THE CYLINDRICAL ANTENNA WITH TAPERED RESISTIVE LOADING

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By

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SUMMARY

The current, the input impedance and the far field pattern of a cylindrical antenna with resistive loading are determined. The distribution of the resistive loading along the antenna is a particular function multiplied by a constant positive parameter α. The current on the antenna and the field pattern are evaluated for a wide range of lengths with several different α's ranging from 0 to 1 and for positive integers. They are found not critically dependent on the parameter α. For α near or greater than 1, the antenna is non-reflecting.

INTRODUCTION

In a recent paper by Wu and King [1] it is found that if the antenna is made of resistive material such that the internal impedance is a particular function of the position along the antenna, a pure outward traveling wave exists on an antenna of finite length. They found that if

$$z^{i}(z) = \frac{\zeta_o \Psi}{4\pi} \frac{2}{h - |z|} \tag{1}$$

then the zero-order current will be

$$I(z) = C(h - |z|)\, e^{ikz} \tag{2}$$

The constant Ψ in (1) is determined by

$$\Psi\, I(z) \simeq \int_{-h}^{h} I(z') \frac{e^{ik\sqrt{(z-z')^2 + a^2}}}{\sqrt{(z-z')^2 + a^2}}\, dz' \tag{3}$$

In the above equations, z is the axial coordinate, z^i is the internal impedance per unit length, ζ_o is the intrinsic impedance of free space, h is the half length of the antenna, k is the free-space wave number, and a is the radius of the antenna. The time dependence is assumed to be $e^{-i\omega t}$.

It is interesting to see how the traveling wave solution is affected when the distribution of the resistive loading is changed from the value prescribed by (1). A change in the internal impedance involves two problems. First, is the task of finding the new current distribution, i.e., solving the changed differential equation. Secondly, and more importantly, is the requirement that the solution obtained be integrable in the sense that the field pattern can be obtained numerically without too much complication.

It turns out that if the internal impedance is changed to

$$z^i(z) = \frac{\zeta_o \Psi}{4\pi} \frac{2\alpha}{h - |z|} \tag{4}$$

where α is a positive constant, the current distribution is the product of the linear decaying traveling wave function (2) and a confluent hypergeometric function. Moreover, and by a new method, the field pattern can be cast into a form which is readily evaluated numerically.

It is clear that when $\alpha = 0$ the current distribution is expected to be identical with the zero-order current of an ordinary dipole antenna. When $\alpha = 1$, the current should yield King and Wu's result. Thus, two references are on hand.

THE DIFFERENTIAL EQUATION AND ITS SOLUTION

The differential equation to be solved is the following:

$$\left(\frac{d^2}{dz^2} + k^2 + \frac{2i\alpha k}{h - |z|}\right) I(z) = \frac{i4\pi k}{\zeta_o \Psi} V_o^e \, \delta(z) \tag{5}$$

For $z > 0$, let

$$I(z) = A e^{ikz} (h - z) \, \Phi[2ik(h - z)] \tag{6}$$

then, from (5)

$$y\frac{d^2}{dy^2}\Phi(y) + (2-y)\frac{d}{dy}\Phi(y) + (\alpha-1)\Phi(y) = 0 \tag{7}$$

where $y = 2ik(h - z)$. A comparison of (7) with equation (6.1.2) of Reference 2 shows that $\Phi(y) = B\,\Phi(1-\alpha,2;y)$. Since α is to vary from 0 to any positive number, it is convenient to define $\Phi(y)$ by a contour integral as follows:

$$\Phi(y) = \oint_C e^{yu}\left(\frac{1-u}{u}\right)^{\alpha} du \tag{8}$$

Substitute (8) into (7), and the result is

$$\left[y\frac{d^2}{dy^2} + (2-y)\frac{d}{dy} - (1-\alpha)\right]\int_C e^{yu}\left(\frac{1-u}{u}\right)^{\alpha} du \equiv -\int_C \frac{d}{du}\left[e^{yu}\,u^{1-\alpha}(1-u)^{1+\alpha}\right]du \tag{9}$$

One of the possible choices of C to make the right hand side of (9) vanish is shown in Fig. 1. Such a choice also makes $\Phi(y)$ finite when $y \to 0$ so that it satisfies the boundary condition that $I(h) = 0$.

Thus, the current distribution is formally expressed as

$$I(z) = A\,e^{ik|z|}(h-|z|)\,\Phi[2ik(h-|z|)] \tag{10}$$

The constant A can be found from the equation

$$\left.\frac{dI(z)}{dz}\right|_{z=0} = \frac{1}{2}\,\frac{4\pi ik\,V_o^e}{\zeta_o\Psi}$$

It is found to be

$$A = \frac{1}{2}\,\frac{4\pi ik\,V_o^e}{\zeta_o\Psi}\,\frac{1}{(ikh-1)\,\Phi(2ikh) - 2ikh\,\Phi'(2ikh)} \tag{11}$$

Note that the confluent hypergeometric function is defined as

$$\Phi(a;c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \tag{12}$$

and has the integral representation

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 e^{xu}\,u^{a-1}(1-u)^{c-a-1}\,du \tag{13}$$

Evidently (13) cannot hold for $a < 0$. But if $\Phi(a,c;x)$ is defined in terms of a different contour, say along C in Fig. 1 rather than from 0 to 1, as follows:

$$\tilde{\Phi}(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_C e^{xu}\, u^{a-1}\,(1-u)^{c-a-1}\, du \tag{14}$$

then $\tilde{\Phi}$ reduces to Φ (except for a complex constant) for $a > 0$ and $c-a-1 > 0$. It can be shown that (14) is equivalent to the series representation (12) even for $a < 0$ and $c-a-1 > 0$. This is accomplished by means of the following relation

$$\frac{d}{dx}\tilde{\Phi}(a,c;x) = \frac{a}{c}\tilde{\Phi}(a+1,c+1;x) \tag{15}$$

Repeated use of (15)would finally make the first argument of $\tilde{\Phi}$ on the right in (15) positive, and whenever it is positive it is equivalent to the series representation (12). The series is then integrated as many times as the relation (15) was used. This completes the proof.

Since in the present case $a = 1-\alpha$, it follows that if α is a positive integer, $\Phi(a,c;x)$ is a polynomial as seen from (12).

EVALUATION OF Ψ

Ψ is determined by the following equation:

$$\Psi = \frac{\int_{-h}^{h} e^{ik|z'|}(h-|z'|)\,\Phi[2ik(h-|z'|)]\frac{e^{ik\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}}\,dz'}{e^{ik|z|}\,(h-|z|)\,\Phi[2ik(h-|z|)]} \tag{16}$$

when $z = 0$,

$$\Psi = \frac{2\int_{0}^{h} e^{ikz}(1-\frac{z}{h})\,\Phi[2ik(h-z)]\frac{e^{ik\sqrt{z^2+a^2}}}{\sqrt{z^2+a^2}}\,dz}{\Phi(2ikh)} \tag{17}$$

If the integral representation (8) for Φ is substituted into (17), and $\sqrt{z^2+a^2}$ is equated approximately to z, the result is

$$\Psi \approx \frac{2\int_0^h \frac{e^{2ikr_o}}{r_o}\int_C e^{2ik(h-r_o)u}\left(\frac{1-u}{u}\right)^\alpha du\,dz - \frac{1}{h}\int_0^h e^{2ikz}\int_C e^{2ikh(h-z)u}\left(\frac{1-u}{u}\right)^\alpha du\,dz}{\int_C e^{2ikhu}\left(\frac{1-u}{u}\right)^\alpha du} \tag{18}$$

where $r_o = \sqrt{z^2+a^2}$. A change in the order of integration of (18) gives

$$\Psi \approx \frac{2\int_C \left(\frac{1-u}{u}\right)^\alpha e^{2ikhu}\left[\sinh^{-1}\frac{h}{a} - C(2A,2H) + iS(2A,2H) - \frac{i}{H}(1-e^{i2H})\right] du}{\int_C \left(\frac{1-u}{u}\right)^\alpha e^{2ikhu}\,du} \tag{19}$$

where $A = ka(1-u)$, $H = kh(1-u)$, and C and S are tabulated functions [3, Appendix] whose definitions are as follows:

$$C(a,x) = \int_0^x \frac{1-\cos\sqrt{u^2+a^2}}{\sqrt{u^2+a^2}}\,du \qquad S(a,x) = \int_0^x \frac{\sin\sqrt{u^2+a^2}}{\sqrt{u^2+a^2}}\,du \tag{20}$$

Thus finally

$$\Psi = 2\sinh^{-1}\frac{h}{a} + \gamma \tag{21}$$

where

$$\gamma = 2\,\frac{\int_C \left(\frac{1-u}{u}\right)^\alpha e^{2ikhu}\left[iS(2A,2H) - C(2A,2H) - \frac{i}{H}(1-e^{i2H})\right] du}{\int_C \left(\frac{1-u}{u}\right)^\alpha e^{2ikhu}\,du} \tag{22}$$

The integral in (22) can be evaluated numerically.

THE FIELD PATTERN

Let

$$\mathcal{F}(\zeta) = \int_0^h I(z)\, e^{i\zeta z}\, dz \tag{23}$$

then the field pattern is essentially $\mathcal{F}(\zeta) + \mathcal{F}(-\zeta)$. When $z \geq 0$, (5) gives

$$(h-z)I''(z) + k^2(h-z)I(z) + 2i\alpha k\, I(z) = \bar{V}\,(h-z)\,\delta(z) \tag{24}$$

where

$$\bar{V} = \frac{i4\pi k\, V_o^e}{\zeta_o \Psi}$$

The following relations are known:

$$\int_0^h I(z)(h-z)\, e^{i\zeta z}\, dz = h\,\mathcal{F}(\zeta) + i\mathcal{F}'(\zeta) \tag{25a}$$

$$\int_0^h I''(z)(h-z)e^{i\zeta z}\, dz = -hI'(0)+I(0)(-1+ih\zeta) - \int_0^h I(z)[\zeta^2(h-z)+2i\zeta]e^{i\zeta z}\, dz$$

$$= -i\zeta^2\,\mathcal{F}'(\zeta) - \zeta(2i+\zeta h)\,\mathcal{F}(\zeta) - (1-i\zeta h)\, I(0) \tag{25b}$$

Clearly $I'(0)$ is zero since $I(z)$ is an even function. The next step is to integrate both sides of (24) from 0 to h with respect to z, to get

$$i(k^2-\zeta^2)\mathcal{F}'(\zeta) + [(k^2-\zeta^2)h + 2i(\alpha k-\zeta)]\mathcal{F}(\zeta) = \frac{\bar{V}h}{2} + (1-i\zeta h)\, I(0) \tag{26}$$

Note that the δ-function has been taken care of properly. The integrating factor of (26) is

$$\vartheta = \exp\int d\zeta\, \frac{(k^2-\zeta^2)h + 2i(\alpha k-\zeta)}{i(k^2-\zeta^2)} = e^{-ih\zeta}\,(k+\zeta)^{1+\alpha}\,(k-\zeta)^{1-\alpha}$$

Thus the solution of (26) is essentially

$$\mathcal{F}(\zeta)\, e^{-ih\zeta}(k+\zeta)^{1+\alpha}\,(k-\zeta)^{1-\alpha} - \mathcal{F}(0)\, k^2$$

$$= -i\int_0^{\zeta} \left[\frac{\bar{V}h}{2} + (1-i\zeta' h)\, I(0)\right] e^{-ih\zeta'}\,(k+\zeta')^{\alpha}\,(k-\zeta')^{-\alpha}\, d\zeta' \tag{27}$$

$$F(y) = e^{ikhy} \left(\frac{1-y}{1+y}\right)^{\alpha} \frac{1}{1-y^2} [a_o + a_1 I_1(y) + a_2 I_2(y)] \tag{28a}$$

where $F(y) = \mathcal{F}(ky)$, $a_o(kh) = \mathcal{F}(0)$, $a_1(kh) = \frac{-i}{k} [\frac{h\overline{V}}{2} + I(0)]$, $a_2(kh) = -h\ I(0)$.

$$I_1(y) = \int_0^y e^{-ikhy'} \left(\frac{1+y'}{1-y'}\right)^{\alpha} dy'$$

$$I_2(y) = \int_0^y y'\ e^{-ikhy'} \left(\frac{1+y'}{1-y'}\right)^{\alpha} dy'$$

It follows with (10) and (11) and the fact that $F(\pm 1)$ is bounded, that

$$a_1(kh) = -Ah^2 \int_C (2u-1)\ e^{i2khu} \left(\frac{1-u}{u}\right)^{\alpha} du$$

$$a_2(kh) = -Ah^2 \int_C e^{2ikhu} \left(\frac{1-u}{u}\right)^{\alpha} du \tag{28b}$$

$$a_o(kh) = -a_1(kh)\ I_1(1) - a_2(kh)\ I_2(1)$$

Usually the field pattern is obtained from a direct integration of (23). If (10) is substituted into (23), this becomes

$$\mathcal{F}(\zeta) = A \int_0^h e^{ikz} (h-z)\ \Phi[2ik(h-z)]\ e^{i\zeta z}\ dz$$

$$= A\ e^{i(\zeta+k)h} \int_0^h dy\ y\ e^{-i(\zeta+k)y} \int_C e^{2ikyu} \left(\frac{1-u}{u}\right)^{\alpha} du$$

$$= A \int_C du \left(\frac{1-u}{u}\right)^{\alpha} \left[\frac{ih\ e^{i2khu}}{\zeta+k\ -2ku} + \frac{e^{i2khu} - e^{i(\zeta+k)h}}{(\zeta+k\ -2ku)^2}\ \right]$$

If the second term is integrated by parts the result is

$$\mathcal{F}(\zeta) = \int_C \frac{e^{ih(\zeta+k)} - e^{i2khu}}{\zeta+k-2uk} \frac{d}{du}\left(\frac{1-u}{u}\right)^{\alpha} du \tag{29}$$

It can be shown that (29) satisfies the differential equation (26). The procedure involves only straight forward substitution although the algebra is tedious. The form of (29) is inconvenient for numerical evaluation. However, it can be used to calculate $\mathcal{F}(\zeta)$ when α is a positive integer.

NUMERICAL CALCULATIONS

A. The Field Pattern, $0< \alpha < 1$

The electric field pattern is obtained from (28), since the electric field is $(1-y^2)^{1/2}[F(y) + F(-y)]$. Thus,

$$F_{tot} = \sqrt{1-y^2}\left\{\frac{-a_1-a_2y}{1+y} + e^{-ikh(1-y)}\left(\frac{1-y}{1+y}\right)^{\alpha}\frac{1}{1-y^2}[a_1 P_1(y) + a_2 P_2(y)]\right.$$
$$\left. + e^{ikh(1-y)}\left(\frac{1+y}{1-y}\right)^{\alpha}\frac{(1-\alpha)}{1-y^2}[a_1 P_3(y) + a_2 P_4(y)]\right\}$$

where $y = \cos\theta$

$$a_1(kh) = [-P_4(0,kh)(1-\alpha)\, e^{ikh} + e^{-ikh} P_2(0,kh)]\, F_o \tag{30a}$$

$$a_2(kh) = [1 - e^{-ikh} P_1(0,kh) + (1-\alpha)\, e^{ikh} P_3(0,kh)]\, F_o \tag{30b}$$

$$P_1(y,kh) = \int_0^{1-y} z^{1-\alpha}\, e^{ikhz} \frac{(2ikh - \alpha - ikhz)}{(2-z)^{1-\alpha}}\, dz \tag{31a}$$

$$P_2(y,kh) = \int_0^{1-y} z^{1-\alpha} \frac{e^{ikhz}[ikh(2-z)(1-z) - \alpha(1-z)-(2-z)]}{(2-z)^{1-\alpha}} dz \qquad (31b)$$

$$P_3(y,kh) = \int_0^{1-y} z^{\alpha} \frac{e^{-ikhz}}{(2-z)^{\alpha}} dz \qquad (31c)$$

$$P_4(y,kh) = \int_0^{1-y} z^{\alpha} \frac{(z-1)\, e^{-ikhz}}{(2-z)^{\alpha}} dz \qquad (31d)$$

$$F_o = \frac{-A\, h^2}{2(1-\alpha)}\, e^{ikh}$$

B. The Current, $0 < \alpha < 1$

From (10) and (28b) the current is expressed in the following form

$$I(z) = -\frac{1}{h^2}\, e^{ik|z|} (h-|z|)\; a_2[kh(1-\frac{|z|}{h})]$$

Or, as seen from (30b),

$$I(z) = \frac{A}{2(1-\alpha)}\, e^{ikh} (h-|z|)\{1-e^{ikh(1-\frac{|z|}{h})} P_1[0,kh(1-\frac{|z|}{h})]+(1-\alpha)e^{ikh(1-\frac{|z|}{h})} P_3[0,kh(1-\frac{|z|}{h})]\} \qquad (32)$$

The numerical integration for the P-functions cannot be done by standard methods such as Simpson's rule. This is due to the fact that the integrand has no Taylor's series expansion at one of the integration limits, namely, $z = 0$. A formula for numerical integration near such a point is developed in the Appendix and is tried here for calculating P-functions.

C. The Current, when α is a positive integer

From Kummer's series (12), it follows that

$$I(z) = A\, e^{ikz} (h-z) \qquad \text{for } \alpha = 1$$

$$I(z) = A\, e^{ikz} (h-z)[1 - ik(h-z)] \quad \text{for } \alpha = 2$$

$$I(z) = A\, e^{ikz} (h-z)[1 - 2ik(h-z) - \tfrac{2}{3}k^2(h-z)^2] \quad \text{for } \alpha = 3$$

And so on.

D. The Field Pattern, when α is a positive integer

From (29), the electric field pattern is obtained easily.

$$F_{tot} = \sqrt{1-y^2}\, [F(y) + F(-y)]$$

$$F(y) = \frac{ikh}{1+y} + \frac{1 - e^{i(1+y)kh}}{(1+y)^2} \quad \text{for } \alpha = 1$$

$$F(y) = \frac{-2kh(kh+i)}{1+y} + \frac{-2+4ikh+2e^{i(1+y)kh}}{(1+y)^2} + \frac{4(1-e^{i(1+y)kh})}{(1+y)^3} \quad \text{for } \alpha = 2$$

$$F(y) = \frac{1}{1+y}(6ikh + 12k^2h^2 - 4ik^3h^3) + \frac{1}{(1+y)^2}(6 - 6e^{i(1+y)kh} - 24ikh - 12k^2h^2)$$
$$+ \frac{1}{(1+y)^3}(-24 + 24e^{i(1+y)kh} + 24ikh) + \frac{1}{(1+y)^4}(24 - 24e^{i(1+y)kh}) \quad \text{for } \alpha = 3.$$

And so on.

PERTURBATION OF α ON THE CURRENT

For α near 1, the P-functions (31) can be expanded into powers of $(1-\alpha)$.

$$P_1(0,y) = \int_0^1 \left(\frac{z}{2-z}\right)^{1-\alpha} e^{iyz} [i2y - iyz - 1 + (1-\alpha)]\, dz \qquad (33)$$

Let

$$\left(\frac{z}{2-z}\right)^{1-\alpha} = e^{(1-\alpha)\ln(z/2-z)} \simeq 1 + (1-\alpha)\ln\left(\frac{z}{2-z}\right)$$

and substitute this into (33). It becomes

$$P_1(0,y) = P_{10}(y) + (1-\alpha)\, P_{1\alpha}(y)$$

where

$$P_{10}(y) = \int_0^1 e^{iyz}\,[i2y - iyz - 1]\,dz = e^{iy} - 2$$

$$P_{1\alpha}(y) = \int_0^1 \left(\ln \frac{z}{2-z}\right) \frac{d}{dz}\left[e^{iyz}(2-z)\right] dz + \int_0^1 e^{iyz}\,dz$$

$$= 2\ln 2 + 2C(0,y) - 2iS(0,y) + \frac{1}{iy}\left[e^{iy} - 1\right]$$

The following relation has been used in the above integration:

$$\int_0^1 \ln z \frac{d}{dz}\left(e^{iyz}\right) dz = \left(e^{iyx} - 1\right)\ln x + C(0,yx) - iS(0,yx)$$

Similarly, we expand $P_3(0,y)$ but only its first term is needed as can be seen from (32).

$$P_3(0,y) = \int_0^1 z^{\alpha} \frac{e^{-iyz}}{(2-z)^{\alpha}}\,dz \simeq \int_0^1 \frac{z\,e^{-iyz}}{2-z}\,dz$$

Thus,

$$P_3(0,y) \simeq 2e^{-i2y}\left[C(0,y) - C(0,2y) - iS(0,y) + iS(0,2y) + \ln 2\right] - \frac{1}{iy}\left[1 - e^{-iy}\right]$$

Under these approximations, the current becomes

$$I(z) = \frac{A}{(1-\alpha)}\left[J_o(z) + (1-\alpha)J_{\alpha}(z)\right]$$

where

$$J_o(z) = (h - |z|)\,e^{ik|z|}$$

$$J_{\alpha}(z) = -e^{ik|z|}\left[C\big(0,2k(h-|z|)\big) - iS\big(0,2k(h-|z|)\big)\right](h-|z|) - \frac{e^{ikh}}{k}\sin k(h-|z|)$$

In order to get first order perturbation terms, we have to expand the coefficient A.

$$\Phi(2ikh) = \frac{I(0)}{Ah} \simeq \frac{1}{1-\alpha}\left\{1 - (1-\alpha)\left[C(0,2kh) - iS(0,2kh) + \frac{e^{ikh}}{kh}\sin kh\right]\right\}$$

$$\Phi'(2ikh) = -\frac{1}{2ikh}\left[1 + \frac{1 - e^{2ikh}}{2ikh}\right]$$

$$A = \frac{2\pi ikV_o^e}{\zeta_o\Psi}\;\frac{1}{(ikh-1)\{\frac{1}{1-\alpha} - [C(0,2kh)-S(0,2kh)+\frac{e^{2ikh}-1}{2ikh}]\} + 1 + \frac{1-e^{2ikh}}{2ikh}}$$

Let, $A = A_o + A_\alpha$, then

$$A_o = \frac{2\pi ikV_o^e}{\zeta_o\Psi}\;\frac{(1-\alpha)}{ikh-1}$$

$$A_\alpha = A_o(1-\alpha)\left\{\frac{(ikh-1)[C(0,2kh) - iS(0,2kh)] + \frac{e^{2ikh}-3}{2}}{(ikh - 1)}\right\}$$

Let, $I(z) = I_o(z) + I_\alpha(z)$, then

$$I_o(z) = A_oJ_o = \frac{2\pi\;V_o^e}{\zeta_o\Psi(1+i/kh)}\,(1-\frac{|z|}{h})\;e^{ik|z|} \tag{34a}$$

$$I_\alpha(z) = J_oA_\alpha + A_oJ_\alpha = \frac{2\pi\;V_o^e(1-\alpha)}{\zeta_o\Psi(1+i/kh)}\left\{(1-\frac{|z|}{h})e^{ik|z|}[C(0,2kh) - C\big(0,2k(h-|z|)\big)\right.$$

$$\left. - iS(0,2kh) + iS\big(0,2k(h-|z|)\big) + \frac{e^{2ikh}-3}{2(ikh-1)}] - \frac{e^{ikh}}{kh}\sin k(h-|z|)\right\} \tag{34b}$$

Note that (34a) is exactly what has been obtained by Wu and King [ref. 1, eq. 21].

When $\alpha \simeq 0$, the expansions are as follows

$$P_1(0,y) = \int_0^1 (\frac{z}{2-z})^{1-\alpha}\;e^{iyz}\;(2iy - iyz - \alpha)\;dz$$

$$P_1(0,y) = P_{10}(y) + \alpha\;P_{1\alpha}(y)$$

where

$$P_{10}(y) = \int_0^1 \frac{z}{2-z}\;e^{iyz}\;(2iy - iyz)\;dz = e^{iy} - \frac{1}{iy}\;(e^{iy}- 1)$$

$$P_{1\alpha}(y) = \int_0^1 z\;\ln(\frac{2-z}{z})\frac{d}{dz}(e^{iyz})dz - \int_0^1 \frac{z}{2-z}\;e^{iyz}\;dz$$

$$= \frac{1}{iy}\left\{e^{iy}-1+C(0,y)-iS(0,y)+ e^{i2y}[(e^{-i2y}- 1)\ln 2 +C(0,2y)+iS(0,2y)\right.$$

$$\left. - C(0,y)\;-iS(0,y)]\right\}$$

$$P_3(0,y) = \int_0^1 \left(\frac{z}{2-z}\right)^{\alpha} e^{-iyz}\, dz$$

$$P_3(0,y) \simeq P_{30}(y) + \alpha\, P_{3\alpha}(y)$$

where

$$P_{30}(y) = \int_0^1 e^{-iyz}\, dz = \frac{1 - e^{-iy}}{iy}$$

$$P_{3\alpha}(y) = \int_0^1 \ln\left(\frac{z}{2-z}\right) e^{-iyz}\, dz$$

$$= \frac{-1}{iy}\Big\{C(0,y) + iS(0,y) + e^{-i2y}\big[(e^{i2y}-1)\ln 2 + C(0,2y) - iS(0,2y)$$

$$- C(0,y) + iS(0,y)\big]\Big\}$$

When these approximations are substituted into (32), the current becomes

$$I(z) = \frac{A}{(1-\alpha)}\,[K_o(z) + \alpha\, K_\alpha(z)]$$

where

$$K_o(z) = \frac{e^{ikh}}{k}\sin k(h-|z|)$$

$$K_\alpha(z) = \frac{-e^{ikh}}{k}\Big\{\sin k(h-|z|) + S\big(0,2k(h-|z|)\big)\sin k(h-|z|)$$

$$- iC\big(0,2k(h-|z|)\big)\cos k(h-|z|)\Big\}$$

$$\Phi(2ikh) \simeq \frac{e^{2ikh}-1}{2ikh} - \alpha\,\frac{1}{2ikh}\Big\{[S(0,2kh)+C(0,2kh)]e^{2ikh} + C(0,2kh) - S(0,2kh)\Big\}$$

$$\Phi'(2ikh) \simeq \frac{1}{2ikh}\left[e^{2ikh} - \frac{e^{2ikh}-1}{2ikh}\right] - \alpha\Bigg\{\frac{[S(0,2kh)+C(0,2kh)]e^{2ikh}}{2ikh}$$

$$+ \frac{e^{2ikh}[\sin 2kh + 1 - \cos 2kh - S(0,2kh) - C(0,2kh)] + [1 - \cos 2kh - \sin 2kh - C(0,2kh) + S(0,2kh)]}{(2ikh)^2}\Bigg\}$$

$$A \simeq A_o + \alpha\, A_\alpha$$

where

$$A_o = \frac{2\pi i k V_o^e}{\zeta_o \Psi} (-e^{ikh} \cos kh)$$

$$A_\alpha = A_o \frac{1}{e^{2ikh}+1} \left\{ e^{2ikh} [S(0,2kh) + C(0,2kh)] + S(0,2kh) - C(0,2kh) + \frac{e^{2ikh}(\sin 2kh - \cos 2kh + 1) + (1 - \cos 2kh - \sin 2kh)}{ikh} \right\}$$

Let

$$I(z) \simeq I_o(z) + \alpha\, I_\alpha(z)$$

then

$$I_o(z) = A_o K_o = \frac{-2\pi V_o^e i}{\zeta_o \Psi} \frac{\sin k(h-|z|)}{\cos kh} \tag{35a}$$

$$I_\alpha(z) = A_\alpha K_o + A_o K_o + A_o K_\alpha \tag{35b}$$

Note that (35a) is exactly what has been obtained by King, [ref. 3, eq. II-18.5].

The first order perturbation terms are in terms of tabulated functions; thus hand calculation of the current distribution is possible for α either near 1 or near zero. The following table shows the comparison of the current obtained by numerical integration (carried out by computer) of the exact formula (32), denoted in the table as I_A and Φ_A, with that obtained by perturbation formulas (34) and (35), denoted as I_B and Φ_B.

$\alpha = 0.75$, $kh = 2\pi$

$\frac{z}{h}$	0.00	0.25	0.50	0.75
I_A	1.000	0.818	0.589	0.371
I_B	1.000	0.812	0.588	0.343
Φ_A	0.000	1.564	3.129	4.658
Φ_B	0.000	1.559	3.120	4.651

$\alpha = 0.25, \quad kh = 2\pi$

$\frac{z}{h}$	0.00	0.25	0.50	0.75
I_A	1.000	1.450	0.754	1.260
I_B	1.000	1.230	0.782	1.170
Φ_A	0.000	1.561	3.132	4.661
Φ_B	0.000	1.570	3.141	4.710

EFFICIENCY

To calculate the efficiency, the power lost on the resistive antenna is compared with the power radiated. The radiated power is obtained by integrating the Poynting vector over a large spherical surface in the far field rather than by calculating the input impedance which is only of zero-order accuracy in the present theory.

The dissipated power can be found directly from the differential equation (5) as will be shown later. However, in general, the radiated power cannot be obtained except by numerical integration. The dissipated power P_h on the antenna is

$$P_h = 2\int_0^h \frac{1}{2}\,|I(z)|^2\,[\mathrm{Re}\; z^i(z)]\,dz \tag{36}$$

and since

$$\left(\frac{d^2}{dz^2} + k^2 + \frac{2i\alpha k}{h-z}\right)(I_R + iI_I) = \frac{4\pi k\, V_o^e}{\zeta_o(-i\Psi)}\;\delta(z) \tag{37}$$

$$\left(\frac{d^2}{dz^2} + k^2\right)I_R - \frac{2\alpha k}{h-z}\, I_I = \overline{V}_h \Psi_I\;\delta(z) \tag{38}$$

$$\left(\frac{d^2}{dz^2} + k^2\right)I_I + \frac{2\alpha k}{h-z}\, I_R = \overline{V}_h \Psi_R\;\delta(z) \tag{39}$$

where subscripts R and I denote respectively the real and imaginary parts of the quantity subscripted and

$$\overline{V}_h = \frac{4\pi k\, V_o^e}{\zeta_o|\Psi|^2} \tag{40}$$

From (38) and (39)

$$I_R \frac{d^2}{dz^2} I_R + I_I \frac{d^2}{dz^2} I_I + k^2|I|^2 = \bar{V}_h(\Psi_I I_R + \Psi_R I_I)\ \delta(z) \tag{41}$$

When (41) is substituted into (36), it becomes

$$P_h = \int_0^h \frac{\zeta_o \Psi_R}{4\pi} \frac{2\alpha}{h-z} \frac{1}{k^2} [\bar{V}_h(\Psi_I I_R + \Psi_R I_I)\delta(z) - I_R \frac{d^2}{dz^2} I_R - I_I \frac{d^2}{dz^2} I_I]dz \tag{42}$$

Let

$$Q = \int_0^h \frac{2\alpha k}{h-z} [I_R \frac{d^2}{dz^2} I_R + I_I \frac{d^2}{dz^2} I_I]dz \tag{43}$$

Then substitute (38) and ~~(39) into~~ (43). The result is

$$\begin{aligned} Q &= \int_0^h \left\{ [\bar{V}_h\ \Psi_R\delta(z) - (\frac{d^2 I_I}{dz^2} + k^2 I_I)]\frac{d^2}{dz^2} I_R + [(\frac{d^2}{dz^2} I_R + k^2 I_R) - \bar{V}_h \Psi_I \delta(z)]\frac{d^2}{dz^2} I_I \right\} dz \\ &= \int_0^h [\bar{V}_h \Psi_R \delta(z) \frac{d^2 I_R}{dz^2} - \bar{V}_h \Psi_I \delta(z) \frac{d^2 I_I}{dz^2}]dz + k^2 \int_0^h [I_I \frac{d^2 I_R}{dz^2} - I_R \frac{d^2 I_I}{dz^2}]dz \end{aligned} \tag{44}$$

The second integral of (44) vanishes so that (42) becomes

$$P_h = \frac{\zeta_o \Psi_R}{4\pi} \frac{\bar{V}_h}{k^3} \left\{ \int_0^h -\Psi_R [\frac{d^2 I_R}{dz^2} - \frac{2\alpha k I_I}{h-z}]\delta(z)\ dz + \int_0^h \Psi_I [\frac{d^2 I_I}{dz^2} + \frac{2\alpha k I_R}{h-z}]\delta(z)\ dz \right\}$$

$$= \frac{\zeta_o \Psi_R}{8\pi} \frac{\bar{V}_h}{k} (I_R(0)\Psi_R - I_I(0)\Psi_I)$$

Finally,

$$P_h = \frac{V_o^e}{2} \frac{\Psi_R}{|\Psi|^2} [\Psi_R I_R(0) - \Psi_I I_I(0)]$$

Usually Ψ is taken to be real, so that

$$P_h = \frac{V_o^e}{2} I_R(0) \equiv \frac{1}{2} |I(0)|^2 R_o$$

where R_o is the real part of the input impedance. Thus it is seen that the real part of the zero-order input impedance takes into account only the ohmic loss on the antenna.

The power radiated P_r is

$$P_r = \frac{1}{2\zeta_o} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \, |E(\theta,\varphi)|^2 \, r_o^2 = \frac{k^2\zeta_o}{8\pi} \int_0^1 |F_{tot}|^2 \, dy$$

Thus,

$$\frac{P_h}{P_r} = \frac{\frac{V_o^e h}{2} [\mathrm{Re}\, A\Phi(2ikh)]}{\frac{k^2\zeta_o}{8\pi} \int_0^1 |F_{tot}|^2 \, dy}$$

Using the relations (11) and (28b) the above expression is put into final form as follows:

$$\frac{P_h}{P_r} = -\frac{2\,\Psi[a_{1R}(kh)a_{2R}(kh) + a_{1I}(kh)a_{2I}(kh)]}{k^2h^2 \int_0^1 |F_{tot}|^2 \, dy} \tag{45}$$

where $a_1(kh)$ and $a_2(kh)$ are given in (30).

$$\text{Efficiency} = \frac{P_r}{P_h + P_r}$$

In the particular case $\alpha = 1$, it can be shown that (45) reduces to

$$\frac{P_h}{P_r} = \frac{2\int_0^1 |F|^2 \, dy}{\Psi} \tag{46}$$

where F is given in (38b) of Reference 1 [See also Ref. 5]. It can also be shown that the dissipated power approaches zero as $\alpha \to 0$. The results of the numerical calculation agree with the above statements.

The efficiency has been calculated for $\alpha = 1$ with kh ranging from $\pi/2$ to 40π (Fig. 10) and for $kh = 2\pi$ with α varying from 0 to 1 (Fig. 11). In Fig. 11 the value of Ψ is chosen to be $|\Psi(\alpha=1)|$ in one curve and Ψ_{K1} [Ref. 3, Table II-20.1] in the other. It is expected that should $|\Psi(\alpha)|$ be used, which could be obtained from (22) by numerical integration, the curve would be between these two curves.

CONCLUSION

The field pattern of the dipole antenna with tapered resistive loading is not critically dependent on the parameter α as long as α stays between 0 and 1. This brightens the aspect of the usefulness of the resistive antenna as a broadband directional communication device. An experimental study of the resistive antenna is in progress. Results will be reported in another report.

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APPENDIX I

Modified Simpson's Rule

In order to calculate numerically the integral of the form $\int_0^B x^\alpha F(x)\,dx$, where $F(x)$ can be expanded in a Taylor series around $x = 0$, a method is developed which is very similar to the celebrated Simpson's rule.

Let $f(x)$ be a polynomial of degree 2 which coincides with $F(x)$ at points $x_o < x_1 < x_2$. Then

$$f(x) = \sum_{k=0}^{2} \ell_k(x)\; F(x_k) \tag{A-1}$$

where

$$\ell_o(x) = \frac{(x-x_1)(x-x_2)}{(x_o-x_1)(x_o-x_2)}$$

$$\ell_1(x) = \frac{(x-x_o)(x-x_2)}{(x_1-x_o)(x_1-x_2)}$$

$$\ell_2(x) = \frac{(x-x_o)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

See [Ref. 4, p. 71]. Let $x_1 - x_o = x_2 - x_1 = h$, $x = 2mh + hS$ where m is a constant while S is a new variable. It follows that

$$\ell_o(x) = \frac{1}{2}(S-1)(S-2)$$

$$\ell_1(x) = -S(S-2)$$

$$\ell_2(x) = \frac{1}{2}S(S-1)$$

$$dx = h\,dS$$

Hence,

$$\int_{x_o}^{x_2} x^\alpha f(x)\,dx = \int_0^2 h^\alpha\,(2m+S)^\alpha\,f(2mh+hS)\,h\,dS$$

$$= h^{1+\alpha}[a_o f(2mh) + a_1 f(2mh+h) + a_2 f(2mh+2h)]$$

where

$$a_o = \frac{1}{2}\int_0^2 (2m+S)^\alpha \, (S-1)(S-2) \, dS$$

$$a_1 = -\int_0^2 (2m+S)^\alpha \, S(S-2) \, dS$$

$$a_2 = \frac{1}{2}\int_0^2 (2m+S)^\alpha \, S(S-1) \, dS$$

The coefficients a can be calculated easily. Thus, after repeated use of the above formula,

$$\int_0^B x^\alpha f(x)\,dx = \frac{h^{1+\alpha}}{2(1+\alpha)(2+\alpha)(3+\alpha)}\Big[b_o(0)f(0) - b_o(N)f(B) + \sum_{k=1}^{N} b_1(k-1)f\big((2k-1)h\big) + b_2(k-1)f(2kh)\Big] \qquad \text{(A-2)}$$

where

$$b_o(k) = (2k+2)^{2+\alpha}(4k+1-\alpha) - (2k)^{1+\alpha}[8k^2 + (3+\alpha)(6k+4+2\alpha)]$$

$$b_1(k) = 4(2k)^{2+\alpha}(2k+3+\alpha) - 4(2k+2)^{2+\alpha}(2k-1-\alpha)$$

$$b_2(k) = -(2k)^{2+\alpha}(3+\alpha+4k) - 12(\alpha+3)(k+1)(2k+2)^{1+\alpha} + (2k+4)^{2+\alpha}(4k+5-\alpha)$$

$$h = B/2N$$

If F(x) is approximated by f(x), it follows that the integral that involves F(x) should be given approximately by (A-2). Note that since when k is large the coefficients $b_n(k)$ involve two or three nearly equal numbers that are subtracted from one another, the error would be large if they were calculated by a computer. It is safe to use the modified Simpson's rule only at points near the singularity and to use Simpson's rule elsewhere.

APPENDIX II

The figures shown are the result of numerical caluclations. The field pattern is normalized to the maximum field and the current is referred to the driving point.

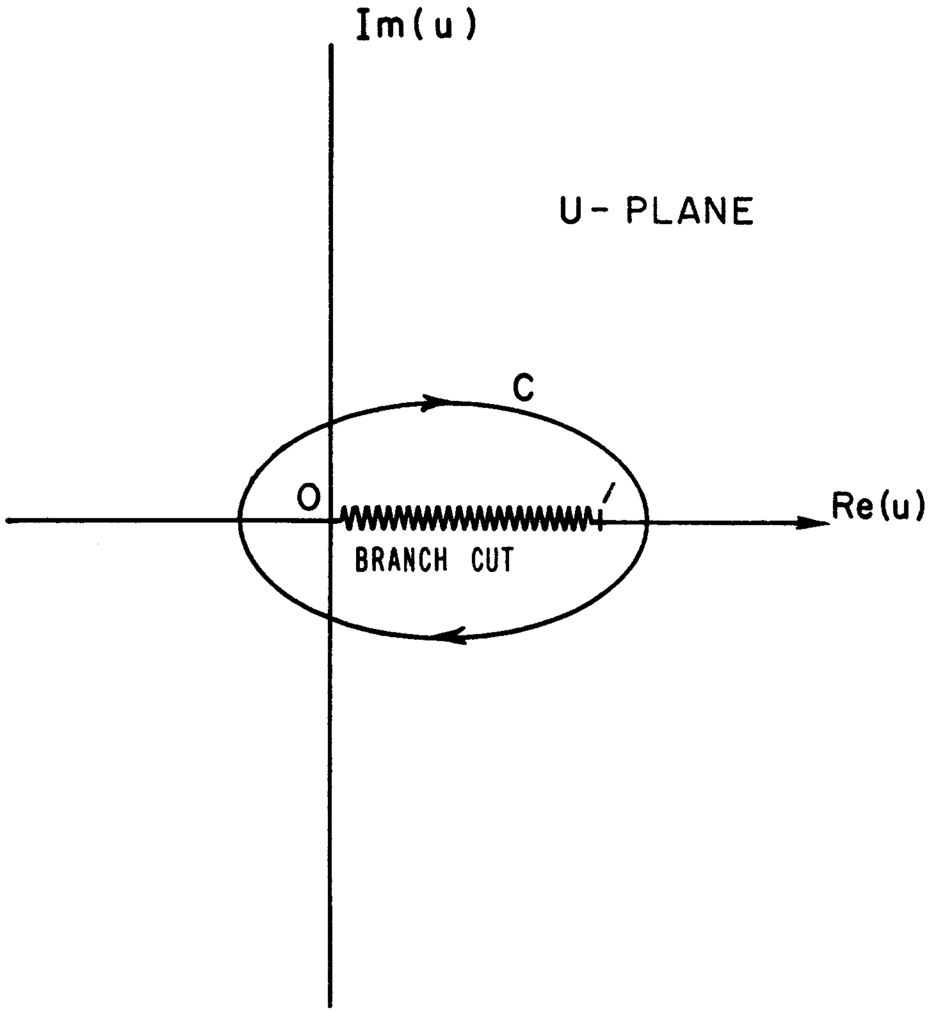

FIG. 1 THE CONTOUR IN THE DEFINITION OF $\Phi(y)$

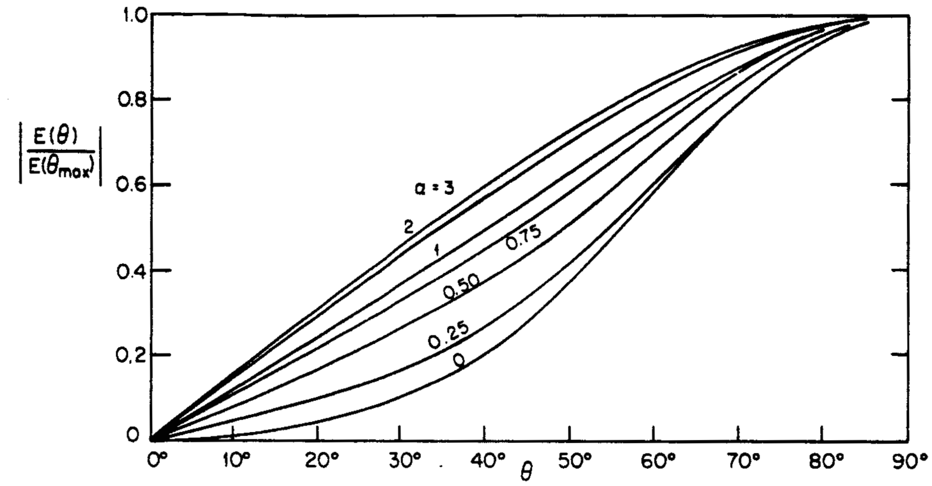

FIG. 2 FIELD PATTERN, kh = π

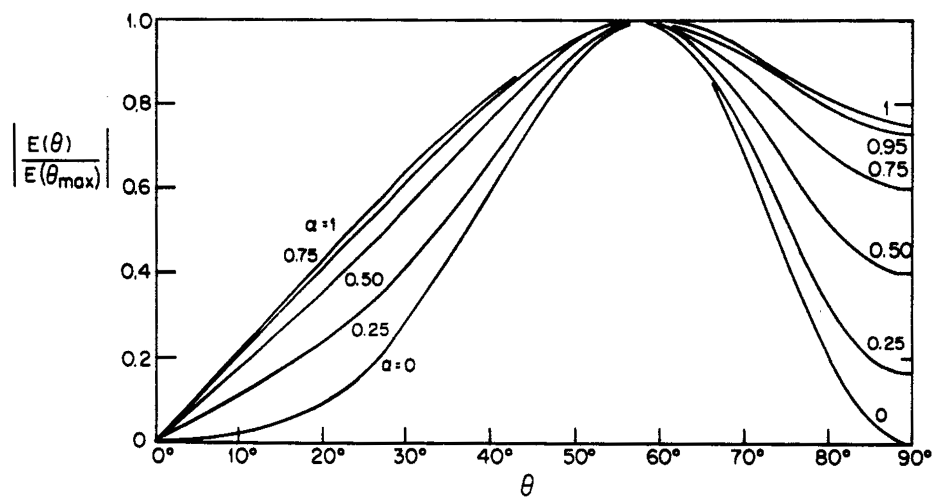

FIG. 3 FIELD PATTERN, kh = 2π

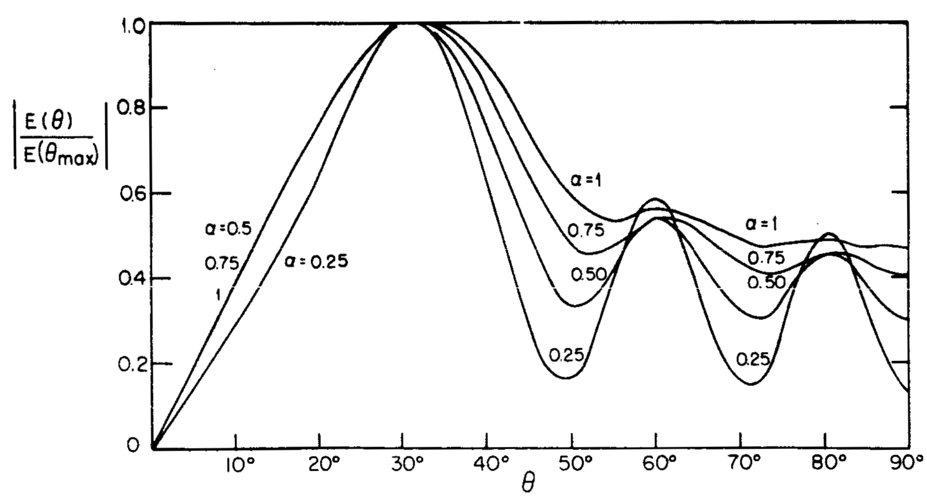

FIG. 4 FIELD PATTERN, kh = 6π

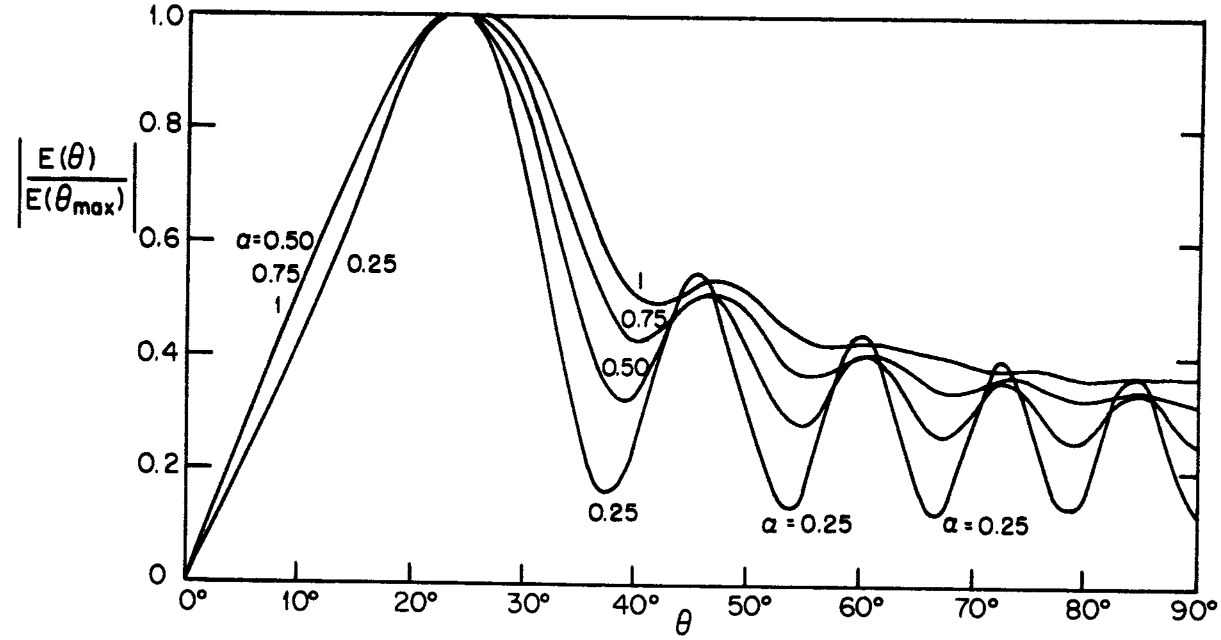

FIG. 5 FIELD PATTERN, $kh = 10\pi$

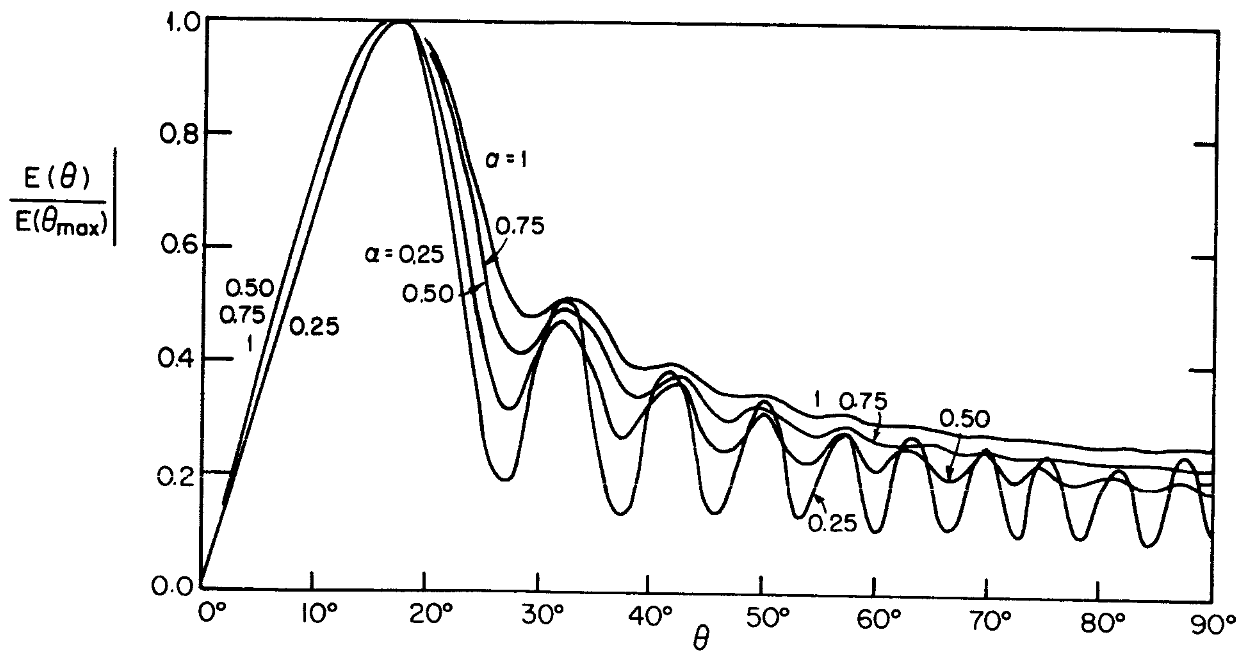

FIG. 6 FIELD PATTERN, $kh = 20\pi$

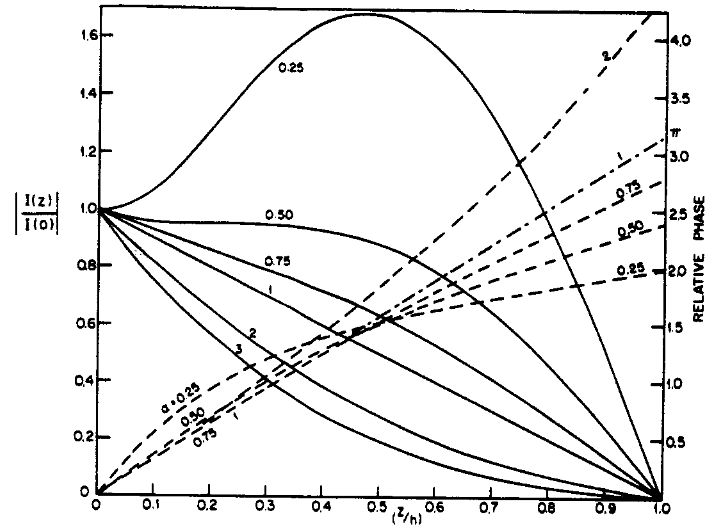

FIG. 7 CURRENT DISTRIBUTION, kh = π

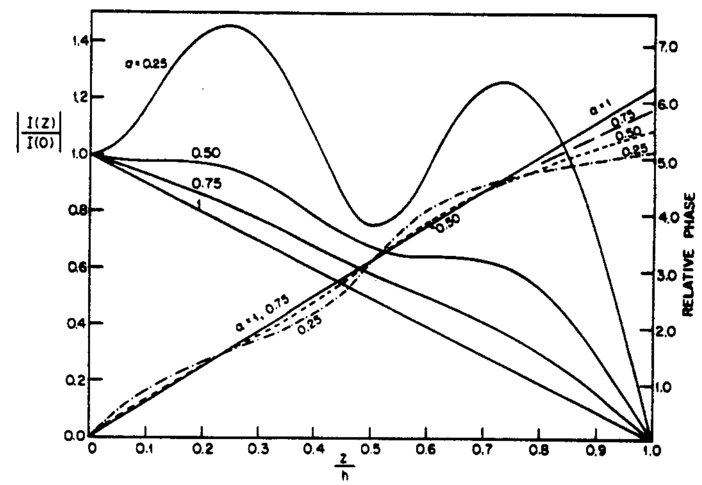

FIG. 8 CURRENT DISTRIBUTION, kh = 2π

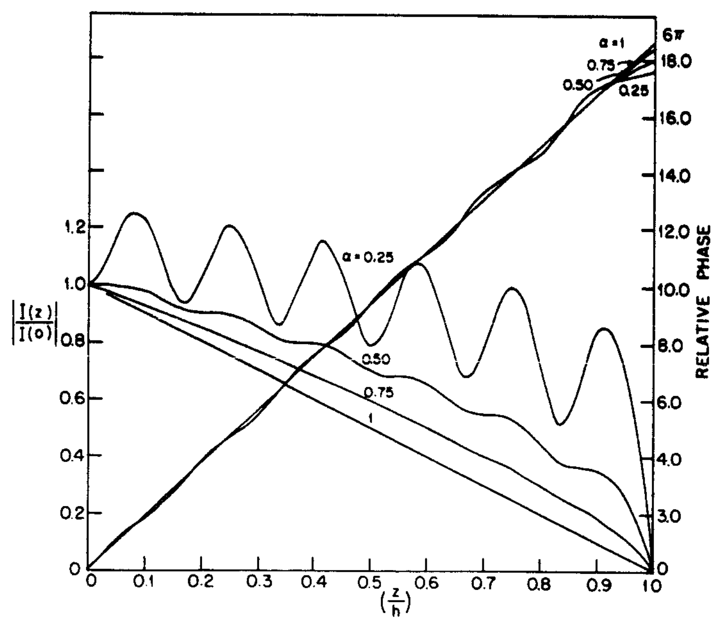

FIG. 9 CURRENT DISTRIBUTION, kh = 6π

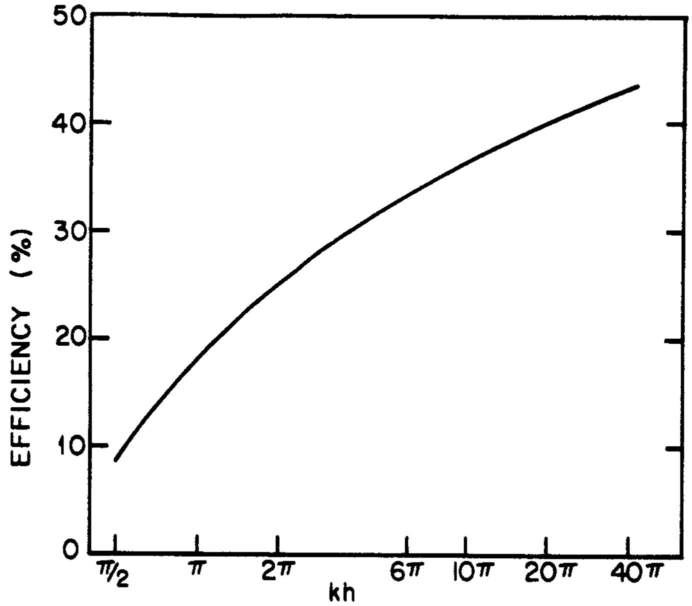

FIG. 10 EFFICIENCY, $\alpha=1$, $ka = 0.021$

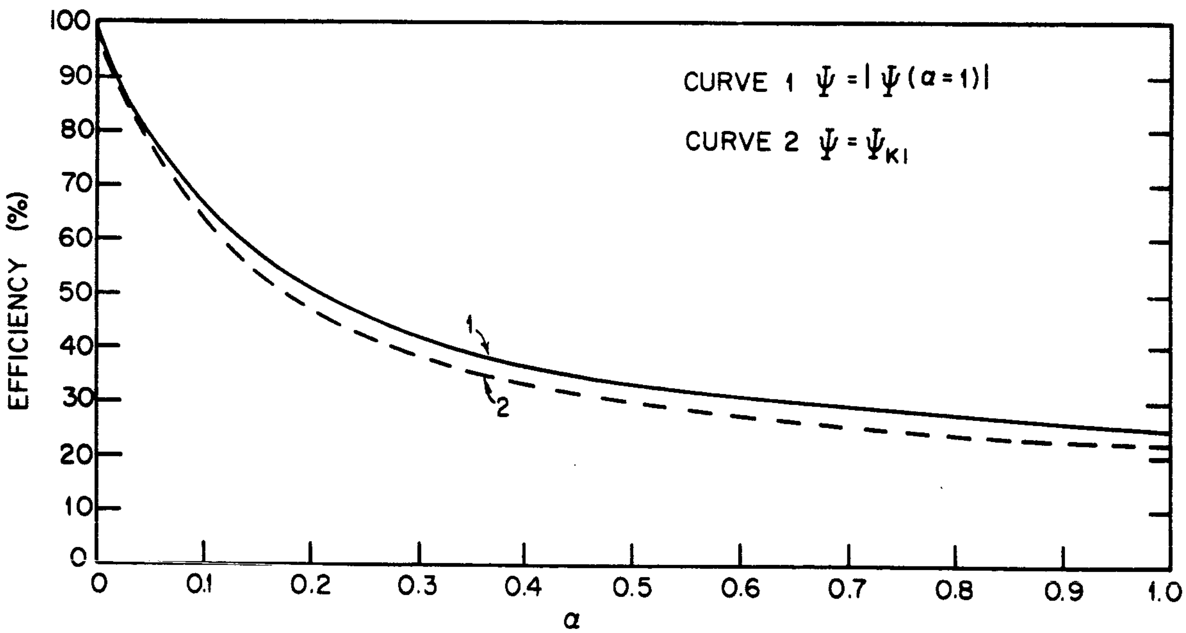

FIG. 11 EFFICIENCY, $kh = 2\pi$, $ka = 0.021$