ANALYSIS AND IMPROVEMENT OF ITERATION METHODS FOR SOLVING AUTOMATIC CONTROL EQUATIONS

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This report covers the period from February to August, 1966. During two months of this period a graduate research assistant was employed making the total time spent on this contract 4.25 man months.

The major effort has been on an intensive study of approaches taken by Gustafson (6) in approximation techniques and to Krishnamurthy (4) on matrix iteration techniques for finding roots. Some improvements have been made on Krishnamurthy's method. These are reported in the following pages. A summary of Gustafson's approach is also included.

In the remaining research period further work will not be done on the matrix approach. Most of the work will be in evaluation of approximation techniques as they are concerned with different design specifications of automatic control systems.

KRISHNAMURTHY APPROACH

E. V. Krishnamurthy in reference (4) presents a matrix approach to the problem of root finding. Given a polynomial, first the companion matrix of this polynomial is formed, then the eigen values of the matrix are found. Since the eigenvalues of the companion matrix are equal to the roots of the polynomial, the root finding problem is converted to an algebraic eigenvalue problem.

Krishnamurthy suggests using the matrix power method to find the eigenvalues. This method, basically an iteration process, will converge on the real eigenvalue with the largest modulus. An arbitrary trial vector is multiplied by the companion matrix. The resulting vector is then examined to see if it differs from the original trial vector by a constant multiplier. If so, the constant multiplier is the sought after eigenvalue. If not, as is usually the case with the first iterations, the resulting vector is then multiplied by the companion matrix and the process is continued until it "converges" on an eigenvector and eigenvalue. Inspecting the method, we see that if the process converges after M iterations, then the original trial vector has been effectively multiplied by the companion matrix M times. Krishnamurthy suggests using the Caley-Hamilton theorem to represent the high power matrix in terms of the first N-1 powers of the matrix for an Nth order polynomial. The Caley-Hamilton theorem states that any square matrix (the companion matrix is square) satisfies its own characteristic equation; so for a given polynomial:

$$S^n + An\text{-}1\ S^{n-1} + An\text{-}2\ S^{n-2} + \ldots + A_1 S + A_0 = 0$$

the companion matrix:

$$A = \begin{bmatrix} 0\ 0 \ldots\ldots 0 & -A_0 \\ 1\ 0 \ldots\ldots 0 & -A_1 \\ 0\ 1 \ldots\ldots 0 & -A_2 \\ \ldots\ldots\ldots & \ldots \\ 0\ 0 \ldots\ldots 1 & -A_{n-1} \end{bmatrix}_{n \times n}$$

satisfies the above polynomial. Therefore:

$$A^n = -(A_{n-1} A^{n-1} + \ldots + A_1 A + A_0 I) \tag{1}$$

or for $m = nr$:

$$A^m = (A^n)^r = (-1)^r (A_{n-1} A^{n-1} + \ldots + A_1 A + A_0 I)^r \tag{2}$$

and any terms of order higher than N-1 resulting from the expansion of the right side of equation (2) are reduced to a linear combination of the first N-1 powers of A by successive substitutions of equation (1). Once the high power matrix is formed, then the multiplication of an arbitrary vector by the matrix will yield approximately the eigenvector corresponding to the highest modulus real eigenvalue, times some constant (not the eigenvalue). This vector is then multiplied by the companion matrix A to find the eigenvalue.

The Krishnamurthy process has been shortened by the following modifications.

An efficient method has been developed to obtain the algebraic expression for the high power matrix using the coefficients of the given polynomial.

The need for constructing the high power matrix itself has been eliminated. Noting the following properties of the high power matrix:

A) If the given polynomial is of order N, the first N-1 powers of the companion matrix A will have the first column consisting of one "1" and the rest zeros.

B) The columns of the high power matrix are linearly dependent, i.e., they differ only by constant multiples. These columns are in fact equal to constant multiples of the eigenvector corresponding to the highest modulus eigenvalue. This is apparent for two reasons:

   1) From observation of matrices formed from specific third and fourth order equations.

   2) From the observation that the only construct of a matrix that will transform any arbitrary (non-zero) vector into a certain eigenvector (times a constant) is that the columns of the matrix be constant multiples of that eigenvector.

C) If the elements of the high power matrix are designated as Aij and the algebraic expression for the high power matrix is:

$$A^m = C_n A^{n-1} + C_{n-1} A^{n-2} + \dots + C_2 A + C_1 I$$

Then: $A_{i1} = C_i$ $(i = 1, 2, \dots, N)$. The first column of the high power matrix is composed of the coefficient of the algebraic expression for the high power matrix.

Thus the eigenvector corresponding to the eigenvalue being sought is

$$\begin{Bmatrix} C_1 \\ C_2 \\ \dots, \\ C_{n-1} \\ C_n \end{Bmatrix}$$

Multiplication of this vector by the comparitively simple companion matrix will yield the eigenvalue (root).

An example of finding the highest modulus real root of a fourth order equation using the matrix approach is given in appendix A.

If the eigenvalues with the largest modulus are a complex conjugate pair, then the method described above will not converge, i. e., the iteration would oscillate. Krishnamurthy suggests two procedures for obtaining complex eigenvalues: (1) using the knowledge of three successive high order iterates to form two simultaneous equations whose solutions give the real and imaginary parts of the sought-after eigenvalues (5). Note that in this case the entire high power matrix, A^m, has to be formed, and then multiplied twice by A to form three successive high power iterates, $A^m X^o$, $A^{m+1} X^o$, $A^{m+2} X^0$. (2) The second method involves iteration with an assumed <u>complex</u> vector. During the iteration, though, each trial vector has to be normalized so that the highest modulus element of the vector is of the form $1 + 0j$. In this case there is no shortcut; all the iterations have to be carried out (i. e., $A^1 X^{(o)}$, AX^1, . . . AX^m).

If the highest modulus roots are equal roots, then convergence to the roots using the matrix power method may occur as it does for the single root or it may be extremely slow. The convergence depends on the properties of the matrix (i. e., whether or not there is a linearly independent eigenvector associated with each of the repeated eigenvalues). Note that here the entire high power matrix need not be formed. The first column alone may be used as the eigenvector; but it may be required to use extremely high powers.

For roots of equal modulus (i. e., one real and two complex conjugate roots of the same modulus), the iteration will oscillate as in the case of the complex conjugate roots. But the successive iterates will equal modulus multiplicity. If a constant real matrix μI is added to

A this will cause the moduli to separate and the iteration would converge on the real root. The eigenvalues would change from: λ_1, $Re\lambda_2 + iIm\lambda_2$, $Re\lambda_2 - iIm\lambda_2$ to: $\lambda_1 + \mu_1$, $(Re\lambda_2 + \mu) + iIm\lambda_2$, $(Re\lambda_2 + \mu) - iIm\lambda_2$, thus the real eigenvalue now has the highest modulus. Note that if the new matrix is formed $(A + \mu I)$, the properties of this new matrix have not been investigated to find a shortcut to the iteration.

After the highest modulus root(s) have been found, the given polynomial may be reduced by dividing found root(s) out. A new companion matrix of lower order is formed and the process is continued.

GUSTAFSON

The Gustafson approach to approximating control equations (6) is to produce two second order transfer functions whose time response approximates the time response of the nth order system.

Given a system transfer function (with no zeros) of the form:

$$F(S) = \frac{b_o}{A_n S^n + \ldots + A_2 S^2 + A_1 S + A_o} \tag{1}$$

Gustafson constructs, directly from this, an approximating function called the truncated function of the second order:

$$T_2(S) = \frac{b_o}{A_2 S^2 + A_1 S + A_o} \quad .$$

He shows that the zeroeth, first, and second time moments of the impulse response of $T_2(S)$ are identical with those of equation (1) above.

Now the $T_2(S)$ function would not include an oscillating high frequency mode that might be present in the $F(S)$ function. To measure the effect of any high frequency "buzz" the Integral of the Squared Impulse Response (ISIR) is computed for the $F(S)$ function and compared to the ISIR for the $T_2(S)$ function. Gustafson shows that the ISIR of the $F(S)$ function will always be greater than the ISIR of the $T_2(S)$ function. The closer these two values are, the closer the $T_2(S)$ response approximates the system response.

It is shown that the ISIR of the $F(S)$ function can be computed from the last two elements of the Routh Array for $F(S)$. The Routh Array (modified by Moore (3)) for $F(S)$ is shown in Figure 1.

$$A_n \;\Big|\; A_{n-1} \qquad A_{n-2} \qquad A_{n-3} \qquad \ldots\ldots A_2 \qquad A_1 \qquad A_o$$

$$\left(\frac{A_n A_{n-3}}{A_{n-1}}\right)$$

$$\underline{\underline{}}\;R_1$$

$$\left(\frac{A_{n-1}}{R_1}\right)$$

$$\underline{\underline{}}\;R_2$$

$$\ldots\ldots\left(\frac{R_{n-4}}{R_{n-3}}\right)(A_1)$$

$$\underline{\underline{\phantom{R_{n-2}}}}\;\boxed{R_{n-2}}$$

$$\left(\frac{R_{n-3}}{R_{n-2}}\right)(A_o)$$

$$\underline{\underline{\phantom{R_{n-1}}}}\;\boxed{R_{n-1}}$$

$$\underline{\underline{}}\;\boxed{A_o}$$

Another approximating function called the associated function of the second order is constructed.

$$A_2(S) = \frac{b_o}{R_{n-2}S^2 + R_{n-1}S + A_o} .$$

Then the ISIR of the $F(S)$ function is identical with that of the $A_2(S)$ function. So computing the ISIR's from the Routh Array

$$\text{ISIR } F(S) = \text{ISIR } A_2(S) = \frac{1}{2}\,\frac{b_o^2}{R_{n-1}A_o}$$

and: ISIR $T_2(S) = \frac{1}{2} \frac{b_o^2}{A_1 A_o}$

then defining the energy ratio:

$$E_1 = \frac{\text{ISIR } F_2(S)}{\text{ISIR } T_2(S)} = \frac{A_1}{R_{n-1}} \quad .$$

This ratio indicates how closely $T_2(S)$ approximates $F(S)$.

The $A_2(S)$ function is shown to have zeroeth, first, and second frequency moments of its spectral energy distribution identical to the corresponding moments of the $F(S)$ function. Also the natural frequency of the $A_2(S)$ approximation is shown to be equal to the W_{rms} of the $F(S)$ function.

Gustafson then shows the following step response properties of the $T_2(S)$ and $A_2(S)$ approximating functions.

T_2S --- The step response of $T_2(S)$ has about the same mean time delay as the system response (resulting from identical first time moments) and is generally a lower bound on overshoot.

$A_2(S)$-- The step response of $A_2(S)$ has about the same rise time as the system response (resulting from identical W_{rms}) and is generally an upper bound on overshoot.

Thus the system response is seen to be approximated by two known second order responses. The accuracy of the approximations $A_2(S)$ and $T_2(S)$ can be computed or fixed by the designer using the E_1 ratio. Gustafson found that a tolerance of 1.4 to 1.6 for E_1 yields good results.

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APPENDIX A

EXAMPLE OF MATRIX APPROACH TO FIND A REAL ROOT

Given the polynomial $X^4 + 5X^3 + 9X^2 + 7X + 2 = 0$, whose roots are (-1, -1, -1, -2). Find the highest modulus <u>real</u> root using the matrix approach.

First generate algebraic representation of the high power matrix.

$$X^4 = -(5X^3 + 9X^2 + 7X + 2) \qquad (1)$$

"normalizing" $1/5\, X^4 = -(X^3 + 1.80X^2 + 1.4X + .4)$. Then using the coefficients and "synthetic squaring"

				1.00	1.80	1.40	.40	
				1.00	1.80	1.40	.40	
				.8(1.00)	.8(1.00)	.8(1.40)	.16	
			2.80	(2.8)1.80	$(1.40)^2$	----		
	1.00	3.60	$(1.80)^2$	----	----			
$\frac{1}{25} X^8 =$	1.00	3.60	6.04	5.84	3.40	1.12	.16	(2)

and for a constant k: $kX^8 = 1.00X^6 + 3.60X^5 + 6.04X^4 + 5.84X^3 + 3.40^2 + 1.12X + .16$ but from (1) above:

$$\begin{aligned} 1.00X^6 &= -X^2\ (5X^3 + 9X^2 + 7X + 2) \\ &= -(5X^5 + 9X^4 + 7X^3 + 2X^2) \end{aligned}$$

so:

$$\begin{aligned} kX^8 &= (-5X^5) - 9X^4 - 7X^3 - 2X^2) + \\ &\quad \underline{3.60X^5 + 6.04X^4 + 5.84X^3 + 1.12X + .16} \\ kX^8 &= -1.40X^5 - 2.96X^4 - 1.16X^3 + 1.40X^2 + 1.12X + .16 \end{aligned}$$

or $\quad kX^8 = -1.00X^5 - 2.11X^4 - 8.24X^3 + 1.00X^1 + .80X + .114$

then again from (1) above we see that:

$$-1.00X^5 = X(5X^3 + 9X^2 + 7X + 2).$$

And the process continues--

But an algorithm can be used to do these successive substitutions more efficiently. Going back to line (2).

X^8 =	1.00	3.60	6.04	5.84	3.40	1.12	.16
substituting		-5.00	-9.00	-7.00	-2.00		
subtracting		-1.40	-2.96	-1.16	+1.40		
normalizing		-1.00	-2.11	- .829	+1.00	+ .80	+ .114
substituting			+5.00	+9.00	+7.00	+2.00	
subtracting			2.89	+8.171	+8.00	+1.80	
normalizing			1.00	+2.83	+2.77	+ .969	+ .0394
substituting				-5.00	-9.00	-7.00	-2.000
subtracting				-2.17	-6.23	-6.031	-1.9606
normalizing			k_3X^8 =	1.00	+2.87	+2.78	+ .904
				1.00	+2.87	+2.78	+ .904
squaring				1.81	+5.19	+5.03	+ .817
			5.56	+15.96	+7.73	----	
	1.00	5.74	8.24	-----	----		
k^4X^{16}	1.00	+5.74	+13.80	+17.77	+12.92	+5.03	+ .817
substituting		-5.00	- 9.00	- 7.00	- 2.00		
subtracting		.74	+ 4.80	+10.77	+10.92		
normalizing		1.00	+ 6.49	+14.55	+14.76	+6.80	+1.104
substituting			- 5.00	- 9.00	- 7.00	-2.00	
subtracting			1.49	+ 5.55	+ 7.76	+4.80	
normalizing			1.00	3.73	5.21	3.22	.741
substituting				- 5.00	- 9.00	-7.00	-2.00
subtracting				- 1.27	- 3.79	-3.78	-1.26
normalizing k^5X^{16} =				1.00	2.98	2.98	.992

Now assuming that sixteen "iterations" are enough we will find the eigenvalue. The "eigenvector" is found from coefficients of the algebraic expression for the high power (16) matrix.

$$\begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{bmatrix} \underbrace{\begin{Bmatrix} .992 \\ 2.98 \\ 2.98 \\ 1.00 \end{Bmatrix}}_{\text{eigenvector}} = \begin{Bmatrix} -2 \\ -7 + .99 \\ -9 + 2.98 \\ -5 + 2.98 \end{Bmatrix} = \begin{Bmatrix} -2 \\ -6.01 \\ 6.02 \\ 2.02 \end{Bmatrix}$$

then removing the constant.

$$\text{Eigenvalue (root)} = -2.02 \quad \begin{Bmatrix} .991 \\ 2.98 \\ 2.98 \\ 1.00 \end{Bmatrix}$$

The correctness of the eigenvalue can be checked by comparing the multiplied vector with the product vector to see if they are equal. If not, the % of difference in the two vectors is approximately the % of error in the eigenvalue. If the error is too large, a higher power matrix expression can be used. The next step in this case would be the 32nd power.

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