NASA CONTRACTOR REPORT

NASA CR-568

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by Z. M. Elias

Prepared by
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, Mass.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1966

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1. Introduction

The basic equations of the linear theory of thin elastic shells have had little modification since Love's first approximation theory. The differences between various formulations pertain mainly to the form of the stress strain relations.

Methods of solution of the basic equations are usually accompanied by their reduction to a system of a lesser number of equations. This reduction may depend on the particular shell or the particular class of shells. A general method, valid for an arbitrary shell which may be called the displacement method consists in reducing the equilibrium equations to a system of equations for the displacements. Such a method is not peculiar to shell structures but is found generally in problems of continuum mechanics from the simple beam theory to the three dimensional theory of elasticity. By contrast, the force method which is well known and used in beam theory and in the problem of stretching of plates has had relatively less attention in shell theory than the displacement method. It involves the use of stress functions in a way similar to the use of Airy's stress function in the plane stress problem. In shell theory stress functions and the equations of compatibility of strain were introduced (1) by Goldenweiser and Lur'e (2, 3, 4). The basis of a general formulation of the shell problem in terms of differential equations for stress functions consists in expressing the equations of compatibility of strain in terms of the stress functions by means of the stress-strain relations. Although the basic equations for performing this are available in the literature, the actual equations for the stress functions have not apparently been obtained for an arbitrary shell. The reason for this preference of the displacement method over the force method may lie in the possibility of expressing all types of boundary conditions and all the dependent variables of the shell problem in terms of the displacements. While it may not be readily apparent how displacement boundary conditions can be explicitly expressed in terms of stress functions, this is possible by obtaining equivalent boundary conditions in terms of strains or as natural conditions of a variational formulation (5). The problem of obtaining the displacements from the stress resultants and stress couples requires in general integration of the stress-displacement relations. It may be interesting to mention here that for a spherical shell the displacements are expressible in terms of the stress functions without the necessity of integration (6).

It seems therefore desirable to formulate the force method of shell theory and to examine its possible advantages. This in fact becomes more interesting in view of the static-geometric analogy that exists in the basic equations of shell theory (7) and that is the basis of a duality between the displacement and the force methods.

A concise presentation of the basic equations of thin shells using mainly E. Reissner's notation for geometric quantities, stress resultants, stress couples, strains and stress functions (8) will be followed by a presentation of a system of stress strain relations and by a discussion on the static-geometric analogy and its implications. The derivation of the compatibility equations in terms of the stress resultants and stress couples is accompanied by a general discussion concerning the different states of stress of a shell. Finally, the differential equations for the stress functions are obtained.

2. Geometric Notation

The principal lines of curvature of the middle surface are chosen as coordinate lines of a system of curvilinear coordinates (ξ_1, ξ_2). The first and second fundamental forms are written, respectively, in the form

$$d\bar{r} \cdot d\bar{r} = \alpha_1^2 \, d\xi_1^2 + \alpha_2^2 \, d\xi_2^2 \qquad (1)$$

$$d\bar{r} \cdot d\bar{n} = \frac{\alpha_1^2}{R_1} \, d\xi_1^2 + \frac{\alpha_2^2}{R_2} \, d\xi_2^2 \qquad (2)$$

where $\bar{r}$ is the position vector and $\bar{n}$ a unit vector normal to the middle surface. A right handed local reference frame is defined through the unit vectors, Fig. 1,

$$\bar{t}_1 = \frac{\bar{r},_1}{\alpha_1} \qquad 3\text{-}1$$

$$\bar{t}_2 = \frac{\bar{r},_2}{\alpha_2} \qquad 3\text{-}2$$

$$\bar{n} = \bar{t}_1 \times \bar{t}_2 \qquad 3\text{-}3$$

Differentiation formulas for the unit vectors may be written in the form

$$\bar{t}_{1,1} = -\frac{\alpha_{1,2}}{\alpha_2}\bar{t}_2 - \frac{\alpha_1}{R_1}\bar{n} \qquad 4\text{-}1 \qquad\qquad \bar{t}_{1,2} = \frac{\alpha_{2,1}}{\alpha_1}\bar{t}_2 \qquad 4\text{-}4$$

$$\bar{t}_{2,1} = \frac{\alpha_{1,2}}{\alpha_2}\bar{t}_1 \qquad 4\text{-}2 \qquad\qquad \bar{t}_{2,2} = -\frac{\alpha_{2,1}}{\alpha_1}\bar{t}_1 - \frac{\alpha_2}{R_2}\bar{n} \qquad 4\text{-}5$$

$$\bar{n},_1 = \frac{\alpha_1}{R_1}\bar{t}_1 \qquad 4\text{-}3 \qquad\qquad \bar{n},_2 = \frac{\alpha_2}{R_2}\bar{t}_2 \qquad 4\text{-}6$$

α_1, α_2, R_1 and R_2 satisfy the Gauss Codazzi relations which are obtained by requiring the equalities $\bar{t}_{1,12} = \bar{t}_{1,21}$ etc.

$$\left(\frac{\alpha_1}{R_1}\right),_2 = \frac{\alpha_{1,2}}{R_2} \qquad 5\text{-}1$$

$$\left(\frac{\alpha_2}{R_2}\right),_1 = \frac{\alpha_{2,1}}{R_1} \qquad 5\text{-}2$$

$$\left(\frac{\alpha_{2,1}}{\alpha_1}\right),_1 + \left(\frac{\alpha_{1,2}}{\alpha_2}\right),_2 + \frac{\alpha_1\alpha_2}{R_1R_2} = 0 \qquad 5\text{-}3$$

3. Differential Equations of Equilibrium, Stress Functions.

The stress resultant vectors $\bar{N}_1$, $\bar{N}_2$ and the stress couple vectors $\bar{M}_1$, $\bar{M}_2$ are written in terms of their components. Fig. 2, according to the relations

$$\overline{N}_1 = N_{11}\overline{t}_1 + N_{12}\overline{t}_2 + N_{13}\overline{n} \qquad 6\text{-}1$$

$$\overline{N}_2 = N_{21}\overline{t}_1 + N_{22}\overline{t}_2 + N_{23}\overline{n} \qquad 6\text{-}2$$

$$\overline{M}_1 = - M_{12}\overline{t}_1 + M_{11}\overline{t}_2 + M_{13}\overline{n} \qquad 6\text{-}3$$

$$\overline{M}_2 = - M_{22}\overline{t}_1 + M_{21}\overline{t}_2 + M_{23}\overline{n} \qquad 6\text{-}4$$

It is known that

$$M_{13} = M_{23} = 0 \qquad 7$$

M_{12} and M_{23} are called by Reissner couple stress stress couples. They are kept in the equations as convenient devices in the static geometric analogy.

Letting $\overline{p}$ denote the surface load intensity per unit area of the middle surface, the equilibrium conditions of an infinitesimal element of volume such as represented in Fig. 3 lead to the differential equations

$$(a_1\overline{N}_2),_2 + (a_2\overline{N}_1),_1 + a_1a_2\overline{p} = 0 \qquad 8\text{-}1$$

$$(a_1\overline{M}_2),_2 + (a_2\overline{M}_1),_1 + a_1a_2(\overline{t}_1 \times \overline{N}_1 + \overline{t}_2 \times \overline{N}_2) = 0 \qquad 8\text{-}2$$

A particular solution of these equations may be taken as a particular solution of the membrane theory equations. For the homogeneous problem, $\overline{p} = 0$, eqs. 8 may be identically satisfied by letting

$$a_2\overline{N}_1 = \overline{F},_2 \qquad 9\text{-}1$$

$$a_1\overline{N}_2 = - \overline{F},_1 \qquad 9\text{-}2$$

$$a_2\overline{M}_1 = \overline{G},_2 + a_2\overline{t}_2 \times \overline{F} \qquad 9\text{-}3$$

$$a_1\overline{M}_2 = - \overline{G},_1 - a_1\overline{t}_1 \times \overline{F} \qquad 9\text{-}4$$

where

$$\overline{F} = F_1\overline{t}_1 + F_2\overline{t}_2 + F_3\overline{n} \qquad 10\text{-}1$$

and

$$\overline{G} = G_1\overline{t}_1 + G_2\overline{t}_2 + G_3\overline{n} \qquad 10\text{-}2$$

are two arbitrary vector stress functions. The components of $\overline{F}$ and $\overline{G}$ are 6 arbitrary scalar stress functions of which only 4 remain arbitrary upon setting

$$M_{13} = \overline{n}.\overline{M}_1 = \frac{\overline{n}.\overline{G},_2}{\alpha_2} - F_1 = 0 \qquad 11\text{-}1$$

$$M_{23} = \overline{n}.\overline{M}_2 = -\frac{\overline{n}.\overline{G},_1}{\alpha_1} - F_2 = 0 \qquad 11\text{-}2$$

The 6 scalar equilibrium equations take the form

$$(\alpha_2 N_{11}),_1 + (\alpha_1 N_{21}),_2 - \alpha_{2,1}N_{22} + \alpha_{1,2}N_{12} + \frac{\alpha_1\alpha_2}{R_1}N_{13} + \alpha_1\alpha_2 p_1 = 0 \qquad 12\text{-}1$$

$$(\alpha_1 N_{22}),_2 + (\alpha_2 N_{12}),_1 - \alpha_{1,2}N_{11} + \alpha_{2,1}N_{21} + \frac{\alpha_1\alpha_2}{R_2}N_{23} + \alpha_1\alpha_2 p_2 = 0 \qquad 12\text{-}2$$

$$(\alpha_2 N_{13}),_1 + (\alpha_1 N_{23}),_2 - \alpha_1\alpha_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - p_3\right) = 0 \qquad 12\text{-}3$$

$$(\alpha_2 M_{11}),_1 + (\alpha_1 M_{21}),_2 - \alpha_{2,1}M_{22} + \alpha_{1,2}M_{12} + \frac{\alpha_1\alpha_2}{R_2}M_{23} - \alpha_1\alpha_2 N_{13} = 0 \qquad 12\text{-}4$$

$$(\alpha_1 M_{22}),_2 + (\alpha_2 M_{12}),_1 - \alpha_{1,2}M_{11} + \alpha_{2,1}M_{21} - \frac{\alpha_1\alpha_2}{R_1}M_{13} - \alpha_1\alpha_2 N_{23} = 0 \qquad 12\text{-}5$$

$$(\alpha_2 M_{13}),_1 + (\alpha_1 M_{23}),_2 + \alpha_1\alpha_2\left(N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2}\right) = 0 \qquad 12\text{-}6$$

The stress-stress function relations that satisfy identically the above equations when $\overline{p} = 0$ take the form

$$N_{11} = \frac{1}{\alpha_1\alpha_2}(\alpha_1 F_{1,2} - \alpha_{2,1}F_2) \qquad 13\text{-}1$$

$$N_{22} = \frac{-1}{\alpha_1\alpha_2}(\alpha_2 F_{2,1} - \alpha_{1,2}F_1) \qquad 13\text{-}2$$

$$N_{12} = \frac{1}{\alpha_1\alpha_2}(\alpha_1 F_{2,2} + \alpha_{2,1}F_1) + \frac{F_3}{R_2} \qquad 13\text{-}3$$

$$N_{21} = \frac{-1}{\alpha_1\alpha_2}(\alpha_2 F_{1,1} + \alpha_{1,2}F_2) - \frac{F_3}{R_1} \qquad 13\text{-}4$$

$$N_{13} = \frac{F_{3,2}}{\alpha_2} - \frac{F_2}{R_2} \qquad 13\text{-}5$$

$$N_{23} = -\frac{F_{3,1}}{\alpha_1} + \frac{F_1}{R_1} \qquad 13\text{-}6$$

$$M_{11} = \frac{1}{\alpha_1\alpha_2}(\alpha_1 G_{2,2} + \alpha_{2,1}G_1) + \frac{G_3}{R_2} \qquad 13\text{-}7$$

$$M_{22} = \frac{1}{\alpha_1\alpha_2}(\alpha_2 G_{1,1} + \alpha_{1,2}G_2) + \frac{G_3}{R_1} \qquad 13\text{-}8$$

$$M_{12} = -\frac{1}{\alpha_1\alpha_2}(\alpha_1 G_{1,2} - \alpha_{2,1}G_2) - F_3 \qquad 13\text{-}9$$

$$M_{21} = -\frac{1}{\alpha_1\alpha_2}(\alpha_2 G_{2,1} - \alpha_{1,2}G_1) + F_3 \qquad 13\text{-}10$$

$$M_{13} = \frac{G_{3,2}}{\alpha_2} - \frac{G_2}{R_2} - F_1 \qquad 13\text{-}11$$

$$M_{23} = -\frac{G_{3,1}}{\alpha_1} + \frac{G_1}{R_1} - F_2 \qquad 13\text{-}12$$

by letting $M_{13} = M_{23} = 0$ the independent stress functions are reduced to 4. These may be chosen as G_1, G_2, G_3 and F_3 or as F_1, F_2, F_3, and G_3.

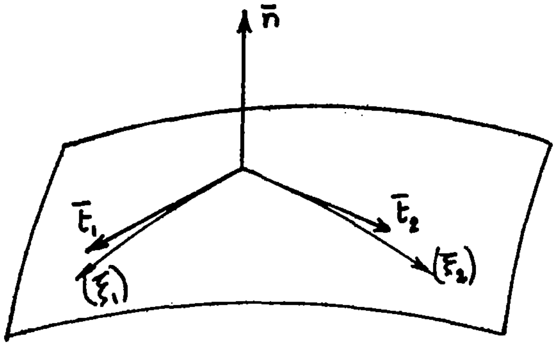

Figure 1

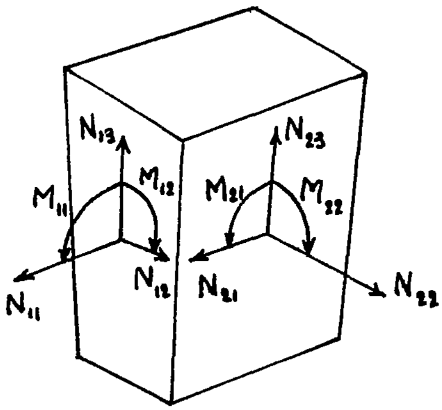

Figure 2

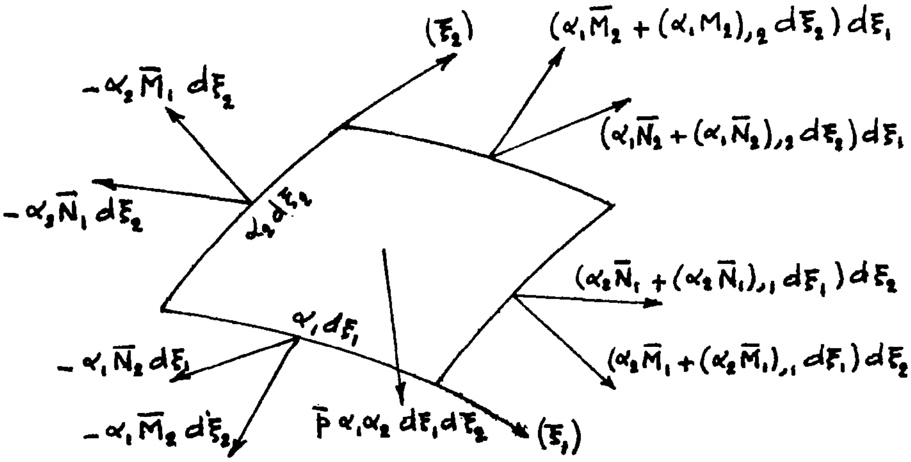

Figure 3

4. Strain Displacement Relations

A method of obtaining linear strain-displacement relations consists in requiring the validity of a virtual work equation for the shell as a whole (9). A method based on the same idea but dealing with an infinitesimal element of volume of the shell avoids the integration by parts that has to be performed in the first method.

Consider an element of volume of the shell, as represented in Fig. 3, experiencing a virtual translational displacement $\bar{u}$ and a virtual rotation $\bar{\omega}$ both functions of ξ_1 and ξ_2. The virtual work of the external forces acting on the element of volume is, except for infinitesimals of higher order, and per unit area of the middle surface

$$\frac{1}{\alpha_1 \alpha_2} \Big\{ \alpha_2 \bar{N}_1 \cdot \bar{u},_1 + \alpha_1 \bar{N}_2 \cdot \bar{u},_2 + \left[(\alpha_2 \bar{N}_1),_1 + (\alpha_1 \bar{N}_2),_2 \right] \cdot \bar{u} +$$

$$\alpha_2 \bar{M}_1 \cdot \bar{\omega},_1 + \alpha_1 \bar{M}_2 \cdot \bar{\omega},_2 + \left[(\alpha_2 \bar{M}_1),_1 + (\alpha_1 \bar{M}_2),_2 \right] \cdot \bar{\omega} +$$

$$\alpha_1 \alpha_2 \bar{p} \cdot \bar{u} \Big\}$$

Taking account of the vector equilibrium equations the expression of the virtual work reduces to

$$\bar{N}_1 \cdot \frac{\bar{u},_1 + \alpha_1 \bar{t}_1 \times \bar{\omega}}{\alpha_1} + \bar{N}_2 \cdot \frac{\bar{u},_2 + \alpha_2 \bar{t}_2 \times \bar{\omega}}{\alpha_2} + \bar{M}_1 \cdot \frac{\bar{\omega},_1}{\alpha_1} + \bar{M}_2 \cdot \frac{\bar{\omega},_2}{\alpha_2} \quad (14)$$

The internal virtual work per unit area of the middle surface is written in the form

$$N_{11} \varepsilon_{11} + N_{12} \varepsilon_{12} + N_{13} \varepsilon_{13} + N_{21} \varepsilon_{21} + N_{22} \varepsilon_{22} + N_{23} \varepsilon_{23}$$
$$+ M_{11} \chi_{11} + M_{12} \chi_{12} + M_{13} \chi_{13} + M_{21} \chi_{21} + M_{22} \chi_{22} + M_{23} \chi_{23} \quad (15)$$

where the ε_{ij}'s and the χ_{ij}'s are the desired strain quantities. The above expression suggests its representation as a sum of dot products in the form

$$\bar{N}_1 \cdot \bar{\varepsilon}_1 + \bar{N}_2 \cdot \bar{\varepsilon}_2 + \bar{M}_1 \cdot \bar{\chi}_1 + \bar{M}_2 \cdot \bar{\chi}_2 \tag{16}$$

where

$$\bar{\varepsilon}_1 = \varepsilon_{11}\bar{t}_1 + \varepsilon_{12}\bar{t}_2 + \varepsilon_{13}\bar{n} \qquad 17\text{-}1$$

$$\bar{\varepsilon}_2 = \varepsilon_{21}\bar{t}_1 + \varepsilon_{22}\bar{t}_2 + \varepsilon_{23}\bar{n} \qquad 17\text{-}2$$

$$\bar{\chi}_1 = -\chi_{12}\bar{t}_1 + \chi_{11}\bar{t}_2 + \chi_{13}\bar{n} \qquad 17\text{-}3$$

$$\bar{\chi}_2 = -\chi_{22}\bar{t}_1 + \chi_{21}\bar{t}_2 + \chi_{23}\bar{n} \qquad 17\text{-}4$$

The equality between the external work, after taking account of equilibrium, and the internal work for arbitrary $\bar{N}_1$, $\bar{N}_2$, $\bar{M}_1$ and $\bar{M}_2$ yields

$$\bar{\varepsilon}_1 = \frac{\bar{u},_1}{a_1} + \bar{t}_1 \times \bar{\omega} \qquad 18\text{-}1$$

$$\bar{\varepsilon}_2 = \frac{\bar{u},_2}{a_2} + \bar{t}_2 \times \bar{\omega} \qquad 18\text{-}2$$

$$\bar{\chi}_1 = \frac{\bar{\omega},_1}{a_1} \qquad 18\text{-}3$$

$$\bar{\chi}_2 = \frac{\bar{\omega},_2}{a_2} \qquad 18\text{-}4$$

By requiring the continuity relations $\bar{u},_{12} = \bar{u},_{21}$ and $\bar{\chi},_{12} = \bar{\chi},_{21}$ to hold, the vector equations of compatibility of strain are obtained in the form

$$(a_2\bar{\chi}_2),_1 - (a_1\bar{\chi}_1),_2 = 0 \qquad 19\text{-}1$$

$$(a_2\bar{\varepsilon}_2),_1 - (a_1\bar{\varepsilon}_1),_2 + a_1a_2(\bar{t}_1 \times \bar{\chi}_2 - \bar{t}_2 \times \bar{\chi}_1) = 0 \qquad 19\text{-}2$$

Eqs. 18 yield 12 strain displacement relations in the form

$$\varepsilon_{11} = \frac{1}{a_1a_2}\,(a_2u_{1,1} + a_{1,2}u_2) + \frac{u_3}{R_1} \qquad 20\text{-}1$$

$$\varepsilon_{22} = \frac{1}{a_1 a_2}(a_1 u_{2,2} + a_{2,1} u_1) + \frac{u_3}{R_2} \qquad 20\text{-}1$$

$$\varepsilon_{12} = \frac{1}{a_1 a_2}(a_2 U_{2,1} - a_{1,2} u_1) - \omega_3 \qquad 20\text{-}3$$

$$\varepsilon_{21} = \frac{1}{a_1 a_2}(a_1 u_{1,2} - a_{2,1} u_2) + \omega_3 \qquad 20\text{-}4$$

$$\varepsilon_{13} = \frac{u_{3,1}}{a_1} - \frac{u_1}{R_1} + \omega_2 \qquad 20\text{-}5$$

$$\varepsilon_{23} = \frac{u_{3,2}}{a_2} - \frac{u_2}{R_2} - \omega_1 \qquad 20\text{-}6$$

$$\chi_{11} = \frac{1}{a_1 a_2}(a_2 \omega_{2,1} - a_{1,2} \omega_1) \qquad 20\text{-}7$$

$$\chi_{22} = -\frac{1}{a_1 a_2}(a_1 \omega_{1,2} - a_{2,1} \omega_2) \qquad 20\text{-}8$$

$$\chi_{12} = -\frac{1}{a_1 a_2}(a_2 \omega_{1,1} + a_{1,2} \omega_2) - \frac{\omega_3}{R_1} \qquad 20\text{-}9$$

$$\chi_{21} = \frac{1}{a_1 a_2}(a_1 \omega_{2,2} + a_{2,1} \omega_1) + \frac{\omega_3}{R_2} \qquad 20\text{-}10$$

$$\chi_{13} = \frac{\omega_{3,1}}{a_1} - \frac{\omega_1}{R_1} \qquad 20\text{-}11$$

$$\chi_{23} = \frac{\omega_{3,2}}{a_2} - \frac{\omega_2}{R_2} \qquad 20\text{-}12$$

Eqs. 19 yield 6 scalar compatibility equations relating the 12 components of $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\chi}_1$, and $\bar{\chi}_2$ in the form

$$(a_2 \chi_{22})_{,1} - (a_1 \chi_{12})_{,2} - a_{2,1} \chi_{11} - a_{1,2} \chi_{21} - \frac{a_1 a_2}{R_1} \chi_{23} = 0 \qquad 21\text{-}1$$

$$(a_1 \chi_{11})_{,2} - (a_2 \chi_{21})_{,1} - a_{1,2} \chi_{22} - a_{2,1} \chi_{12} + \frac{a_1 a_2}{R_2} \chi_{13} = 0 \qquad 21\text{-}2$$

$$(a_2 \chi_{23})_{,1} - (a_1 \chi_{13})_{,2} + a_1 a_2 \left(\frac{\chi_{22}}{R_1} + \frac{\chi_{11}}{R_2} \right) = 0 \qquad 21\text{-}3$$

$$(a_2 \varepsilon_{22})_{,1} - (a_1 \varepsilon_{12})_{,2} - a_{2,1} \varepsilon_{11} - a_{1,2} \varepsilon_{21} - \frac{a_1 a_2}{R_2} \varepsilon_{13} - a_1 a_2 \chi_{23} = 0 \qquad 21\text{-}4$$

$$(a_1 \varepsilon_{11})_{,2} - (a_2 \varepsilon_{21})_{,1} - a_{1,2} \varepsilon_{22} - a_{2,1} \varepsilon_{12} - \frac{a_1 a_2}{R_1} \varepsilon_{23} + a_1 a_2 \chi_{13} = 0 \qquad 21\text{-}5$$

$$(a_2 \varepsilon_{23})_{,1} - (a_1 \varepsilon_{13})_{,2} + a_1 a_2 \left(\chi_{21} - \chi_{12} - \frac{\varepsilon_{21}}{R_1} + \frac{\varepsilon_{12}}{R_2} \right) = 0 \qquad 21\text{-}6$$

If χ_{13} and χ_{23} are eliminated from these equations there results the 4 compatibility equations of reference (8).

A physical interpretation of the strain quantities defined above seems desirable and may be made by studying geometrically the deformation of the middle surface taking $\bar{u}$ as the translation vector and $\bar{\omega}$ as the rotation vector of the normal to the shell displacing as a rigid body. Within the framework of linear strain displacement relations it is found that ε_{11} and ε_{22} are extensional strains in the directions of $\bar{t}_1$ and $\bar{t}_2$, respectively. The shear strain between these 2 directions is $(\varepsilon_{12} + \varepsilon_{21})$. The transverse shear strains between the normal and the directions of $\bar{t}_1$ and $\bar{t}_2$ are ε_{13} and ε_{23}, respectively. If normals to the undeformed middle surface are assumed to displace into normals to the deformed middle surface, then $\varepsilon_{13} = \varepsilon_{23} = 0$ and $\chi_{11} - \varepsilon_{11}/R_1$ and $\chi_{22} - \varepsilon_{22}/R_2$ are the changes of the normal curvatures of the middle surface in the directions of $\bar{t}_1$ and $\bar{t}_2$, respectively. The twist is $\chi_{12} + \varepsilon_{21}/R_1 = \chi_{21} + \varepsilon_{12}/R_2$. For inextensible deformations, such that $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{21} = 0$, χ_{11}, χ_{22} and $\chi_{12} = \chi_{21}$ become changes of curvature and twist, respectively, and are often referred to as such notwithstanding the inextensibility assumption. It may be noted that the "physical" strains mentioned above do not depend on the component of rotation ω_3 about the normal. The strain quantities χ_{13} and χ_{23} do not appear in the stress strain relations of shell theory and have not apparently reveived a physical interpretation in the literature. They may be identified with the quantities

ξ_1 and ξ_2, respectively, in ref. (7) eqs. 19.1, p. 52. It is shown in what follows that for inextensional deformations and with normals remaining normal, χ_{13} and χ_{23} may be interpreted as changes of the geodesic curvatures of the coordinate lines (ξ_1) and (ξ_2), respectively. If (C) is a curve drawn on the middle surface with arclength s, the normal curvature K_n in the direction of (C) and the geodesic curvature K_g of (C) are obtained through the relation

$$\frac{d^2\bar{r}}{ds^2} = - K_n \bar{n} + K_g \bar{t} \tag{22}$$

where $\bar{t}$ is some unit vector in the plane tangent to the surface. Taking as curve (C) the coordinate line (ξ_1) the above equation takes the form

$$\frac{1}{a_1}\left(\frac{\bar{r},_1}{a_1}\right),_1 = \frac{1}{a_1}\bar{t}_{1,1} = -\frac{\bar{n}}{R_1} - \frac{a_{1,2}}{a_1 a_2}\bar{t}_2 \tag{23}$$

$\left(-\frac{a_{1,2}}{a_1 a_2}\right)$ is thus the geodesic curvature of the line (ξ_1). In order to perform the same calculations for the deformed line (ξ_1) it is first noted that the assumption of inextensibility allows setting $\varepsilon_{12} = 0$ and interpreting ω_3 as the rotation about the normal of the tangent to the line (ξ_1). Assuming also that normals remain normal we can write

$$\bar{\varepsilon}_1 = \frac{\bar{u},_1 + a_1 \bar{t}_1 \times \bar{\omega}}{a_1} = 0$$

Writing the equation similar to eq. 22 for the deformed line (ξ_1), obtain

$$\frac{1}{a_1}\left(\frac{(\bar{r} + \bar{u}),_1}{a_1}\right),_1 = \frac{1}{a_1}(\bar{t}_1 - \bar{t}_1 \times \bar{\omega}),_1$$

or after performing the differentiation

$$\frac{1}{a_1}\left(\frac{(\bar{r} + \bar{u}),_1}{a_1}\right),_1 = -\left(\frac{1}{R_{11}} + \chi_{11}\right)(\bar{n} + \bar{n} \times \bar{\omega}) - \frac{a_{1,2}}{a_1 a_2}(\bar{t}_2 - \omega_1 \bar{n} + \omega_3 \bar{t}_1) + \chi_{13}\bar{t}_2 \tag{24}$$

It is apparent now by comparing eqs. 23 and 24 that except for non linear terms in the rotations the geodesic curvature of the deformed line (ξ_1) is $(-\frac{\alpha_{1,2}}{\alpha_1\alpha_2} + \chi_{13})$.

5. Stress-Strain Relations

It is known that different systems of stress-strain relations for thin elastic and isotropic shells may be found in the literature. None of these systems is presently recognized to be preferable to the others. It is claimed by Goldenweiser (10) that the choice of a system of stress-strain relations may best be decided upon on the basis of the particular problem at hand. The discrepancies between different systems of stress strain relations involve generally terms of relative order h/R and are generally unimportant. Some desirable features, however, of a system of stress strain relations to be used for general derivations are the following

a) no contradiction between the 6th non differential equilibrium equation (with $M_{13} = M_{23} = 0$) and the stress strain relations. Such a contradiction occurs for example when the stress strain relations include the 2 relations $N_{12} = N_{21}$ and $M_{12} = M_{21}$.

b) A form of the stress strain relations similar to that of 3 dimensional elasticity and insuring the validity of general energy theorems. The form proposed by Reissner namely,

$$\varepsilon_{ij} = \frac{\partial W^*}{\partial N_{ij}}$$

$$\chi_{ij} = \frac{\partial W^*}{\partial M_{ij}} \qquad i, j \neq 3 \qquad (25)$$

where W* is the complementary strain energy density, satisfies the above requirement.

c) A functional form of W* that is invariant under a change of the curvilinear coordinates of the middle surface.

More than one form of W* satisfying the 3 above requirements is possible. It is proposed to use

$$W^* = \frac{1}{2Eh}\Big[(N_{11} + N_{22})^2 - 2(1+\nu)(N_{11}N_{22} - N_{12}N_{21})\Big] + \frac{6}{Eh^3}\Big[(M_{11} + M_{22})^2 - 2(1+\nu)(M_{11}M_{22} - M_{12}M_{21})\Big] \tag{26}$$

and to adopt as stress strain relations those obtained from W* through eqs. 25

$$\varepsilon_{11} = \frac{1}{Eh}(N_{11} - \nu N_{22}) \qquad 27\text{-}1 \qquad \chi_{11} = \frac{12}{Eh^3}(M_{11} - \nu M_{22}) \qquad 27\text{-}5$$

$$\varepsilon_{12} = \frac{1+\nu}{Eh} N_{21} \qquad 27\text{-}2 \qquad \chi_{12} = \frac{12(1+\nu)}{Eh^3} M_{21} \qquad 27\text{-}6$$

$$\varepsilon_{21} = \frac{1+\nu}{Eh} N_{12} \qquad 27\text{-}3 \qquad \chi_{21} = \frac{12(1+\nu)}{Eh^3} M_{12} \qquad 27\text{-}7$$

$$\varepsilon_{22} = \frac{1}{Eh}(N_{22} - \nu N_{11}) \qquad 27\text{-}4 \qquad \chi_{22} = \frac{12}{Eh^3}(M_{22} - \nu M_{11}) \qquad 27\text{-}8$$

The form of W* implies a shell rigid with regard to transverse shear deformation i. e.

$$\varepsilon_{13} = \varepsilon_{23} = 0 \tag{28}$$

The strain energy density W has a form completely analogous to W*, namely

$$W = \frac{Eh}{2(1-\nu^2)}\Big[(\varepsilon_{11} + \varepsilon_{22})^2 - 2(1-\nu)(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21})\Big] + \frac{Eh^3}{24(1-\nu^2)}\Big[(\chi_{11} + \chi_{22})^2 - 2(1-\nu)(\chi_{11}\chi_{22} - \chi_{12}\chi_{21})\Big] \tag{29}$$

with the property

$$N_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \qquad i,j \neq 3$$

$$M_{ij} = \frac{\partial W}{\partial \chi_{ij}} \tag{30}$$

or

$$N_{11} = \frac{Eh}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}) \quad 31\text{-}1 \qquad M_{11} = \frac{Eh^3}{12(1-\nu^2)}(\chi_{11} + \nu\chi_{22}) \quad 31\text{-}5$$

$$N_{12} = \frac{Eh}{1+\nu}\varepsilon_{21} \quad 31\text{-}2 \qquad M_{12} = \frac{Eh^3}{12(1+\nu)}\chi_{21} \quad 31\text{-}6$$

$$N_{21} = \frac{Eh}{1+\nu}\varepsilon_{12} \quad 31\text{-}3 \qquad M_{21} = \frac{Eh^3}{12(1+\nu)}\chi_{12} \quad 31\text{-}7$$

$$N_{22} = \frac{Eh}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}) \quad 31\text{-}4 \qquad M_{22} = \frac{Eh^3}{12(1-\nu^2)}(\chi_{22} + \nu\chi_{11}) \quad 31\text{-}8$$

6. Static Geometric Analogy

Inspection of the equilibrium equations, eqs. 8, for $\bar{p} = 0$, and of the compatibility equations, eqs. 19, in vector form, shows that one set of equations is transformed into the other by the correspondence indicated below

$$\bar{N}_1 = N_{11}\bar{t}_1 + N_{12}\bar{t}_2 + N_{13}\bar{n} \qquad \bar{\chi}_2 = -\chi_{22}\bar{t}_1 + \chi_{21}\bar{t}_2 + \chi_{23}\bar{n} \quad 32\text{-}1$$

$$\bar{N}_2 = N_{21}\bar{t}_1 + N_{22}\bar{t}_2 + N_{23}\bar{n} \qquad -\bar{\chi}_1 = \chi_{12}\bar{t}_1 - \chi_{11}\bar{t}_2 - \chi_{13}\bar{n} \quad 32\text{-}2$$

$$\bar{M}_1 = -M_{12}\bar{t}_1 + M_{11}\bar{t}_2 + M_{13}\bar{n} \qquad \bar{\varepsilon}_2 = \varepsilon_{21}\bar{t}_1 + \varepsilon_{22}\bar{t}_2 + \varepsilon_{23}\bar{n} \quad 32\text{-}3$$

$$\bar{M}_2 = -M_{22}\bar{t}_1 + M_{21}\bar{t}_2 + M_{23}\bar{n} \qquad -\bar{\varepsilon}_1 = -\varepsilon_{11}\bar{t}_1 - \varepsilon_{12}\bar{t}_2 - \varepsilon_{13}\bar{n} \quad 32\text{-}4$$

Also, the stress-stress function relations eqs. 9 are transformed into the strain-displacement relations, eqs. 18, by the correspondence

$$\bar{F} = F_1\bar{t}_1 + F_2\bar{t}_2 + F_3\bar{n} \qquad \bar{\omega} = \omega_1\bar{t}_1 + \omega_2\bar{t}_2 + \omega_3\bar{n} \quad 32\text{-}5$$

$$\bar{G} = G_1\bar{t}_1 + G_2\bar{t}_2 + G_3\bar{n} \qquad \bar{u} = u_1\bar{t}_1 + u_2\bar{t}_2 + u_3\bar{n} \quad 32\text{-}6$$

In particular the following analogies are of interest

$M_{13} = M_{23} = 0$	$\varepsilon_{13} = \varepsilon_{23} = 0$	32-7
Effect of $M_{12} = M_{21}$ on stress functions	Effect of $\varepsilon_{12} = \varepsilon_{21}$ on displacements	
4 equations obtained by eliminating N_{13} and N_{23} from the 6 scalar equilibrium equations	4 equations obtained by eliminating χ_{13} and χ_{23} from the 6 scalar compatibility equations	

The static geometric analogy indicated above may be extended to the stress strain relations if the correspondence indicated below is adopted.

ν	$-\nu$	33-1
$\dfrac{12}{Eh^3}$	$\dfrac{Eh}{1-\nu^2}$	33-2
h^2	h^2	33-3

Then the analogy may be completed as follows

$W^*(N_{ij}, M_{ij})$		$W(\varepsilon_{ij}, \chi_{ij})$		33-4
$\varepsilon_{ij} = \dfrac{\partial W^*}{\partial N_{ij}}$		$M_{ij} = \dfrac{\partial W}{\partial \chi_{ij}}$		33-5
$\chi_{ij} = \dfrac{\partial W^*}{\partial M_{ij}}$	$i, j \neq 3$	$N_{ij} = \dfrac{\partial W}{\partial \varepsilon_{ij}}$	$i, j \neq 3$	33-6

Further, the reduction of the number of stress functions to 4 through the requirement $M_{13} = M_{23} = 0$ is analogous to the reduction of the number of displacement unknowns to 4 through the requirement $\varepsilon_{13} = \varepsilon_{23} = 0$. Thus W expressed in terms of u_1, u_2, u_3 and ω_3 is analogous to W* expressed in terms of G_1, G_2, G_3, and F_3.

$$W^*(G_1, G_2, G_3, F_3) \qquad\qquad W(u_1, u_2, u_3, \omega_3) \qquad 33\text{-}7$$

It may also be noted that F_1, F_2, F_3 and G_3 may be chosen as the 4 independent stress functions and correspondingly ω_1, ω_2, ω_3, and u_3 may be chosen as the 4 independent displacements. Finally if the 6th equilibrium equation is used to determine ω_3 in terms of the 3 remaining displacements by means of the stress-strain relations, the analogue of this is the use of the 6th compatibility equation to determine F_3 in terms of the 3 remaining stress functions. In reducing the basic equations of thin shells to a system of differential equations for the displacements the 6th equilibrium equation may be ignored and ω_3 may be determined instead by the relation

$$\varepsilon_{12} = \varepsilon_{21}$$

The analogue of this in obtaining equations for the stress functions is the deletion of the 6th compatibility equation and the determination of the stress function F_3 through the relation

$$M_{12} = M_{21}$$

The correspondence between $Eh/1 - \nu^2$ and $12/Eh^3$ in the static geometric analogy may be avoided through homogenization of the analogous quantities. The stress couples are replaced by non-dimensional quantities and the stress resultants replaced by quantities having the dimension of a curvature according to the relations

$$T_{ij} = \sqrt{12(1 - \nu^2)}\ \frac{N_{ij}}{Eh^2} \qquad 34\text{-}1$$

$$H_{ij} = \sqrt{12(1 - \nu^2)}\ \frac{M_{ij}}{Eh^2} \qquad 34\text{-}2$$

The stress functions corresponding to T_{ij} and H_{ij} are, in vector form,

$$\overline{P} = \sqrt{12(1 - \nu^2)}\ \frac{\overline{F}}{Eh^2} \qquad 35\text{-}1$$

$$\overline{Q} = \sqrt{12(1-\nu^2)}\ \frac{\overline{G}}{Eh^2} \qquad 35\text{-}2$$

$\overline{P}$ is non dimensional and $\overline{Q}$ has the dimension of a length. W* takes the form

$$W^* = \frac{Eh^3}{24(1-\nu^2)}\left[(T_{11}+T_{22})^2 - 2(1+\nu)(T_{11}T_{22} - T_{12}T_{21})\right]$$

$$+ \frac{Eh}{2(1-\nu^2)}\left[(H_{11}+H_{22})^2 - 2(1+\nu)(H_{11}H_{22} - H_{12}H_{21})\right] \qquad (36)$$

and is the analogue of W, eq. 29, by changing ν into $-\nu$ and applying the static geometric analogy. The stress strain relations take the form

$$\varepsilon_{ij} = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}\ \frac{\partial W^*}{\partial T_{ij}}$$

$$\chi_{ij} = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}\ \frac{\partial W^*}{\partial H_{ij}} \qquad i,j \neq 3 \qquad (37)$$

They are the analogues of the relations

$$H_{ij} = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}\ \frac{\partial W}{\partial \chi_{ij}}$$

$$T_{ij} = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}\ \frac{\partial W}{\partial \varepsilon_{ij}} \qquad i,j \neq 3 \qquad (38)$$

From the preceding it may be stated that results obtained in terms of displacements may be transformed into results for the stress functions by changing ν into $-\nu$ and using the homogeneous static geometric analogy. In particular, differential equations for the displacements may be directly transformed into differential equations for the stress functions. The extension of the applicability of the static geometric analogy to the subject of boundary conditions should be of great interest if it can be shown that the solution of a specific problem may be obtained from the solution of the analogous but physically different problem. Only a general consideration of this question will be attempted here. The analogues of assigned displacement boundary values are assigned stress-function boundary values. The displacement boundary value problem can be transformed into a physically equivalent problem where strains of the boundary surface are assigned. The analogous problem is one in which stress resultants and stress couples are assigned at the boundary. These are expressed in terms of stress functions as the analogous strains are expressed in terms of the displacements. It appears therefore that the static geometric analogy may be applied to 2 physically different but mathematically analogous problems and allows obtaining the solution of one from the solution of the other.

7. Compatibility Equations in Terms of the Stress Resultants and Stress Couples. General Considerations on States of Stress.

$$(\alpha_1 N_{11})_{,2} - (\alpha_2 N_{12})_{,1} - \alpha_{1,2} N_{22} - \alpha_{2,1} N_{21}$$

$$- \nu \left[(\alpha_1 N_{22})_{,2} + (\alpha_2 N_{12})_{,1} - \alpha_{1,2} N_{11} + \alpha_{2,1} N_{21} \right] = -Eh\, \alpha_1 \alpha_2 \chi_{13} \qquad 39\text{-}1$$

$$(\alpha_2 N_{22})_{,1} - (\alpha_1 N_{21})_{,2} - \alpha_{2,1} N_{11} - \alpha_{1,2} N_{12}$$

$$- \nu \left[(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{21})_{,2} - \alpha_{2,1} N_{22} + \alpha_{1,2} N_{12} \right] = Eh \alpha_1 \alpha_2 \chi_{23} \qquad 39\text{-}2$$

$$(\alpha_1 M_{11})_{,2} - (\alpha_2 M_{12})_{,1} - \alpha_{1,2} M_{22} - \alpha_{2,1} M_{21}$$

$$- \nu \left[(\alpha_1 M_{22})_{,2} + (\alpha_2 M_{12})_{,1} - \alpha_{1,2} M_{11} + \alpha_{2,1} M_{21} \right] = - \frac{Eh^3}{12} \alpha_1 \alpha_2 \frac{\chi_{13}}{R_2} \qquad 39\text{-}3$$

$$(\alpha_2 M_{22})_{,1} - (\alpha_1 M_{21})_{,2} - \alpha_{2,1} M_{11} - \alpha_{1,2} M_{12}$$

$$- \nu \left[(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{21})_{,2} - \alpha_{2,1} M_{22} + \alpha_{1,2} M_{12} \right] = \frac{Eh^3}{12} \alpha_1 \alpha_2 \frac{\chi_{23}}{R_1} \qquad 39\text{-}4$$

$$\alpha_1 \alpha_2 \left(\frac{M_{22} - \nu M_{11}}{R_1} + \frac{M_{11} - \nu M_{22}}{R_2} \right) = \frac{Eh^3}{12} \left[(\alpha_1 \chi_{13})_{,2} - (\alpha_2 \chi_{23})_{,1} \right] \qquad 39\text{-}5$$

$$M_{12} - M_{21} + \frac{h^2}{12} \left(\frac{N_{21}}{R_2} - \frac{N_{12}}{R_1} \right) = 0 \qquad 39\text{-}6$$

The 4 first equations may be transformed through use of the equilibrium equations into the form

$$(1 + \nu) \frac{\alpha_2 Q_2}{R_2} + N,_2 = - Eh\alpha_2 \chi_{13} - (1 + \nu)\alpha_2 p_2 \qquad 40\text{-}1$$

$$(1 + \nu) \frac{\alpha_1 Q_1}{R_1} + N,_1 = Eh\alpha_1 \chi_{23} - (1 + \nu) \alpha_1 p_1 \qquad 40\text{-}2$$

$$(1 + \nu) \alpha_2 Q_2 - M,_2 = \frac{Eh^3}{12} \frac{\alpha_2 \chi_{13}}{R_2} \qquad 40\text{-}3$$

$$(1 + \nu)\alpha_1 Q_1 - M,_1 = - \frac{Eh^3}{12} \frac{\alpha_1 \chi_{23}}{R_1} \qquad 40\text{-}4$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} \right) M - (1 + \nu)\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2} \right) = \frac{Eh^3}{12} \frac{(\alpha_1 \chi_{13}),_2 - (\alpha_2 \chi_{23}),_1}{\alpha_1 \alpha_2} \qquad 40\text{-}5$$

$$M_{12} - M_{21} + \frac{h^2}{12} \left(\frac{N_{21}}{R_2} - \frac{N_{12}}{R_1} \right) = 0 \qquad 40\text{-}6$$

where the following notation was introduced

$N = N_{11} + N_{22}$ $\qquad$ $M = M_{11} + M_{22}$ $\qquad$ 41-1

$Q_1 = N_{13}$ $\qquad$ $Q_2 = N_{23}$ $\qquad$ 41-2

Eliminating χ_{13} and χ_{23} from the first 5 equations and letting in the coefficients of Q_1 and Q_2

$$1 + \frac{h^2}{12R_1^2} \approx 1 + \frac{h^2}{12R_2^2} \approx 1$$

obtain

$$(1 + \nu)\alpha_2 Q_2 - M,_2 = - \frac{h^2}{12R_2} (N,_2 + (1 + \nu)\alpha_2 p_2) \qquad 42\text{-}1$$

$$(1 + \nu)\alpha_1 Q_1 - M,_1 = - \frac{h^2}{12R_1} (N,_1 + (1 + \nu)\alpha_1 p_1) \qquad 42\text{-}2$$

$$\Delta N + \frac{12}{h^2}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)M - (1+\nu)\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right)\right] =$$

$$- \frac{1+\nu}{a_1 a_2}\left[\left(a_1\left(\frac{Q_2}{R_2} + p_2\right)\right)_{,2} + \left(a_2\left(\frac{Q_1}{R_1} + p_1\right)\right)_{,1}\right] \qquad 42\text{-}3$$

where Δ is the Laplace operator

$$\Delta(\ ) = \frac{1}{a_1 a_2}\left[\left(\frac{a_2(\ )_{,1}}{a_1}\right)_{,1} + \left(\frac{a_1(\ )_{,2}}{a_2}\right)_{,2}\right] \qquad 43$$

The 3 equations above agree except for negligible terms with those obtained in ref. (11) through a different system of stress-strain relations. Eqs. 42 may be simplified through an analysis of the relative orders of magnitude of the terms involved. In order to do this the curvilinear coordinates will be considered as non dimensional variables such as the Lamé parameters a_1 and a_2 have the dimensions of lengths of the same order of magnitude as the radii of curvature of the coordinate lines. It will be assumed for the purposes of this discussion that R denotes the order of magnitude of R_1, R_2, a_1 and a_2 or, if needed, of the smallest of these quantities. 3 cases corresponding to different behaviors of a thin shell are of interest.

a) Membrane Solution

Assuming that differentiation with regard to ξ_1 and ξ_2 does not increase the order of magnitude, eqs. 42 may be satisfied by stress resultants and stress couples related through the order of magnitude relations

$$Q = O\left(\frac{M}{R}\right) = O\left(\frac{h^2(N + Rp)}{R^2}\right) \qquad 44\text{-}1$$

where Q, N, M and p are generic symbols indicating transverse shears, in-plane stress resultants, stress couples and surface load, respectively. The above relation corresponds to the case where the different terms in each of eqs. 42 are of the same order of magnitude except the right hand side of eq. 42-3 which is negligible as being of relative order h^2/R^2.

The relation $Q = O(M/R)$ is consistent with the moment equilibrium equations, whereas the relation $Q = O(h^2 N/R^2)$ allows neglecting Q_1 and Q_2 in the force equilibrium equations and the relation $M/R = O(h^2 N/R^2)$ allows neglecting M_{12} and M_{21} in the 6th equilibrium equation. The result is the equilibrium equations of the membrane theory. For the validity of the original assumptions the displacements of the membrane theory must imply in plane strains and changes of curvature satisfying the order of magnitude relation

$$\chi = O(\varepsilon/R) \qquad 44\text{-}2$$

This will be the case if the inextensional bending that arises from the homogeneous solution of the strain displacement relations of the membrane theory is made to agree with eq. 44-2 and if smoothness of load and geometry of shell satisfy the requirement that differentiation does not change the order of magnitude.

b) <u>Inextensional Bending Solution</u>

The displacements of the membrane solution include the general solution for the displacements of the equations $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{21} = 0$. These displacements yield, through use of the stress-strain relations, zero in plane stress resultants and non zero stress couples which, on the basis of the compatibility equations, satisfy the relation

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right)M - (1+\nu)\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) = 0 \qquad 45\text{-}1$$

$$M_{12} = M_{21} \qquad 45\text{-}2$$

The transverse shears are obtained through the moment equilibrium equations in the form

$$Q_1 = \frac{M_{,1}}{(1+\nu)a_1} \qquad 46\text{-}1$$

$$Q_2 = \frac{M_{,2}}{(1+\nu)a_2} \qquad 46\text{-}2$$

If the geometric parameters of the middle surface are not rapidly varying it may be assumed that differentiation does not increase the order of magnitude. From eqs. 46 it is possible to write

$$Q = O(\frac{M}{R}) \qquad 47\text{-}1$$

With transverse shears satisfying eq. 47-1 the stress resultants needed to satisfy force equilibrium and the 6th equilibrium equation have the order of magnitude

$$N = O(\frac{M}{R}) \qquad 47\text{-}2$$

i.e. they produce stresses that are negligible with regard to the bending stresses as being of relative order h/R. Eq. 47-2 is in accordance with setting $\mathcal{E}_{11} = \mathcal{E}_{22} = \mathcal{E}_{12} = \mathcal{E}_{21} = 0$ in the compatibility equations.

It may be mentioned here that for a spherical shell the inextensional solution is an exact solution of the original equations (6). For shells of positive Gaussian curvature such that $(1/R_1 - 1/R_2)$ is small compared to $1/R_1$ and $1/R_2$, eq. 45-1 shows that $M = M_{11} + M_{22}$ is small compared to M_{11} and M_{22}. These are then of the same order of magnitude but have different signs. Q_1 and Q_2 are then smaller than what is implied by eq. 47-1 and accordingly the in-plane stress resultants needed to satisfy equilibrium are smaller than implied by 47-2.

For shells of negative Gaussian curvature such that $1/R_1 + 1/R_2$ is small compared to $1/R_1$ and $1/R_2$, M_{11} and M_{22} tend to be of the same order of magnitude and of the same sign. There is, however, no reduction in the order of magnitude of Q_1, Q_2 and the in-plane stress resultants. Before considering a third type of state of stress it may be interesting to show the analogy between the membrane solution for the case of zero surface load and the inextensional bending solution. This analogy is summarized below.

Solution of equilibrium equations with $\bar{p} = 0$ and $M_{11} = M_{22} = M_{21} = Q_1 = Q_2 = 0$.	Solution of compatibility equations with $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{21} = \chi_{13} = \chi_{23} = 0$
Displacements u_1, u_2, u_3, ω_3 as obtained by integrating the stress strain relations between in-plane strains and stress resultants	Stress functions G_1, G_2, G_3, F_3 as obtained by integrating the stress strain relations between stress couples and changes of curvature.
Changes of curvature due to u_1, u_2, u_3, ω_3.	In-plane stress resultants due to G_1, G_2, G_3, F_3.

If the analogy above is set in terms of the dimensionally homogeneous quantities defined in sec. 2.5 then it may be extended to the order of magnitude relationships in each solution. These should not be affected by the change of ν into $-\nu$. Thus if $\chi = O(\varepsilon/R)$ in the membrane solution, then $N = O(M/R)$ in the inextensional solution.

c) Edge Zone Solution or Boundary Layer

If it is assumed that differentiation with regard to at least one coordinate increases the order of magnitude by a factor of order $\leq \sqrt{h/R}$ then it is generally possible to satisfy the shell equations by bending stresses of an order of magnitude equal to or larger than that of the membrane stresses i.e. $M/h \geq O(N)$ 48

In that case the right hand sides of eqs. 42-1, 2 are negligible as being of relative order $\leq h/R$. The transverse shears are thus related to M as in b) through the relations

$$(1+\nu)\alpha_2 Q_2 = M,_2 \qquad 49\text{-}1$$

$$(1+\nu)\alpha_1 Q_1 = M,_1 \qquad 49\text{-}2$$

In eq. 42-3 the transverse shear terms are negligible as being of relative order $\leq h/R$ but ΔN which involves double differentiation may be of the same order of magnitude as the stress couple term. In the force equilibrium equations the transverse shear terms may be of relative order $\sqrt{h/R}$ in the first 2 equations but of relative order unity in the third equation.

It may be noted here that neglecting the transverse shear terms in the first 2 equilibrium equations as is done for shallow shells may also be done for non shallow shells if $\sqrt{h/R}$ is negligible with regard to unity. This latter approximation is made in obtaining the so-called Mustari-Vlassow equations (12). Finally the assumption concerning the behavior of the dependent variables under differentiation is consistent with the type of differential equations.

The 3 states of stress described above are treated in the literature. The orders of magnitude of the errors associated with their extraction from the general equations were the object of the above discussion. By superimposing them it is generally possible to satisfy 4 arbitrary boundary conditions. They represent then the complete solutions of the original equations.

From the preceding discussion it appears that in cases (b) and (c) eqs. 42 may be simplified by deleting their right hand sides. In case (a) eqs. 42 are not used; but remembering that the membrane solution may be formally obtained by letting $Q_1 = Q_2 = M_{12} = M_{21} = 0$ in the equilibrium equation it is possible to write in all cases

$$(1 + \nu)\alpha_2 Q_2 - M,_2 = 0 \qquad 50\text{-}1$$

$$(1 + \nu)\alpha_1 Q_1 - M,_1 = 0 \qquad 50\text{-}2$$

$$\Delta N + \frac{12}{h^2}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)M - (1 + \nu)\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right)\right] = 0 \qquad 50\text{-}3$$

The first 2 equations may be replaced by

$$(\alpha_2 Q_2),_1 - (\alpha_1 Q_1),_2 = 0 \qquad 51\text{-}1$$

$$(\alpha_1 Q_2),_2 + (\alpha_2 Q_1),_1 - \frac{\alpha_1 \alpha_2}{1 + \nu}\,\Delta M = 0 \qquad 51\text{-}2$$

Finally upon taking account of the third equilibrium equation we obtain the system of equations

$$(\alpha_2 Q_2)_{,1} - (\alpha_1 Q_1)_{,2} = 0 \qquad 52\text{-}1$$

$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - p_3 - \frac{1}{1+\nu}\Delta M = 0 \qquad 52\text{-}2$$

$$\Delta N + \frac{12}{h^2}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)M - (1+\nu)\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right)\right] = 0 \qquad 52\text{-}3$$

It may be noted that eq. 40-6 which is sometimes called a seventh equilibrium equation did not enter into the preceding discussion. It is known that it may be replaced with a relative error not exceeding O(h/R) by

$$M_{12} - M_{21} = 0 \qquad 52\text{-}4$$

8. Differential Equations for the Stress Functions

The general solution of the 6 equilibrium equations will be considered as the sum of a particular solution and of the general solution of the homogeneous equations. The particular solution will be taken as a particular solution of the equilibrium equations of the membrane theory and will be denoted by

$$N_{11}^*, \ N_{22}^*, \ N_{12}^* = N_{21}^*$$

The general solution of the homogeneous equilibrium equations consists of the stress-stress functions relations eqs. 13. In expressing eqs. 52 in terms of the stress functions, the following expressions are obtained

$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - p_3 = \frac{1}{\alpha_1 \alpha_2}\left[\left(\frac{\alpha_1}{R_1} F_1\right)_{,2} - \left(\frac{\alpha_2 F_2}{R_2}\right)_{,1}\right] \qquad 53\text{-}1$$

$$M = M_{11} + M_{22} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)G_3 + \frac{(\alpha_1 G_2)_{,2} + (\alpha_2 G_1)_{,1}}{\alpha_1 \alpha_2} \qquad 53\text{-}2$$

$$\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2} = \frac{2G_3}{R_1R_2} + \frac{1}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_1 G_2}{R_1}\right)_{,2} + \left(\frac{\alpha_2 G_1}{R_2}\right)_{,1}\right] \qquad 53\text{-}3$$

$$N = N_{11} + N_{22} = \frac{(\alpha_1 F_1)_{,2} - (\alpha_2 F_2)_{,1}}{\alpha_1\alpha_2} + N^* \qquad 53\text{-}4$$

Eqs. 52-1, 2, 3 take then the form

$$\alpha_1\alpha_2 \,\Delta F_3 - \left(\frac{\alpha_2 F_1}{R_1}\right)_{,1} - \left(\frac{\alpha_1 F_2}{R_2}\right)_{,2} = 0 \qquad 54\text{-}1$$

$$\left(\frac{\alpha_1 F_1}{R_1}\right)_{,2} - \left(\frac{\alpha_2 F_2}{R_2}\right)_{,1} - \frac{\alpha_1\alpha_2}{1+\nu}\,\Delta\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right) G_3 + \frac{(\alpha_1 G_2)_{,2} + (\alpha_2 G_1)_{,1}}{\alpha_1\alpha_2}\right] = 0$$

54-2

$$\Delta\left[\frac{(\alpha_1 F_1)_{,2} - (\alpha_2 F_2)_{,1}}{\alpha_1\alpha_2}\right] + \frac{12}{h^2}\left\{\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2 - \frac{2(1+\nu)}{R_1R_2}\right] G_3 + \right.$$

$$\left.\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{(\alpha_1 G_2)_{,2} + (\alpha_2 G_1)_{,1}}{\alpha_1\alpha_2} - \frac{1+\nu}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_1 G_2}{R_1}\right)_{,2} + \left(\frac{\alpha_2 G_1}{R_2}\right)_{,1}\right]\right\} + \Delta N^* = 0$$

54-3

In these equations F_1 and F_2 are related to G_1, G_2, G_3 through the relations

$$F_1 = \frac{G_{3,2}}{\alpha_2} - \frac{G_2}{R_2} \qquad 55\text{-}1$$

$$F_2 = -\frac{G_{3,1}}{\alpha_1} + \frac{G_1}{R_1} \qquad 55\text{-}2$$

and if eq. 40-6 is replaced by $M_{12} = M_{21}$, F_3 is determined in terms of G_1 and G_2 through the relation

$$F_3 = \frac{1}{2}\,\frac{(\alpha_2 G_2)_{,1} - (\alpha_1 G_1)_{,2}}{\alpha_1\alpha_2} \qquad 55\text{-}3$$

In terms of G_1, G_2, G_3 eqs. 54 take the form

$$\frac{\alpha_1\alpha_2}{2}\,\Delta\left[\frac{(\alpha_2 G_2),_1 - (\alpha_1 G_1),_2}{\alpha_1\alpha_2}\right] - \left(\frac{G_{3,2}}{R_1} - \frac{\alpha_2 G_2}{R_1 R_2}\right),_1 + \left(\frac{G_{3,1}}{R_2} - \frac{\alpha_1 G_1}{R_1 R_2}\right),_2 = 0 \qquad 56\text{-}1$$

$$DG_3 - \frac{1}{1+\nu}\,\Delta\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)G_3 + \frac{(\alpha_1 G_2),_2 + (\alpha_2 G_1),_1}{\alpha_1\alpha_2}\right] - \frac{1}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_2 G_1}{R_1 R_2}\right),_1 + \left(\frac{\alpha_1 G_2}{R_1 R_2}\right),_2\right] = 0 \qquad 56\text{-}2$$

$$\Delta\Delta G_3 - \Delta\left(\frac{\left(\frac{\alpha_1 G_2}{R_2}\right),_2 + \left(\frac{\alpha_2 G_1}{R_1}\right),_1}{\alpha_1\alpha_2}\right) + \frac{12}{h^2}\left\{\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2 - \frac{2(1+\nu)}{R_1 R_2}\right] G_3 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{(\alpha_1 G_2),_2 + (\alpha_2 G_1),_1}{\alpha_1\alpha_2} - \frac{1+\nu}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_1 G_2}{R_1}\right),_2 + \left(\frac{\alpha_2 G_1}{R_2}\right),_1\right]\right\} + \Delta N^* = 0 \qquad 56\text{-}3$$

where

$$D(\quad) = \frac{1}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_2(\quad),_1}{\alpha_1 R_2}\right),_1 + \left(\frac{\alpha_1(\quad),_2}{\alpha_2 R_1}\right),_2\right] \qquad 57$$

9. Summary and Conclusion

A formulation of the equations of thin elastic shells including a discussion of the static geometric analogy and of the different types of states of stress was presented. The differential equations for the stress functions were also obtained. The proposed system of stress strain relations has not apparently been used before. It was derived, however, from an accepted form, of the complementary strain energy function and lead to results in accordance with those of other established formulations. It has a simple form and is invariant in a change of curvilinear coordinates.

In the continuation of this work it is proposed to investigate the equations describing the particular states of stress, their possible simplification and their specialization to certain particular shells. A more thorough investigation of the static geometric analogy is also proposed in view of the possibility of directly relating solutions in terms of displacements to solutions of different problems in terms of stress functions and also in view of combining both displacements and stress functions in one system of differential equations.

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NASA-Langley, 1966 CR-568

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