choose a Cartesian coordinate system in which the cosmic ray moves along the x-axis with speed u $(> c/n)$. If the particle has a charge, Q, then the electrostatic potential, Φ, satisfies the equation

$$\nabla^2 \Phi - \frac{n^2 \partial^2 \Phi}{c^2 \partial t^2} = 4\pi Q \,\delta(x-ut)\,\delta(y)\,\delta(z). \qquad (1)$$

where $\delta(\xi)$ is the usual Dirac δ-function.

The electromagnetic potential, $\underset{\sim}{A}$, satisfies a similar equation, namely

$$\nabla^2 \underset{\sim}{A} - \frac{n^2 \partial^2 \underset{\sim}{A}}{c^2 \partial t^2} = 4\pi \underset{\sim}{j}/c \quad . \qquad (2)$$

where the current density, $\underset{\sim}{j}$, is given by

$$\underset{\sim}{j} = u Q \hat{x}\, \delta(x-ut)\,\delta(y)\,\delta(z).$$

Thus the electromagnetic potential is in the $\hat{x}$ direction only and

$$\underset{\sim}{A} = \hat{x} A = \hat{x}\,(u/c)\,\Phi \, .$$

Hence the electric field which is formally given by

$$\underset{\sim}{E} = -\nabla\Phi - \frac{\partial \underset{\sim}{A}}{c\,\partial t} \tag{3}$$

becomes

$$\underset{\sim}{E} = -\nabla\Phi - \hat{x}\,\frac{u\,\partial\Phi}{c^2\,\partial t} \, . \tag{4}$$

It is clear that in order to find the electric and magnetic fields it is sufficient to find the scalar electrostatic potential. It is well known that we can write

$$\delta(x-ut)\,\delta(y)\,\delta(z) = (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i\left[k(x-ut)+ly+mz\right] dk\,dl\,dm \tag{5}$$

In like manner we Fourier analyze the potential Φ so that

$$\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(k,l,m)\exp i\left[k(x-ut)+ly+mz\right] dk\,dl\,dm. \tag{6}$$

Substituting for (5) and (6) into (1) leads to the equation

$$\Psi = \frac{-Q}{2\pi^2\left[k^2(1-n^2u^2c^{-2})+\ell^2+m^2\right]} \quad . \tag{7}$$

Since we are assuming that $nu > c$ we may set

$$\cos\alpha = c/(nu) \quad . \tag{8}$$

We recognize that (8) is the usual definition of the Cerenkov angle α .

Using (7) and (8) it can easily be seen that the Fourier component of the potential which varies as $e^{ik(x-ut)}$, say φ , is given by

$$\varphi = -Q(2\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(\ell y+mz)}\,d\ell\,dm}{(\ell^2+m^2-k^2\tan^2\alpha)} \quad . \tag{9}$$

In order to evaluate the double integral in equation (9) we transform to polar coordinates in (ℓ, m) space. Thus we let $\ell = \mu\cos\theta$, $m = \mu\sin\theta$. We also let $y = r\sin\zeta$, $z = r\cos\zeta$.

Then (9) becomes

$$\varphi = -Q(2\pi^2)^{-1}\int_0^{2\pi}d\theta\int_0^{\infty}\frac{\mu\, e^{ir\mu\sin(\zeta+\theta)}\,d\mu}{(\mu^2-k^2\tan^2\alpha)} \tag{10}$$

Since the integral over θ in (10) is periodic with period 2π then we can set $\zeta = 0$ in (10) without loss of generality. Upon doing so and making use of the standard expansion theorem

$$e^{ir\mu\sin\theta} = \sum_{s=-\infty}^{\infty} J_s(r\mu)\, e^{is\theta} \tag{11}$$

where $J_s(\xi)$ is the Bessel function of the first kind of order s and argument ξ, we find that (10) can be written

$$\varphi = -Q\pi^{-1}\int_0^{\infty} \frac{x J_0(x)\, dx}{(x^2 - k^2 r^2 \tan^2\alpha)} \quad . \tag{12}$$

Some care is needed in evaluating the integral in (12) because of the singularity in the integrand when $x = kr\tan\alpha$. If we compute the integral purely as a principal value integral as shown in Figure 1a, i.e. we ignore the pole, then the integral is a simple Hilbert transform which has been evaluated elsewhere (Erdelyi et al., 1954). Thus

$$\mathcal{P}\int_0^{\infty} \frac{x J_0(x)\, dx}{(x^2 - k^2 r^2 \tan^2\alpha)} = -\frac{1}{2}\pi Y_0(kr\tan\alpha) \quad . \tag{13}$$

However, if we deform the path of integration off the real x-axis as shown in Figure 1b, then the integral becomes

$$\int_0^\infty \frac{xJ_0(x)\,dx}{(x^2 - k^2r^2\tan^2\alpha)} = -\frac{1}{2}\pi\left(Y_0(kr\tan\alpha) + i\,J_0(kr\tan\alpha)\right). \qquad (14)$$

Likewise if we evaluate the integral along the path shown in Figure 1c we obtain

$$\int_0^\infty \frac{xJ_0(x)\,dx}{(x^2 - k^2r^2\tan^2\alpha)} = -\frac{1}{2}\pi\left(Y_0(kr\tan\alpha) - i\,J_0(kr\tan\alpha)\right). \qquad (15)$$

To decide which path of integration is appropriate in our case we must employ a physical argument. At large distances in the plane perpendicular to the track of the primary we require that the potential become a progressive wave propagating outwards away from the particle. Thus the phase factor of the wave must have the dependence $\exp ik(r\tan\alpha - ut)$. Now it is well known that the Hankel function of the first kind has the property that, as its argument becomes large compared to unity, the asymptotic expansion has the above form. The Hankel function of the first kind of order zero and argument ξ is defined by

$$H_0^{(1)}(\xi) = J_0(\xi) + i\,Y_0(\xi).$$

Thus it is now clear that the appropriate path of integration is that given in Figure 1c and hence (12) becomes

$$\varphi = -\frac{1}{2}iQ\;H_0^{(1)}(kr\tan\alpha). \qquad (16)$$

Thus the Fourier component of the electric field with wave number k, say $\underset{\sim}{\mathcal{E}}(k,r)$, is given by

$$\mathcal{E}_x = -\frac{1}{2} kQ(1-u^2c^{-2}) H_0^{(1)}(kr\tan\alpha) \, . \tag{17a}$$

$$\mathcal{E}_r = \frac{1}{2} \, ikQ\tan\alpha \; H_0^{(1)'}(kr\tan\alpha) \, . \tag{17b}$$

where the prime denotes differentiation with respect to the argument.

We can also evaluate the magnetic field since $\underset{\sim}{H} = \nabla\wedge\underset{\sim}{A} \equiv \nabla\wedge(\hat{x}A) = (u/c)\nabla\wedge(\hat{x}\Phi)$. The Fourier component of the magnetic field with wave-number k, say $\underset{\sim}{h}(k,r)$, is

$$\underset{\sim}{h}(k,r) = \frac{1}{2} \, ik\tan\alpha \; Q(u/c) \, H_0^{(1)'}(kr\tan\alpha) \, \hat{\zeta} \, . \tag{18}$$

3. <u>Energy Considerations</u>

If the linear scale size of the system is L we can use a quasi-plane wave approximation provided

$$k \gtrsim L^{-1} \, . \tag{19}$$

In our case with the appropriate scale size being about $r\tan\alpha$ we can use the approximation provided

$$k r \gtrsim \cot\alpha \ . \tag{20}$$

Assuming that the observation site is chosen sufficiently far from the primary cosmic ray for this to hold then the total amount of energy received per unit area, say P, can be written

$$P = c/(8\pi) \int \underset{\sim}{E} \cdot \underset{\sim}{E}^{*} dt$$

$$= c/(8\pi) \iiint \underset{\sim}{\mathcal{E}}(k,r) . \underset{\sim}{\mathcal{E}}^{*}(k',r)\, e^{ix(k-k')} e^{iut(k-k')} dk\,dk'\,dt \ .$$

$$= c/(4u) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underset{\sim}{\mathcal{E}}(k,r) . \underset{\sim}{\mathcal{E}}^{*}(k',r)\, \delta(k-k')\, dk\,dk' \ .$$

$$= c/(4u) \int_{-\infty}^{\infty} | \underset{\sim}{\mathcal{E}}(k,r) |^{2} dk \ .$$

$$= c/(2u) \int_{0}^{\infty} | \underset{\sim}{\mathcal{E}}(k,r) |^{2} dk \ .$$

If we choose to work with the frequency, f, instead of the wave number, k, then the amount of energy received per unit area can be written

$$P = \pi c u^{-2} \int_0^\infty |\underset{\sim}{\xi}(f,r)|^2 df \quad . \tag{21}$$

where the frequency, f , has been expressed in terms of the wave number, k , through the relation $uk = 2\pi f$.

Since most detectors have a fixed bandwidth, the amount of energy received per unit area in a given bandwidth is normally the interesting and observable quantity. Calling this quantity $p(f)\,\delta f$ we see that

$$p(f) = \pi c u^{-2} \, |\underset{\sim}{\xi}(f,r)|^2 \quad . \tag{22}$$

Using the expression for the electric field given in (17) we see that

$$p(f) = \pi^3 c Q^2 f^2 u^{-4} \left[\tan^2\alpha \, |H_0^{(1)\prime}(2\pi f r \tan\alpha \, u^{-1})|^2 + (1-u^2/c^2)^2 \, |H_0^{(1)}(2\pi f r \tan\alpha \, u^{-1})|^2 \right] . \tag{23}$$

Since we are in the region where the quasi-plane wave approximation is valid we may write

$$|H_0^{(1)}(\xi)| = |H_0^{(1)\prime}(\xi)| = \sqrt{\left(\frac{2}{\pi \xi}\right)} \quad .$$

Upon doing so we see that (23) becomes

$$p(f) \simeq \pi c Q^2 \cot\alpha\, f (r u^3)^{-1} \left(\tan^2\alpha + (1 - u^2/c^2)^2\right). \tag{24}$$

The polarization of the radiation in this approximation at a given frequency is given by

$$\left| \mathcal{E}_x(f,r) / \mathcal{E}_r(f,r) \right| = \cot\alpha \, . \, (1 - u^2/c^2) \quad . \tag{25}$$

It is also clear that the particle always moves ahead of the radiation field. This can easily be seen by looking at a point of stationary phase on the wave which will satisfy the equation

$$r = ut\cot\alpha$$

The position of the particle satisfies $x = ut$. If we transform to coordinates in which the particle is at rest it can easily be seen that the point of stationary phase satisfying the transformed equations lags with respect to the position of the primary cosmic ray. Thus the particle is always in advance of the Cerenkov shock.

4. Numerical Estimates

To obtain some idea of the direction of polarization of the Cerenkov radiation and also of the amount of emitted energy per unit area we apply the above results to the case where we take the refractive index of the air to be

$n = 1 + 3.10^{-4}$.

We can take the polarization of the electric vector as being purely radial provided that

$$n^2u^2 - c^2 \gtrsim c^2 - 2u^2 + u^4c^{-2}, \qquad \text{i.e.} \quad u \gtrsim (2-n)\,c. \tag{26}$$

But we already have assumed that $u > cn^{-1} \simeq (2-n)\,c$ so that (26) is automatically satisfied. Hence to a very good approximation we can assume that the radiation is polarized radially in the plane normal to the path of the primary cosmic ray.

We can therefore write that

$$p(f) \simeq \pi c Q^2 \tan\alpha \, f\,(ru^3)^{-1} \; .$$

Since $u \simeq c$ we may replace $\tan\alpha$ by $\sqrt{(n^2-1)}$ and then

$$p(f) \simeq \pi Q^2 f \sqrt{(n^2-1)}\;(rc^2)^{-1} \; . \tag{27}$$

Assuming the primary cosmic ray is a proton and inserting the appropriate numerical constants it can easily be shown that

$$p(f) \simeq 2.10^{-41}\,(f/r)\ \text{ergs.cm.}^{-2}\,\text{sec.} \tag{28}$$

Converting to the normal radio astronomical measurements in terms of flux units (1 flux unit $\equiv 10^{-26}\,W.m^{-2}\,(c/s)^{-1}$) we find that (28) becomes

$$p(f) \simeq 2.10^{-18}\,(f/r) \quad f.u.\,sec. \tag{29}$$

Since we are using a quasi-plane wave approximation we know that (29) is only valid for $2\pi f r \gtrsim u \cot\alpha$ i.e. $r \gtrsim 2.10^{11} f^{-1}$ cm. Let us therefore set $r = u \cot\alpha\,(2\pi f)^{-1} s$ where $s \gtrsim 1$. Then

$$p(f) \simeq 10^{-29} f^2/s \qquad f.u.\,sec.$$

The total amount of energy received per unit area in a bandwidth δf is just $p(f)\delta f$.

From the dependence of the radiation on frequency it is clear that we obtain the most energy at the shortest wavelengths. However using a radio telescope means that we must remain on the long wavelength side of 2 cm. since shorter wavelengths are absorbed by water molecules in the atmosphere. Therefore let us choose to observe the radiation at a wavelength of 3 cm. and choose the bandwidth to be $\delta f = 10^{-1} f$. Then the total observed energy per unit area is just

$$p(f)\delta f \simeq s^{-1} \quad f.u. \tag{30}$$

If we let s = 10 to ensure that the quasi-plane wave approximation is valid then we should observe about 1/10 f.u. from such a primary at an observing frequency of 10^4 Mc/s. With a bandwidth of 10^3 Mc/s we have to take into account the noise signal. Suppose the overall noise temperature from all effects at a given frequency is $T\,^\circ K$ and that the receiver has a linear size D, then the amount of energy received per unit area due solely to noise, say N, is given by

$$N = \kappa T D^{-2}\, 10^{23}\ \text{f.u.}$$

where κ is Boltzmann's constant.

We can allow the signal strength to be small compared with the mean noise level provided the 'spikes' due to r.m.s. fluctuations in the noise are small compared with the signal 'spike'.

Let us assume that the receiver averages over a time interval τ and that reception is uniform over the bandwidth δf and nothing is received outside this band. Then it can be shown (Bracewell, 1965) that size of the r.m.s. fluctuations above the mean noise level is

$$\kappa T D^{-2} (\tau \delta f)^{-1/2}\, 10^{23}\ \text{f.u.}$$

The time the cosmic ray takes to travel one scale height of the atmosphere (~ 8 km.) moving at $u \simeq c$ is about 2.10^{-5} secs. Thus we do not wish to integrate the signal for longer than this time or else the assumption that we could neglect spatial variations in the refractive index becomes invalid. Hence we set $\tau = 2.10^{-5}$ secs. Knowing also the bandwidth δf ($\equiv$ 10^3 Mc/s) we see that in order that the noise spikes do not swamp the signal spike (10^{-1} f.u.)

we require that

$$\mathcal{D}^2 \gtrsim 10^{24} \kappa T \, (\tau \delta f)^{-1/2} \ \mathrm{cm}^2. \tag{31}$$

At a wavelength of 3 cm. it is possible to ensure that the noise temperature from all sources (e.g. background sky, Earth, receiver) is at most about 20° K. Thus inserting the appropriate numerical factors we see that a receiver of linear size $D \gtrsim 45$ meters is required in order for the signal to be observable above the noise fluctuations. If we demand that the noise spikes be a factor three less than the signal to ensure signal detection this raises the size of telescope required to $D \gtrsim 80$ meters.

This size of telescope is not unreasonable and so we conclude that it is possible to detect primary cosmic rays using radio telescopes.

To obtain an estimate of the expected number of counts per second picked up by the radio telescope we can argue as follows. At the top of the atmosphere the flux of cosmic rays is estimated to be about 1 $\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$. Now the radio telescope of linear size $\mathcal{D}$ has an effective receiving area of about $\pi \mathcal{D}^2$. Using $\mathcal{D} \simeq 80$ meters, this amounts to about $2.10^8\ \mathrm{cm}^2$. Thus at first sight we might expect about 2.10^8 counts per second. However in passing through the atmosphere the flux of primary cosmic rays is attenuated by absorption and the probability that a particle reach the ground is $\sim \exp(-x/x_0)$. where x_0 is the scale height of the atmosphere and x is the thickness of the atmosphere. Assuming that $x \simeq 120$ km and $x_0 \sim 8$ km this gives a decay of $\sim e^{-15}$.

Hence the number of expected counts now becomes

$$2.10^{8}\, e^{-15} \quad \text{counts}\,.\,\text{sec}^{-1}.$$

which amounts to some 60 counts per second. Since the integration time is only about 20 microseconds it is clear that we can discern the Cerenkov radiation from each individual cosmic ray without any danger that the pulses will overlap.

It should be remarked that this calculation is only applicable to primary cosmic rays provided they do not undergo shower formation, nuclear spallation, Coulomb scattering or any other process which prevents them moving in straight line motion. It also places a lower limit on the energy of such particles in that their Larmor radii must be greater than a scale height of the atmosphere.

When a primary cosmic ray undergoes shower formation this calculation can be applied to such a shower in a semi-quantitative manner. Several large showers composed of perhaps a million electrons and an equal number of positrons show a 10% deviation from charge neutrality. This is due to the fact that the electrons have longer lifetimes than positrons in the atmosphere. We can treat the radiation from the charge excess as an incoherent emission problem at 3 cm. wavelength since the shower thickness is normally of the order of a meter. The front of the shower is normally about 50 meters in radius. Thus provided we are sufficiently far from the shower, say 100 meters from the core, we can estimate the shower radiation by multiplying (30) by N, where N is the number of excess electrons ($\sim 10^{5}$). Thus for a shower the incoherent emission amounts to

$$p(f)\,\delta f \simeq O(N/s) \quad f.u. \tag{32}$$

Since we wish to be about 100 meters away this requires that $s \simeq 5000$. Thus the amount of energy received per unit area is of the order of 20 f.u. . This must be compared with about 6 f.u. from the noise background using an 80 meter telescope. Thus the shower to noise ratio is about 3:1 and thus the shower should be detectable by means of the incoherent emission.

5. Conclusion

It has been shown that the Cerenkov radiation from an individual cosmic ray is detectable at radio frequencies provided a large enough receiver is used and provided the noise temperature can be reduced to around 20° K.

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Figure Captions

Figure 1. The possible paths of integration for the integral in Equation (12).

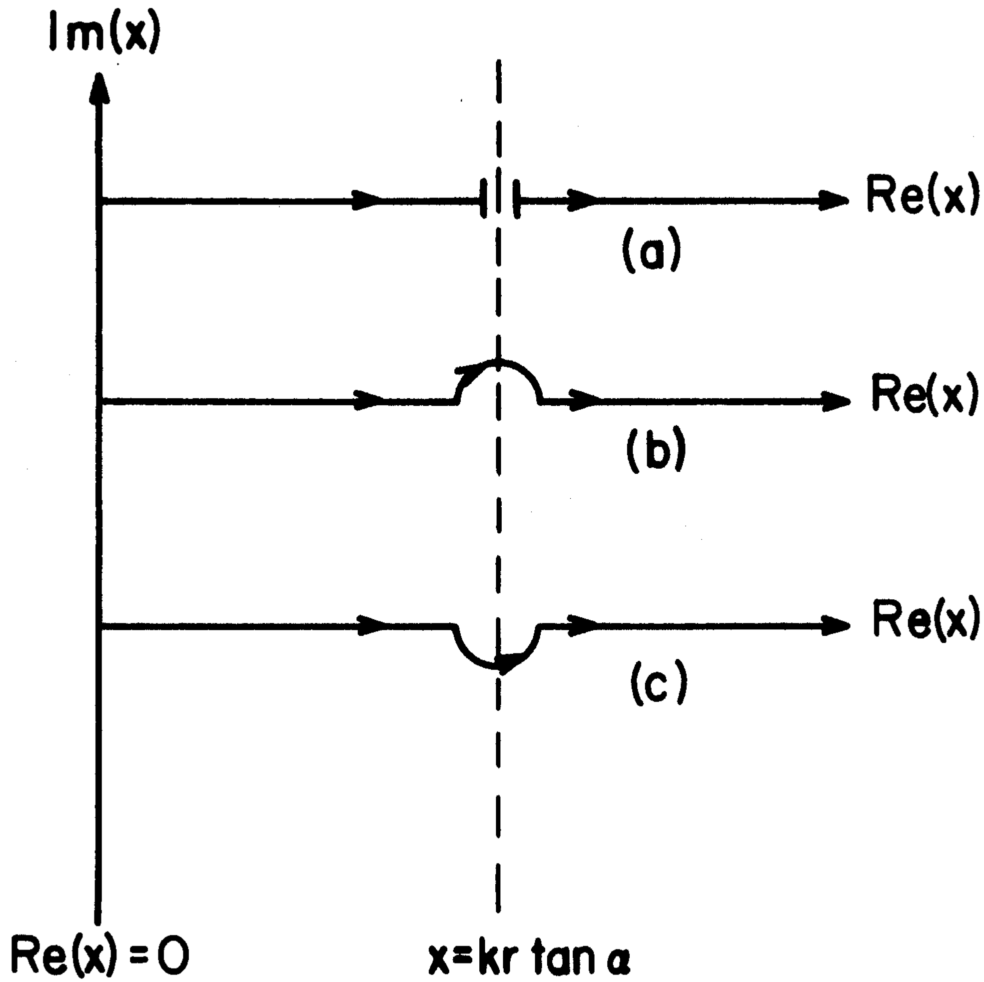

Fig. 1