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RENSSELAER POLYTECHNIC INSTITUTE
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SUMMARY

In many situations plasma phenomena vary rapidly in time. The theoretical formalism required in these cases must be capable of solving the revelant initial value problem. This paper discusses certain transport phenomena in plasmas in terms of initial value problems. Plasma diffusion is discussed from two points of view. If the moments of the Boltzmann equation are taken there results the macroscopic equations. This approach is shown to yield a diffusion equation which differs from the conventional equation which is based on Fick's law. Some gross aspects of the appropriate initial value problem are thereby revealed. A second point of view is developed by solving the Boltzmann equation using Fourier and Laplace transform techniques which permit the initial conditions to be inserted into the solution. This latter approach is shown to be the superior one. It is also used to study plasma transport in the presence of an applied electric field. In treating the collision operator a Lorentz gas model of the plasma is chosen. The solution is obtained in terms of odd and even components of the distribution function. It is shown that the initial condition propagates along a single particle orbit in phase space but the number of particles is depleted at the rate exp $(-\nu t/2)$ where ν is the electron collision frequency. Those particles which have suffered a collision are deposited in a residue in phase space. An asymptotic formula for the solution for relatively long times has been obtained. The collisional residue then contains the following elements: 1. the Maxwell-Boltzmann distribution, 2. a term describing the diffusion in configuration space independent of the electric field and 3. a term describing the heating of the electron gas by the electric field and the resulting diffusion in velocity space.

INTRODUCTION

The theory of transport phenomena in gases is largely based upon certain linear constitutive relations such as Fick's law,

$$\bar{\Gamma} = - D \nabla N \tag{1}$$

which relate transported quantities to gradients in the state variables. The validity of such laws has been well established for steady flows due to constant gradients.

However, many current plasma applications involve situations that may vary very rapidly in time, and there seems to have been little effort devoted toward determining the range of applicability of the linear laws or to extending the theory beyond this range.

For example, a common description of the time varying characteristics of the diffusion phenomenon is obtained by combining Fick's law with the equation of continuity to yield the well known diffusion equation,

$$\frac{\partial N}{\partial t} = D \nabla^2 N \tag{2}$$

Recently, this result has been criticized on the basis of the macroscopic plasma equations (Refs. 1-3). In order to review this approach we note that the first moment of the Boltzmann equation is commonly written in the form

$$\frac{\partial \Gamma}{\partial t} + \nabla \cdot \overline{\overline{\Pi}} = - \nu \Gamma \tag{3}$$

Simplifying the pressure tensor on the basis of a locally Maxwellian distribution and a uniform temperature (for example, see Huchital and Holt in ref. 1) yields

$$\frac{\partial \Gamma}{\partial t} + 2 \frac{\epsilon}{m} \nabla N = - \nu \Gamma \tag{4}$$

which, combined with the continuity equation, results in

$$\frac{\partial^2 N}{\partial t^2} + \nu \frac{\partial N}{\partial t} = 2 \frac{\epsilon}{m} \nabla^2 N \tag{5}$$

Equation (5) is then a description of the diffusion process in terms of the telegrapher's equation. To more clearly illustrate the difference between equations (2) and (5), they have been solved as initial value problems for a specific initial condition as shown in Figure 1. The solutions are

Equation (2)

$$N(x,t) = \frac{n}{\sqrt{\pi}} \left[\operatorname{erf}\left(\frac{x + a}{2\sqrt{Dt}}\right) - \operatorname{erf}\left(\frac{x - a}{2\sqrt{Dt}}\right) \right] \tag{6}$$

Equation (5)

$$N(x,t) = \frac{1}{2} e^{-\nu t/2} \left[N(x - vt, 0) + N(x + vt, 0) \right] + \frac{1}{2} e^{-\nu t/2} \int_{x - vt}^{x + vt} \left[N(\gamma, 0) \frac{\partial}{\partial t} J_o(\zeta) + \nu N(\gamma, 0) J_o(\zeta) \right] d\gamma \tag{7}$$

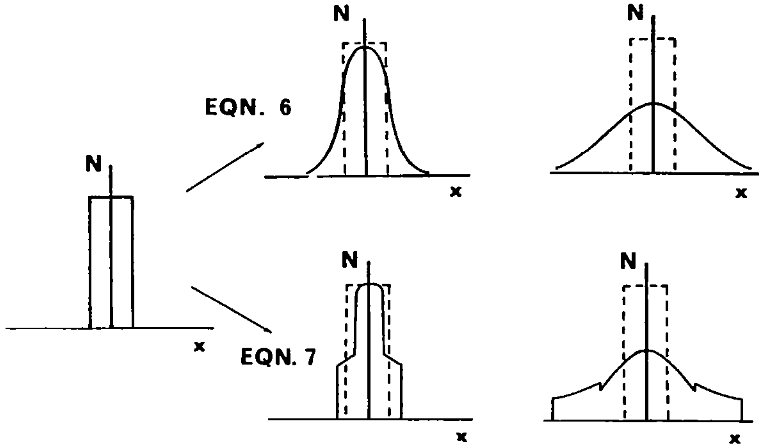

Figure 1. Comparison of the Solutions of the Diffusion Equation and the Telegrapher's Equation.

where

$$v = (2\epsilon/m)^{1/2} \quad \text{and} \quad \zeta = \frac{\nu}{2}\sqrt{\frac{m}{2\epsilon}}\sqrt{(x - \gamma)^2 - \frac{2\epsilon}{m}t^2}$$

The major points to be gathered from the solution for equation (5) are

1) the initial distribution splits and propagates with a finite speed along the positive and negative x axis, and

2) these propagating groups leave behind a bell-shaped residue which flattens out slowly.

3) the effect of collisions is to remove particles from the propagating groups at a rate $e^{-\nu t/2}$ and deposit them in the residue. It should be noted that the residue vanishes if $\nu = 0$.

While the description afforded by these analyses represents an improvement over the conventional theory, it is still open to objections. We have interpreted the initial propagating groups as consisting of particles that stream directly through the background gas. However, the solution, equation (7), implies that this group propagates intact at a velocity $(2\epsilon/m)^{1/2}$ even though we have postulated a distribution of velocities among the gas particles. The reasons for this deficiency are apparent when one considers the nature of the flow equation. The chief problem in obtaining a solution was the assumption necessary to reduce the variable $\overline{\overline{\Pi}}$. While the macroscopic approach can probably be improved to a certain extent by a more careful truncation of the series of equations, the point we wish to make here is that while these equations are attractive because they deal with physical quantities, the approach as a whole is difficult to apply to the problem at hand.

An alternative, which has remained relatively unexplored, is to study the Boltzmann equation itself and to obtain solutions for the macroscopic variables by integration of the result. It appears to us that a microscopic approach, directly from the distribution function, is considerably more promising. The Boltzmann equation is, after all, a single equation in one unknown. To be sure, it is also a nonlinear, integro-differential equation, but in this case, it requires fewer simplifying assumptions to reduce it to a tractable form than are required by the moment equations.

SOLUTION OF THE BOLTZMANN EQUATION FOR FREE DIFFUSION OF AN ELECTRON GAS

The problem of transport variables is directly related to the symmetry or anti-symmetry of the distribution function in velocity space. No flow can result in, say, the x direction if the distribution function is symmetric about the v_x axis.

For gradients only in the x direction, the force free Boltzmann equation becomes

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} = J(f) \quad (8)$$

Let us split the distribution function into odd and even components with respect to v_x, i.e.,

$$f(v_x) = f^o(v_x) + f^e(v_x) \quad (9)$$

We can then write the Boltzmann equation in the form

$$\frac{\partial f^o}{\partial t} + \frac{\partial f^e}{\partial t} + v_x \frac{\partial f^o}{\partial x} + v_x \frac{\partial f^e}{\partial x} = J(f^o + f^e) \quad (10)$$

and we can show by expansion in spherical harmonics that

$$J(f^o) \approx -\nu f^o$$

$$J(f^e) \approx 0$$

We now equate even and odd components of equation (10) to obtain

$$\frac{\partial f^o}{\partial t} + v_x \frac{\partial f^e}{\partial x} = -\nu f^o \quad (11)$$

$$\frac{\partial f^e}{\partial t} + v_x \frac{\partial f^o}{\partial x} = 0 \quad (12)$$

Equations (11) and (12) enable us to solve the Boltzmann equation as an initial value problem.

We can combine equations (11) and (12) to obtain partial differential equations in f^o and f^e individually, of the form

$$\frac{\partial^2 f}{\partial t^2} + \nu \frac{\partial f}{\partial t} = v_x^2 \frac{\partial^2 f}{\partial x^2} \quad (13)$$

However, it is important to note that f^O and f^e are still related by the initial conditions. If these conditions specify the values of $f^O(x,v_x,0)$, $f_e^{\ e}(x,v_x,0)$ $\lim_{t \to 0} \partial/\partial t\, f^O(x,v_x,t)$, and $\lim_{t \to 0} \partial/\partial t\, f^e(x,v_x,t)$ their relationship to each other is defined by equations (11) and (12).

Let us consider the problem of an initially symmetric distribution function. Then $f^O(x,v_x,0) = 0$ and $f^e(x,v_x,0) = F(x,v_x)$.

The remaining initial conditions are determined from equations (11) and (12) to be

$$\lim_{t \to 0} \partial/\partial t\, f^O(x,v_x,t) = -v_x\, \partial/\partial x\, F(x,v_x)$$

$$\lim_{t \to 0} \partial/\partial t\, f^e(x,v_x,t) = 0$$

Equation (13) is a form of the telegrapher's equation. Combining the results for f^O and f^e yields

$$f(x,v_x,t) = e^{-\nu t/2}\, F(x - v_x t, v_x) + \frac{\nu e^{-\nu t/2}}{2 v_x} \int_{x-v_x t}^{x+v_x t} F(\gamma, v_x) \left[QI_1(\Lambda) + I_0(\Lambda) \right] d\gamma$$

where

$$Q = \sqrt{\frac{v_x t - x + \gamma}{v_x t + x + \gamma}} \qquad \Lambda = \frac{\nu}{2v_x} \sqrt{v_x^{\,2} t^2 - (x - \gamma)^2} \qquad (14)$$

In contrast with the previous result, equation (7), it may appear that equation (14) predicts propagation of the free streaming group only along the positive x axis. In this connection, it is necessary to recall that v_x takes both positive and negative value so that propagation in both directions is implied. Second and most important, equation (14) predicts that each velocity class diffuses at its own intrinsic speed. Therefore, a diffusing gas smears out due to a distribution of initial velocities. In addition, we note that the rate of development of the diffusion phenomenon is determined by the initial distribution of velocities, $F(v_x)$, so that it becomes questionable to try to express the expansion of a diffusing gas in terms of macroscopic parameters.

A further implication of equation (14) that might, under some circumstances, be important, is the point that "hot" particles diffuse more quickly than do "cool" ones. Therefore, temperature gradients are immediately set up in a diffusing gas so that it is unrealistic, though not necessarily inaccurate, to discuss density gradients in the nonsteady state without considering the associated temperature gradients.

SOLUTION OF THE BOLTZMANN EQUATION FOR AN ELECTRON GAS IN AN ELECTRIC FIELD

If we include the effects of an external field on the previous model, the Boltzmann equation takes the form

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + R\frac{\partial f}{\partial v_x} = J(f) \tag{15}$$

where $R = eE/m$. By a method entirely analogous to that used in the previous section, we find we can write

$$\frac{\partial f^e}{\partial t} + R\frac{\partial f^0}{\partial v_x} = -v_x \frac{\partial f^0}{\partial x} \tag{16}$$

$$\frac{\partial f^0}{\partial t} + R\frac{\partial f^e}{\partial v_x} = -v_x \frac{\partial f^e}{\partial v_x} - \nu f^0 \tag{17}$$

So that if we take the initial condition to be

$$f^e(x,v_x,0) = F(x,v_x);\ f^0(x,v_x,0) = 0$$

we find

$$f(x,v_x,t) = e^{-\nu t/2} F(x - v_x t - \frac{Rt^2}{2}, v_x - Rt)$$

$$+ \frac{\nu e^{-\nu t/2}}{2R} \int_{v_x - Rt}^{v_x + Rt} F(x - \frac{v_x{}^2}{2R} - \frac{\gamma^2}{2R}) \left[I_0(\Lambda) + QI_1(\Lambda) \right] d\gamma \tag{18}$$

where

$$\Lambda = \frac{\nu}{2R}\sqrt{R^2t^2 - (v_x - \gamma)^2}\ ; \qquad Q = \frac{v_x - \gamma + Rt}{\left[(v_x - \gamma)^2 + R^2t^2\right]^{1/2}} \tag{19}$$

Equation (18) demonstrates the role of collisions in flow in an accelerating field. We note that the initial condition migrates along a single particle orbit in phase space but that the number of particles in the class is depleted at a rate $e^{-\nu t/2}$. These particles appear in a phase space residue described by the two integral terms.

Asymptotic Formula for Long Time

In order to discuss equation (18) more fully, we will perform an asymptotic expansion in the neighborhood of the origin in phase space. Let us simplify the process by writing $F(x,v_x) = n\,\delta(x)G(v_x)$. Making use of the fact that $G(v_x) = G(-v_x)$ and the well-known asymptotic formulas for modified Bessel functions of large argument, we find

$$f(x,v_x,t) \simeq n\sqrt{\frac{2}{R}}\,G\left(v_x - \frac{Rx}{v_x}\right)\left(\frac{\nu}{\pi t}\right)^{1/2} \times \left[\exp\left(-\frac{\nu x^2}{4v_x{}^2 t}\right) + \exp\left(-\frac{\nu v_x{}^2}{R^2 t}\right)\right] \tag{20}$$

It is interesting to note that the distribution function is now even in velocity space because

$$G(v_x - Rx/v_x) = G\left[(-v_x - Rx/(-v_x)\right]$$

The situation is somewhat more difficult to discuss in configuration space as the most interesting characteristics of the solution are contained in the term $G(v_x - Rx/v_x)$. In order to illustrate these points, let us choose the specific example of a Gaussian initial velocity distribution, that is

$$G(v_x) = A\exp(-Kv_x{}^2)$$

Then

$$f(x,v_x,t) \sim nA\left(\frac{2\nu}{\pi R t}\right)^{1/2}\exp\left[2KRx - Kv_x{}^2 - KR^2x^2/v_x{}^2\right] \times \left[\exp\left(-\frac{\nu x^2}{4v_x{}^2 t}\right) + \exp\left(-\frac{\nu v_x{}^2}{R^2 t}\right)\right]$$

The first term in square brackets is the normal diffusion term referring to the thinning out of the distribution in space due to its initial thermal energy. The term in $\exp(2KRx)$ is most interesting as it is odd in x whereas all other terms are even. Therefore the asymptotic expansion implies more particles for positive x than negative x. This is certainly a reasonable conclusion, as the external force has been postulated in the positive x direction. Of course, for large x, the solution is dominated by terms in e^{-x^2} so the distribution becomes vanishingly small at large distances from the origin. We have attempted to illustrate these characteristics in Figure 2 where we have considered the initial condition of a rectangular pulse in configuration space with a Gaussian distribution of velocities. We have illustrated the result in terms of the particle density,

$$N(x,t) = \int f dv_x$$

and the total velocity distribution

$$Z(v_x,t) = \int f dx$$

Energy Considerations

The asymptotic formula for the residue can be put in terms of familiar quantities if we consider the second term in square brackets of equation (20). This is essentially a description of the modification of the distribution in velocity space by the electric field. Now we can estimate the average power delivered by the field to the electrons by the formula

$$P = \sigma_{gas} E^2$$

where σ_{gas} is the conductivity of the electron gas. As an order of magnitude estimate, we take

$$\sigma_{gas} = \frac{Ne^2}{m \nu}$$

so that the energy delivered per particle by the electric field is

$$\langle E \rangle = \frac{mR^2}{\nu} t$$

We can therefore write the second term in square brackets of equation (20) as

$$\exp\left[-\frac{m v_x^2}{E}\right] - \exp\left[-\frac{2 \text{ K.E.}}{\langle E \rangle}\right]$$

We see, then, that the distribution falls off according to the ratio of kinetic energy to energy delivered by the field. Now since all the particles in the residue have suffered collisions, the velocities, in this central portion of phase space are completely random. We can therefore define a new thermal energy, $\Theta(t)$, as being equal to the energy delivered by the field into random motion of the particles.

We can continue this procedure to put the entire residue in terms of particle energies. Let us define the reference level of potential energy at $x = 0$, so that

$$\text{potential energy} = \text{P. E.} = - mRx$$

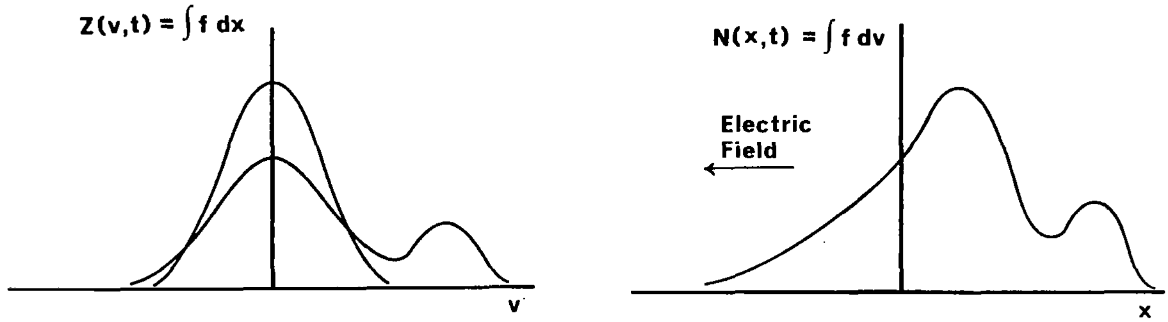

Figure 2. Development of the Initial Condition $F(x,v_x) = n\,\delta(x)\exp(-Kv_x^2)$ in an Electric Field.

Then if we write the initial distribution of velocities in the form

$$e^{-Kv_x^2} = \exp\left(-\frac{m}{2\theta_o}v_x^2\right)$$

where θ_o is the initial thermal energy, we can write the complete expression for the residue as

$$f \sim n\,A\sqrt{\frac{2\nu}{\pi R t}}\;\exp\left[-\frac{K.E. + P.E.}{\theta_o}\right] \times$$

$$\exp\left(-\frac{K\,R^2X^2}{V_x^2}\right)\left[\exp\left(\frac{-\nu x^2}{4\,v_x^2\,t}\right) + \exp\left(-\frac{2\,K.E.}{\theta(t)}\right)\right] \qquad (20)$$

We note that the first element of this result is the conventional Maxwell-Boltzmann distribution, that is exp(-total energy/thermal energy) where the thermal energy is the value determined by the initial conditions. And as noted above, the last term is a modified Maxwellian distribution referred to a monotonically increasing thermal energy representing the randomization of energy delivered by the field.

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