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SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY MEANS OF LIE SERIES

by F. Cap, D. Floriani, W. Gröbner, A. Schett, and J. Weil

Prepared by
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Preface

Lie series are special series containing differential operators. Name and concept ot these series are due to W.Groebner, Institute for Mathematics, Innsbruck University. These series were invented by Groebner to solve special problems in algebraic geometry. However it was found that these series, named by Groebner after S.Lie, were very useful to solve differential equations. My friend Groebner offered this new tool to theoretical physics. The usefullness of the new method was shown in several papers by W.Groebner and F.Cap. In celestial mechanics the new method could compete with other modern methods. So H.Knapp, Innsbruck, calculated the orbit of the eight satellite of Jupiter using a special version of the new method and J.Kovalevsky in Paris calculated the same problem using Cowells method. The two results showed excellent agreement.

In the US Dr.Wilson from the Applied Mathematics Section of NASA recognized first the advantages of the new method. Thanks to his understanding and interest NASA offered a research grant for further investigations on the new method. The results of the investigations are presented in this monograph. The authors would like to express their deep gratitude to NASA and to Dr.Wilson - without their help and encouragement this book would never have been written.

Innsbruck (Austria) Ferdinand Cap

the University

June 1966

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Chapter I

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In the last few years, Lie series have proved to be an useful tool for solving differential equations. Based on the work of W.GRÖEBNER, /1/, Department of Mathematics at the Universitiy of Innsbruck, a lot of further theoretical development and physical applications has been published /2-16/.

A series of the following kind:

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu f(z) = f(z) + tDf(z) + \frac{t^2 D^2 f(z)}{2!} + \ldots\ldots \qquad (I,1)$$

is here called Lie series; $f(z)$ is any function which depends on the complex variables z_1, z_2, ..., z_n and is holomorphic in the neighborhood of the point z_0. D is a linear differential operator, defined by:

$$D = \delta_1(z)\frac{\partial}{\partial z_1} + \delta_2(z)\frac{\partial}{\partial z_2} + \ldots\ldots + \delta_n(z)\frac{\partial}{\partial z_n} \qquad (I,2)$$

the coefficients $\delta_i(z)$ represent functions of the complex variables z_1, z_2, ..., z_n, which are all assumed to be holomorphic in the neighborhood of the point z_0.

The convergence of the Lie series (I,1) was proved by GROEBNER /1/, using the method of CAUCHY'S majorants.

The following theorem holds:

The Lie series (I,1) converges absolutely at every point of the z-space, where the operator D, i.e. all functions $\delta_K(z)$ as well as the function $f(z)$ are holomorphic; in every such point in the z-space a positive number T can be given in such a manner that the Lie series converges absolutely at least for $|t| < T$.

The series (I,1) can be used to solve differential equations. We consider an ordinary differential equation of the order n which is given by:

$$Z^{(n)}(t) = \delta\ (t, Z, Z', Z'', \ldots\ldots\ldots, Z^{(n-1)}) \tag{I,3}$$

Eq.(I,3) can be written in the form:

$$\begin{aligned} Z'_1 &= \delta_1(t,\ Z_1,\ Z_2,\ \ldots,\ Z_n) \\ Z'_2 &= \delta_2(t,\ Z_1,\ Z_2,\ \ldots,\ Z_n) \\ &\text{-------------------------} \\ Z'_n &= \delta_n(t,\ Z_1,\ Z_2,\ \ldots,\ Z_n) \end{aligned} \tag{I,4}$$

Assuming the functions δ_i $(i = 1, 2, \ldots, n)$ to be analytic the equation holds /1/:

$$\delta_i(t,\ Z_1,\ Z_2,\ \ldots\ldots,\ Z_n) =$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu \delta_i(t_o,\ z_1,\ z_2,\ \ldots,\ z_n) \tag{I,5}$$

$$i = 1, 2, \ldots, n$$

and

$$Z'_i = \frac{d}{dt} Z_i(t) = \frac{d}{dt} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_i =$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^{\nu+1} z_i = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu D z_i =$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu \delta_i(t_o,\ z_1,\ z_2,\ \ldots,\ z_n) \tag{I,6}$$

$$(i = 1, 2, \ldots, n)$$

where t_o, z_1, z_2, ..., z_n indicate that after applying the operator D ν-times, t, Z_1, Z_2, Z_3, ..., Z_n have to be replaced by the initial values t_o, z_1, z_2, ..., z_n.

Eqs. (I,5) and (I,6) show that the Lie series $Z_i(t)=\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_i$ solve Eq.(I,4) and Eq.(I,3) respectively.

Lie Series are not only suitable to solve initial value problems /2,3/, but also to solve boundary value problems /4,5/.

Irrespective of their theoretical significance, Lie series are interesting from a numerical point of view /5, 6, 7/. Thus, e.g. F.CAP and J.MENNIG /7/ and F.CAP and A.SCHETT /5/ have applied Lie series to reactor theory. While F.CAP and J.MENNIG were concerned with shielding theory i.e. treated an initial value problem, F.CAP and A.SCHETT's work is a boundary value problem as it comprises aspects of shielding and reactor core. Mathematically speaking, the modified Bessel functions and its derivatives which appear in the conventional calculations assume so large values, with increasing distance from the core that overflow of the computer may occur. In contrast to this Lie series are appropriate to this problem as they are broken off if the values become too large and expanded anew. Furthermore, this stepwise method is favorable insofar as the reactor itself consists of coaxial zones which are appropriately treated by such repeated expansions of Lie series.

In this monograph (chapter II,III) we investigate the general linear homogeneous second order differential equation which is the most general type of equations containing all 33 equations resulting from a separation of the Helmholtz equation in 11 coordinate systems. We succeeded in developing two alternative forms of solutions, one of them still containing the D-operators, the other one splitting off

known functions from the total solutions and determining the remaining part by means of recurrence formulas and integral represention respectively.
In chapter IV, applications to various spezific cases of the differential equations resulting from the separation of the Helmholtz are considered.
In chapter V, the general investigations of Chapter II and III are applied to some physical problems.
Theoretical and numerical investigations on Mathieu functions are presented in chapter VI.
In chapter VII Webers function of the parabolic cylinder are treated.

It should be mentioned, that GROEBNER and collaborators used Lie series to solve partial differential equations. So the Cauchy Problem of linear nonhomogeneous partial differential equations of any arbitrary order with non constant coefficients and of a system of simultaneous partial linear differential equations of first order was solved. Also boundary value problems of ordinary differential equations have been solved and methods to improve the numerical convergence of the Lie solutions were found.

Chapter II

The Solution of the General Homogeneous Linear Differential Equation of Second Order.

1) Solution by Recurrence Formulas by A.SCHETT and J.WEIL.

We treat the equation:

$$Z''(t) - f_1(t)Z'(t) - f_2(t)Z(t) = 0 \qquad (II,1)$$

This equation represents the most general type of the equations resulting from a separation of the Helmholtz equation:

$$\Delta\Phi \pm \varkappa^2\Phi = 0 \qquad (II,2)$$

This equation is known to be separable in 11 coordinate systems /17/.

Equation (II,1) may be replaced by the following system of first order equations:

$$t' = Z_0' = 1$$

$$Z' = Z_1' = Z_2 \qquad (II,3)$$

$$Z'' = Z_2' = f_1 Z_2 + f_2 Z_1$$

This system is solved formally by /1/

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z, \qquad (II,4)$$

D beeing a differential operator which is given by:

$$D = \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + (f_1 z_2 + f_2 z_1) \frac{\partial}{\partial z_2} \qquad (II,5)$$

in our case /1/.

Evidently,

$$D^0 z_1 = z_1 \qquad (II,6)$$

and, applying the operator, respectively, once and twice, we obtain:

$$D^1 z_1 = z_2 \qquad (II,7)$$

$$D^2 z_1 = f_1 z_2 + f_2 z_1 \qquad (II,8)$$

As we are allowed to split up the operator powers /1/, the following identities hold:

$$D^{\nu} z_1 = D^{\nu-2} (D^2 z_1) = D^{\nu-2} (f_1 z_2 + f_2 z_1) \; ; \; \nu = 2,3, \dots \qquad (II,9)$$

$$D^{\nu} z_1 = D^{\nu-1} (D z_1) = D^{\nu-1} z_2 \qquad (II,10)$$

$$D^{\nu} z_1 = D^{\nu} (D^0 z_1) =$$

Having in mind the formation of a recurrence formula for $D^{\nu} z_1$, our interest is centered on (II,9), the first term of which may be represented in the form:

$$D^{\nu-2}(f_1 z_2) = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_1 D^{\nu-2-\varrho} z_2 = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_1 \cdot D^{-1-\varrho}(D^{\nu-1} z_2),$$

using a generalization of the Leibniz rule proved in /1/.
With the help of (II,7) we have

$$D^{\nu-2}(f_1 z_2) = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_1 D^{\nu-1-\varrho} z_1 \qquad (II,11)$$

and, in analogy,

$$D^{\nu-2}(f_2 z_1) = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_2 D^{\nu-2-\varrho} z_1 \qquad (II,12)$$

From (II,9), (II,11) and (II,12) we obtain

$$D^{\nu} z_1 = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} (D^{\varrho} f_1 D^{\nu-1-\varrho} z_1 + D^{\varrho} f_2 D^{\nu-2-\varrho} z_1)$$

or, in view of

$$D^{\varrho} f_1(z_o) = f_1^{(\varrho)}(z_o),$$

$$D^{\nu} z_1 = \sum_{\rho=0}^{\nu-2} \binom{\nu-2}{\rho} \left(f_1^{(\rho)}(z_o)\, D^{\nu-1-\rho} z_1 + f_2^{(\rho)}(z_o)\, D^{\nu-2-\rho} z_1 \right) \qquad (II,13)$$

which allows all $D^{\nu} z_1$ to be determined by recurrence, as $D^{o} z_1$ and $D^{1} z_1$ may easily be calculated in a direct way (see (II,6), (II,7)). Consequently, the solution of (II,1) reads:

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu} z_1 = \sum_{\nu=2}^{\infty} \frac{t^{\nu}}{\nu!} \sum_{\rho=0}^{\nu-2} \binom{\nu-2}{\rho} \left(f_1^{(\rho)}(z_o) D^{\nu-1-\rho} z_1 + \right.$$

$$\left. + f_2^{(\rho)}(z_o) D^{\nu-2-\rho} z_1 \right) + z_1 + t z_2 \qquad (II,14)$$

This form of solution may be used for the numerical calculation of $Z(t)$ in a computer; nevertheless, we attempt to find an alternative way by splitting this form of solution into well-known functions and remaining terms for which recurrence formulas will have to be obtained.

At first, we will show that it is generally possible to splitt sin and cos or sinh and cosh, from the total solution of Eq.(II,1).

According to (II,4), the solution may be written:

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu} z_1 = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} (D_1 + D_2)^{\nu} z_1$$

For any decomposition of D, we may write /1/:

$$\sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu} z_1 = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} (D_1 + D_2)^{\nu} z_1 = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D_1^{\nu} z_1 + \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} \sum_{i=1}^{\nu} D_1^{\nu-i} D_2 D^{i-1} z_1$$

$$(1 \leq i \leq \nu)$$

In our case, D reads (II,5):

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} + (f_1(z_o) z_2 + f_2(z_o) z_1) \frac{\partial}{\partial z_2}$$

The operator D_1 chosen in the following decomposition $D = D_1 + D_2$ is known to generate the trigonometric (hyperbolic) functions when applied to z_1, /1/:

The total solution, therefore, is given by:

$$Z(t) = z_1 \cosh(t\sqrt{f_2}) + \frac{z_2}{\sqrt{f_2}} \sinh(t\sqrt{f_2}) + \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \sum_{i=1}^{\nu} D_1^{\nu-i} D_2 D^{i-1} z_1$$

If $f_1 = -f_1^*$ and $f_2 = -f_2^*$, i.e. $\sqrt{f_1}$ and $\sqrt{f_2}$ are purely imaginary, cosh and sinh are to be replaced by cos and sin. Q.e.d.

For the term

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \sum_{i=1}^{\nu} D_1^{\nu-i} D_2 D^{i-1} z_1$$

it will be necessary to obtain recurrence formulas.

Replacing $D^{i-1} z_1$ by (II,13) we get:

$$\sum_{\nu=3}^{\infty} \frac{t^\nu}{\nu!} \sum_{i=1}^{\nu} D_1^{\nu-i} D_2 \sum_{\rho=0}^{i-3} \binom{i-3}{\rho} \left(f_1^{(\rho)} z_0\right) D^{i-2-\rho} z_1 + f_2^{(\rho)} D^{i-3-\rho} z_1)$$

$$+ z_1 + t z_2 + D^2 z_1 = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 \qquad (1 \leq i \leq \nu)$$

In order to avoid the evaluation of this threefold sum which seems to be rather horrible, we will follow another way of deriving recurrence formulas:

Using (II,9) we obtain:

$$D^\nu z_1 = D^{\nu-2} (f_1 z_2 + f_2 z_1) \qquad \nu = 2, 3, \ldots$$

where the first term of the right side may be written as

$$D^{\nu-2}(f_1 z_2) = f_1 D^{\nu-2} z_2 + \sum_{\rho=1}^{\nu-2} \binom{\nu-2}{\rho} D^\rho f_1 D^{\nu-2-\rho} z_2$$

and the second term:

$$D^{\nu-2}(f_2 z_1) = f_2 D^{\nu-2} z_1 + \sum_{\rho=1}^{\nu-2} \binom{\nu-2}{\rho} D^\rho f_2 D^{\nu-2-\rho} z_1$$

so that:

$$D_1 = z_2 \frac{\partial}{\partial z_1} + f_2(z_o) z_1 \frac{\partial}{\partial z_2}$$

$$D_2 = \frac{\partial}{\partial z_o} + f_1(z_o) z_2 \frac{\partial}{\partial z_2}$$

In so doing, we calculate $\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu z_1$, the first terms of which are:

$$D_1^{\,o} z_1 = z_1$$

$$D_1^{\,1} z_1 = z_2$$

$$D_1^{\,2} z_1 = f_2(z_o) z_1$$

$$D_1^{\,3} z_1 = f_2(z_o) z_2$$

$$D_1^{\,4} z_1 = f_2^{\,2}(z_o) z_1$$

$$D_1^{\,5} z_1 = f_2^{\,2}(z_o) z_2$$

or, generally:

$$D_1^{\,2\nu} z_1 = f_2^{\nu}(z_o) z_1$$

$$D_1^{\,2\nu+1} z_1 = f_2^{\nu}(z_o) z_2,$$

so that:

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu z_1 = \sum_{\nu=0}^{\infty} \frac{t^{2\nu}}{(2\nu)!} D_1^{\,2\nu} z_1 + \sum_{\nu=0}^{\infty} \frac{t^{2\nu+1}}{(2\nu+1)!} D_2^{\,2\nu+1} z_1 =$$

$$= \sum_{\nu=0}^{\infty} \frac{t^{2\nu}}{(2\nu)!} f_2^{\nu}(z_o) z_1 + \sum_{\nu=0}^{\infty} \frac{t^{2\nu+1}}{(2\nu+1)!} f_2^{\nu}(z_o) z_2$$

Evidently, these series represent:

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1 \; z_1 = z_1 \cosh\left(t\sqrt{f_2}\right) + \frac{z_2}{\sqrt{f_2}} \sinh\left(t\sqrt{f_2}\right)$$

$$D^{\nu} z_1 = f_1 D^{\nu-2} z_2 + f_2 D^{\nu-2} z_1 + \sum_{\varrho=1}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_1 D^{\nu-2-\varrho} z_2 +$$

$$+ \sum_{\varrho=1}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_2 D^{\nu-2-\varrho} z_1 =$$

$$= f_1 D^{\nu-2} z_2 + f_2 D^{\nu-2} z_1 + R_o = f_1 D^{\nu-1} z_1 + f_2 D^{\nu-2} z_1 + R_o$$

Applying the formula for $D^{\nu} z_1$ to $D^{\nu-1} z_1$ and $D^{\nu-2} z_1$, we obtain:

$$D^{\nu} z_1 = f_1^{\,2} D^{\nu-3} z_2 + f_2^{\,2} D^{\nu-4} z_1 + R_1 = f_1^{\,2} D^{\nu-2} z_1 + f_2^{\,2} D^{\nu-4} z_1 + R_1 \qquad \text{(II,15)}$$

where sums and terms with products of f_1 and f_2 are understood to be contained in R_1. As we require the exponents of the operator powers to be equal in the general recurrence formula, we have to show that is generally possible to obtain expressions with equal exponents in the powers of D. Applying procedure of (II,9) to the first term of (II,15) and denoting all sums and terms with "mixed" products by R_2, we obtain:

$$D^{\nu} z_1 = f_1^{\,3} D^{\nu-4} z_2 + f_2^{\,2} D^{\nu-4} z_1 + R_2$$

in the same way:

$$D^{\nu} z_1 = f_1^{\,5} D^{\nu-6} z_2 + f_2^{\,3} D^{\nu-6} z_1 + R_4$$

and, generally:

$$D^{\nu} z_1 = f_1^{\,2k-1} D^{\nu-2k} z_2 + f_2^{\,k} D^{\nu-2k} + R_{2k-2}$$

Putting $\nu = 2\lambda$, we obtain with a slight change in the notation of the remaining terms:

$$D^{2\lambda} z_1 = f_1^{\,2k-1} D^{2\lambda-2k} z_2 + f_2^{\,k} D^{2\lambda-2k} z_1 + S_{2\lambda}$$

which, with $\lambda = k$, becomes:

$$D^{2\lambda}z_1 = f_1^{2\lambda-1}z_2 + f_2^{\lambda}z_1 + S_{2\lambda} \qquad \text{(II,16)}$$

$$2\lambda = 2, 4, 6, \ldots .$$

Similarly for odd $\nu = 2\lambda + 1$ we obtain:

$$D^{2\lambda+1}z_1 = f_1^{2\lambda}z_2 + f_1^{2\lambda-1}f_2z_1 + f_2^{\lambda}z_2 + S_{2\lambda+1} \qquad \text{(II,17)}$$

With the help of these results, the general solution reads:

$$Z(t) = \sum_{\varrho=0}^{\infty} \frac{t^{2\varrho}}{(2\varrho)!} D^{2\varrho}z_1 + \sum_{\varrho=0}^{\infty} \frac{t^{2\varrho+1}}{(2\varrho+1)!} D^{2\varrho+1}z_1 =$$

$$= \sum_{\varrho=2}^{\infty} \frac{t^{2\varrho}}{(2\varrho)!} (f_1^{2\varrho-1}z_2 + f_2^{\varrho}z_1 + S_{2\varrho}) + \sum_{\varrho=2}^{\infty} \frac{t^{2\varrho+1}}{(2\varrho+1)!} (f_1^{2\varrho}z_2 +$$

$$+ f_1^{2\varrho-1}f_2z_1 + f_2^{\varrho}z_2 + S_{2\varrho+1}) + z_1 + tz_2$$

$$= z_2 \cdot \frac{1}{f_1}\left[\cosh(tf_1) - 1 - \frac{(tf_1)^2}{2}\right] + z_1\left[\cosh(t\sqrt{f_2}) - 1 - \frac{t^2 f_1}{2}\right] \qquad \text{(II,18)}$$

$$+ z_2 \frac{1}{f_1}\left[\sinh(tf_1) - tf_1 - \frac{(tf_1)^3}{3!}\right] + z_1 \frac{f_2}{f_1^2}\left[\sinh(tf_1) - tf_1 - \frac{(tf_1)^3}{3!}\right]$$

$$+ \frac{z_2}{\sqrt{f_2}}\left[\sinh(t\sqrt{f_2}) - t\sqrt{f_2} - \frac{(t\sqrt{f_2})^3}{3!}\right] + \sum_{\varrho=2}^{\infty} \frac{t^{\varrho}}{\varrho!} S_{\varrho}(z_0, z_1, z_2, f_1, f_2)$$

$$+ z_1 + tz_2$$

If $f_1 = -\bar{f}_1$ and $f_2 = -\bar{f}_2$

$$Z(t) = z_2 \frac{-1}{\bar{f}_1}\left[\cosh(t\bar{f}_1) - 1 - \frac{(t\bar{f}_1)^2}{2}\right] + z_1\left[\cos(t\sqrt{\bar{f}_2}) - 1 + \frac{t^2\bar{f}_2}{2}\right] +$$

$$z_2 \frac{1}{\bar{f}_1}\left[\sinh(t\bar{f}_1) - t\bar{f}_1 - \frac{(t\bar{f}_1)^3}{3!}\right] + z_1 \frac{\bar{f}_2}{\bar{f}_1^{\,2}}\left[\sinh(t\bar{f}_1) - t\bar{f}_1 - \frac{(t\bar{f}_1)^3}{3!}\right] \qquad \text{(II,19)}$$

$$+ \frac{z_2}{\sqrt{\bar{f}_2}}\left[\sin(t\sqrt{\bar{f}_2} - t\sqrt{\bar{f}_2} - \frac{(t\sqrt{\bar{f}_2})^3}{3!}\right] + \sum_{\varrho=2}^{\infty} \frac{t^{\varrho}}{\varrho!} S_{\varrho}(z_0, z_1, z_2, \bar{f}_1, \bar{f}_2) + z_1 + tz_2$$

With the help of (II,16). (II,17), (II,13) the following equation is seen to hold:

$$S_{2\lambda} + f_1^{2\lambda-1} z_2 + f_2^{\lambda} z_1 = \sum_{\varrho=0}^{\lambda-2} \binom{2\lambda-2}{\varrho} f_1^{(2\varrho)} D^{2\lambda-2-2\varrho} z_2 + \sum_{\varrho=0}^{\lambda-2} \binom{2\lambda-2}{2\varrho+1} f_1^{(2\varrho+1)} D^{2\lambda-3-2\varrho} z_2 + \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-2}{2\varrho} f_2^{(2\varrho)} D^{2\lambda-2-2\varrho} z_1 + \sum_{\varrho=0}^{\lambda-2} \binom{2\lambda-2}{2\varrho+1} f_2^{(2\varrho+1)} D^{2\lambda-3-2\varrho} z_1 \qquad (II,20)$$

where

$$D^{2\lambda-2-2\varrho} z_2 = D^{2\lambda-1-2\varrho} z_1 = f_1^{2\lambda-2-2\varrho} z_2 + f_1^{2\lambda-3-2\varrho} f_2 z_1 + + f_1^{2\lambda-3-2\varrho} f_2 z_1 + f_2^{-1-\varrho} z_2 + S_{2\lambda-1-2\varrho} \qquad (II,21)$$

$$D^{2\lambda-3-\varrho} z_2 = D^{2\lambda-2-2\varrho} z_1 = f_1^{2\lambda-3-2\varrho} z_2 + f_2^{\lambda-1-\varrho} z_1 + S_{2\lambda-2-2\varrho} \qquad (II,22)$$

$$D^{2\lambda-3-2\varrho} z_1 = f_1^{2\lambda-4-2\varrho} z_2 + f_1^{2\lambda-5-2\varrho} f_2 z_1 + f_2^{\lambda-2-\varrho} z_2 + S_{2\lambda-3-2\varrho} \qquad (II,23)$$

$$D^{2\lambda-2-2\varrho} z_1 = f_1^{2\lambda-3-2\varrho} z_2 + f_2^{\lambda-1-\varrho} z_1 + S_{2\lambda-2-2\varrho} \qquad (II,24)$$

With the help of these formulas $S_{2\lambda}$ becomes:

$$S_{2\lambda} = \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-2}{2\varrho} f_1^{(2\varrho)} f_1^{2\lambda-2-2\varrho} z_2 + \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-2}{2\varrho} f_1^{(2\varrho)} \left(f_1^{2\lambda-3-2\varrho} f_2 z_1 + f_2^{\lambda-1-\varrho} z_2 + S_{2\lambda-1-2\varrho}\right) + \sum_{\varrho=0}^{\lambda-2} \binom{2\lambda-2}{2\varrho+1} f_1^{(2\varrho+1)} \left(f_1^{2\lambda-3-2\varrho} z_2 + f_2^{\lambda-1-\varrho} z_1 + S_{2\lambda-2-2\varrho}\right) +$$

$$+ \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-2}{2\varrho} f_2^{(2\varrho)} f_2^{\lambda-1-\varrho} z_1 +$$

$$+ \sum_{\varrho=0}^{\lambda-2} \binom{2\lambda-2}{2\varrho} f_2^{(2\varrho)} (f_1^{2\lambda-3-2\varrho} z_2 + S_{2\lambda-2-2\varrho}) +$$

$$+ \sum_{\varrho=0}^{2\lambda-2} \binom{2\lambda-2}{2\varrho+1} f_2^{(2\varrho+1)} (f_1^{2\lambda-4-2\varrho} + f_1^{2\lambda-5-2\varrho} f_2 z_1 + \qquad \text{(II,25)}$$

$$+ f_2^{\lambda-2-\varrho} z_2 + S_{2\lambda-3-2\varrho})$$

Similarly, we obtain for $S_{2\lambda+1}$:

$$S_{2\lambda+1} + f_1^{2\lambda} z_2 + f_1^{2\lambda-1} f_2 z_1 + f_2^{\lambda} z_2 = \qquad \text{(II,26)}$$

$$= \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho} f_1^{(2\varrho)} D^{2\lambda-1-2\varrho} z_2 + \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho+1} f_1^{(2\varrho+1)} D^{2\lambda-2-2\varrho} z_2$$

$$+ \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho} f_2^{(2\varrho)} D^{2\lambda-1-2\varrho} z_1 + \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho+1} f_2^{(2\varrho+1)} D^{2\lambda-2-2\varrho} z_1$$

where

$$D^{2\lambda-1-2\varrho} z_2 = D^{2\lambda-2\varrho} z_1 = f_1^{2\lambda-2\varrho-1} z_2 + f_2^{\lambda-\varrho} z_1 + S_{2\lambda-2\varrho} \qquad \text{(II,27)}$$

while for

$D^{2\lambda-2-2\varrho} z_2 = D^{2\lambda-1-2\varrho} z_1$ ... see (II,21)

$D^{2\lambda-2-2\varrho} z_1$ ... see (II,22)

and

$D^{2\lambda-1-2\varrho} z_1$ ... see (II,21) again.

Consequently, $S_{2\lambda+1}$ is given by:

$$S_{2\lambda+1} = \sum_{\varrho=1}^{\lambda-1} \binom{2\lambda-1}{2\varrho} f_1^{(2\varrho)} f_1^{2\lambda-2\varrho-1} z_2 +$$

$$+ \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho} f_1^{(2\varrho)} (f_2^{\lambda-\varrho} z_1 + S_{2\lambda-2\varrho}) +$$

$$+ \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho+1} f_1^{(2\varrho+1)} (f_1^{2\lambda-2-2\varrho} z_2 + f_1^{2\lambda-3-2\varrho} f_2 z_1 +$$

$$+ f_2^{\lambda-1-\varrho} z_2 + S_{2\lambda-1-2\varrho}) + \sum_{\varrho=1}^{\lambda-1} \binom{2\lambda-1}{2\varrho} f_2^{(2\varrho)} f_2^{\lambda-1-\varrho} z_2 +$$

$$+ \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho} f_2^{(2\varrho)} (f_1^{2\lambda-2-2\varrho} z_2 + f_1^{2\lambda-3-2\varrho} f_2 z_1 +$$

$$+ S_{2\lambda-1-2\varrho}) + \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-1}{2\varrho+1} f_2^{(2\varrho+1)} (f_1^{2\lambda-3-2\varrho} z_2 +$$

$$+ f_2^{\lambda-1-\varrho} z_1 + S_{2\lambda-2-2\varrho}) - f_1^{2\lambda-1} f_2 z_1 . \qquad \text{(II,28)}$$

The first three S_ϱ which have to be calculated in a direct way are:

$$S_o = - \frac{z_2}{f_1} \qquad \text{(II,29)}$$

$$S_1 = - \frac{f_2}{f_1} z_1 - z_2$$

$$S_2 = 0$$

By means of (II,18), (II,25), (II,28), (II,29) the solution of (II,1) may be calculated numerically.

In conclusion we found two alternative representations of the solution, one of them still containing the D-operators and the other one split up into trigonometric (hyperbolic) functions and remaining terms for which recurrence formulas could be derived.

Remark: In the course of our investigations, we attempted to solve Eq. (II,1) by using the method of Laplace transformation. But as it turned out, this method is not advantageous in our case, according to the general theory given in /18/.

2) Solution by Iterative Method, by W.GROEBNER.

GROEBNER split the Lie operator D up in the following way:

$$D = D_1 + D_2$$

where

$$D_1 = z_2 \frac{\partial}{\partial z_1} + (f_2(z_o)z_1 + f_1(z_o)z_2) \frac{\partial}{\partial z_2}$$

$$D_2 = \frac{\partial}{\partial z_o}$$

the philosophy being his intention to put most of the operator into the main part of the solution and to keep only $D_2 = \frac{\partial}{\partial z_o}$ for the formation of the remaining terms.

Using matrix formalism, D_1 reads:

$$D_1 = (z_1, z_2) \begin{pmatrix} 0 & f_2 \\ 1 & f_1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} = (z_1, z_2)\, A \boldsymbol{\nabla} \qquad \text{(II,30)}$$

where:

$$A = \begin{pmatrix} 0 & f_2 \\ 1 & f_1 \end{pmatrix} \qquad \text{(II,31)}$$

and

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right)^T \qquad \text{(II,32)}$$

Consequently,

$$D_1(z_1, z_2) = (z_1, z_2)A\nabla(z_1, z_2) = (z_1, z_2)A \tag{II,33}$$

as $\nabla(z_1, z_2) = 1$

Applying D_1 once more, we obtain:

$$D_1^2(z_1,z_2) = (z_1,z_2)A\nabla(z_1,z_2)A = (z_1,z_2)A^2 \tag{II,34}$$

and generally:

$$D_1^{\nu}(z_1,z_2) = (z_1,z_2)A^{\nu} \tag{II,35}$$

The main part of the solution is, therefore, given by:

$$e^{tD_1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D_1^{\nu} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} A^{\nu T} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{II,36}$$

Now we have to calculate A^{ν}. We assume A to be diagonizable, i.e.,

$$T^{-1}AT = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{II,37}$$

As is well known, λ_1 and λ_2 are obtained by solving the secular equation:

$$|\lambda E - A| = \begin{vmatrix} \lambda & -f_2 \\ -1 & \lambda - f_1 \end{vmatrix} = 0 \tag{II,38}$$

$$\lambda_{1,2} = \frac{f_1}{2} \pm \sqrt{\frac{f_1^2}{4} + f_2} \tag{II,39}$$

Supposing $\lambda_1 \neq \lambda_2$, T and T^{-1} can be calculated:

$$T = \begin{pmatrix} f_2 & f_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} ; \quad (T^{-1})^T = \frac{1}{f_2(\lambda_2 - \lambda_1)} \cdot \begin{pmatrix} \lambda_2 & -\lambda_1 \\ -f_2 & f_2 \end{pmatrix} \tag{II,40}$$

From (II,37), we have

$$A = T\Lambda T^{-1} \tag{II,41}$$

and

$$A^{\nu} = (T\Lambda T^{-1}) = (T\Lambda T^{-1})(T\Lambda T^{-1})\ldots\ldots\ldots\ldots (T\Lambda T^{-1}) =$$

$$= T\Lambda^{\nu} T^{-1} \tag{II,42}$$

With the help of (II,42), (II,36) reads:

$$\begin{pmatrix}\hat{z}_1\\ \hat{z}_2\end{pmatrix} = e^{tD_1}\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\nu!}(T^{-1})^T\Lambda^{\nu}\; T^T\begin{pmatrix}z_1\\ z_2\end{pmatrix} = (T^{-1})^T\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\nu!}\Lambda^{\nu}T^T\begin{pmatrix}z_1\\ z_2\end{pmatrix} =$$

$$= (T^{-1})^T\begin{pmatrix}\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\nu!}\lambda_1^{\nu} & 0\\ 0 & \sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\nu!}\lambda_2^{\nu}\end{pmatrix} T^T \begin{pmatrix}z_1\\ z_2\end{pmatrix} =$$

$$= (T^{-1})^T\begin{pmatrix}e^{t\lambda_1} & 0\\ 0 & e^{t\lambda_2}\end{pmatrix} T^T \begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

Consequently according to /1/ the solution may be written:

$$\begin{pmatrix}Z_1(t)\\ Z_2(t)\end{pmatrix} = e^{tD}\begin{pmatrix}z_1\\ z_2\end{pmatrix} = e^{t(D_1+D_2)}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

$$= \begin{pmatrix}\hat{Z}_1\\ \hat{Z}_2\end{pmatrix} + \sum_{\alpha=0}^{\infty}\int_0^t \frac{(t-\tau)^{\alpha}}{\alpha!}\left[D_2 D^{\alpha}\begin{pmatrix}z_1\\ z_2\end{pmatrix}\right]_{\bar{a}} d\tau$$

This integral can be evaluated by an iteration method, according to /1/. The symbol $\bar{a}$ added after the bracket is to indicate that after application of the D-operators z_1, z_2 have to be replaced by $e^{tD_1}z_1$ and $e^{tD_1}z_2$, respectively.

Chapter III

The Solution of the General Inhomogeneous Linear Differential Equation of Second Order by A.SCHETT and J.WEIL.

1) Solution by Iterative Method.

The equation in question reads:

$$Y''(t) - f_1(t)Y'(t) - f_2(t)Y(t) = f_3(t) \qquad \text{(III,1)}$$

where we suppose $f_i(t)$ to be regular in the considered domain.

The following system is equivalent to (III,1)

$$\begin{aligned} Y_0' &= t' = 1 \\ Y &= Y_1 \\ Y_1' &= Y_2 \\ Y_2' &= f_3 + f_2Y_1 + f_1Y_2, \end{aligned} \qquad \text{(III,2)}$$

which is formally solved by /1/:

$$Y(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu y. \qquad \text{(III,3)}$$

In our case, D is given by

$$D = \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_1} + (f_3 + f_2y_1 + f_1y_2) \frac{\partial}{\partial y_2} \qquad \text{(III,4)}$$

The small letters are to indicate that after application of the operator the Y-variables have to be replaced by their initial values. According to GROEBNER /1/, we split the operator in the following way:

$$D = D_1 + D_2 \qquad \text{(III,5)}$$

with

$$D_1 = y_2 \frac{\partial}{\partial y_1} + (f_3 + f_2 y_1 + f_1 y_2) \frac{\partial}{\partial y_2} \qquad \text{(III,5a)}$$

and

$$D_2 = \frac{\partial}{\partial y_o} \qquad \text{(III,5b)}$$

where D_1 will produce the main part and D_2 the correction terms. In view of that, the total solution reads /1/:

$$\begin{pmatrix} Y_2(t) \\ Y_1(t) \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} + \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D^\alpha \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \right]_{\bar{a}} d\tau \qquad \text{(III,6)}$$

where the symbol $\bar{a}$ added after the bracket indicates the fact that after application of the D-operator y_1, y_2 have to be replaced by $e^{tD_1} y_1$ and $d^{tD_1} y_2$, respectively.

We now turn **to an evaluation of the first term at the right side** of (III,6):

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} = e^{t(y_2 \frac{\partial}{\partial y_1} + (f_3 + f_2 y_1 + f_1 y_2) \frac{\partial}{\partial y_2}} \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} .$$

D_1 may be written in matrix form:

$$D_1 = (y_2, y_1, 1) \begin{pmatrix} 1 & f_1 & 0 \\ 0 & f_2 & 0 \\ 0 & f_3 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ 0 \end{pmatrix} =$$

$$= (y_2, y_1, 1) A \nabla \qquad \text{(III,7)}$$

where

$$A = \begin{pmatrix} 1 & f_1 & 0 \\ 0 & f_2 & 0 \\ 0 & f_3 & 1 \end{pmatrix} \qquad \text{(III,8)}$$

and

$$\nabla = \left(\frac{\partial}{\partial y_1}, \ \frac{\partial}{\partial y_2}, \ 0\right)^T \tag{III,9}$$

Applying the operator to the variable, we obtain:

$$D_1(y_2,y_1,1) = (y_2,y_1,1)\, A\nabla\, (y_2,y_1,1) = (y_2,y_1,1)\, B \tag{III,10}$$

where

$$B = A\nabla\,(y_2,y_1,1).$$

Repeating this operation, we get:

$$D_1^2(y_2,y_1,1) = (y_2,y_1,1)A\nabla(y_2,y_1,1)B = (y_2,y_1,1)B^2 \tag{III,10a}$$

and

$$D_1^3(y_2,y_1,1) = (y_2,y_1,1)A\nabla(y_2,y_1,1)B^2 = (y_2,y_1,1)B^3 \tag{III,10b}$$

and generally:

$$D_1^n(y_2,y_1,1) = (y_2,y_1,1)B^n. \tag{III,10c}$$

For the homogeneous case, a repeated application of D_1 results in multiplying the expression by the coefficient matrix A itself, a rôle played by the more complicated matrix B, in our case. The main part of the solution is, therefore, given by:

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B^{\nu^T} \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} \tag{III,11}$$

using $(AB)^T = B^T A^T$.

Now we have to calculate B^n:

$$B = A\nabla(y_2, y_1, 1) =$$

$$= A\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} f_1 & 1 & 0 \\ f_2 & 0 & 0 \\ f_3 & 0 & 0 \end{pmatrix} \qquad \text{(III,12)}$$

Assuming the eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$, we diagonalize B, i.e.:

$$T^{-1}BT = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \text{(III,13)}$$

by solving the secular equation:

$$|\lambda E - B| = 0, \qquad \text{(III,14)}$$

or, in extenso:

$$\begin{vmatrix} \lambda - f_1, & -1, & 0 \\ -f_2, & \lambda, & 0 \\ -f_3, & 0, & \lambda \end{vmatrix} = \lambda(\lambda^2 - \lambda f_1 - f_2) = 0 \qquad \text{(III,14a)}$$

so that

$$\lambda_3 = 0, \quad \lambda_{1,2} = \frac{f_1}{2} \pm \sqrt{\frac{f_1^2}{4} + f_2} \qquad \text{(III,15)}$$

T and T^{-1} are, respectively, given by:

$$T = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 \\ f_2 & f_2 & 0 \\ f_3 & f_3 & 1 \end{pmatrix} \qquad \text{(III,16)}$$

and

$$T^{-1} = \frac{1}{f_2(\lambda_1 - \lambda_2)} \begin{pmatrix} f_2, & -\lambda_2, & 0 \\ -f_2, & \lambda_1, & 0 \\ 0, & -\lambda_1 f_3 + \lambda_2 f_3, & -\lambda_1 f_2 - \lambda_2 f_2 \end{pmatrix} \qquad \text{(III,17)}$$

From (III,13), we have:

$$B = T\Lambda T^{-1} \tag{III,13a}$$

and

$$B^n = (T\Lambda T^{-1})(T\Lambda T^{-1}) \;\ldots..\; (T\Lambda T^{-1}) = T\Lambda^n T^{-1}, \tag{III,13b}$$

so that we obtain for (III,11):

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B^{\nu T} \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} = \tag{III,18}$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} (T^{-1})^T \Lambda^\nu T^T \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} =$$

$$= (T^{-1})^T \begin{pmatrix} \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_1^\nu & 0 & 0 \\ 0 & \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_2^\nu & 0 \\ 0 & 0 & \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_3^\nu \end{pmatrix} T^T \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} = \tag{III,19}$$

$$= (T^{-1})^T \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^T \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix}$$

Consequently, the total solution reads:

$$\begin{pmatrix} Y_2(t) \\ Y_1(t) \\ 0 \end{pmatrix} = e^{tD} \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} = (T^{-1})^T \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^T \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} \tag{III,20}$$

$$+ \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D^\alpha \, , \, \begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix} \right]_{\bar{a}} d\tau$$

where the perturbation integral can be evaluated by an iteration method according to /6/; its evaluation is promising insofar as it offers several possibilities of adaptation, viz., by choosing the numbers of iterations, the step size, and the break-off of the α-summation.

2) Solution by Recurrence Formulas:

Applying D ν times to y_1 and splitting up the operator powers we obtain:

$$D^{\nu} y_1 = D^{\nu-2}(D^2 y_1) = D^{\nu-2}(f_3 + f_1 y_2 + f_2 y_1). \qquad \text{(III,21)}$$

Evidently

$$D^{\nu-2} f_3 = f_3^{(\nu-2)}. \qquad \text{(III,22)}$$

In order to avoid meaningless expressions, we have to define:

$$D^{-\mu} = 0 \quad \text{for} \quad \mu > 0. \qquad \text{(III,23)}$$

Using the well-known Leibniz rule for the D-operators, we obtain for the second and third terms of (III,21), respectively:

$$D^{\nu-2}(f_1 y_2) = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_1 D^{\nu-2-\varrho} y_2 =$$

$$= \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_1 D^{\nu-1-\varrho} y_1 \qquad \text{(III,24)}$$

and

$$D^{\nu-2}(f_2 y_1) = \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} D^{\varrho} f_2 D^{\nu-2-\varrho} y_1, \qquad \text{(III,25)}$$

so that the recurrence formula for D^{ν} is given by:

$$D^{\nu}y_1 = \sum_{\varrho=2}^{\nu-2} \left[f_3^{(\nu-2)} + \binom{\nu-2}{\varrho} \left\{ D^{\varrho} f_1 D^{\nu-1-\varrho} y_1 + D^{\varrho} f_2 D^{\nu-2-\varrho} y_1 \right\} \right]. \qquad \text{(III,26)}$$

With the help of this result, the total solution reads:

$$Y(t) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu} y_1 = \sum_{\nu=5}^{\infty} \frac{t^{\nu}}{\nu!} \sum_{\varrho=2}^{\nu-2} \left[f_3^{(\nu-2)} + \binom{\nu-2}{\varrho} \right.$$

$$\left. \cdot (D^{\varrho} f_1 D^{\nu-1-\varrho} y_1 + D^{\varrho} f_2 D^{\nu-2-\varrho} y_1) \right] + y_1 + t y_2 +$$

$$+ \frac{t^2}{2} D^2 y_1 + \frac{t^3}{3!} D^3 y_1 + \frac{t^4}{4!} D^4 y_1 \qquad \text{(III,27)}$$

This formula is not difficult to code. It is, of course, possible to split known functions from the total solution, also in this case; one of the ways in which this splitting is possible is evidently equivalent to the method used by GROEBNER, his part $e^{tD_1}\begin{pmatrix} y_2 \\ y_1 \\ 1 \end{pmatrix}$ essentially being the hyperbolic (trigonometric) main term of our method. - In contrast to GROEBNER'S method, no way of estimating the error made by breaking off the computation seems to exist for the recurrence formulas, up to now. Nevertheless, they may prove to be superior to the first way, from a physicist's point of view, owing to their easier coding and the fact that an analytic method of error estimating may be replaced by experience on the machine, for practical purposes.

Chapter IV

Solution of the Equations Resulting From a Separation of the Helmholtz Equations in Special Coordinate Systems

by A.SCHETT and J.WEIL.

As we will see below, the equation

$$Z''(t) - f_1(t)Z'(t) - f_2(t)Z(t) = 0$$

represents the most general type of the equations resulting from a separation of the Helmholtz equation:

$$\Delta\Phi \pm \varkappa^2\Phi = 0 \qquad (IV,1)$$

This equation is known to be separable in the following 11 coordinate systems /17/.

a) Rectangular coordinates: the equations resulting from a separation in these coordinates are extremely simple compared to those occurring in the other systems insofar as:

$$\frac{d^2X}{dx^2} - (\alpha_2 + \alpha_3)\,X = 0$$

$$\frac{d^2Y}{dy^2} + \alpha_2 Y = 0$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)Z = 0$$

Evidently,

$$f_1(t) = 0$$

and

$$f_2(t) = \text{const.}$$

the solutions of these equations respectively being trigonometric and hyperbolic functions depending on the sign of the constant.

b) Circular-Cylinder Coordinates (r, Ψ, z):

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{\alpha_2}{r^2} + \alpha_3\right) R = 0 \qquad \text{(IV,2)}$$

$$\frac{d^2\Psi}{d\psi^2} + \alpha_2 \Psi = 0 \qquad \text{(IV,3)}$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)Z = 0 \qquad \text{(IV,4)}$$

In (IV,2)

$$f_1(t) = \frac{-1}{t}$$

and

$$f_2(t) = \frac{\alpha_2}{t^2} + \alpha_3$$

the solutions being Bessel functions. (IV,3) and (IV,4) are analogous to the case of rectangular coordinates.

c) Elliptic-Cylinder Coordinates (η, Ψ, z):

$$\frac{d^2H}{d\eta^2} - (\alpha_2 + \alpha_3\, a^2 \cosh^2\eta)H = 0 \qquad \text{(IV,5)}$$

$$\frac{d^2\Psi}{d\psi^2} + (\alpha_2 + \alpha_3\, a^2 \cos^2\psi)\Psi = 0 \qquad \text{(IV,6)}$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)Z = 0 \qquad \text{(IV,7)}$$

In (IV,5-7)

$$f_1(t) = 0$$

where as

$$f_2(t) = \alpha_2 + \alpha_3 a^2 \cosh^2 t \qquad \text{in (IV,5)}$$

$$f_2(t) = -\alpha_2 - \alpha_3 a^2 \cos t \qquad \text{in (IV,6)}$$

$$f_2(t) = -\varkappa^2 - \alpha_3 = \text{const.} \qquad \text{in (IV,7)}$$

(IV,5) and (IV,6) are solved by Mathieu functions, (IV,7) is analogous to the case of rectangular coordinates.

d) Parabolic-Cylinder Coordinates (μ, ν , z):

$$\frac{d^2M}{d\mu^2} - (\alpha_2 + \alpha_3\mu^2)M = 0 \qquad \text{(IV,8)}$$

$$\frac{d^2N}{d\nu^2} + (\alpha_2 - \alpha_3\nu^2)N = 0 \qquad \text{(IV,9)}$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)Z = 0 \qquad \text{(IV,10)}$$

Evidently, in (IV,8)

$$f_1(t) = 0$$

$$f_2(t) = \alpha_2 + \alpha_3 t^2;$$

In (IV,9):

$$f_1(t) = 0$$

$$f_2(t) = -\alpha_2 + \alpha_3 t^2,$$

and in (IV,10):

$$f_1(t) = 0$$

$$f_2(t) = -\varkappa^2 - \alpha_3 = \text{const.}$$

(IV,8) and (IV,9) are solved by Weber functions; (IV,10) is again analogous to the case of rectangular coordinates.

e) Spherical Coordinates (r, Θ, Ψ):

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(\varkappa^2 - \frac{\alpha_2}{r^2}\right)R = 0 \qquad \text{(IV,11)}$$

$$\frac{d^2\Theta}{d\vartheta^2} + \cot\vartheta\,\frac{d\Theta}{d\vartheta} + \left(\alpha_2 - \frac{\alpha_3}{\sin^2\vartheta}\right)\Theta = 0 \qquad \text{(IV,12)}$$

$$\frac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0 \qquad \text{(IV,13)}$$

In (IV,11) we have:

$$f_1(t) = \frac{-2}{t}$$

$$f_2(t) = -\varkappa^2 + \frac{\alpha_2}{t^2},$$

in (IV,12):

$$f_1(t) = -\cot t$$

$$f_2(t) = -\alpha_2 + \frac{\alpha_3}{\sin^2 t}$$

in (IV,13):

$$f_1(t) = 0$$

$$f_2(t) = -\alpha_3 = \text{const.}$$

(IV,11) is solved by Bessel functions, (IV,12) by Legendre functions and (IV,13) is trivial again.

f) Prolate Spheroidal Coordinates (η, ϑ, ψ):

$$\frac{d^2H}{d\eta^2} + \coth\eta \frac{dH}{d\eta} + (\varkappa^2 a^2 \sinh^2\eta - \alpha_2 - \frac{\alpha_3}{\sinh^2\eta})H = 0 \qquad (IV,14)$$

$$\frac{d^2\Theta}{d\vartheta^2} + \cot\vartheta \frac{d\Theta}{d\vartheta} + (\varkappa^2 a^2 \sin^2\vartheta + \alpha_2 - \frac{\alpha_3}{\sin^2\vartheta})\Theta = 0 \qquad (IV,15)$$

$$\frac{d^2\Psi}{d\psi^2} + \alpha_3 \Psi = 0 \qquad (IV,16)$$

In (IV,14)

$$f_1(t) = -\coth t$$

$$f_2(t) = -\varkappa^2 a^2 \sinh^2 t + \alpha_2 + \frac{\alpha_3}{\sinh^2 t},$$

In (IV,15)

$$f_1(t) = -\cot t$$

$$f_2(t) = -\varkappa^2 a^2 \sin^2 t - \alpha_2 + \frac{\alpha_3}{\sin^2 t}$$

In (IV,16)

$$f_1(t) = 0$$

$$f_2(t) = \alpha_3 = \text{const.}$$

(IV,14) and (IV,15) are solved by Legendre functions, (IV,16) is trivial again

g) Oblate Spherical Coordinates (η, ϑ, ψ):

$$\frac{d^2H}{d\eta^2} + \tanh\eta \frac{dH}{d\eta} + (\varkappa^2 a^2 \cosh^2\eta - \alpha_2 + \frac{\alpha_3}{\cosh^2\eta})H = 0 \qquad (IV,17)$$

$$\frac{d^2\Theta}{d\vartheta^2} + \cot\vartheta \frac{d\Theta}{d\vartheta} + (-\varkappa^2 a^2 \sin^2\vartheta + \alpha_2 - \frac{\alpha_3}{\sin^2\vartheta}).\Theta = 0 \qquad (IV,18)$$

$$\frac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0 \qquad (IV,19)$$

In (IV,17) we have:

$$f_1 = -\tanh t$$

$$f_2 = -\varkappa^2 a^2 \cosh^2 t + \alpha_2 - \frac{\alpha_3}{\cosh^2 t}$$

and in (IV,18)

$$f_1 = -\cot t$$

$$f_2 = \varkappa^2 a^2 \sin^2 t - \alpha_2 + \frac{\alpha_3}{\sin^2 t}$$

and in (IV,19)

$$f_1(t) = 0$$

$$f_2(t) = -\alpha_3$$

(IV,17) and (IV,18) are solved by Legendre functions, (IV,19) is trivial again.

h) Parabolic Coordinates (μ, ν, Ψ):

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + (\varkappa^2\mu^2 - \alpha_2 - \frac{\alpha_3}{\mu^2})M = 0 \qquad (IV,20)$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + (\varkappa^2\nu^2 + \alpha_2 - \frac{\alpha_3}{\nu^2})N = 0 \qquad (IV,21)$$

$$\frac{d^2\Psi}{d\varphi^2} + \alpha^3\Psi = 0 \qquad (IV,22)$$

In (IV,20)

$$f_1(t) = -\frac{1}{t}$$

$$f_2(t) = -\varkappa^2 t^2 + \alpha_2 + \frac{\alpha_3}{t^2}$$

in (IV,21)

$$f_1(t) = -\frac{1}{t}$$

$$f_2(t) = -\varkappa^2 t^2 - \alpha_2 + \frac{\alpha_3}{t^2}$$

in (IV,22)

$$f_1(t) = 0$$

$$f_2(t) = -\alpha_3 = \text{const.}$$

(IV,20) and (IV,21) are solved by Bessel functions, (IV,22) is the trivial trigonometric case.

i) Conical Coordinates (r, θ, Λ):

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + (\varkappa^2 - \frac{\alpha_2}{r^2})R = 0 \qquad (IV,23)$$

$$(\vartheta^2-b^2)(c^2-\vartheta^2)\frac{d^2\Theta}{d\vartheta^2} - (2\vartheta^2 - (b^2+c^2))\frac{d\Theta}{d\vartheta} + (\alpha_2\vartheta^2 - \alpha_3)\Theta = 0 \qquad (IV,24)$$

$$(b^2-\lambda^2)(c^2-\lambda^2)\frac{d^2\Lambda}{d\lambda^2} + \lambda(2\lambda^2-(b^2+c^2))\frac{d\Lambda}{d\lambda} - (\alpha_2\lambda^2-\alpha_3)\Lambda = 0 \qquad (IV,25)$$

In (IV,23)

$$f_1(t) = \frac{-2}{t}$$

$$f_2(t) = -\varkappa^2 + \frac{\alpha_2}{t^2}$$

in (IV,24)

$$f_1(t) = \frac{(2t^2 - (b^2+c^2))}{(t^2-b^2)(c^2-t^2)}$$

$$f_2(t) = \frac{\alpha_3 - \alpha_2 t^2}{(t^2-b^2)(c^2-t^2)}$$

in (IV,25)

$$f_1(t) = \frac{t(b^2+c^2-2t^2)}{(b^2-t^2)(c^2-t^2)}$$

$$f_2(t) = \frac{\alpha_2 t^2 - \alpha_3}{(b^2-t^2)(c^2-t^2)}$$

(IV,24) and (IV,25) are solved by Lamé functions, while the solution of (IV,23) is given by Bessel functions.

j) Ellipsoidal Coordinates:

$$\frac{d^2H}{d\eta^2} + \frac{(2\eta^2 -(b^2+c^2))}{(\eta^2-b^2)(\eta^2-c^2)}\frac{dH}{d\eta} + \frac{(\varkappa^2\eta^4 + \alpha_3\eta^2 + \alpha_2)}{(\eta^2-b^2)(\eta^2-c^2)} H = 0 \qquad \text{(IV,26)}$$

$$\frac{d^2\Theta}{d\vartheta^2} - \frac{\vartheta(2\vartheta^2 - (b^2+c^2))}{(\vartheta^2-b^2)(c^2-\vartheta^2)}\frac{d\Theta}{d\vartheta} - \frac{(\varkappa^2\vartheta^4 + \alpha_3\vartheta^2 + \alpha_2)}{(\vartheta^2-b^2)(c^2-\vartheta^2)}\Theta = 0 \qquad \text{(IV,27)}$$

$$\frac{d^2\Lambda}{d\lambda^2} + \frac{\lambda(2\lambda^2 - (b^2+c^2))}{(b^2-\lambda^2)(c^2-\lambda^2)}\frac{d\Lambda}{d\lambda} + \frac{(\varkappa^2\lambda^4 + \alpha_3\lambda^2 + \alpha_2)}{(b^2-\lambda^2)(c^2-\lambda^2)} = 0 \qquad \text{(IV,28)}$$

In (IV,26)

$$f_1(t) = \frac{(-2t^2 + (b^2 + c^2))}{(t^2 - b^2)(t^2-c^2)}$$

$$f_2(t) = \frac{-\varkappa^2 t^4 - \alpha_3 t^2 - \alpha_2}{(t^2-b^2)(t^2-c^2)}$$

in (IV,27)

$$f_1(t) = \frac{t(2t^2 - (b^2 + c^2))}{(t^2-b^2)(c^2-t^2)}$$

$$f_2(t) = \frac{\varkappa^2 t^4 + \alpha_3 t^2 + \alpha_2}{(t^2-b^2)(c^2-t^2)}$$

in (IV,28)

$$f_1(t) = \frac{t((b^2 + c^2) - 2t^2)}{(b^2-t^2)(c^2-t^2)}$$

$$f_2(t) = \frac{-(\varkappa^2 t^4 + \alpha_3 t^2 + \alpha_2)}{(b^2-t^2)(c^2-t^2)}$$

All of these equations are solved by means of Lamé functions.

k) Paraboloid Coordinates (μ, ν, λ):

$$\frac{d^2M}{d\mu^2} + \frac{1}{2}\frac{(2\mu - (b+c))}{(\mu-b)(\mu-c)}\frac{dM}{d\mu} + \frac{(\varkappa^2\mu^2 + \alpha_3\mu - \alpha_2)}{(\mu-b)(\mu-c)} M = 0 \qquad \text{(IV,29)}$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{2}\frac{(2\nu - (b + c))}{(b-\nu)(c-\nu)}\frac{dN}{d\nu} + \frac{(\varkappa^2\nu^2 + \alpha_3\nu - \alpha_2)}{(b-\nu)\ c-\nu)} N = 0 \qquad \text{(IV,30)}$$

$$\frac{d^2\Lambda}{d\lambda^2} + \frac{1}{2}\frac{(2\lambda - (b + c))}{(b-\lambda)(\lambda-c)}\frac{d\Lambda}{d\lambda} - \frac{\varkappa^2\lambda^2 + \alpha_3\lambda - \alpha_2}{(b-\lambda)(\lambda-c)} = 0 \qquad \text{(IV,31)}$$

In (IV,29) we have:

$$f_1(t) = \frac{1}{2}\frac{(b + c - 2t)}{(t - b)(t - c)}$$

$$f_2(t) = \frac{-\varkappa^2 t^2 - \alpha_3 t + \alpha_2}{(t-b)\ (t-c)}$$

in (IV,30)

$$f_1(t) = \frac{1}{2}\frac{(b + c - 2t)}{(b-t)\ (c-t)}$$

$$f_2(t) = \frac{-\kappa^2 t^2 - \alpha_3 t + \alpha_2}{(b-t)\ (c-t)}$$

and in (IV, 31)

$$f_1(t) = \frac{1}{2}\frac{(2t - (b+c))}{(b-t)\ (t-c)}$$

$$f_2(t) = \frac{\kappa^2 t^2 + \alpha_3 t - \alpha_2}{(b-t)\cdot(t-c)}$$

The solutions are Baer functions.

Apparently, some of these equations have singularities; as the Lie solution to be discussed is only valid for regular functions f_1 and f_2, we have to exclude these singularities. In the following general derivations we restrict consideration to regular domains.
We are going to apply the methods presented in Chapter II and III to the equations resulting from a separation of the Helmholtz equation in 11 coordinate systems /17/.
In presenting our results, we are going to adopt the following principles of ordering: the invidual types of equations are subsumed under somewath generalized types for which the two formalisms are carried out until a reasonable vicinity to numerical evaluation seems to be reached. Under each type the special cases in which it appears are mentioned.

<u>Type I:</u>

$$Z''(t) - cZ(t) = 0,$$

c being a constant.

This type contains all three separation equations in rectangular coordinates, two of the equations in circular-cylinder coordinates and one equation among the equations in elliptic-cylinder, parabolic-cylinder, spherical, prolate spheroidal, oblate spherical, and parabolic coordinates, respectively.

The equation can be written in the form:

$$t' = Z_0' = 1$$

$$Z' = Z_1' = Z_2$$

$$Z'' = Z_2' = cZ_1$$

while the Lie operator is given by:

$$D = \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + cz_1 \frac{\partial}{\partial z_2}$$

The solution of the systems is given by /1/:

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 ,$$

the evaluation of which is extremely simple in this case, as

$$D^0 z_1 = z_1$$

$$D^1 z_1 = z_2$$

$$D^2 z_1 = cz_1$$

$$D^3 z_1 = cz_2$$

$$D^4 z_1 = c^2 z_1 ,$$

or generally:

$$D^{2\nu} z_1 = c^\nu z_1$$

$$D^{2\nu+1} z_1 = c^\nu z_2 ,$$

so that we may write the solution:

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^{2\nu}}{(2\nu)!} c^{\nu} z_1 + \sum_{\nu=0}^{\infty} \frac{t^{2\nu+1}}{(2\nu+1)!} c^{\nu} z_2 =$$

$$= z_1 \cosh (t\sqrt{c}) + \frac{z_2}{\sqrt{c}} \sinh (t\sqrt{c})$$

or for $c = -c^*$:

$$Z(t) = z_1 \cos (t\sqrt{c^*}) + \frac{z_2}{\sqrt{c}} \sin (t\sqrt{c^*}),$$

respectively, depending on the sign of c. Evidently, no recurrence formulas are necessary in this case, as the solution reduces to its "main part" split-off from the total expression.

Using the method presented in Chapter II,2 we obtain in our case

$$D_1 = (z_1, z_2) \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} = (z_1, z_2) A\nabla$$

with

$$A = \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix},$$

$$\nabla^T = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right),$$

and

$$D_2 = \frac{\partial}{\partial z_o} .$$

Evidently,

$$D_1^{\nu} (z_1, z_2) = (z_1, z_2) A^{\nu},$$

while the part of the solution due to D_1 is given by:

$$\begin{pmatrix}\hat{Z}_1(t)\\ \hat{Z}_2(t)\end{pmatrix} = e^{tD_1}\begin{pmatrix}z_1\\ z_2\end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \; A^T \begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

Diagonalizing A by

$$T^{-1}AT = \Lambda,$$

the eigenvalues resulting from

$$|\lambda E - A| = \begin{vmatrix}\lambda & -c\\ -1 & \lambda\end{vmatrix} = 0$$

are

$$\lambda_{1,2} = \pm\sqrt{c}$$

T^T and T^{-1} are given by:

$$T^T = \begin{pmatrix}c & c\\ \lambda_1 & \lambda_2\end{pmatrix}$$

and

$$T^{-1} = \frac{1}{c(\lambda_2-\lambda_1)} \begin{pmatrix}\lambda_2 & -\lambda_1\\ -c & c\end{pmatrix},$$

respectively.

The total solution is given by:

$$\begin{pmatrix}Z_1(t)\\ Z_2(t)\end{pmatrix} = (T^{-1})\begin{pmatrix}e^{t\sqrt{c}} & 0\\ 0 & e^{-t\sqrt{c}}\end{pmatrix} T^T \begin{pmatrix}z_1\\ z_2\end{pmatrix} +$$

$$+ \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D \begin{pmatrix}z_1\\ z_2\end{pmatrix}\right]_\alpha d\tau$$

Evidently, this perturbation integral vanishes as there is no z_o-dependence in the operator D.

Type II (The Bessel Equation):

$$Z''(t) + \frac{a}{t} Z'(t) - (\frac{b}{t^2} + c) Z(t) = 0 \qquad \text{(IV,32)}$$

a, b, and c being constants.

This equation appears among the equations of the circular-cylinder coordinates (a = 1) and of the spherical coordinates (a = 2) and conical coordinates. In t = 0 the equation is singular. We solve the equation for the domain t > o.

We again replace this equation by the following system:

$$t' = Z_0' = 1$$

$$Z' = Z_1' = Z_2$$

$$Z'' = Z_2' = -\frac{a}{t} Z_2 + (\frac{b}{t^2} + c) Z_1,$$

while the Lie operator is given by:

$$D = \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + \left[-\frac{a}{z_0} z_2 + (\frac{b}{z_0^{\ 2}} + c)\right] z_1 \frac{\partial}{\partial z_2}$$

The solution is given by the following formula derived in Chapter II,1

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} \Big(f_1^{(\varrho)}(z_0) D^{\nu-1-\varrho} z_1 +$$

$$+ f_2^{(\varrho)}(z_0) D^{\nu-2-\varrho} z_1\Big) + z_1 + t z_2$$

With the help of:

$$f_1(t) = -\frac{a}{t}$$

$$f_2(t) = \frac{b}{t^2} + c$$

$$f_1^{(\varrho)}(t) = (-1)^{\varrho+1} \varrho! \ a t^{-(\varrho+1)}$$

$$f_2^{(\varrho)}(t) = (-1)^{\varrho} (\varrho+1)! t^{-(\varrho+2)} + c\delta_{o\varrho}$$

where $\delta_{o\varrho}$ is the Kronecker symbol,

we have in our case:

$$Z(t) = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} \Big\{ (-1)^{\varrho+1} \varrho! a z_o^{-(\varrho+1)} D^{\nu-1-\varrho} z_1 +$$

$$+ (-1)^{\varrho}(\varrho+1)! z_o^{-(\varrho+2)} + c\delta_{o\varrho}\, D^{\nu-2-\varrho} z_1 \Big\} + z_1 + t z_2,$$

or, splitting off known functions from the total solution:

$$Z(t) = z_2 \left[\left(\frac{-z_o}{a}\right) \cosh \left(t \frac{-a}{z_o} - 1\right) \right] + z_1 \left[\cosh \sqrt{\left(t \frac{b}{z_o^2} + c\right)} - 1 \right] +$$

$$\frac{b}{2} + c$$

$$+ z_2 \left[\left(\frac{-z_o}{a}\right) \sinh \left(t\left(\frac{-a}{z_o}\right)\right) - 1 \right] + z_1 \frac{z_o}{\left(-\frac{a}{z_o}\right)^2} \cdot \left[\sinh \left(t\left(\frac{-a}{z_o}\right)\right) - 1 \right] +$$

$$+ \frac{z_2}{\sqrt{\frac{b}{z_o^2} + c}} \left(t \sqrt{\frac{b}{z_o^2} + c}\right) - 1 \sqrt{\frac{b}{z_o^2} + c} + \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} S_\varrho(z_o, z_1, z_2, f_1, f_2)$$

With $S_{2\lambda}$ and $S_{2\lambda+1}$, respectively, given by

$$S_{2\lambda} = \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-2}{2\varrho} (-1)^{2\varrho+1} (2\varrho)! a z_o^{-(2\varrho+1)} \left(-\frac{a}{z_o}\right)^{2\lambda-2-2\varrho} z_2 +$$

$$+ \sum_{\varrho=0}^{\lambda-1} \binom{2\lambda-2}{2\varrho} (-1)^{2\varrho+1} (2\varrho)! a z_o^{-(2\varrho+1)} \left(\frac{-a}{z_o}\right)^{2\lambda-3-2\varrho} \left(\frac{b}{z_o^2} + c\right) z_1 +$$

$$+ \left(\frac{b}{z_o^2} + c\right)^{\lambda-1-\varrho} z_2 + S_{2\lambda-1-2\varrho} +$$

$$+ \sum_{\varrho=0}^{\lambda-2} \binom{2\lambda-2}{2\varrho+1} (-1)^{2\varrho} (2\varrho+1)! a z_o^{-(2\varrho+2)} \cdot \left(\left(\frac{-a}{z_o}\right)^{2\lambda-3-2\varrho} z_2 +\right.$$

$$\left. + \left(\frac{b}{z_o^2} + c\right)^{\lambda-1-\varrho} z_1 + S_{2\lambda-2-2\varrho}\right) +$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-2}{2\varrho}\left[(-1)^{2\varrho}(2\varrho+1)!t^{-(2\varrho+2)}+c\delta_{o\varrho}\right]\cdot(\frac{b}{z_o^2}+c)^{\lambda-1-\varrho}z_1+$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-2}{2\varrho}\left[(-1)^{2\varrho}(2\varrho+1)!t^{-(2\varrho+2)}+c\delta_{o\varrho}\right]\cdot((-\frac{a}{z_o})^{2\lambda-3-2\varrho}z_2+$$

$$+S_{2\lambda-2-2\varrho})+$$

$$+\sum_{\varrho=0}^{\lambda-2}\binom{2\lambda-2}{2\varrho+1}\left[(-1)^{2\varrho+1}(2\varrho+2)!t^{-(2\varrho+3)}+c\delta_{o\varrho}\right]\cdot((-\frac{a}{z_o})^{2\lambda-4-2\varrho}+$$

$$+(-\frac{a}{z_o})^{2\lambda-5-2\varrho}(\frac{b}{z_o^2}+c)z_1+(\frac{b}{z_o^2}+c)^{\lambda-2-\varrho}z_2+S_{2\lambda-3-2\varrho})$$

and

$$S_{2\lambda+1}=\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-1}{2\varrho}(-1)^{2\varrho+1}(2\varrho)!at^{-(2\varrho+1)}.(-\frac{a}{z_o})^{2\lambda-2\varrho-1}z_2+$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-1}{2\varrho}(-1)^{2\varrho+1}(2\varrho)!at^{-(2\varrho+1)}.((\frac{b}{z_o^2}+c)^{\lambda-\varrho}z_1+S_{2\lambda-2\varrho})+$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-1}{2\varrho+1}(-1)^{2\varrho}(2\varrho+1)!at^{-(2\varrho+2)}.((-\frac{a}{z_o})^{2\lambda-2-2\varrho}z_2+$$

$$+(\frac{a}{z_o})^{2\lambda-3-2\varrho}(\frac{b}{z_o^2}+c)z_1+(\frac{b}{z_o^2}+c)^{\lambda-1-\varrho}z_2+S_{2\lambda-1-2\varrho})+$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-1}{\varrho}(-1)^{2\varrho}(2\varrho+1)!t^{-(2\varrho+2)}+c\delta_{o\varrho}\cdot(\frac{b}{z_o^2}+c)^{\lambda-1-\varrho}z_2+$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-1}{2\varrho}(-1)^{2\varrho}(2\varrho+1)!t^{-(2\varrho+2)}+c\delta_{o\varrho}\cdot((-\frac{a}{z_o})^{2\lambda-2-2\varrho}z_2+$$

$$+(\frac{-a}{z_o})^{2\ \ -3-2\varrho}(\frac{b}{z_o^2}+c)z_1+S_{2\lambda-1-2\varrho}+$$

$$+\sum_{\varrho=0}^{\lambda-1}\binom{2\lambda-1}{2\varrho+1}\left[(-1)^{(2\varrho+1)}(2\varrho+2)!t^{-(2\varrho+3)}+c\delta_{o\varrho}\right]\cdot\left(\left(\frac{-a}{z_o}\right)^{2\lambda-3-2\varrho}z_2+\right.$$

$$\left.+\left(\frac{b}{z_o^2}+c\right)^{\lambda-1-\varrho}z_1+S_{2\lambda-2-2\varrho}\right)-\left(-\frac{a}{z_o}\right)^{2\lambda-1}\left(\frac{b}{z_o^2}+c\right)z_1 .$$

The first three S_ϱ which have to be calculated in a direct way are:

$$S_o = + z_2 \frac{z_o}{a},$$

$$S_1 = \frac{b/z_o^2 + c}{a/z_o} - z_2,$$

$$S_2 = 0.$$

Using the iterative method we obtain for D_1

$$D_1 = (z_1, z_2)\begin{pmatrix} 0 & b/z_o^2 \\ 1 & -a/z_o \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} = (z_1, z_2)\, A\nabla$$

and

$$D_2 = \frac{\partial}{\partial z_o} .$$

The eigenvalues of A are given by:

$$\lambda_{1,2} = -\frac{a}{2z_o} \pm \sqrt{\frac{a^2 + 4b}{4z_o^2} + c}$$

T^T and T^{-1} are given by:

$$T^T = \begin{pmatrix} \frac{b}{z_o^2} + c & ; & \frac{b}{z_o^2} + c \\ -\frac{a}{2z_o} + \sqrt{\frac{a^2 + 4b + 4z_o^2 c}{4z_o^2}} & ; & -\frac{a}{2z_o} - \sqrt{\frac{a^2 + 4b + 4z_o^2 c}{4z_o^2}} \end{pmatrix}$$

and

$$T^{-1} = \frac{1}{2\left(\frac{b}{z_o^2} + c\right)\sqrt{\frac{a^2+4b+4z_o^2c}{4z_o^2}}} \begin{pmatrix} -\frac{a}{2z_o} + \sqrt{\frac{a^2+4b+4z_o^2c}{4z_o^2}} \; ; & \frac{a}{2z_o} + \sqrt{\frac{a^2+4b+4z_o^2c}{4z_o^2}} \\ -\frac{b}{z_o^2} - c & ; \; \frac{b}{z_o^2} - c \end{pmatrix}$$

respectively.

The total solution is given by

$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (T^{-1})^T \begin{pmatrix} e^{t\left(-\frac{a}{2z_o} + \sqrt{\frac{a^2 + 4b + 4z_o^2c}{4z_o^2}}\right)}, & 0 \\ 0, & e^{t\left(-\frac{a}{2z_o} - \sqrt{\frac{a^2 + 4b + 4z_o^2c}{4z_o^2}}\right)} \end{pmatrix} T^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} +$$

$$+ \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[\frac{\partial}{\partial z_o} \left((z_1, z_2) \begin{pmatrix} 0, & \frac{b}{z_o^2} + c \\ 1, & -\frac{a}{z_o} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} + \frac{\partial}{\partial z_o} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right]_{\bar{a}} d\tau$$

We now consider the equation $Z''(t) + \frac{1}{t_o+t} Z'(t) + \kappa^2 Z(t) = 0$ (IV,33)

This equation results from equation (IV,32) setting:

$$t \to t_o + t, \; a = 1, \; -c = \kappa^2$$

The solution of equation (IV,33) reads /7/, /8/:

$$Z(t) = z_1 \cosh \kappa t + \frac{z_2}{\kappa} \sinh \kappa t - \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} (f_{1\nu} Z_1 + f_{2\nu} Z_2)$$

where

$$f_{j\nu} = -\frac{\nu-1}{t_o} t_{j,\nu-1} + 2\kappa^2 f_{j,\nu-2} + \frac{f_{j,\nu-3}}{t_o} \kappa^2 (2\nu-5) - \kappa^4 f_{j,\nu-4} -$$

$$- \frac{\nu-4}{t_o} \kappa^4 f_{j,\nu-3}, \qquad j = 1,2 \qquad \text{(IV,34)}$$

$f_{10} = 0$

$f_{11} = 0$ $\qquad$ $f_{20} = 0$

$f_{12} = 0$ $\qquad$ $f_{21} = 0$

$f_{13} = \frac{\varkappa^2}{t_o}$ $\qquad$ $f_{22} = \frac{1}{t_o}$

Type III

$$Z''(t) + (a + bt^2)Z(t) = 0$$

a, b being constants.

This equation appears as the μ- and ν-equation when separating the Helmholtz equation in parabolic-cylinder coordinates.

The equation is equivalent to the following system:

$$t' = Z'_o = 1$$

$$Z' = Z'_1 = Z_2$$

$$Z'' = Z'_2 = -(a + bt^2)Z_1$$

while the Lie operator is given by:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} + \left[-(a + bt^2)\right] z_1 \frac{\partial}{\partial z_2}$$

Using the recurrence formulas the solution is again, given by the general formula (II,14), where $f_i(t)$ are specialized to:

$$f_1(t) = 0$$

$$f_2(t) = -(a + bt^2),$$

in our case.

The derivatives of $f_i(t)$ are extremely simple, in this case:

$$f_1^{(0)}(t) = 0$$

$$f_2^{(1)}(t) = -2bt$$

$$f_2^{(2)}(t) = -2b$$

$$f_2^{(3)}(t) = 0$$

so that (II,14) reduces to:

$$Z(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_1 = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Big[\tbinom{\nu-2}{0}(-(a + bz_o^2)) . D^{\nu-2} z_1 + \\ + \tbinom{\nu-2}{1}(-2bz_o) \, D^{\nu-3} z_1 + \tbinom{\nu-2}{2}(-2b) \, D^{\nu-4} z_1 \Big] + z_1 + tz_2 =$$

$$= \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \Big[(-a - bz_o^2) \, D^{\nu-2} z_1 + (\nu - 2)(-2bz_o) . D^{\nu-3} z_1 + \\ + \frac{(\nu-2)(\nu-3)}{2} (-2b) D^{\nu-4} z_1 \Big] + z_1 + tz_2$$

Using the iterative method (Chapter II,2) we again put

$$D = D_1 + D_2$$

with

$$D_1 = (z_1, z_2) \begin{pmatrix} 0, & -(a+bz_o^2) \\ a, & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} = (z_1, z_2) \, A\nabla$$

and

$$D_2 = \frac{\partial}{\partial z_o}$$

The eigenvalues of A are given by:

$$\lambda_{1,2} = \pm \sqrt{f_2} = \pm \sqrt{-(a+bz_o^2)},$$

while T^T and T^{-1} are given by:

$$T^T = \begin{pmatrix} -a-bz_o^2 \; , & -a-bz_o^2 \\ + \sqrt{-a-bz_o^2}, & - \sqrt{-a-bz_o^2} \end{pmatrix}$$

$$T^{-1} = \frac{1}{(-a-bz_o^2)(-2)\sqrt{-a-bz_o^2}} \begin{pmatrix} -\sqrt{-a-bz_o^2}, & -\sqrt{-a-bz_o^2} \\ a+bz_o^2, & -a-bz_o^2 \end{pmatrix}$$

The total solution is given by:

$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (T^{-1}) \begin{pmatrix} e^{t\sqrt{-a-bz_o^2}}, & 0 \\ 0 & e^{-t\sqrt{-a-bz_o^2}} \end{pmatrix} T^T \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

$$+ \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[\frac{\partial}{\partial z_o} \left((z_1, z_2) \begin{pmatrix} 0, & -a-bz_o^2 \\ 1, & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} + \frac{\partial}{\partial z_o} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right]_{\bar{a}} d\tau$$

Type IV:

$$Z''(t) - (\alpha_2 + \alpha_3 a^2 \cosh t)\, Z(t) = 0 \qquad \text{(IV,35)}$$

α_2, α_3 and a being constants.

This equation appears among the equations resulting from a separation of the Helmholtz equation in elliptic-cylinder-coordinates. The solution functions are Mathieu functions /17/, /19/, /22/, /33/.

Eq. (IV,35) can be written in the form:

$$Z_0' = 1 = t'$$

$$Z' = Z_1' = Z_2$$

$$Z'' = Z'_2 = (\alpha_2 + \alpha_3 a^2 \cosh t) Z_1$$

The Lie operator D is given by:

$$D = \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + (\alpha_2 + \alpha_3 a^2 \cosh t) z_1 \qquad \text{(IV,36)}$$

The solution of Eq.(IV,35) is given by Eq.(II,14). In our case, f_1 and f_2 in Eq.(II,14) are given by the relations:

$$f_1 = 0$$

$$f_2 = \alpha_2 + \alpha_3 a^2 \cosh t$$

The ϱ^{th} derivative of f_2 is given by:

$$f^{(\varrho)} = \begin{cases} \alpha_3 a^2 \cosh t + \alpha_2 \delta_{o\varrho} & (\varrho \text{ even}) \\ \alpha_3 a^2 \sinh t & (\varrho \text{ odd}) \end{cases}$$

$\delta_{\rho o}$ is the Kronecker symbol

As shown in chapter II one can splitt off known functions from the total solution (Eqs.(II,18), (II,19). This representation is, however disadvantageous insofar as it is much more difficult to code than the recurrence formula representation (Eq.(II,14)).

Using the iteration method, λ_1, λ_2, T and D_2 in Eq.(II,44) are given by the following relations, in our case;

$$\lambda_{1,2} = \pm \sqrt{\alpha_2 + \alpha_3 a^2 \cosh t}$$

$$T = \begin{pmatrix} \frac{\alpha_2 + \alpha_3 a^2 \cosh t}{\lambda_1}, & \frac{\alpha_2 + \alpha_3 a^2 \cosh t}{\lambda_2} \end{pmatrix}$$

and

$$D_2 = \frac{\partial}{\partial z_o}$$

<u>Type V:</u>

$$Z''(t) + \frac{1}{t} Z'(t) + (\varkappa^2 t^2 - \alpha_2 - \frac{\alpha_3}{t^2}) Z(t) = 0 \qquad \text{(IV,37)}$$

$\varkappa^2$, α_2 and α_3 being constants.

This equation appears among the equations in parabolic coordinates. Eq.(IV,37) is solved by Bessel functions /33/, /36/, /39/, /40/. For t = 0, Eq. (IV,37) is singular and therefore cannot be solved by Lie formalism. /1/. By means of the transformation $t \rightarrow t_o + t$, $t_o > o$, one can avoid these singularities. Taking into account this transformation, the Lie operator D reads:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} - \left(\frac{z_2}{t_o - z_o} + \varkappa^2 (t_o - z_o)^2 + \alpha_2 + \frac{\alpha_3}{(t_o - z_o)} z_1 \right) \frac{\partial}{\partial z_2}$$

Eq. (IV,37) is solved by Es.(II,14) where

$$f_1 = \frac{1}{t_o + t}$$

$$f_2 = \varkappa^2 (t_o - t)^2 - \alpha_2 - \frac{\alpha_3}{(t_o - t)^2}$$

The ϱ^{th} derivative of f_1 and f_2 is given by:

$$f_1^{(\varrho)} = (-1)^{\varrho} \frac{\varrho!}{(t_o+t)^{\varrho+1}}$$

$$f_2^{(\varrho)} = \varkappa^2(t_o - t)^2\delta_{\varrho o} + \varkappa^2 \cdot 2(t_o - t)\delta_{\varrho 1} - 2\varkappa^2\delta_{\varrho 2} - \frac{\alpha_3}{(t_o-t)^{\varrho+1}}(-1)^{\varrho}\varrho!$$

where $\delta_{\rho i}$ is the Kronecker symbol

λ_1, λ_2, T and D_2 in Eq.(II,44) are given by:

$$\lambda_{1,2} = \frac{1}{z_o+t} \pm \sqrt{\frac{[\varkappa^2(z_o-t^2)]^2}{4} + \varkappa^2(z_o - t)^2 - \alpha_2 - \frac{\alpha_3}{(z_o-t)^2}}$$

$$T = \begin{pmatrix} \varkappa^2(z_o-t)^2 - \alpha_2 - \frac{\alpha_3}{(z_o-t)^2}, & \varkappa^2(z_o-t)^2 - \alpha_2 - \frac{\alpha_3}{(z_o-t)^2} \\ \lambda_1 & , \quad \lambda_2 \end{pmatrix}$$

$$D_2 = \frac{\partial}{\partial z_o}$$

Type VI:

$$Z''(t) + \coth t\, Z'(t) + (\varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t})Z(t) = 0 \qquad \text{(IV,38)}$$

$\varkappa$, a, α_2 and α_3 being constants.

This equation appears among the equations in prolate spheroidal coordinates. The solution functions are Legendre functions /17/, /33/, /36/, /39/. At t = o, Eq.(IV,38) is singular. We solve Eq.(IV,38) in the domain $|t|>o$. In this case, we can use Lie series to solve Eq.(IV,38). The Lie operator reads:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} - (z_2 \coth t + (\varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t})z_1)\frac{\partial}{\partial z_2}$$

The solution of Eq.(IV,38) is again given by Eq.(II,14), where

$$f_1 = \coth t$$

and

$$f_2 = \varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t}$$

The ϱ^{th} derivatives of f_1 and f_2 are given by:

$$f_1^{(\varrho)}(t) = (-1)^{\varrho} \frac{\varrho!}{t^{\varrho+1}} + \sum_{K=1}^{\infty} \frac{2^{2K} B_{2K}}{2K!} \frac{(2K-1)!}{(2K-(\varrho+1))!} . t^{2K-(\varrho+1)}$$

$o < t^2 < \pi^2$

$\varrho = 1, 2, \ldots\ldots$

where B_{2K} are the Bernoulli numbers.

$$f_2^{(\rho)}(t) = \sum_{K=1}^{\infty} \frac{2^{2K} B_{2K}}{(2K)!} \frac{(2K-1)(2K-2)!}{(2K-(\rho+2))!} t^{2K-(\rho+2)} + \frac{1}{t^{\rho+2}}(-1)^{\rho+1}.(\rho+1)! +$$

$$+ \begin{cases} 2^{\rho-1} \cosh 2t - \delta_{\rho o}(\frac{1}{2} + \alpha_2) & \text{for even } \rho \\ 2^{\rho-1} \sinh 2t & \text{for odd } \rho \end{cases}$$

where $\delta_{\rho o}$ is the Kronecker symbol

$(o < t^2 < \pi^2)$

Using the iteration method, λ_1, λ_2, T and D_2 in Eq.(II,44) are given by:

$$\lambda_{1,2} = \frac{\coth t}{2} \pm \sqrt{\frac{\coth^2 t}{4} + \varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t}}$$

$$T = \begin{pmatrix} \varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t}, & \varkappa^2 a^2 \sinh^2 t - \alpha_2 - \frac{\alpha_3}{\sinh^2 t} \\ \lambda_1 & , \quad \lambda_2 \end{pmatrix}$$

$$D_2 = \frac{\partial}{\partial z_o}$$

<u>Type VII:</u>

$$Z''(t) + \cot t \, Z'(t) + (\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t}) \, Z(t) = 0 \qquad \text{(IV,39)}$$

$\varkappa$, a, α_2 and α_3 being constants.

This equation appears among the equations in prolate spheroidal coordinates, spherical coordinates (a=o) and oblate spherical coordinates. Eq.(IV,39) is solved by Legendre functions /17/, /33/, /36/, /39/. Evidently, for $t = n\pi$ ($n = 1,2, \ldots$) Eq.(IV,39) is singular. We solve Eq.(IV,39) formally for the regular domain $t \neq n\pi$.

The Lie operator reads:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} + (-z_2 \cot t - (\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t}) z_1) \frac{\partial}{\partial z_2}$$

In our case, f_1 and f_2 in Eq.(II,14) are given by:

$$f_1 = -\cot t = \frac{1}{t} + \sum_{K=1}^{\infty} \frac{2^{2K} B_{2K}}{(2K)!} t^{2K-1}$$

$$f_2 = -(\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t}) = -(\frac{\varkappa^2 a^2}{2} (1-\cos 2t) + \alpha_2 + \alpha_3 \frac{d}{dt} \operatorname{ctg} t =$$

$$= -(\frac{\varkappa^2 a^2}{2} (1-\cos 2t + \alpha_2 + \alpha_3 (-\frac{1}{t^2} + \sum_{K=1}^{\infty} \frac{2^{2K} B_{2K}}{(2K)!} (2K-1) t^{2K-2})$$

$$(o < t^2 < \pi^2)$$

For the numerical evaluation of Eq.(II,14) we need the ϱ^{th} derivatives of f_1 and f_2, which are given by:

$$f_1^{(\varrho)} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n} B_{2n}}{(2n)!} \cdot \frac{(2K-1)!}{(2K-(\varrho+1))!} \cdot t^{2K-(\varrho+1)} + (-1)^{\varrho} \frac{\varrho!}{t^{\varrho+1}}$$

$(|t| < \pi)$

$\varrho = 1, 2, 3, \ldots\ldots$

$$f_2^{(\rho)} = \alpha_3 \left((-1)^{\rho+1} \frac{(\rho+1)!}{t^{\rho+2}} + \sum_{K=1}^{\infty} \frac{2^{2K} B_{2K}}{(2K)!} \frac{(2K-1)(2K-2)!}{(2K-(\rho+2))!} t^{2K-(2+\rho)} + \right.$$

$$+ \begin{matrix} (-1)^{\frac{\rho+1}{2}} \quad 2^{\rho-1} \sin 2t & \rho \text{ odd} \\ (-1)^{\frac{\rho}{2}} \;.\; 2^{\rho-1} \cos 2t - \frac{\varkappa^2 a^2}{2} \delta_{\rho o} - \alpha_2 \delta_{\rho o} & \end{matrix}$$

where B_{2K} are the Bernoulli numbers and $\delta_{\rho o}$ is the Kronecker symbol

$(o < t^2 < \pi)$

Considering Eq.(II,44) λ_1, λ_2, T and D_2 are given by:

$$\lambda_{1,2} = \frac{-\cot t}{2} \pm \sqrt{\frac{(\cot t)^2}{2} - \varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t}}$$

$$T = \begin{pmatrix} -\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t} \;; & -\varkappa^2 a^2 \sin^2 t + \alpha_2 - \frac{\alpha_3}{\sin^2 t} \\ \lambda_1 \;; & \lambda_2 \end{pmatrix}$$

and

$$D_2 = \frac{\partial}{\partial z_o}$$

Type VIII:

$$Z''(t) + \text{tangh } t\, Z'(t) + (\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t})\, Z(t) = 0 \qquad \text{(IV,40)}$$

$\varkappa$, a, α_2 and α_3 being constants.

This equation appears among the oblate spherical coordinates and is solved by Legendre functions /17/, /33/, /36/, /39/.

The Lie operator is given by:

$$D = \frac{\partial}{\partial z_0} + z_2 \frac{\partial}{\partial z_1} + (-z_2 \tanh t - (\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t}) z_1 \frac{\partial}{\partial z_2}$$

f_1 and f_2 in the general solution of Eq.(II,14) are given by:

$$f_1 = -\tanh t = \sum_{K=1}^{\infty} \frac{2^{2K}(2^{2K}-1)}{(2K)!} B_{2K} t^{2K-1}$$

$$(t^2 < \frac{\pi^2}{4})$$

B_{2K} are the Bernouilli numbers.

$$f_2 = -(\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t}) = -(\frac{\varkappa^2 a^2}{2} (\cosh 2t + 1) - \alpha_2 +$$

$$+ \alpha_3 \frac{d}{dt} (\tanh t + C))$$

$$f_1^{(\varrho)} = -\sum_{K=1}^{\infty} \frac{2^{2K}(2^{2K}-1)}{(2K)!} \frac{(2K-1)!}{(2K-(\rho+1))!}\, t^{2K-(\rho+1)} B_{2K}$$

where B_{2K} are the Bernoulli numbers

$$(t^2 < \frac{\pi^2}{4})$$

$\varrho = 1, 2, 3, \ldots\ldots$

$$f_2^{(\rho)} = \alpha_3 \sum_{K=1}^{\infty} \frac{2^{2K}(2^{2K}-1)B_{2K}}{(2K)!} \frac{(2K-1)(2K-2)!}{(2K-(\rho+2))!} t^{2K-(\rho+2)} +$$

$$+ \begin{cases} -\frac{\varkappa^2 a^2}{2} 2^{\rho-1} \sinh 2t & \text{for odd } \rho \\ -\frac{\varkappa^2 a^2}{2} 2^{\rho-1} \cosh 2t & \text{for even } \rho \end{cases}$$

where B_{2K} are the Bernoulli numbers

$$(t^2 < \frac{\pi^2}{4}) \qquad \rho = 1, 2, 3, \ldots\ldots.$$

Using the iteration method one obtains for λ_1, λ_2, T and D_2 in Eq.(II,44)

$$\lambda_{1,2} = -\frac{\tanh t}{2} \pm \sqrt{\frac{\tanh^2 t}{4} - \varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t}}$$

$$T = \begin{pmatrix} -\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t}, & -\varkappa^2 a^2 \cosh^2 t - \alpha_2 + \frac{\alpha_3}{\cosh^2 t} \\ \lambda_1, & \lambda_2 \end{pmatrix}$$

and

$$D_2 = \frac{\partial}{\partial z_0}$$

Type IX:

$$Z''(t) + \frac{(2t^2-(b^2+c^2))}{(t^2-b^2)(t^2-c^2)} Z'(t) + \frac{(\varkappa^2 t^4 + \alpha_3 t^2 + \alpha_2)}{(t^2-b^2)(t^2-c^2)} Z(t) = 0 \qquad \text{(IV,41)}$$

b, c, $\varkappa$, α_2 and α_3 being constants.

This equation appears among the equations in conical coordinates and ellipsoidal coordinates and is solved by Lamé functions /17/, /20/, /33/, /34/, /41/. Eq.(IV,41) is singular for $t^2 = b^2$ and $t^2 = c^2$.

We solve Eq.(IV,41) for the regular domain: $t^2 \neq b^2$, $t^2 \neq c^2$. For this regular domain we can use Lie series in order to represent the solution.

The Lie operator reads:

$$D = \frac{\partial}{\partial z_o} + z_2 \frac{\partial}{\partial z_1} - \left[\frac{2t^2-(b^2+c^2)}{(t^2-b^2)(t^2-c^2)} + \frac{\varkappa^2 t^4+\alpha_3 t^2+\alpha_2}{(t^2-b^2)(t^2-c^2)}\right] \frac{\partial}{\partial z_2}$$

f_1 and f_2 in Eq.(II,14) are given by the relations:

$$f_1 = - \frac{(2t^2-(b^2+c^2))}{(t^2-b^2)(t^2-c^2)}$$

$$f_2 = - \frac{\varkappa^2 t^4+\alpha_3 t^2+\alpha_2}{(t^2-b^2)(t^2-c^2)}$$

or

$$f_1 = \frac{A_1}{t-b} + \frac{A_2}{t+b} + \frac{A_3}{t-c} + \frac{A_4}{t+c}$$

and

$$f_2 = \frac{B_1}{t-b} + \frac{B_2}{t+b} + \frac{B_3}{t-c} + \frac{B_4}{t+c}$$

A_i, B_i ($i = 1, 2, 3, 4,$) being constants.

The ϱ-th derivatives of f_1 and f_2 are given by

$$f_1^{(\varrho)} = (-1)^\varrho \varrho! \left\{ \frac{A_1}{(t-b)^{\varrho+1}} + \frac{A_2}{(t+b)^{\varrho+1}} + \frac{A_3}{(t-c)^{\varrho+1}} + \frac{A_4}{(t+c)^{\varrho+1}} \right\}$$

and

$$f_2^{(\varrho)} = (-1)^\varrho \varrho! \left\{ \frac{B_1}{(t-b)^{\varrho+1}} + \frac{B_2}{(t+b)^{\varrho+1}} + \frac{B_3}{(t-c)^{\varrho+1}} + \frac{B_4}{(t+c)^{\varrho+1}} \right\}$$

$|t| \lessgtr c, \quad |t| \lessgtr b$

If one solves Eq.(IV,41) by iteration method, one has to put the following expressions for λ_1, λ_2, T and D_2 in Eq.(II,44)

$$\lambda_{1,2} = -\frac{2t^2-(b^2+c^2)}{(t^2-b^2)(t^2-c^2)} \pm \sqrt{\left[\frac{2t^2-(b^2+c^2)}{(t^2+b^2)(t^2-c^2)}\right]^2 - \frac{\varkappa^2t^4+\alpha_3t^2+\alpha_2}{(t^2-b^2)(t^2-c^2)}}$$

$$T = \begin{pmatrix} \frac{\varkappa^2t^4+\alpha_3t^2+\alpha_2}{(t^2-b^2)(t^2-c^2)} & , & \frac{\varkappa^2t^4+\alpha_3t^2+\alpha_2}{(t^2-b^2)(t^2-c^2)} \\ \lambda_1 & , & \lambda_2 \end{pmatrix}$$

and

$$D_2 = \frac{\partial}{\partial z_o}$$

Type X:

$$Z''(t) + \frac{1}{2}\frac{(2t-(b+c))}{(t-b)(t-c)}\, Z'(t) + \frac{(\varkappa^2t^2+\alpha_3t-\alpha_2)}{(t-b)(t-c)}\, Z(t) = 0 \qquad (IV,42)$$

This equation results from the separation of the Helmholtz equation in paraboloidal coordinates. The solutionsare Baer functions /17/. Eq. (IV,42) is singular at t=b, t=c. We solve the equation in the region $t \neq b$, $t \neq c$. In this regular domain, we can represent the solution of Eq.(IV,42) by Lie series.

The Lie operator for Eq.(IV,42) reads:

$$D = \frac{\partial}{\partial z_o} + z_2\frac{\partial}{\partial z_1} + \left(\frac{1}{2}\frac{(b+c-2t)}{(t-b)(t-c)}\, z_2 - \frac{\varkappa^2t^2+\alpha_3t-\alpha_2}{(t-b)(t-c)}\, z_1\right)\frac{\partial}{\partial z_2}$$

The general solution of Eq.(IV,42) is given by Eq.(II,14) where

$$f_1 = \frac{1}{2}\frac{(b+c-2t)}{(t-b)(t-c)}$$

$$f_2 = \frac{-\varkappa^2 t^2 - \alpha_3 t + \alpha_2}{(t-b)(t-c)}$$

The ϱ^{th} derivatives of f_1 and f_2 can be obtained by an analogous procedure as under type IX.

Using the iteration method to solve Eq.(IV,42) one has to put the following expressions for λ_1, λ_2, T and D_2 in Eq.(II,44)

$$\lambda_{1,2} = \frac{1}{2}\frac{b+c-2t}{(t-b)(t-c)} \pm \sqrt{\frac{1}{4}\left[\frac{b+c-2t}{(t-b)(t-c)}\right]^2 + \frac{-\varkappa^2 t^2 - \alpha_3 t + \alpha_2}{(t-b)(t-c)}}$$

$$T = \begin{pmatrix} \frac{-\varkappa^2 t^2 - \alpha_3 t + \alpha_2}{(t-b)(t-c)} , & \frac{-\varkappa^2 t^2 - \alpha_3 t + \alpha_2}{(t-b)(t-c)} \\ \lambda_1 , & \lambda_2 \end{pmatrix}$$

$$D_2 = \frac{\partial}{\partial z_o}$$

Chapter V

Applications in Physics by A.SCHETT and J.WEIL

An Example From Rigid Body Mechanics:

The equation of motion of a plane mathematical pendulum of length 1 and mass m in the case of a suspensory point vibrating in vertical direction according to the law a $\cos\omega t$ (a,ω constant, g gravitational acceleration) is given by /19/:

$$\ddot{\alpha} + \delta(t)\dot{\alpha} + \left(\frac{g}{1} - \frac{a}{1}\omega^2 \cos\omega t\right)\alpha = f_3(t)$$

if the elongation α is sufficiently small.

In this case:

$$f_1(t) = -\delta(t)$$

$$f_2(t) = -\frac{g}{1} + \frac{a}{1}\omega^2 \cos\omega t$$

the solution is given by:

$$Y(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu y$$

Assuming $\lambda_1 \neq \lambda_2 \neq \lambda_3$, we obtain the following eigenvalues of the matrix B (chapter III):

$$\lambda_{1,2} = \frac{f_1}{2} \pm \sqrt{\frac{f_1^2}{4} + f_2} =$$

$$= \frac{-\delta(t)}{2} \pm \sqrt{\frac{\delta^2(t)}{4} - \left(\frac{g}{1} - \frac{a}{1}\omega^2 \cos\omega t\right)}$$

and

$$\lambda_3 = 0,$$

so that the T_{ij} (the components of transformation matrix) are given by:

$$t_{11} = -\frac{\delta(t)}{2} + \sqrt{\frac{\delta^2(t)}{4} - \left(\frac{g}{1} - \frac{a}{1}\omega^2\cos\omega t\right)}$$

$$t_{12} = -\frac{\delta(t)}{2} - \sqrt{\frac{\delta^2(t)}{4} - \left(\frac{g}{1} - \frac{a}{1}\omega^2\cos\omega t\right)}$$

$$t_{13} = 0$$

$$t_{21} = -\left(\frac{g}{1} - \frac{a}{1}\omega^2\cos\omega t\right)$$

$$t_{22} = -\left(\frac{g}{1} - \frac{a}{1}\omega^2\cos\omega t\right)$$

$$t_{23} = 0$$

$$t_{31} = f_3$$

$$t_{32} = f_3$$

$$t_{33} = 1$$

The solution is given by (III,20).

Using recurrence formulas, the solution reads:

$$Y(t) = \sum_{\nu=5}^{\infty} \frac{t^\nu}{\nu!} \sum_{\varrho=2}^{\nu-2} \Bigg[f_3^{(\nu-2)} - \binom{\nu-2}{\varrho} D^\varrho \delta_1(t) D^{\nu-1-\varrho} y_1 +$$

$$+ D^\varrho \left[-\frac{g}{1} + \frac{a}{1}\omega^2\cos\omega t\right] D^{\nu-2-\varrho} y_1 \Bigg] +$$

$$+ y_1 + ty_2 + \frac{t^2}{2} D^2 y_1 + \frac{t^3}{3!} D^3 y_1 + \frac{t^4}{4!} D^4 y_1 .$$

An Example From Electricity:

Another problem described by the following circuital equation may be solved within our formalism:

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC(t)} = \frac{E(t)}{L}$$

i.e., a circuit containing a driving e.m.f. an inductance L in series with acapacitance C varying with time, and a constant resistance R. Assuming $C(t) = C_o(1 + \epsilon\cos 2\omega_1 t) = \frac{C_o}{1-\epsilon\cos 2\omega_1 t}$, (if ϵ is small enough) and writing $Q = Y$, $1/LC_o = \omega^2_o$, $z=\omega_1 t$, $\frac{R}{\omega_1 L} = 2\alpha$, $\bar{a} = (\omega_o/\omega_1)^2$ and $E(t)/\omega_1{}^2 L = f_3(z)$, we obtain the following inhomogeneous Mathieu equation:

$$Y'' + 2\alpha Y' + (\bar{a} - 2q\cos 2z)Y = f_3(z)$$

where

$$f_1 = -2\alpha$$

$$f_2(z) = 2q\cos 2z-\bar{a}$$

(z corresponds to t in the general treatment)
The formal solution is given by:

$$Y(t) = \sum_{y=0}^{\infty} \frac{z^\nu}{\nu!} D^\nu y$$

The eigenvalues of the matrix B (chapter III) are given by:

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 + 2q\cos 2z-\bar{a}}$$

$$\lambda_3 = 0$$

while the elements of the transformation matrix are given by:

$$t_{11} = -\alpha + \sqrt{\alpha^2 + 2q\cos 2z - \bar{a}}$$

$$t_{12} = -\alpha - \sqrt{\alpha^2 + 2q\cos 2z - \bar{a}}$$

$t_{13} = 0$

$t_{21} = 2q\cos 2z - \bar{a}$

$t_{22} = 2q\cos 2z - \ddot{a}$

$t_{23} = 0$

$t_{31} = f_3(z)$

$t_{32} = f_3(z)$

$t_{33} = 1$

Using these results, the general solution may be written in the way given in chapter III Eq.(III,20).

<u>Gravity Gradient Stabilization of Artificial Satellites.</u>

As Rumyantse showed in Athens at the International Astronautical Congress, summer 1965, the problem of gravity gradient stabilization of artificicial satellites leads to Mathieu functions. We did not hear this lecture, but the principal features of this problem may be contained in the simple model of a pendulum with a vibrating suspensory point the theory of which is given in many textbooks, e.g., /19/. The motion of the pendulum is described by the following equation:

$$\ddot{\Psi}(t) + f_1(t)\Psi + f_2(t)\Psi = f_3(t),$$

where $f_2(t)$ is given by:

$$f_2(t) = \left(\frac{g}{1} - \frac{a}{1}\omega^2 \cos\omega t\right)$$

In this equation Ψ, the elongation, is assumed to be restricted to small values, ω is the frequency of the vibrating suspensory point, g the gravitational acceleration and ω is the frequency of the vibrating suspensory point; the functions $f_i(t)$ are supposed to be regular. Evidently, this equation is of the Mathieu type and, consequently, belongs to those types of differential equations which have been treated by both lines of research of our institute: the general one which has set up a program for solving homogeneous and inhomogeneous second order differential equations and the special one whose efforts are focused on treating the Mathieu equation.

Calculation of the Strongly Focusing Synchrotron:

The srongly focusing synchrotron, a device proposed by Courant, Livingston and Snyder /19/, is a high energy accelerator which has an even number A of magnets along the circle representing the "ideal" path of particles. The motion of the particles is described by the following Mathieu equations for ϱ and z, the radial and axial deviations from an ideal circle:

$$\frac{d^2\varrho}{du} + \frac{16}{N^2}(1-a-b \cos 2u)\varrho = \frac{\delta}{\omega}$$

$$\frac{d^2z}{du^2} + \frac{16}{N^2}(a + b \cos 2u) z = 0$$

where $\frac{N}{2}\cdot\varrho = 2u$, $\varrho = \omega t$, and ω is the sychrotron frequency as may be seen, e.g., in /19/. Evidently, these are also equations of the Mathieu type whose treatment lies on the line of our investigations.

The Problem of the Heavy Asymmetrical Top.(Gyroscope).

Another promising physical application of Lie methods is the problem of the heavy asymmetrical top. Usually such bodies - e.g., space vehicles - are treated as symmetrical ones showing only a slight deviation from symmetry which can satisfactorily taken into account by a suitable perturbation calculation. Lie series formalism, however, allows the total asymmetrical problem to be solved. In particular, owing to the favorable decomposability of the Lie operator $D = D_1 + D_2$, all deviations from symmetrical construction may be put into D_2. In this way, we will be able to check the present-day perturbation calculations from the view-point of the general theory.

Chapter VI

Representation and Computation of Mathieu Functions by Means of Lie Series

As a first example to the previous investigations the Mathieu equation

$$y''(x) - (2q.\cos 2x - \lambda).y(x) = 0$$

was studied in order to demonstrate the usefulness of the Lie series method for solving (1). According to their periodic nature, the Mathieu functions - at least those of the 1st kind - are not very appropriate to a representation by power series, as obtained in the described way. This unfavorable example was chosen on purpose as we wanted to establish the limits of this method as quickly as possible. The fact that in spite of that satisfying results can be obtained attests to the usefulness of Lie series formalism. In so doing, investigations on the remaining term, better convergence (in this respect, Dr. Knapp's iteration method might be very advantageous) and on computer times (e.g., compared to Fourier series representations) were put off, as above all, we want to show that useful, partly even numerical representations of the solution of (1) can be obtained by means of the method worked out. (See also the concluding discussion of this example).

(6.1) Theory

(F. Cap and Floriani D.)

(6.11) General results:

(6.111) In the following, we shall summarize the method of solving

$$v''(u) - h(u).v(u) = g(u) \qquad \text{(VI,1)}$$

in a way which is somewhat different from that chosen in the previous chapters since the formulas derived in this connected will be needed in what follows:

$$Z_o = u, \qquad Z_1 = v, \qquad Z_2 = \frac{d}{du} v(u) \tag{VI,2}$$

From (VI,1) and (VI,2), we obtain the following system:

$$\begin{aligned} \dot{Z}_o &= 1 \\ \dot{Z}_1 &= Z_2 \\ \dot{Z}_2 &= h(Z_o).Z_1 + g(Z_o) \end{aligned} \tag{VI,3}$$

and the Lie operator

$$D = \frac{\partial}{\partial z_o} + z_2 \cdot \frac{\partial}{\partial z_1} + \left[z_1.h(z_o) + g(z_o)\right] \cdot \frac{\partial}{\partial z_2} \tag{VI,4}$$

With help of the "commutation theorem" (proof, for example, in Ref. 1):

$$f(Z_o,Z_1,Z_2) = \exp(tD).f(z_o,z_1,z_2) = \sum_{o}^{\infty} \frac{t^\rho}{\rho!} \cdot D^\rho f(z_o,z_1,z_2) \tag{VI,5}$$

and, particularly

$$Z_1 = \exp(tD).z_1 = \sum_{o}^{\infty} \frac{t^\rho}{\rho!} \cdot D^\rho z_1 \tag{VI,6}$$

and after introducing the following new functions:

$$\begin{aligned} \Psi_{2k} &= D^{2k} z_1 - g^{(2k-2)}(z_o) - z_1.h^k(z_o) \\ \Psi_{2k+1} &= D^{2k+1} z_1 - g^{(2k-1)}(z_o) - z_2.h^k(z_o) \end{aligned} \tag{VI,7}$$

$$(k = o,1,2,\ldots)$$

$$\begin{gathered} g^{(-1)}(u) = \int_{u_o}^{u} g(\alpha)d\alpha, \quad g^{(-2)}(u) = \int_{u_o}^{u} g^{(-1)}(\alpha)d\alpha = G(u) \\ g^{(\rho)}(z_o) = G^{(\rho+2)}(z_o) \end{gathered} \tag{VI,7a}$$

we obtain for the general solution:

$$Z_1 = \sum_0^{\infty} \frac{t^{2\rho}}{(2\rho)!}\cdot\left[z_1\cdot\sqrt{h(z_o)}^{2\rho} + g^{(2\rho-2)}(z_o) + \Psi_{2\rho}\right] +$$

$$+ \sum_0^{\infty} \frac{t^{2\rho+1}}{(2\rho+1)!}\cdot\left[\frac{z_2}{\sqrt{h(z_o)}}\cdot\sqrt{h(z_o)}^{2\rho+1} + g^{(2\rho-1)}(z_o) + \Psi_{2\rho+1}\right] = \tag{VI,8}$$

$$= z_1\cdot \mathrm{Ch}(t\cdot\sqrt{h(z_o)}) + \frac{z_2}{\sqrt{h(z_o)}}\cdot \mathrm{Sh}(t\cdot\sqrt{h(z_o)}) + \sum_0^{\infty}\frac{t^{\rho}}{\rho!}\cdot\left[G^{(\rho)}(z_o) + \Psi_{\rho}\right].$$

Because of

$$z_n = Z_n(t=o), (n = o,1,2,\ldots) \tag{VI,9}$$

the following relation holds within the radius of convergence:

$$\sum_0^{\infty} \frac{t^{\rho}}{\rho!}\cdot G^{(\rho)}(z_o) = G(Z_o) = \iint_{z_o}^{Z_o} g(\alpha)d\alpha^2 \tag{VI ,1o}$$

Furthermore, (VI,4) yields:

$$D^1 z_1 = z_2, \quad D^2 z_1 = g(z_o) + z_1\cdot h(z_o) \tag{VI,11}$$

as well as

$$D^1 h(z_o) = \frac{d}{dz_o}h(z_o), \quad D^1 g(z_o) = \frac{d}{dz_o} g(z_o)$$

and

$$D^{\rho} h(z_o) = h^{(\rho)}(z_o), \quad D^{\rho} g(z_o) = g^{(\rho)}(z_o), \tag{VI,12}$$

respectively.

With

$$D^{n+2} z_1 = D^n(D^2 z_1) = D^n\left[g(z_o) + z_1\cdot h(z_o)\right] =$$

$$= g^{(n)}(z_o) + D^n\left[h(z_o)\cdot z_1\right] \tag{VI,13}$$

we have:

$$\Psi_{2k+2} = D^{2k}\left[z_1 . h(z_o)\right] - z_1 . h^{k+1}(z_o)$$

$$\Psi_{2k+3} = D^{2k+1}\left[z_1 . h(z_o)\right] - z_2 . h^{k+1}(z_o). \tag{VI,14}$$

Using the Leibniz rule we obtain:

$$D^{2k}\left[z_1 . h(z_o)\right] = \left[h(z_o) . G(z_o)\right]^{(2k)} +$$

$$+ \sum_o^k \left\{h^{(2k-2\rho)}(z_o) . \binom{2k}{2\rho} . \left[\Psi_{2\rho} + z_1 . h^\rho(z_o)\right]\right\} +$$

$$+ \sum_o^{k-1} \left\{h^{(2k-2\rho-1)}(z_o) . \binom{2k}{2\rho+1} . \left[\Psi_{2\rho+1} + z_2 . h^\rho(z_o)\right]\right\}$$

(VI,15)

$$D^{2k+1}\left[z_1 . h(z_o)\right] = \left[h(z_o) . G(z_o)\right]^{(2k+1)} +$$

$$+ \sum_o^k \left\{h^{(2k+1-2\rho)}(z_o) . \binom{2k+1}{2\rho} . \left[\Psi_{2\rho} + z_1 . h^\rho(z_o)\right]\right\} +$$

$$+ \sum_o^k \left\{h^{(2k-2\rho)}(z_o) . \binom{2k+1}{2\rho+1} . \left[\Psi_{2\rho+1} + z_2 . h^\rho(z_o)\right]\right\}.$$

From

$$t = u - u_o \tag{VI,16}$$

and (VI,2+9), we have:

$$z_o = u_o$$

$$z_1 = v(u_o) = v_o \tag{VI,17}$$

$$z_2 = v'(u_o) = v_o'$$

$h(z_o) = h(u_o) = h_o$, $\qquad h^{(\rho)}(z_o) = h^{(\rho)}(u_o) = h_o^{(\rho)}$

$g(z_o) = g(u_o) = g_o$, $\qquad g^{(\rho)}(z_o) = g^{(\rho)}(u_o) = g_o^{(\rho)}$

In original notation, the general solution is, then, given by

$$v(u) = v_o.\mathrm{Ch}(u-u_o)\sqrt{h_o} + \frac{v'_o}{\sqrt{h_o}}.\mathrm{Sh}(u-u_o)\sqrt{h_o} + \iint_{u_o}^{u} g(\alpha)d\alpha^2 + \sum_{o}^{\infty} \frac{(u-u_o)^{\rho}}{\rho!}.\Psi_{\rho}(u_o) =$$

$$= v_o.\cos(u-u_o)\sqrt{-h_o} + \frac{v'_o}{\sqrt{-h_o}}.\sin(u-u_o)\sqrt{-h_o} + \iint^{u} g(\alpha)d\alpha^2 + \sum_{o}^{\infty} \frac{(u-u_o)^{\rho}}{\rho!}.\Psi_{\rho}(u_o). \qquad \text{(VI,18)}$$

Its derivative is:

$$v'(u) = +v_o\sqrt{h_o}.\mathrm{Sh}(u-u_o)\sqrt{h_o} + v'_o.\mathrm{Ch}(u-u_o)\sqrt{h_o} + \int_{u_o}^{u} g(\alpha)d\alpha + \sum_{o}^{\infty} \frac{(u-u_o)^{\rho}}{\rho!}.\Psi_{\rho+1}(u_o) =$$

$$= -v_o\sqrt{-h_o}.\sin(u-u_o)\sqrt{-h_o} + v'_o.\cos(u-u_o)\sqrt{-h_o} + \int_{u_o}^{u} g(\alpha)d\alpha + \sum_{o}^{\infty} \frac{(u-u_o)^{\rho}}{\rho!}.\Psi_{\rho+1}(u_o). \qquad \text{(VI,19)}$$

The following recurrence formulas hold for the Ψ_{ρ}:

$$\Psi_{2k+2}(u_o) = \left[h.G\right]_o^{(2k)} + h_o.\Psi_{2k} + \sum_{o}^{k-1}\left\{h_o^{(2k-2\rho)}\binom{2k}{2\rho}.\left[\Psi_{2\rho} + v_o.h_o^{\rho}\right]\right\} + \sum_{o}^{k-1}\left\{h_o^{(2k-2\rho-1)}\binom{2k}{2\rho+1}.\left[\Psi_{2\rho+1} + v'_o.h_o^{\rho}\right]\right\}$$

$$\text{(VI,20)}$$

$$\Psi_{2k+3}(u_o) = \left[h.G\right]_o^{(2k+1)} + h_o.\Psi_{2k+1} + \binom{2k+1}{2k}.h'_o.\left[\Psi_{2k} + v_o.h_o^{k}\right] + \sum_{o}^{k-1}\left\{h_o^{(2k+1-2\rho)}\binom{2k+1}{2\rho}.\left[\Psi_{2\rho} + v_o.h_o^{\rho}\right]\right\} + \sum_{o}^{k-1}\left\{h_o^{(2k-2\rho)}\binom{2k+1}{2\rho+1}.\left[\Psi_{2\rho+1} + v'_o.h_o^{\rho}\right]\right\}$$

(k=1,2,3,...) with (from (VI,7))

$$\Psi_o(u_o) = -g^{(-2)}(u_o) = -G(u_o) = \iint\limits_{u_o}^{u} g(\alpha)d\alpha^2 = 0$$

$$\Psi_1(u_o) = -g^{(-1)}(u_o) = -G'(u_o) = \int\limits_{u_o}^{u} g(\alpha)d\alpha = 0$$

$$\Psi_2(u_o) = g_o + v_o.h_o - g_o - v_o.h_o = 0 \qquad (VI,21)$$

$$\Psi_3(u_o) = g_o' + v_o'.h_o + v_o.h_o' - g_o' - v_o'.h_o = v_o.h_o'$$

(6.112) There is also another way leading to the separation of the sine and cosine terms given in (VI,18): One wants to approximate in the neighbourhood of a point u_o the solution of the homogeneous equation

$$v''(u) = h(u).v(u) \qquad (VI,22)$$

by a function A(u) with

$$A(u_o) = v(u_o)$$
$$A'(u_o) = v'(u_o) \qquad (VI,23)$$

so that for the remaining term

$$R(u) = v(u) - A(u) \qquad (VI,24)$$

with

$$R(u_o) = 0, \quad R'(u_o) = 0 \qquad (VI,25)$$

the following inequality holds:

$$|R| \ll |A| \qquad (VI,26)$$

in this neighborhood. It follows from

$$\delta h(u) = h(u) - h(u_o) \tag{VI,27}$$

and (VI,22):

$$A''(u) = h(u_o).A(u) + \left\{-R'' + \left[h(u_o).R + A.\delta h\right] + \dots\right\} \tag{VI,28}$$

In all cases that not are too unfavorable the curled bracket will be negligible; in this case, A(u) becomes:

$$A(u) = v_o.Ch(u-u_o)\sqrt{h_o} + \frac{v'_o}{\sqrt{h_o}}.Sh(u-u_o)\sqrt{h_o} \tag{VI,29}$$

as is given in (VI,18). Besides (VI,23), the relations

$$\begin{aligned} A''(u_o) &= v''(u_o) \\ R''(u_o) &= 0 \end{aligned} \tag{VI,30}$$

hold true.

(6.113) The application of the same principle to the inhomogeneous equation (VI,1) leads to

$$A''(u) = h(u_o).A(u) + g(u_o) \tag{VI,31}$$

with the solution

$$A(u) = \left[v_o + \frac{g_o}{h_o}\right].Ch(u-u_o)\sqrt{h_o} + \frac{v'_o}{\sqrt{h_o}}.Sh(u-u_o)\sqrt{h_o} - \frac{g_o}{h_o} \tag{VI,32}$$

By virtue of (VI,1), (VI,31) and (VI,32) the following relations

$$\begin{aligned} A(u_o) &= v(u_o) \qquad & R(u_o) &= 0 \\ A'(u_o) &= v'(u_o) \qquad & R'(u_o) &= 0 \\ A''(u_o) &= v''(u_o) \qquad & R''(u_o) &= 0 \end{aligned} \tag{VI,33}$$

are valid. For R(u) we have

$$R(u) = \iint_{u_0}^{u} g(\alpha)d\alpha^2 + \sum_{0}^{\infty} \frac{(u-u_o)^\rho}{\rho!}\cdot\beta_\rho(u_o) \qquad (VI,34)$$

with

$$\beta_o = \beta_1 = 0, \qquad \beta_2 = -g(u_o)$$
$$\beta_3(u_o) = +h'(u_o).v(u_o) \qquad (VI,35)$$

and for the $\beta_n(u_o)$ (n=4,5,6,...) (VI,20) by substituting

$$\Psi\ (u_o) \longrightarrow \beta\ (u_o)$$

and adding

$$+\sum_{\mu=1}^{\left[\frac{n-1}{2}\right]} \binom{n}{2\mu}\cdot\frac{g(u_o)}{h(u_o)}\cdot h^{(n-2\mu)}(u_o).h^{\mu}(u_o)$$

to the right-hand side, ($[x]$ means an integer with $x-1<[x]\leq x$), (VI,36) or:

$$\beta_{2k} = \Psi_{2k} - g_o.h_o^{k-1}, \qquad \beta_{2k-1} = \Psi_{2k-1} \qquad (k=1,2,3,\ldots),\ \text{resp.}$$

In the same sense one may in (VI,18) understand

$$A^*(u) = v_o.Ch(u-u_o)\sqrt{h_o} + \frac{v'_o}{\sqrt{h_o}}.Sh(u-u_o)\sqrt{h_o} + \iint_{u_0}^{u} g(\alpha)d\alpha^2 \qquad (VI,37)$$

as an approximation. The differential equation for A* is given by

$$A^{*\prime\prime}(u) = h(u_o).A^*(u) + g(u) - h(u_o).\iint_{u_0}^{u} g(\alpha)d\alpha^2 \qquad (VI,38)$$

According to (VI,33) and (VI,34):

$$R^*(u) = \sum_{0}^{\infty} \frac{(u-u_o)^\rho}{\rho!}.\Psi_\rho(u_o), \qquad (VI,39)$$

$\Psi_\rho(u_o)$ from (VI,20) and (VI,21), and:

$$A^*\ (u_o) = v(u_o) \qquad R^*\ (u_o) = 0$$

$$A^{*\prime}(u_o) = v'(u_o) \qquad R^{*\prime}(u_o) = 0 \qquad (VI,40)$$

$$A^{*''}(u_o) = v''(u_o) \qquad R^{*''}(u_o) = 0$$

Splitting up $v = A^* + R^*$ seems to be more convenient than $v = A + \bar{R}$, for which reason its derivation was given, in detail, in (6.111).

(6.114) Another representation of the solution of (VI,1):
The relations (VI,21) do not change if $g(u)$ vanishes identically:

$$\Psi_\rho^{(hom)} = \Psi_\rho^{(inh)} \qquad \text{for } \rho = 0,1,2,3; \tag{VI,41}$$

from (VI,2o) the Ψ_ρ for $\rho = 4,5,6,\ldots$ are well determined also for $g(u) = 0$; therefore, with

$$\delta_\rho(u_o) = \Psi_\rho^{(inh)}(u_o) - \Psi_\rho^{(hom)}(u_o) \tag{VI,42}$$

the following equations hold

$$\begin{aligned} &\delta_\rho(u_o) = 0 \qquad \text{for } \rho = 0,1,2,3;; \\ &\delta_{\rho+2}(u_o) = [h.G]_o^{(\rho)} + \sum_0^\rho \binom{\rho}{\sigma}.h_o^{(\rho-\sigma)}.\delta_\sigma(u_o) = \\ &\qquad = \sum_\sigma^\rho \binom{\rho}{\sigma}.h_o^{(\rho-\sigma)}.\left[\delta_\sigma(u_o) + g_o^{(\sigma-2)}\right] \end{aligned} \tag{VI,43}$$

for $\rho = 2,3,4,\ldots$ The first two terms in (VI,18) and (VI,19) do not depend on $g(u)$ and are therefore part of the solution of the homogeneous equation.

Because of this hold the relations

$$v(u) = v_{hom}(u) + G(u) + \sum_{o,(4)}^{\infty} \frac{(u-u_o)^\rho}{\rho!}.\delta_\rho(u_o) \tag{VI,44}$$

and

$$v'(u) = v'_{hom}(u) + G'(u) + \sum_{o,(3)}^{\infty} \frac{(u-u_o)^\rho}{\rho!}.\delta_{\rho+1}(u_o) \tag{VI,45}$$

with the $\delta_\rho(u_o)$ from (VI,43) and G(u) from (VI,7a).

(6.12) <u>Specialization to Mathieu's Equation:</u>

(6.121) We treat this equation in its most usual forms:

$$y''(x) = (2q.\cos 2x - \lambda).y(x) \qquad \text{(VI,46o)}$$

and

$$Y''(x) = (\lambda - 2q.\mathrm{Ch}\ 2x).Y(x) \qquad \text{(VI,46m)}$$

for the ordinary and the modified Mathieu equations, respectively. One obtains the solutions by specializing (VI,18) to (VI,21) as follows:

(6.122) Ordinary Mathieu equation:

$$\begin{aligned} u &= x \\ v(u) &= y(x;q,\lambda) \\ h(u) &= 2q.\cos 2x - \lambda \\ \Psi_\rho(u_o) &= \varphi_\rho(x_o;q,\lambda). \end{aligned} \qquad \text{(VI,47o)}$$

For the derivatives of h(u) we have:

$$\begin{aligned} h_o^{(2\rho-1)} &= 2q.2^{2\rho-1}.(-1)^\rho.\sin 2x_o = 2q.(-4)^\rho.\frac{\sin 2x_o}{2} \\ h_o^{(2\rho)} &= 2q.2^{2\rho}.(-1)^\rho.\cos 2x_o = 2q.(-4)^\rho.\cos 2x_o \end{aligned} \qquad \text{(VI,48o)}$$

and, from this:

$$\Psi_3(u_o) = v_o.h_o' = \varphi_3(x_o;q,\lambda) = -4q.y(x_o).\sin 2x_o = 2q.(-4).y(x_o).\frac{\sin 2x_o}{2}$$

(VI,49o)

(6.123) Modified Mathieu equation:

$$u = x$$
$$v(u) = Y(x;q,\lambda)$$
$$h(u) = \lambda - 2q.\mathrm{Ch}\ 2x \qquad \text{(VI,47m)}$$
$$\Psi_\rho(u_o) = \phi_\rho(x_o;q,\lambda).$$

Here we have for the derivatives $h_o^{(\rho)}$:

$$h_o^{(2\rho-1)} = (-2q).2^{2\rho-1}.\mathrm{Sh}\ 2x_o = (-2q).4^\rho.\frac{\mathrm{Sh}\ 2x_o}{2}$$
$$\text{(VI,48m)}$$
$$h_o^{(2\rho)} = (-2q).2^{2\rho}.\mathrm{Ch}\ 2x_o = (-2q).4^\rho.\mathrm{Ch}\ 2x_o$$

and

$$\Psi_3(u_o) = v_o.h_o' = \phi_3(x_o;q,\lambda) = -4q.Y(x_o).\mathrm{Sh}\ 2x_o = (-2q).4.Y(x_o).\frac{\mathrm{Sh}\ 2x_o}{2}$$
$$\text{(VI,49m)}$$

(6.124) Additionally we have to mehtion that the solution of (VI,46m) also is to be obtained from (VI,46o) by a substitution $x \to ix$, that is $u = ix$ in (VI,47m). After introducing a parameter δ with

$$\delta = \begin{cases} +1 & \text{for the ordinary Mathieu equation} \\ -1 & \text{for the modified Mathieu equation} \end{cases}$$

we get a simultaneous representation of (VI,47o,m) to (VI,49o,m):

$$h_o = \delta.(2q.\cos 2x_o.\sqrt{\delta} - \lambda) \qquad \text{(VI,47)}$$

$$h_o^{(2\rho-1)} = (2q\delta).(-4\delta)^\rho.\frac{\sin 2x_o.\sqrt{\delta}}{2\sqrt{\delta}}$$
$$\text{(VI,48)}$$
$$h_o^{(2\rho)} = (2q\delta).(-4\delta)^\rho.\cos 2x_o.\sqrt{\delta}$$

$$V_3 = (2q\delta).(-4\delta).v_o.\frac{\sin 2x_o\sqrt{\delta}}{2\sqrt{\delta}} \qquad (VI,49)$$

(6.125) As is well known, the solution of

$$v''(u) = h(u).v(u),$$

where $h(u)$ is analytic at a point $u = u_o$, is also an analytic function at the same point. If $h(u)$ can be expanded into a power series within $|u - u_o| < \rho$ the same statement holds for the solution $v(u)$. In the case of the Mathieu equations, $h(u)$ is an analytic function with $\rho = \infty$ in the whole complex plane. The same holds for the solutions, as well as for the approximation $A(u)$. Consequently, the remaining term $R(u) = v(u) - A(u)$ is also an analytic function with the radius of convergence $\rho = \infty$.

(6.13) <u>The Mathieu functions:</u>

(6.131) For any point of the (λ,q)-plane the general solution of the homogeneous Mathieu equation is given by (VI,18) to (VI,21) with (VI,47) to (VI,49). With

$$v_1(0) = 1 \qquad v_1'(0) = 0$$
$$v_2(0) = 0 \qquad v_2'(0) = 1 \qquad (VI,50)$$

one obtains two independent solutions. It is very convenient to use these conditions in our solution (VI,18) to (VI,21). Any solution of (VI,46) is given by

$$v(u) = A.v_1(u) + B.v_2(u) \qquad (VI,51)$$

with

$$A = v(0), \qquad B = v'(0), \qquad (VI,52)$$

v(o) and v'(o) are arbitrary constants. For the fundamental system (VI,5o), we have (see, eg.: Ref. 19: 2.12):

$$v_1(u).v_2'(u) - v_1'(u).v_2(u) = 1$$

$$v_1(-u) = v_1(u) \qquad \text{(VI,53)}$$

$$v_2(-u) = -v_2(u).$$

(6.132) Mathieu functions of nonintegral order:
In the following, we denote the different kinds of Mathieu functions by the symbols used by Whittaker, Watson, McLachlan, Meixner, Schaefke, etc: ce, se, fe, ge and Ce, Se, Fe, Ge. For any given values of λ, q we can compute the characteristic exponent $\mu = \mu(\lambda,q)$ (Ref. 19: 2.13) and obtain two independent solutions $ce_\mu(x;\lambda,q)$ and $se_\mu(x;,\lambda,q)$ resp. $Ce_\mu(x;\lambda,q)$ and $Se_\mu(x;,\lambda,q)$ with:

$$ce_\mu(-x) = ce_\mu(x) \qquad Ce_\mu(-x) = Ce_\mu(x)$$

$$se_\mu(-x) = -se_\mu(x) \qquad Se_\mu(-x) = -Se_\mu(x) \qquad \text{(VI,54)}$$

From (VI,52) and (VI,53) follows then:

$$ce_\mu(x;\lambda,q) = ce_\mu(0;\lambda,q).y_1(x;\lambda,q)$$

$$se_\mu(x;\lambda,q) = se_\mu'(0;\lambda,q).y_2(x;\lambda,q)$$

$$Ce_\mu(x;\lambda,q) = Ce_\mu(0;\lambda,q).Y_1(x;\lambda,q) \qquad \text{(VI,55)}$$

$$Se_\mu(x;\lambda,q) = Se_\mu'(0;\lambda,q).Y_2(x;\lambda,q)$$

for the same λ and q.

(6.133) Mathieu functions of integral order: They are the solutions if (λ,q) lies on the limiting curves between stable and unstable regions.

$$\begin{aligned} &ce_m(x;q): \quad \lambda = a_m(q) \quad (m=0,1,2,\ldots) \\ &se_m(x;q): \quad \lambda = b_m(q) \quad (m=1,2,3,\ldots) \end{aligned} \qquad (VI,56)$$

They are Floquet solutions with the period 2π (or π):

$$\begin{aligned} ce_m(x+\pi) &= (-1)^m \cdot ce_m(x) \\ se_m(x+\pi) &= (-1)^m \cdot se_m(x) \\ ce_m(x+\tfrac{\pi}{2}) &= (-1)^m \cdot ce_m(\tfrac{\pi}{2}-x) \\ se_{m+1}(x+\tfrac{\pi}{2}) &= (-1)^m \cdot se_{m+1}(\tfrac{\pi}{2}-x) \end{aligned} \qquad (VI,57)$$

It is necessary to introduce new non-periodic functions to get an independent second solution:

$$\begin{aligned} &\lambda = a_m(q): \quad fe_m(x;q) \text{ as second solution to } ce_m(x;q) \\ &\lambda = b_m(q): \quad ge_m(x;q) \text{ as second solution to } se_m(x;q) \end{aligned} \qquad (VI,58)$$

resp. $Fe_m(x;q)$ to Ce_m and $Ge_m(x;q)$ to $Se_m(x;q)$. Ce, ce, Ge, ge are even, Se, se, Fe, fe are odd functions. Then it follows from (VI,52):

$$\begin{aligned} ce_n(x;\lambda=a_n,q) &= ce_n(0;q) \cdot y_1(x;\lambda=a_n,q) \\ ge_n(x;\lambda=b_n,q) &= ge_n(0;q) \cdot y_1(x;\lambda=b_n,q) \\ se_n(x;\lambda=b_n,q) &= se_n'(0;q) \cdot y_2(x;\lambda=b_n,q) \\ fe_n(x;\lambda=a_n,q) &= fe_n'(0;q) \cdot y_2(x;\lambda=a_n,q) \end{aligned} \qquad (VI,59)$$

The same holds for Ce_n, Se_n, Fe_n, Ge_n.

Some other necessary results are:

(1) $\lambda = a_n(q)$:

(a) $n = 2k$:

$$ce'_{2k}(0;q) = 0 = Ce'_{2k}(0;q)$$

$$ce'_{2k}(\tfrac{\pi}{2};q) = 0 = Ce'_{2k}(\pm i\tfrac{\pi}{2};q)$$

$$fe_{2k}(0;q) = 0 = Fe_{2k}(0;q) \qquad \text{(VI,60 i)}$$

$$y_1(\pi;q) = y'_2(\pi;q) = +1$$

$$y'_1(\pi) = 0$$

(b) $n = 2k + 1$:

$$ce'_{2k+1}(0;q) = 0 = Ce'_{2k+1}(0;q)$$

$$ce_{2k+1}(\tfrac{\pi}{2};q) = 0 = Ce_{2k+1}(i\tfrac{\pi}{2};q)$$

$$fe_{2k+1}(0;q) = 0 = Fe_{2k+1}(0;q) \qquad \text{(VI,60 ii)}$$

$$y_1(\pi) = y'_2(\pi) = -1$$

$$y'_1(\pi) = 0$$

(2) $\lambda = b_n(q)$:

(a) $n = 2k + 1$:

$$se_{2n+1}(0;q) = 0 = Se_{2n+1}(0;q)$$

$$se'_{2n+1}(\tfrac{\pi}{2};q) = 0 = Se'_{2n+1}(i\tfrac{\pi}{2};q)$$

$$ge'_{2n+1}(0;q) = 0 = Ge'_{2n+1}(0;q) \qquad \text{(VI,60 iii)}$$

$$y_1(\pi) = y'_2(\pi) = -1$$

$$y_2(\pi) = 0$$

(b) $n = 2k + 2$:

$$se_{2n+2}(0;q) = 0 = Se_{2n+2}(0;q)$$

$$se_{2n+2}(\frac{\pi}{2};q) = 0 = Se_{2n+2}(i\frac{\pi}{2};q)$$

$$ge'_{2n+2}(0;q) = 0 = Ge'_{2n+2}(0;q) \qquad \text{(VI,60 iv)}$$

$$y_1(\pi) = y'_2(\pi) = +1$$

$$y_2(\pi) = 0$$

According to (VI,57), it is sufficient to know ce_n, se_n, Ce_n and Se_n within $(0,\frac{\pi}{2})$.

The functions ce_n and se_n are normalized as follows (Ref. 19: 2.71, (4)):

$$\int_0^{2\pi} ce_n^2(x;q)dx = \int_0^{2\pi} se_n^2(x;q)dx = \pi, \qquad \text{(VI, 61)}$$

which yields:

$$\left[ce_n(0;q)\right]^2 = \frac{\pi}{\int_0^{2\pi} y_1^2(x)dx}$$

$$\left[se'_n(0;q)\right]^2 = \frac{\pi}{\int_0^{2\pi} y_2^2(x)dx} \; . \qquad \text{(VI, 62)}$$

(6.2) <u>Discussion of the Computation Process; Flow Diagrams</u>

(Floriani Dietmar)

(6.21) <u>General remarks:</u>

(6.211) All together, the following nine programs were written in the course of the investigations concerning the applicability of Lie series formalism to the Mathieu differential equation:

4 programs for computation of Mathieu functions by means of Lie series (AA, AB, AC, AD)

1 program for computation by means of Fourier series (CA)

1 program for a combination of Fourier and power series with respect to q (DA) (see, e. g.: Ref. 22: 2.13,(2)-(15))

3 programs to calculate the eigenvalue λ of the Mathieu differential equation. (BA, BB, BC)

In the following, let us discuss AC, AD and BC in detail. Remarks on the remaining programs are to be found in (6.4).

The purpose of the work was to master completely the computation of the Mathieu functions. The solution given by equ. (VI,18) to (VI,21) is, above all, suitable for computing the fundamental system (VI,50). If one masters the computation of the eigenvalue λ the functions ce, se, fe, ge and Ce, Se, Fe, Ge can be computed from (VI,59) except for the normalizing factor (VI,62). The main purpose of this work was to examine the usefulness of Lie series to calculate Mathieu functions of the first kind and of integral order. From (VI,57) follows, that it is sufficient to know this functions within $(0,\frac{\pi}{2})$. This result was made use of with coding the equations (VI,18) to (VI,21). Because of the Floquet theorem

$$v(u+\pi) = \sigma . v(u) \qquad \text{(VI,63)}$$

the restriction to $(0,\frac{\pi}{2})$ is also valid for any Floquet solution satisfying (VI,53):

$$v_1(\pi-u) = \sigma . v_1(u)$$
$$v_2(\pi-u) = -\sigma . v_2(u)$$

or, with $u \rightarrow \frac{\pi}{2}-u$:

$$\begin{aligned} v_1(\tfrac{\pi}{2}+u) &= \sigma . v_1(\tfrac{\pi}{2}-u) \\ v_2(\tfrac{\pi}{2}+u) &= -\sigma . v_2(\tfrac{\pi}{2}-u) \end{aligned} \qquad \text{(VI,64)}$$

(6.212) While the programs AA to AC only served to compute the ordinary Mathieu functions and expansion was only performed in the neighborhood of $x = 0$, in AD the equations (VI,20+21) were programmed in complete generality using (VI,47,48,49). The recurrence formulas for the Ψ_ρ's then read as follows:

$$\Psi_{2k+2}(u_o) = (2q\delta).(-4\delta)^k. \sum_{o}^{k-1} (-4\delta)^{-\rho}.\gamma_{2k,\rho} + h_o.\Psi_{2k}$$

$$\Psi_{2k+3}(u_o) = (2q\delta).(-4\delta)^k. \sum_{o}^{k-1} (-4\delta)^{-\rho}.\gamma_{2k+1,\rho} + \qquad \text{(VI,65)}$$

$$+ (2q\delta).(-4\delta).(2k+1).A.\alpha_k + h_o.\Psi_{2k+1}$$

with

$$\gamma_{2k,\rho} = \left\{\binom{2k}{2\rho}.B.\alpha_\rho + \binom{2k}{2\rho+1}.A.\beta_\rho\right\}$$

$$\gamma_{2k+1,\rho} = \left\{(-4\delta).\binom{2k+1}{2\rho}.A.\alpha_\rho + \binom{2k+1}{2\rho+1}.B.\beta_\rho\right\} \qquad \text{(VI,66)}$$

and

$$A = \frac{\sin 2x_o\sqrt{\delta}}{2\sqrt{\delta}} \qquad B = \cos 2x_o\sqrt{\delta}$$

$$\alpha_\rho = (\Psi_{2\rho}+v_o.h_o^\rho) \qquad \beta_\rho = (\Psi_{2\rho+1}+v_o'.h_o^\rho) \qquad \text{(VI,67)}$$

$$\Psi_3 = v_o.h_o' = v_o.(2q\delta).(-4\delta).A \qquad \text{(VI,68)}$$

In contrast to AA, AB, AC, we expanded also in the neighborhood of the points $x = \pm 45^\circ$. While dealing with the individual programs we found it desirable to be able to influence the course of the program during the computation process, in a still more pronounced way. This intention was taken into account in AD to a rather great extent.

(6.213) Short description of AC and BC:

With AC, only expansion round x=0 is carried out. In this case, the equations (VI,20) simplify to:

$$\Psi_{2k+2} = (2q).(-4)^k. \sum_{0}^{k-1} (-4)^{-\rho}.\binom{2k}{2\rho}.(\Psi_{2\rho}+v_o.h_o^{\rho}) + $$
$$+ (2q-\lambda).\Psi_{2k}$$

(VI,65a)

$$\Psi_{2k+3} = (2q).(-4)^k. \sum_{0}^{k-1} (-4)^{-\rho}.\binom{2k+1}{2\rho+1}.(\Psi_{2\rho+1}+v_o'.h_o^{\rho}) +$$
$$+ (2q-\lambda).\Psi_{2k+1}$$

Furthermore we have:

$$\Psi_3 = 0. \qquad \text{(VI,68a)}$$

(VI,65a) is essentially faster to calculate than (VI,65). For $v_o = 0$ all $\Psi_{2k+2} = 0$, for $v_o' = 0$ all $\Psi_{2k+3} = 0$. Corresponding inquiries were built in in AC, a fact by which the computer time becomes less than half of the time needed for (VI,65) in AD.

(6.214) By the appropriate choice of the conditional switch a more or less large number of provisional results can be put out, e.g.: the $\Psi_\rho(u_o)$, each $(u-u_o)^\rho.\Psi_\rho/\rho!$ or only the last two $\Psi_\rho(u_o)$, $(u-u_o)^\rho.\Psi_\rho/\rho!$, A(u), R(u), A'(u), R'(u), the most little terms in the sums for R(u); etc. Usually theyare useful items of information to estimate the accuracy of the final results v(u), v'(u).

(6.215) BC: The machine jumps to BC with "GO TO: (EIGWE)". Unfortunately, our mastering of the eigenvalue calculation is not sufficient.

By lack of time we could not deal with this problem, any more. At first, we make a preliminary and approximate calculation of λ by means of power series with respect to q. (See, e.g., Ref.19: 2.25). In Ref.23 besides radii of convergence the following equation is given for the error estimates:

$$\left|\sum_{k=m+1}^{\infty} \lambda_{nk}\cdot q^k\right| < \left(\frac{|q|}{\rho_n}\right)^{m+1}\cdot\left(1-\frac{|q|^2}{\rho_n^2}\right)^{-1/2}\cdot\left(8\rho_n^2-\sum_{k=1}^{m}\left|\lambda_{nk}\right|^2\cdot\rho_n^{2k}\right)^{1/2} \qquad (VI,69)$$

For this equation a program was written. The extension of BC aimed to the following purpose: If the error limit put-in is exceeded, the computation of the eigenvalue is to be continued by a continued fraction. We consider that as transgressing our problem, by far, and, therefore, we did not deal with this problem, any longer.

(6.22) Short Description of the Computer we used:

The ZUSE Z 23-V digital computer of the computer center of Innsbruck University was available to our investigations. The machine possesses 250 quick access storage registers and 8200 registers on a magnetic drum. Several difficulties in coding were due to the rather limited range of the quick access storage.

As regards the representation of numbers, we distinguish, as it is usual, between floating point numbers (= GKZ, symbol: x, = REAL in ALGOL) and fixed point numbers (= FKZ, symbol: x', = INTEGER in ALGOL). The FKZ representation is only possible up to 10^{12}, GKZ representation between 10^{-39} and 10^{39}. Output is performed by means of a teletyper:

n ZVB (n line feeds)

n ZWI (n intervals)

n SPA (n column jumps)

PRINT (output of data)

or by means of a quick puncher for tapes:

n ZVB2, n ZWI2,

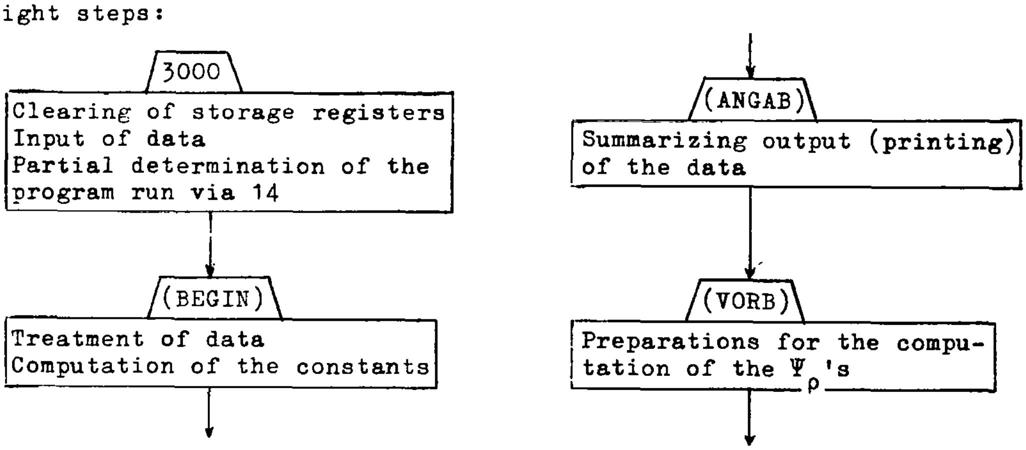
n SPA2, PRINT2.

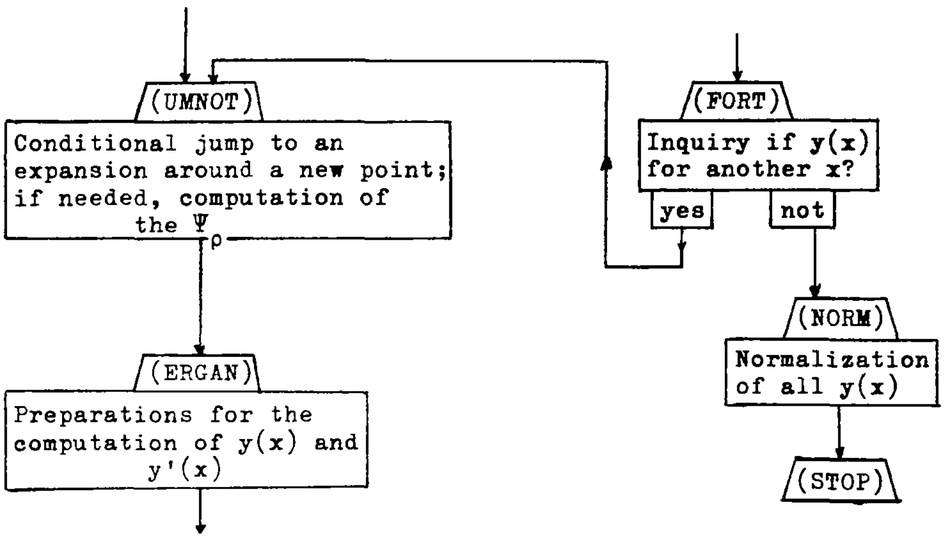
In what follows, the accumulator is denoted by "A". For arithmetic operations in GKZ-representation an auxiliary storage with address 6 (in the following: "S") is needed. n means the address of the storage ($n \leq 255$: quick access storage, $n > 255$; drum), (n) its contents.

Furthermore, a conditional switch with the address 14 is available. It is adjustable between 0' and 31'.

(6.23) Program MNVM-AD for Computing the Ordinary and Modified Mathieu Functions by Means of Lie series:

(6.231) This program has been stored from 3ooo and essentially consists of eight steps:





(6.232) The data which are to be put in were classed in three groups:

a) Data which remain equal in a rather great number of calculations: x_A, δx, x_E.

b) Data varying comparatively often: q, λ.

c) Data which always have to be put in again: v(u=0), v'(u=0), the number of terms N, the parameter δ (ordinary or modified Mathieu functions), printing program.

Accordingly, the data (a) and (b) are taken over into two different registers for the calculation and are always available for new calculations again.

(6.233) Storage plan:

(a) x_A = (252)

δx = (253)

x_E = (254) ($\gtrless x_A$)

Input may be carried out independent of each other as FKZ (degrees) or GKZ (radians)

(b) λ = (250)

q = (251)

(c)

(14)	(237)	(238)	v_o=(248)	v_o'=(249)	(255)
0'	0	0	are read in		0
1'	+1,-	+1,-	+1,-	0	0
2'	+1,-	-1,-	0	+1,-	0
3'	+1,-	0	0	+1,-	+1,-

δ = (239) = +1,-: ordinary Mathieu functions

= -1,-: modified Mathieu functions

(236) = 0: quick printing: only x_w, x_w^o, $y(x_w)$, $y'(x_w)$

(236) = +1: partial printing: some intermediate results additionally

(236) = -1: complete printing: all interesting intermediate results are printed.

(255) = 0: printing, as explained above

(255) = +1: for the purpose of comparison with a less customary form of the Mathieu equations

$$\frac{d^2}{dt^2}w(t) + e.(1+k.\cos t).w(t) = 0$$

$2.y(x_o)$ will be put out, additionally.

For a simplification of the calculation we have additionally put in:

(233) = 180,-, (234) = +1,-, (235) = 10^{32}.

The binomial coefficients are computed by addition in each last line of the Pascal triangle; the Ψ_ρ's are stored from 4000 onwards and transferred into the quick access storage by block transfer.

$$\binom{2n}{\rho} = \binom{2n-1}{\rho-1} + \binom{2n-1}{\rho} = (50+\rho) \qquad \Psi_\rho = (150+\rho)$$

$$\binom{2n+1}{\rho} = \binom{2n}{\rho-1} + \binom{2n}{\rho} = (150+\rho) \qquad \Psi_\rho = (50+\rho)$$

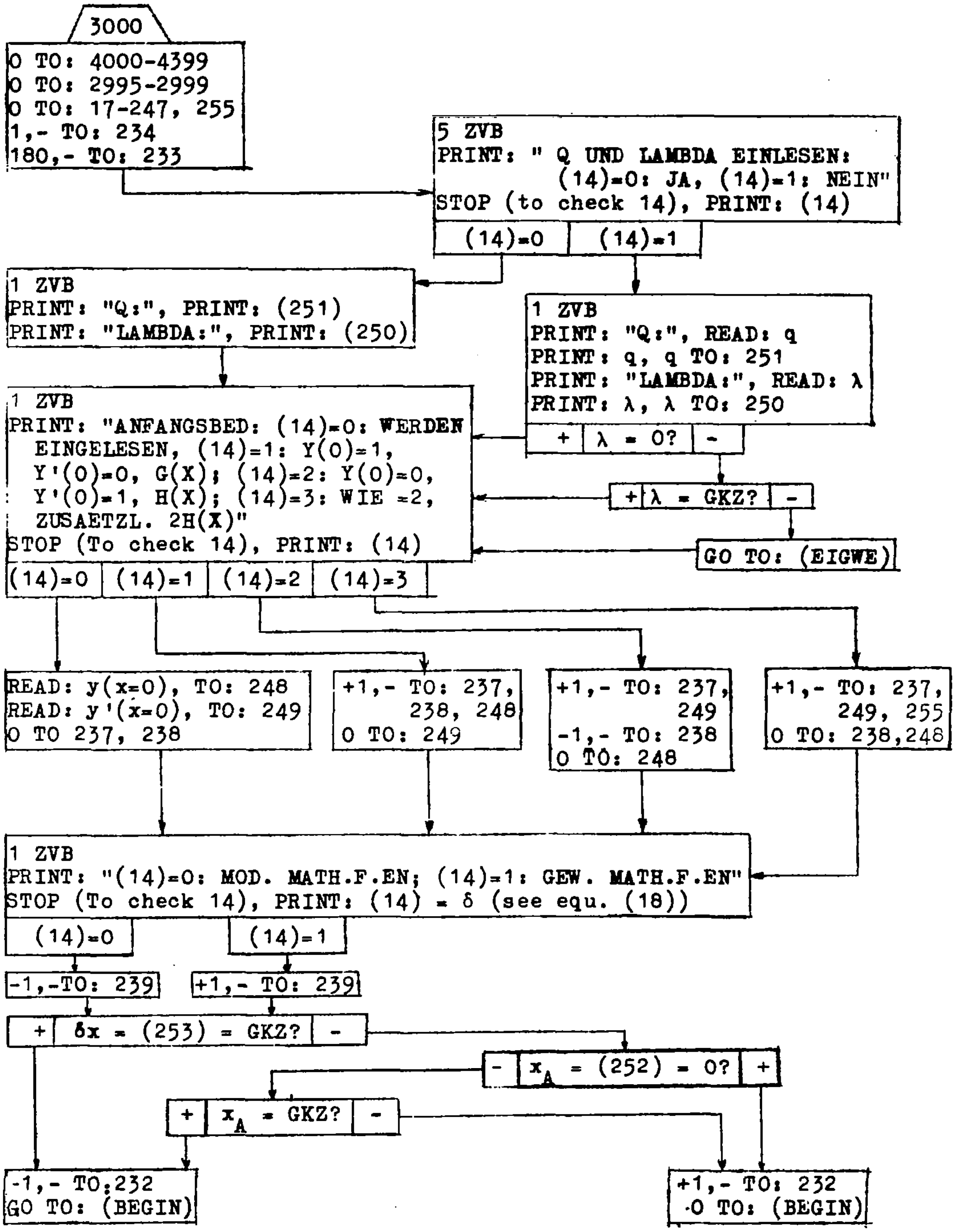
$$\Psi_\rho = (4000+\rho)$$

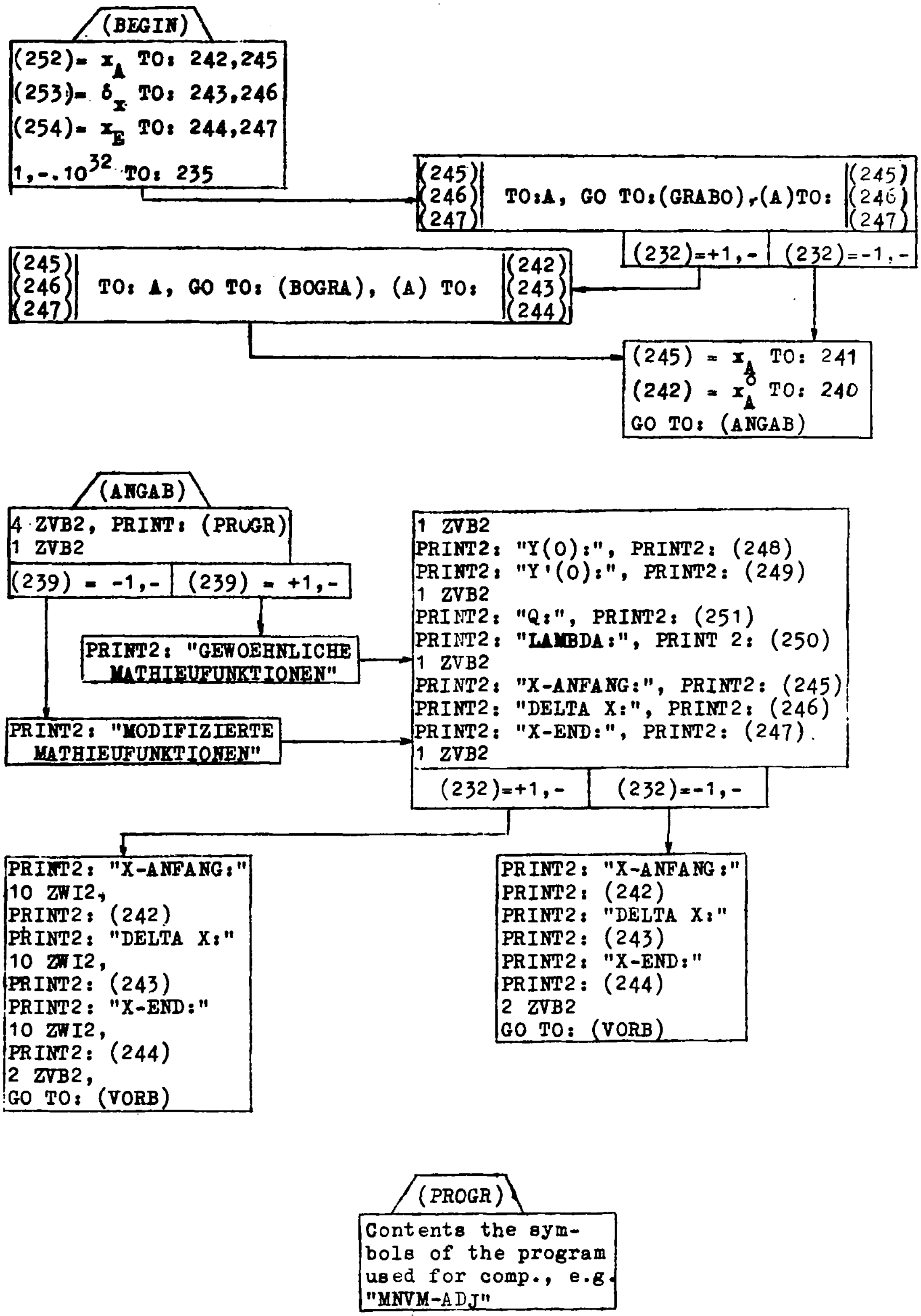
Additionally, we store for the purpose of normalizations to be undertaken later on:

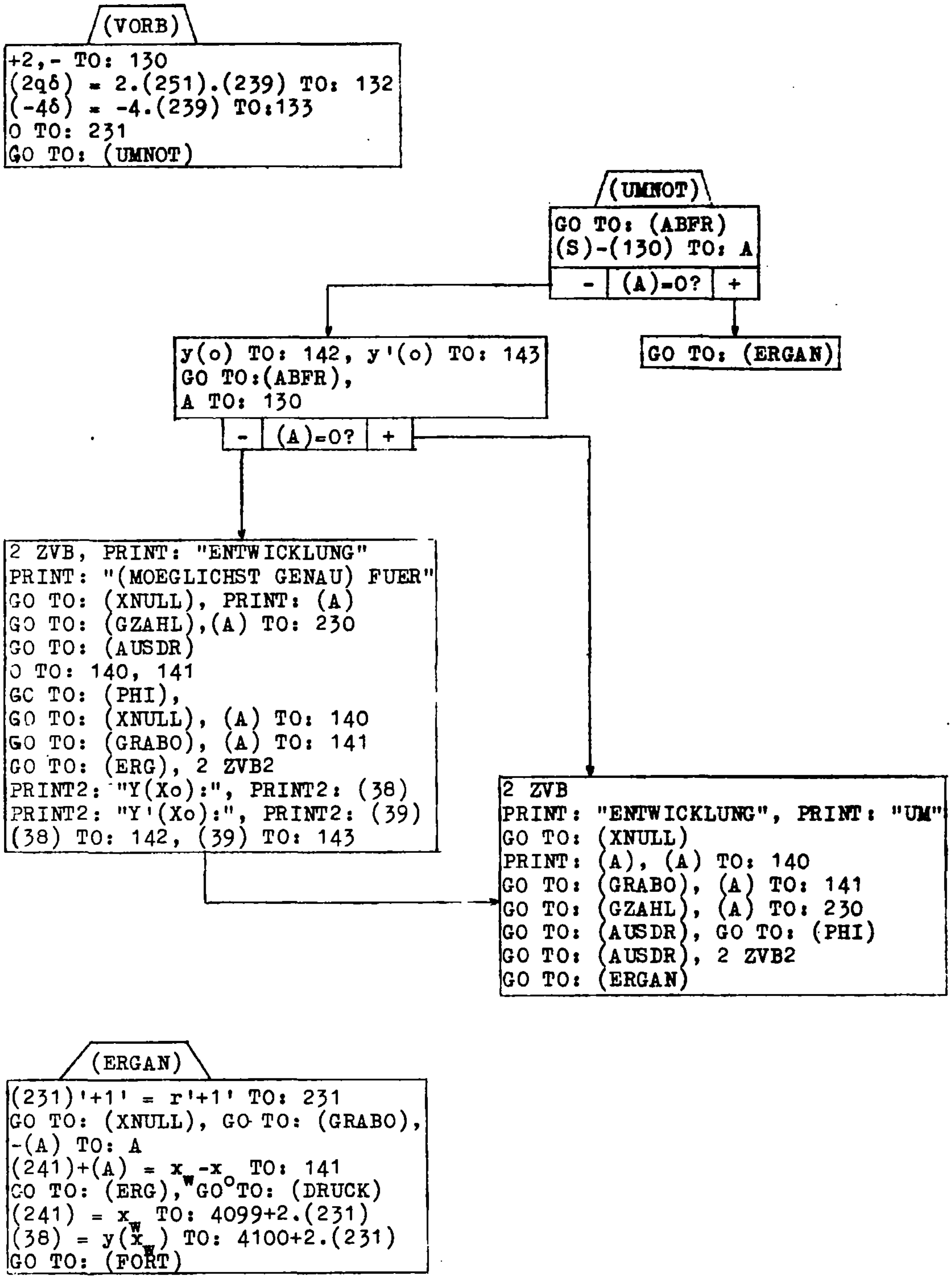
$$x_r = (4099+2r),$$
$$y(x_r) = (4100+2r).$$

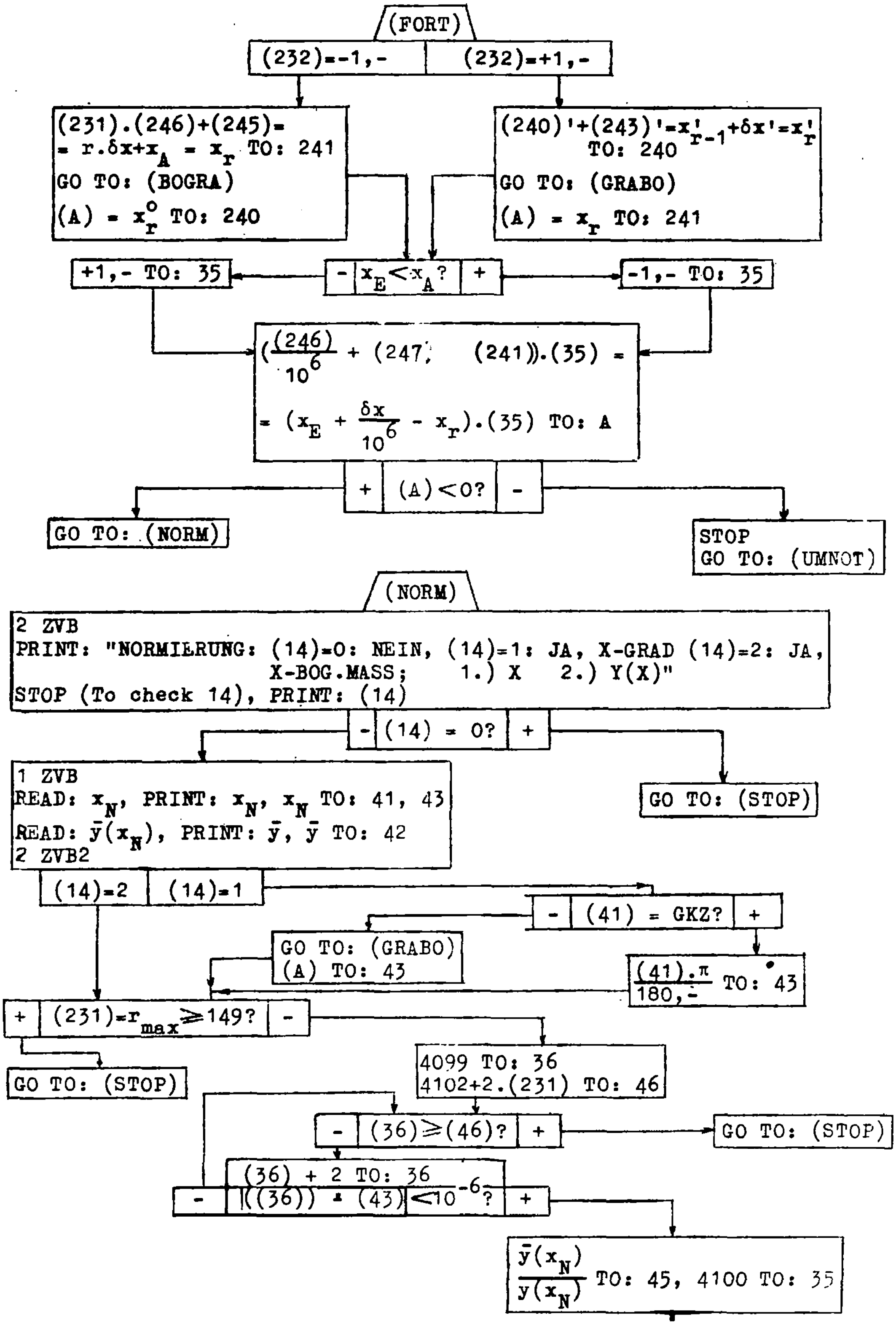
(6.234) **Normalization**: avoid the laborious calculation of the normalization factor from (VI,62), up to now we have always calculated the corresponding solution of the fundamental system (VI,50) for (q,λ) (e.g., like Morse-Feschbach, Ref. 21) and afterwards normalized by putting in a given value $\bar{y}(x_N;q,\lambda)$.

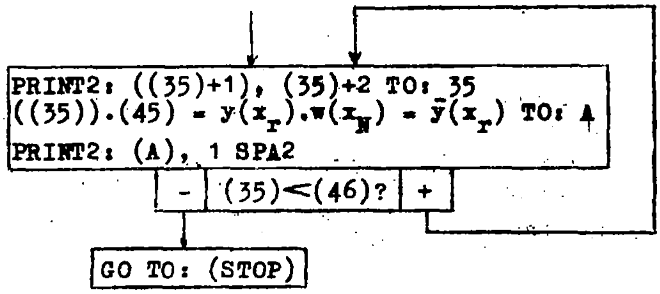
Flow-Diagram to MNVM - AD

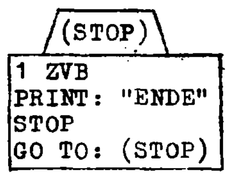



(BEGIN)
(252)= x_A TO: 242,245
(253)= δ_x TO: 243,246
(254)= x_E TO: 244,247
1,-.10^{32} TO: 235
(245) (246) (247) TO:A, GO TO:(GRABO),(A)TO: (245) (246) (247)
(232)=+1,-
(232)=-1,-
(245) (246) (247) TO: A, GO TO: (BOGRA), (A) TO: (242) (243) (244)
(245) = x_A TO: 241
(242) = x_A^δ TO: 240
GO TO: (ANGAB)
(ANGAB)
4 ZVB2, PRINT: (PROGR)
1 ZVB2
(239) = -1,-
(239) = +1,-
PRINT2: "GEWOEHNLICHE MATHIEUFUNKTIONEN"
PRINT2: "MODIFIZIERTE MATHIEUFUNKTIONEN"
1 ZVB2
PRINT2: "Y(0):", PRINT2: (248)
PRINT2: "Y'(0):", PRINT2: (249)
1 ZVB2
PRINT2: "Q:", PRINT2: (251)
PRINT2: "LAMBDA:", PRINT 2: (250)
1 ZVB2
PRINT2: "X-ANFANG:", PRINT2: (245)
PRINT2: "DELTA X:", PRINT2: (246)
PRINT2: "X-END:", PRINT2: (247)
1 ZVB2
(232)=+1,-
(232)=-1,-
PRINT2: "X-ANFANG:"
10 ZWI2,
PRINT2: (242)
PRINT2: "DELTA X:"
10 ZWI2,
PRINT2: (243)
PRINT2: "X-END:"
10 ZWI2,
PRINT2: (244)
2 ZVB2,
GO TO: (VORB)
PRINT2: "X-ANFANG:"
PRINT2: (242)
PRINT2: "DELTA X:"
PRINT2: (243)
PRINT2: "X-END:"
PRINT2: (244)
2 ZVB2
GO TO: (VORB)
(PROGR)
Contents the symbols of the program used for comp., e.g. "MNVM-ADJ"


(VORB)
+2,- TO: 130
(2qδ) = 2.(251).(239) TO: 132
(-4δ) = -4.(239) TO:133
0 TO: 231
GO TO: (UMNOT)
(UMNOT)
GO TO: (ABFR)
(S)-(130) TO: A
-
(A)=0?
+
GO TO: (ERGAN)
y(o) TO: 142, y'(o) TO: 143
GO TO:(ABFR),
A TO: 130
-
(A)=0?
+
2 ZVB, PRINT: "ENTWICKLUNG"
PRINT: "(MOEGLICHST GENAU) FUER"
GO TO: (XNULL), PRINT: (A)
GO TO: (GZAHL),(A) TO: 230
GO TO: (AUSDR)
0 TO: 140, 141
GO TO: (PHI),
GO TO: (XNULL), (A) TO: 140
GO TO: (GRABO), (A) TO: 141
GO TO: (ERG), 2 ZVB2
PRINT2: "Y(Xo):", PRINT2: (38)
PRINT2: "Y'(Xo):", PRINT2: (39)
(38) TO: 142, (39) TO: 143
2 ZVB
PRINT: "ENTWICKLUNG", PRINT: "UM"
GO TO: (XNULL)
PRINT: (A), (A) TO: 140
GO TO: (GRABO), (A) TO: 141
GO TO: (GZAHL), (A) TO: 230
GO TO: (AUSDR), GO TO: (PHI)
GO TO: (AUSDR), 2 ZVB2
GO TO: (ERGAN)
(ERGAN)
(231)'+1' = r'+1' TO: 231
GO TO: (XNULL), GO TO: (GRABO),
-(A) TO: A
(241)+(A) = x_w-x_o TO: 141
GO TO: (ERG), GO TO: (DRUCK)
(241) = x_w TO: 4099+2.(231)
(38) = y(x_w) TO: 4100+2.(231)
GO TO: (FORT)


(FORT)
(232)=-1,-
(232)=+1,-
(231).(246)+(245)=
= r.δx+x_A = x_r TO: 241
GO TO: (BOGRA)
(A) = x_r^o TO: 240
(240)'+(243)'=x'_{r-1}+δx'=x'_r
TO: 240
GO TO: (GRABO)
(A) = x_r TO: 241
+1,- TO: 35
- x_E < x_A ? +
-1,- TO: 35
((246)/10^6 + (247) (241)).(35) =
= (x_E + δx/10^6 - x_r).(35) TO: A
+ (A)<0? -
GO TO: (NORM)
STOP
GO TO: (UMNOT)
(NORM)
2 ZVB
PRINT: "NORMIERUNG: (14)=0: NEIN, (14)=1: JA, X-GRAD (14)=2: JA,
X-BOG.MASS; 1.) X 2.) Y(X)"
STOP (To check 14), PRINT: (14)
- (14) = 0? +
1 ZVB
READ: x_N, PRINT: x_N, x_N TO: 41, 43
READ: ȳ(x_N), PRINT: ȳ, ȳ TO: 42
2 ZVB2
GO TO: (STOP)
(14)=2
(14)=1
- (41) = GKZ? +
GO TO: (GRABO)
(A) TO: 43
(41).π/180,- TO: 43
+ (231)=r_max≥149? -
GO TO: (STOP)
4099 TO: 36
4102+2.(231) TO: 46
- (36)≥(46)? +
GO TO: (STOP)
(36) + 2 TO: 36
- |((36)) - (43)|<10^-6? +
ȳ(x_N)/y(x_N) TO: 45, 4100 TO: 35





Functions of the subprograms used for **MNVM-AD:**

(EIGWE): Berechnung des EIGenWErtes (Calculation of the eigenvalue)

(AUSDR): Wahl des AUSDRuckprogrammes (Choice of print program)

(GRABO): Verwandlung von GRAdmaß in BOgenmaß (Change of degrees to radians)

(BOGRA): Verwandlung von BOgenmaß in GRAdmaß (Change of radians to degrees)

(ABFR): ABFRage, ob Umentwicklung (Inquiry whether expansion around a new point is needed)

(XNULL): Berechnung des Punktes x_o, um den zu entwickeln ist (Calculation of the point x_o around which expansion has to be carried out)

(KONST): Berechnung von KONSTanten (zur Phi-Berechnung) (Calculation of constants (for calculating phi))

(GZAHL): Eingabe der ZAHL der verwendeten Glieder (Input of the number of terms used)

(GERAD): Abfrage, ob (A)' = 2k' (Inquiry if (A)' = 2k')

(KLAMI): Berechnung der Inneren KLAMmern (Calculation of the internal brackets)

(KLAMA): Berechnung der Aeußeren KLAMmern (Calculation of the external brackets)

(PHISU): Fuer ein PHI AufSUmmierung der Rekursionsformel (Summation of the recurrence formulas for one phi)

(BINOM): Berechnung der BINOMialkoeffizienten (Calculation of the binomial coefficients)

(XUFAK): Berechnung einer X-Potenz Und einer FAKultaet (Calculation of an x-power and of the corresponding factorials)

(MINMA): Bestimmung des MINimalen und des MAximalen Gliedes (Determination of the minimum and maximum term)

(AXFAK): Berechnung Aller X-Potenzen und aller FAKultaeten (Calculation of all x-powers and all factorials)

(SUMME): Berechnung der SUMME aller Glieder $(x-x_o)^Q \cdot Y_Q / Q!$ (Calculation of the sum of all these terms)

(UMSP): UMSPeichern (Storing of the intermediate sums)

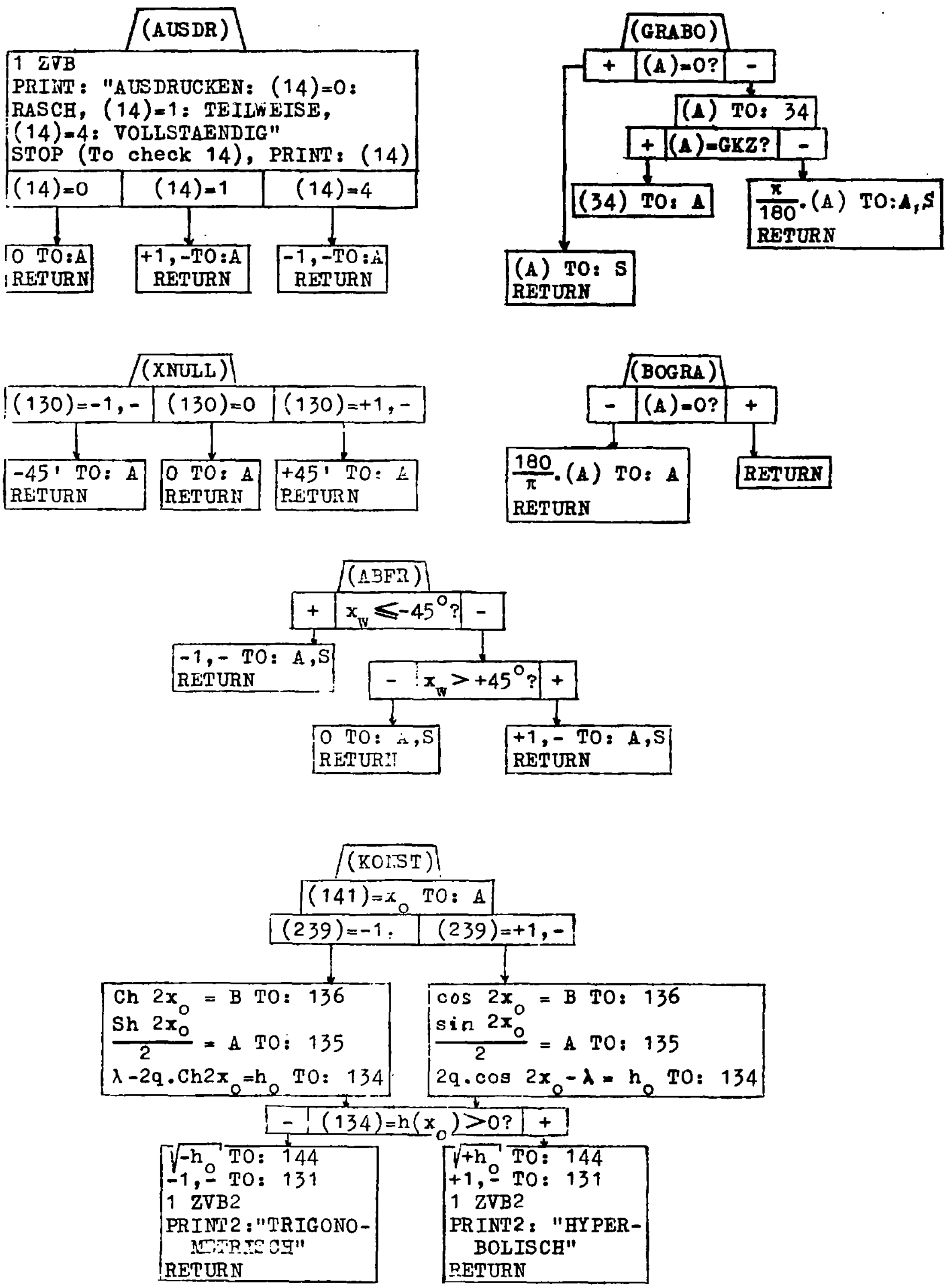
(NAEH): Berechnung der NAEHerungsloesung (= ohne Korrekturglieder) (Calculation of the approximate solution (without correction terms))

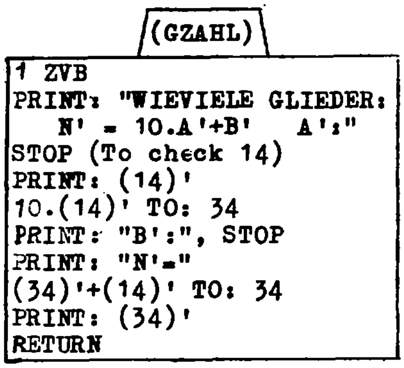
(KORR): Berechnung der KORRekturglieder (Calculation of correction terms)

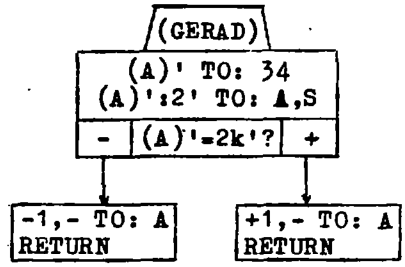
(ERG): Berechnung der ERGebnisse: y(x) und y'(x) (Calculation of results)

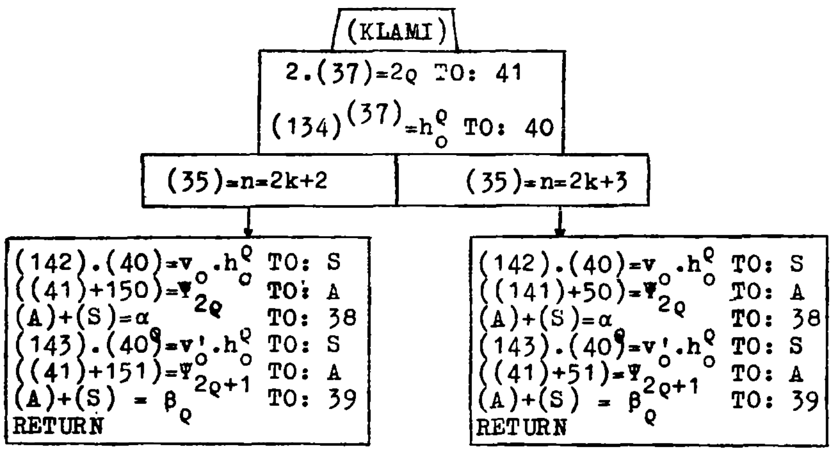
(DRUCK): DRUCKen der Ergebnisse entsprechend (AUSDR) (Printing of results according to (AUSDR))

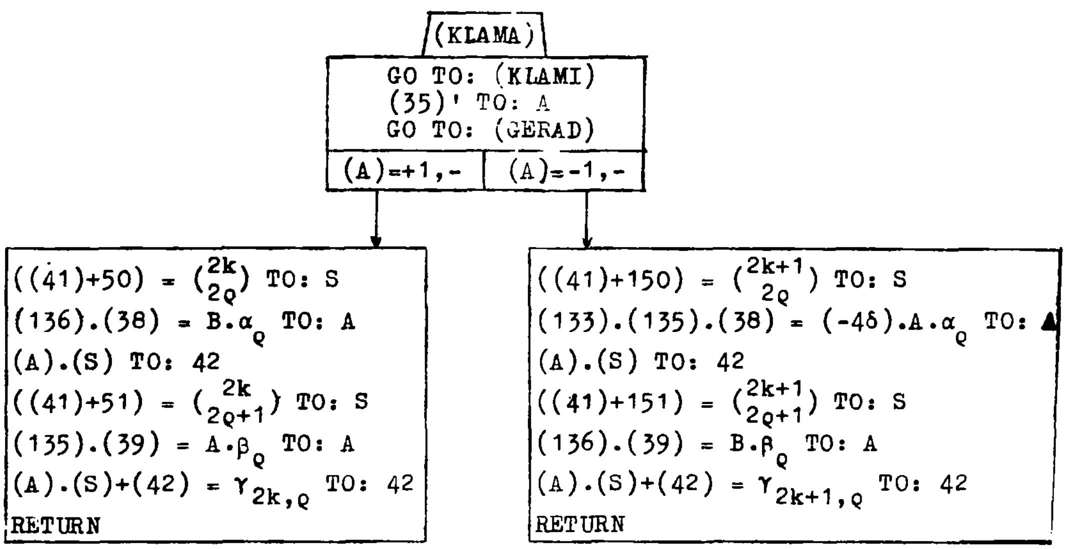
Flow-diagrams of the Subprograms to MNVM-AD:

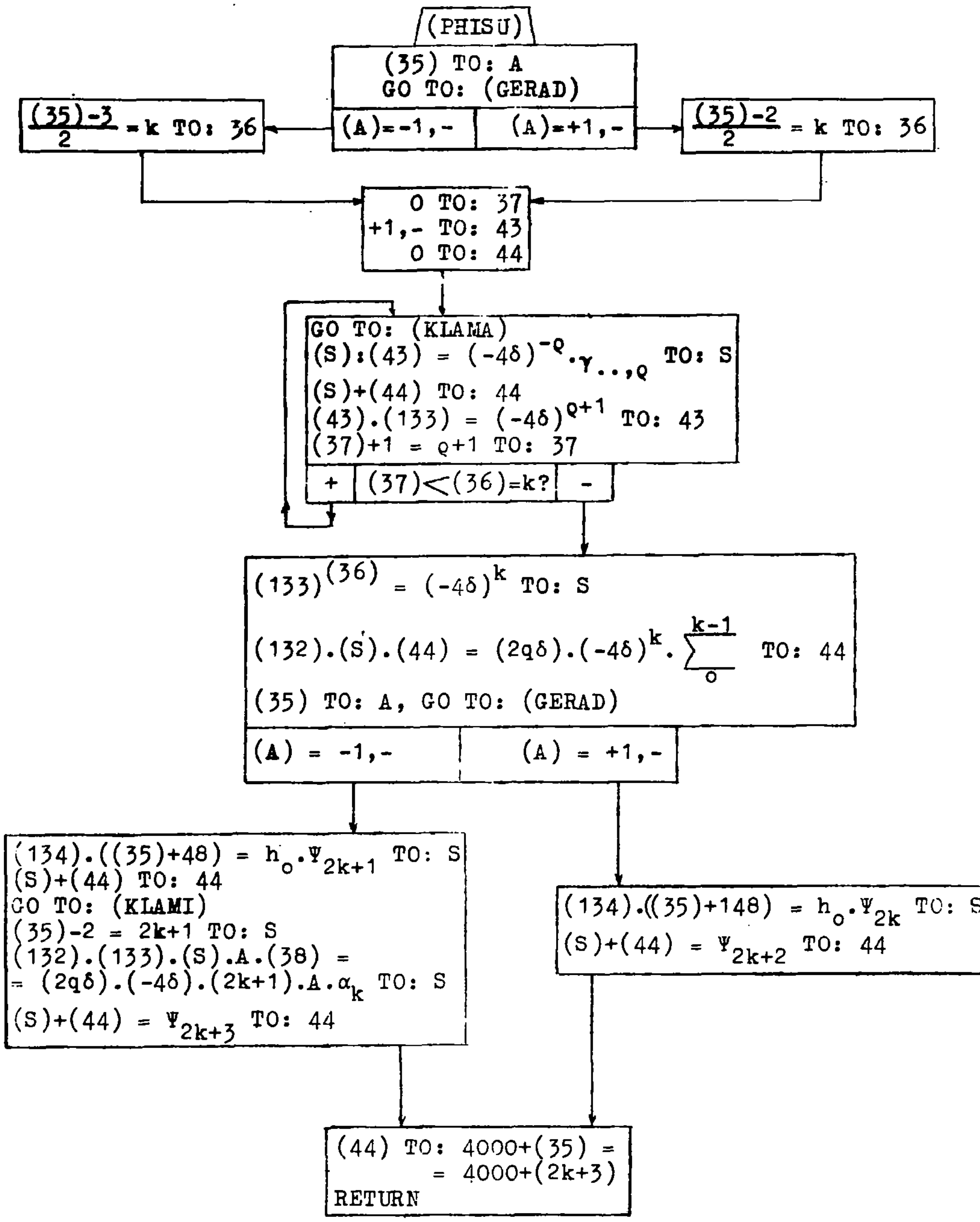


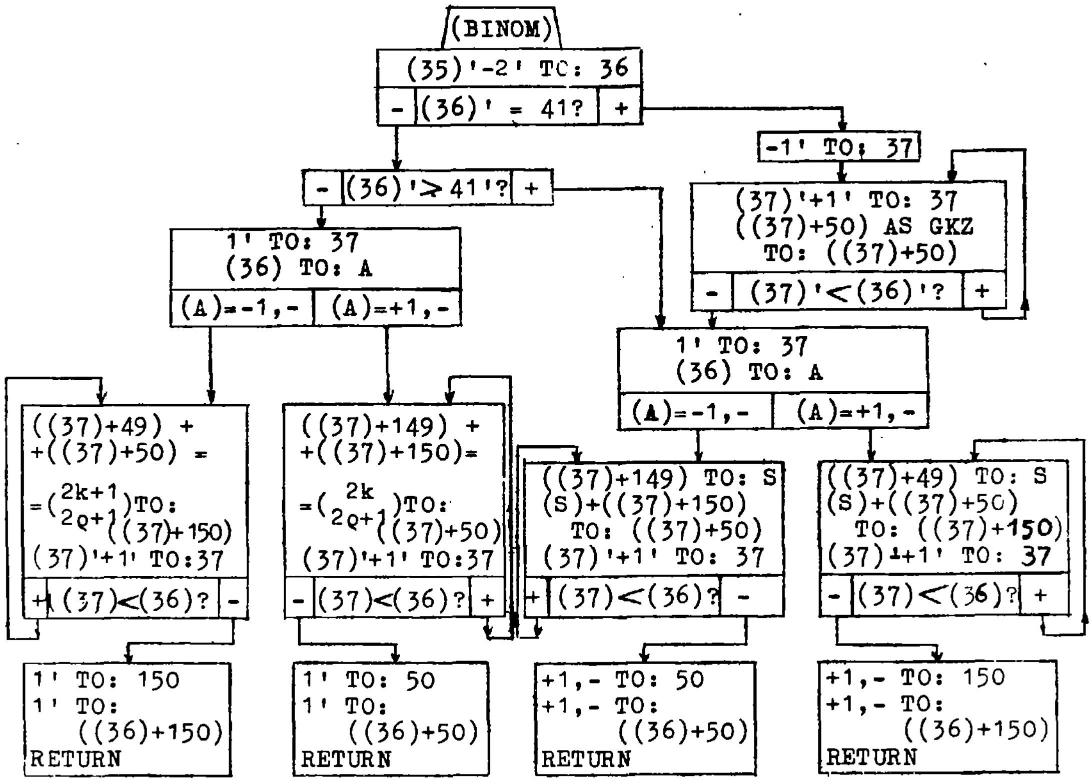

(GZAHL)
1 ZVB
PRINT: "WIEVIELE GLIEDER:
N' = 10.A'+B' A':"
STOP (To check 14)
PRINT: (14)'
10.(14)' TO: 34
PRINT: "B':", STOP
PRINT: "N'="
(34)'+(14)' TO: 34
PRINT: (34)'
RETURN

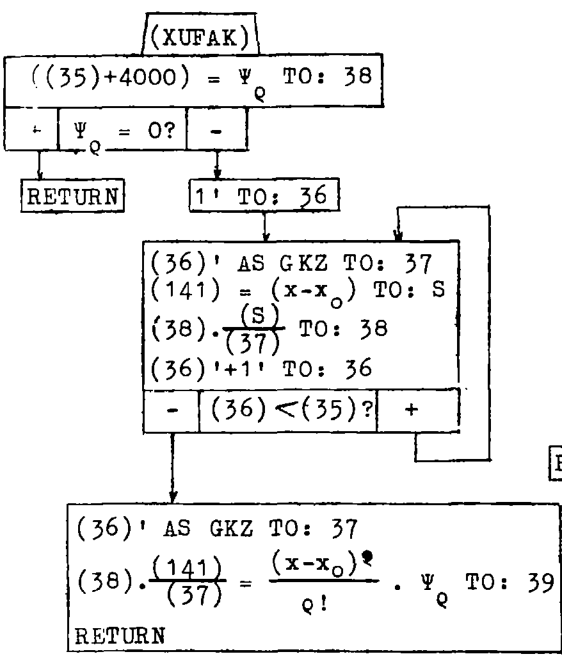

(GERAD)
(A)' TO: 34
(A)':2' TO: A,S
-
(A)'=2k'?
+
-1,- TO: A
RETURN
+1,- TO: A
RETURN

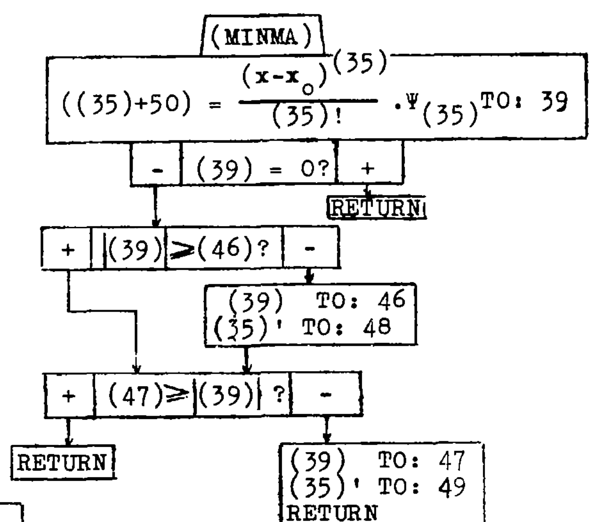

(KLAMI)
2.(37)=2Q TO: 41
(134)^(37)=h_o^Q TO: 40
(35)=n=2k+2
(35)=n=2k+3
(142).(40)=v_o.h_o^Q TO: S
((41)+150)=Ψ_2Q^o TO: A
(A)+(S)=α TO: 38
(143).(40)=v'_o.h_o^Q TO: S
((41)+151)=Ψ_2Q+1^o TO: A
(A)+(S) = β_Q TO: 39
RETURN
(142).(40)=v_o.h_o^Q TO: S
((141)+50)=Ψ_2Q^o TO: A
(A)+(S)=α TO: 38
(143).(40)=v'_o.h_o^Q TO: S
((41)+51)=Ψ_2Q+1^o TO: A
(A)+(S) = β_Q TO: 39
RETURN

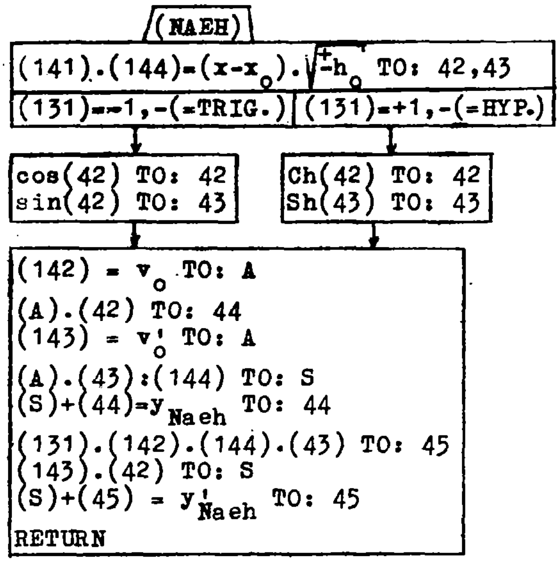

(KLAMA)
GO TO: (KLAMI)
(35)' TO: A
GO TO: (GERAD)
(A)=+1,-
(A)=-1,-
((41)+50) = (2k over 2Q) TO: S
(136).(38) = B.α_Q TO: A
(A).(S) TO: 42
((41)+51) = (2k over 2Q+1) TO: S
(135).(39) = A.β_Q TO: A
(A).(S)+(42) = γ_2k,Q TO: 42
RETURN
((41)+150) = (2k+1 over 2Q) TO: S
(133).(135).(38) = (-45).A.α_Q TO: A
(A).(S) TO: 42
((41)+151) = (2k+1 over 2Q+1) TO: S
(136).(39) = B.β_Q TO: A
(A).(S)+(42) = γ_2k+1,Q TO: 42
RETURN

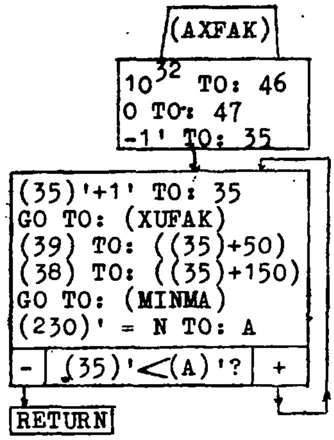

(PHISU)
(35) TO: A
GO TO: (GERAD)
(A)=-1,-
(A)=+1,-
(35)-3 / 2 = k TO: 36
(35)-2 / 2 = k TO: 36
0 TO: 37
+1,- TO: 43
0 TO: 44
GO TO: (KLAMA)
(S):(43) = (-4δ)^-ϱ.γ..,ϱ TO: S
(S)+(44) TO: 44
(43).(133) = (-4δ)^ϱ+1 TO: 43
(37)+1 = ϱ+1 TO: 37
+
(37)<(36)=k?
-
(133)^(36) = (-4δ)^k TO: S
(132).(S).(44) = (2qδ).(-4δ)^k. Σ k-1 0 TO: 44
(35) TO: A, GO TO: (GERAD)
(A) = -1,-
(A) = +1,-
(134).((35)+48) = h_o.Ψ_2k+1 TO: S
(S)+(44) TO: 44
GO TO: (KLAMI)
(35)-2 = 2k+1 TO: S
(132).(133).(S).A.(38) =
= (2qδ).(-4δ).(2k+1).A.α_k TO: S
(S)+(44) = Ψ_2k+3 TO: 44
(134).((35)+148) = h_o.Ψ_2k TO: S
(S)+(44) = Ψ_2k+2 TO: 44
(44) TO: 4000+(35) =
= 4000+(2k+3)
RETURN

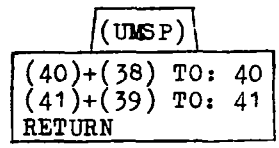

(BINOM)
(35)'-2' TO: 36
- (36)' = 41? +
-1' TO: 37
- (36)'>41'? +
(37)'+1' TO: 37
((37)+50) AS GKZ
TO: ((37)+50)
- (37)'<(36)'? +
1' TO: 37
(36) TO: A
(A)=-1,- (A)=+1,-
1' TO: 37
(36) TO: A
(A)=-1,- (A)=+1,-
((37)+49) +
+((37)+50) =
=(2k+1 / 2Q+1) TO:
((37)+150)
(37)'+1' TO:37
+ (37)<(36)? -
((37)+149) +
+((37)+150)=
=(2k / 2Q+1) TO:
((37)+50)
(37)'+1' TO:37
- (37)<(36)? +
((37)+149) TO: S
(S)+((37)+150)
TO: ((37)+50)
(37)'+1' TO: 37
+ (37)<(36)? -
((37)+49) TO: S
(S)+((37)+50)
TO: ((37)+150)
(37)'+1' TO: 37
- (37)<(36)? +
1' TO: 150
1' TO:
((36)+150)
RETURN
1' TO: 50
1' TO:
((36)+50)
RETURN
+1,- TO: 50
+1,- TO:
((36)+50)
RETURN
+1,- TO: 150
+1,- TO:
((36)+150)
RETURN

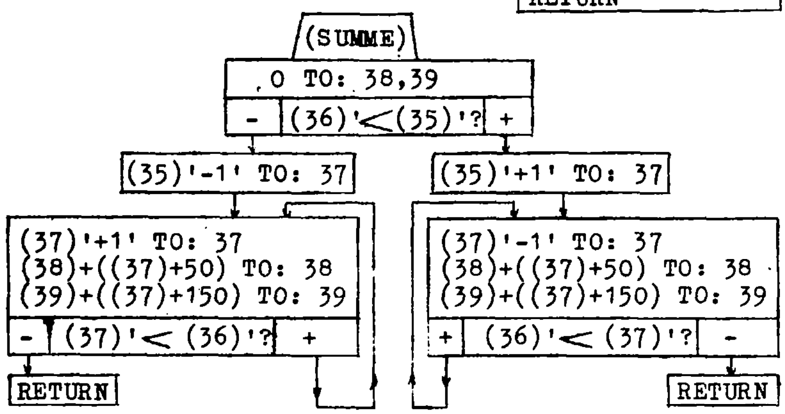

(XUFAK)
((35)+4000) = Ψ_Q TO: 38
+ Ψ_Q = 0? -
RETURN
1' TO: 36
(36)' AS GKZ TO: 37
(141) = (x-x_o) TO: S
(38).(S)/(37) TO: 38
(36)'+1' TO: 36
- (36)<(35)? +
(36)' AS GKZ TO: 37
(38).(141)/(37) = (x-x_o)^Q / Q! . Ψ_Q TO: 39
RETURN

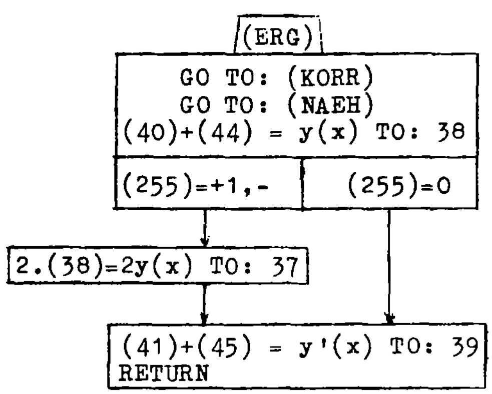

(MINMA)
((35)+50) = (x-x_o)^(35) / (35)! . Ψ_(35) TO: 39
- (39) = 0? +
RETURN
+ |(39)|≥(46)? -
(39) TO: 46
(35)' TO: 48
+ (47)≥|(39)| ? -
RETURN
(39) TO: 47
(35)' TO: 49
RETURN

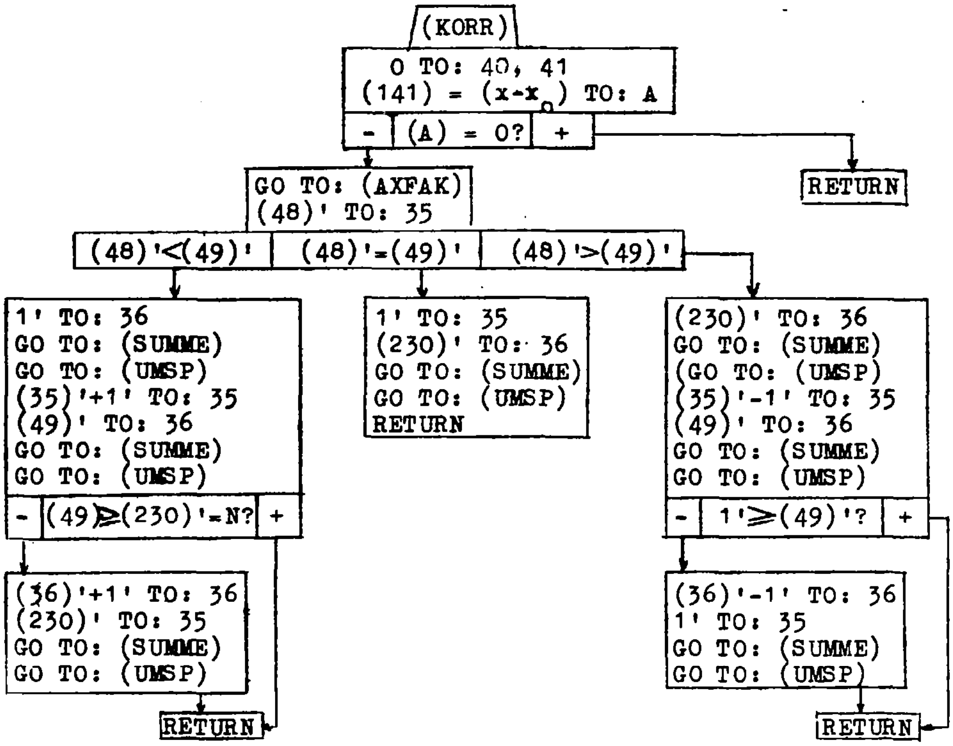

(NAEH)
(141).(144)=(x-x0).√±h0 TO: 42,43
(131)=-1,-(=TRIG.)
(131)=+1,-(=HYP.)
cos(42) TO: 42
sin(42) TO: 43
Ch(42) TO: 42
Sh(43) TO: 43
(142) = v0 TO: A
(A).(42) TO: 44
(143) = v'0 TO: A
(A).(43):(144) TO: S
(S)+(44)=yNaeh TO: 44
(131).(142).(144).(43) TO: 45
(143).(42) TO: S
(S)+(45) = y'Naeh TO: 45
RETURN

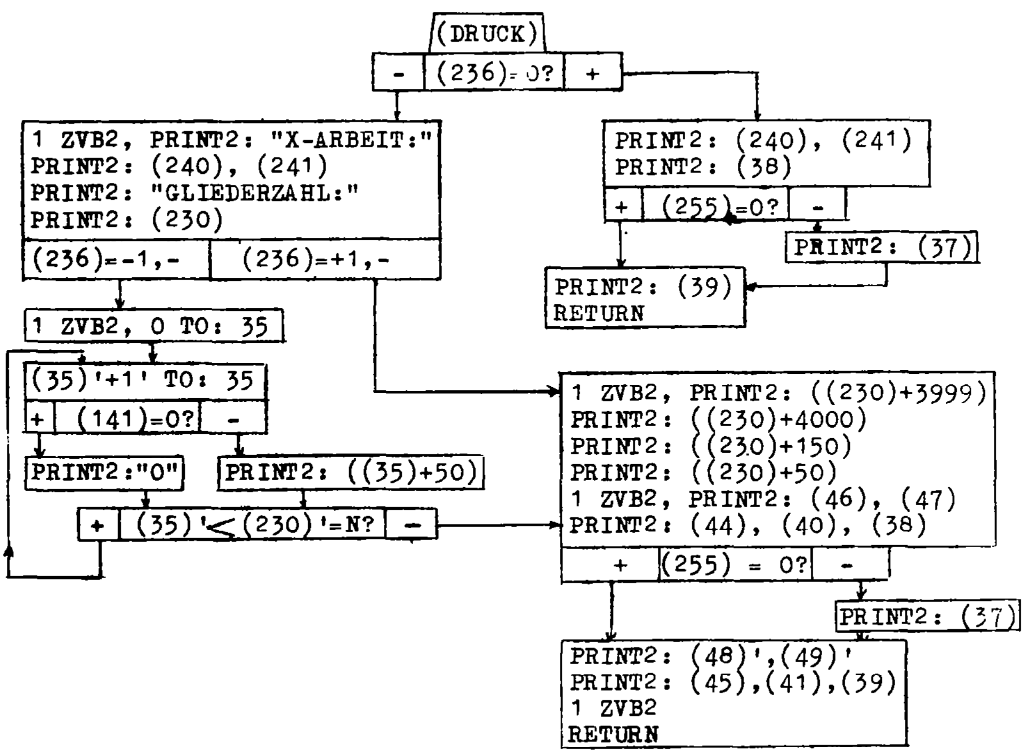

(AXFAK)
10^32 TO: 46
0 TO: 47
-1' TO: 35
(35)'+1' TO: 35
GO TO: (XUFAK)
(39) TO: ((35)+50)
(38) TO: ((35)+150)
GO TO: (MINMA)
(230)' = N TO: A
-
(35)'<(A)'?
+
RETURN


(UMSP)
(40)+(38) TO: 40
(41)+(39) TO: 41
RETURN


(SUMME)
0 TO: 38,39
-
(36)'<(35)'?
+
(35)'-1' TO: 37
(35)'+1' TO: 37
(37)'+1' TO: 37
(38)+((37)+50) TO: 38
(39)+((37)+150) TO: 39
-
(37)'< (36)'?
+
RETURN
(37)'-1' TO: 37
(38)+((37)+50) TO: 38
(39)+((37)+150) TO: 39
+
(36)'< (37)'?
-
RETURN


(ERG)
GO TO: (KORR)
GO TO: (NAEH)
(40)+(44) = y(x) TO: 38
(255)=+1,-
(255)=0
2.(38)=2y(x) TO: 37
(41)+(45) = y'(x) TO: 39
RETURN


(KORR)
0 TO: 40, 41
(141) = (x-x0) TO: A
- (A) = 0? +
RETURN
GO TO: (AXFAK)
(48)' TO: 35
(48)'<(49)'
(48)'=(49)'
(48)'>(49)'
1' TO: 36
GO TO: (SUMME)
GO TO: (UMSP)
(35)'+1' TO: 35
(49)' TO: 36
GO TO: (SUMME)
GO TO: (UMSP)
1' TO: 35
(230)' TO: 36
GO TO: (SUMME)
GO TO: (UMSP)
RETURN
(230)' TO: 36
GO TO: (SUMME)
(GO TO: (UMSP)
(35)'-1' TO: 35
(49)' TO: 36
GO TO: (SUMME)
GO TO: (UMSP)
- (49)≥(230)'=N? +
- 1'≥(49)'? +
(36)'+1' TO: 36
(230)' TO: 35
GO TO: (SUMME)
GO TO: (UMSP)
(36)'-1' TO: 36
1' TO: 35
GO TO: (SUMME)
GO TO: (UMSP)
RETURN
RETURN


(DRUCK)
- (236)=0? +
1 ZVB2, PRINT2: "X-ARBEIT:"
PRINT2: (240), (241)
PRINT2: "GLIEDERZAHL:"
PRINT2: (230)
(236)=-1,-
(236)=+1,-
PRINT2: (240), (241)
PRINT2: (38)
+ (255)=0? -
PRINT2: (37)
PRINT2: (39)
RETURN
1 ZVB2, 0 TO: 35
(35)'+1' TO: 35
+ (141)=0? -
PRINT2:"0"
PRINT2: ((35)+50)
+ (35)'<(230)'=N? -
1 ZVB2, PRINT2: ((230)+3999)
PRINT2: ((230)+4000)
PRINT2: ((230)+150)
PRINT2: ((230)+50)
1 ZVB2, PRINT2: (46), (47)
PRINT2: (44), (40), (38)
+ (255) = 0? -
PRINT2: (37)
PRINT2: (48)',(49)'
PRINT2: (45),(41),(39)
1 ZVB2
RETURN

(6.3) Numerical Results

(Floriani Dietmar)

(6.31) In order to demonstrate the usefulness of the Lie series method, we reproduce four examples, the ordinary functions se_1, ce_2, ce_3 and the modified function Ce_4. As told above, we calculated the fundamental system (VI,50) (like Ref. 21). If a value normalized according to (VI,62) was available we multiplied our results by the normalization factor $\bar{y}(x_N)/y(x_N)$.

Explanations referring to the columns:

1) Ce_4: Column 1: x in radians

Column 2: y(x); below the normalized value (see remark); below, the value of comparison from Ref. 24.

Column 3: y'(x) (not normalized)

Column 4: Order of magnitude of the smallest term used in the sum (VI,39).((-n) means 10^{-n}, of course)

Column 5: The same for the derivative of (VI,39).

The values $\bar{y}(x)$ were obtained by normalizing y(x=0,1) to 1,1330 (taken from Ref. 24). Consequently, the values $\bar{y}(x)$ of the second line of column 2 may be wrong from the fifth digit while the unnormalized values y(x) (in the first line) have the original accuracy.

2) ce_3: In this example the values of y(x) and y'(x) are not normalized.

Column 1: x in degrees

Column 2: y(x) calculated for 30, below it for 17, below it for 10 terms, in (39); below these values, the value from Morse-Feshbach (Ref.21) is given for the sake of comparison

Column 3: The same for y'(x) (without value for comparison)

Column 4, 5: as in 1)

$Ce_4(x;10)$

0	,100000000/+01	0		
0,1	,100519363/+01 ,113300000/+01 ,11330 /+01	,972570370/-01	(-37)	(-36)
0,2	,101676300/+01 ,114604036/+01 ,11460 /+01	,113613849/+00	(-29)	(-27)
0,3	,102231891/+01 ,115230268/+01 ,11523 /+01	-,381249934/-01	(-23)	(-21)
0,4	,100052404/+01 ,112773668/+01 ,11277 /+01	-,447630695/+00	(-19)	(-17)
0,5	,921920767/+00 ,103913931/+01 ,10391 /+01	-,118059090/+01	(-15)	(-13)
0,6	,754107536/+00 ,849989300/+00 ,84996 /+00	-,221632845/+01	(-13)	(-11)
0,7	,475140199/+00 ,535552378/+00 ,53553 /+00	-,334757849/+01	(-10)	(-09)
0,8	,974910743/-01 ,109886675/+00 ,10988 /+00	-,408189917/+01	(-40)	(-40)
0,9	-,302955275/+00 -,341474829/+00 -,34146 /+00	-,366919362/+01	(-27)	(-23)
1,0	-,575802785/+00 -,649013811/+00 -,64899 /+00	-,147121582/+01	(-19)	(-17)

$ce_3(x;2)$

0°	,100000000/+01	0		
10°	,918721565/+00	-,925242223/+00	(-30)	(-28)
	,918721565/+00	-,925242223/+00	(-15)	
	,918721565/+00	-,925242225/+00	(-09)	(-07)
	,9187			
20°	,681985475/+00	-,176294217/+01	(-21)	(-19)
	,681985475/+00	-,176294217/+01	(-10)	
	,681985379/+00	-,176294546/+01	(-06)	(-04)
	,6820			
30°	,317080590/+00	-,236160093/+01	(-16)	(-14)
	,317080587/+00	-,236160105/+01	(-07)	
	,317068498/+00	-,236187557/+01	(-04)	(-03)
	,3171			
40°	-,115417896/+00	-,249712071/+01	(-12)	(-10)
	-,115418488/+00	-,249713584/+01	(-05)	
	-,115783378/+00	-,250329561/+01	(-03)	(-02)
	-,1154			
50°	-,515119951/+00	-,196064023/+01	(-38)	(-35)
	-,515134036/+00	-,196074153/+01	(-20)	(-18)
	-,518409426/+00	-,198043166/+01	(-11)	(-08)
	-,5151			
60°	-,758697177/+00	-,729353347/+00	(-24)	(-22)
	-,758725800/+00	-,729413240/+00	(-12)	(-10)
	-,764743045/+00	-,740099783/+00	(-06)	(-04)
	-,7587			
70°	-,746682435/+00	,886775360/+00	(-17)	(-15)
	-,746715920/+00	,886782194/+00	(-08)	(-06)
	-,753429435/+00	,890114724/+00	(-04)	(-02)
	-,7467			
80°	-,461980548/+00	,228150895/+01	(-13)	(-11)
	-,462006955/+00	,228157557/+01	(-06)	(-04)
	-,466545243/+00	,230532689/+01	(-02)	(-01)
	-,4620			
90°	,372529029/-08	,283447515/+01	(-09)	(-08)
	-,347094610/-04	,283401283/+01	(-04)	(-02)
	,375638250/-02	,291895629/+01	(-01)	(+00)
	,0000			

3) se_1 and ce_2:

Column 1: x in degrees

Column 2: y(x); below the same quantity normalized according to (VI,62) (see remark)

Column 3: y'(x)

Column 4, 5: as in 1)

In order to normalize se_1 and ce_2 we took eight-digit-values $\bar{y}$ at $x=90^{\circ}$ from Ref. 42. Before this we estimate the accuracy of the normalized values to seven digits.

$se_1(x;5)$

0°	0	,100000000/+01		
10°	,188695561/+00 ,329604726/-01	,124535882/+01	(-37)	(-34)
20°	,466132532/+0(,814218865/-01	,203286757/+01	(-25)	(-23)
30°	,935811049/+00 ,163463169/-00	,346129214/+01	(-19)	(-17)
40°	,171063116/+01 ,298805182/+00	,549981492/+01	(-14)	(-13)
50°	,286744961/+01 ,500872908/+00	,772154247/+01	(-40)	(-39)
60°	,436030518/+01 ,761638051/+00	,914730588/+01	(-25)	(-23)
70°	,594110387/+01 ,103776470/+01	,854072594/+01	(-18)	(-16)
80°	,718240068/+01 ,125458871/+01	,524849689/+01	(-13)	(-11)
90°	,765668132/+01 ,133743390/+01	-,178813934/-06	(-09)	(-07)

$ce_2(x;5)$

0°	,100000000/+01	0		
10°	,103753674/+01 ,762894862/+00	,414834533/+00	(-35)	(-33)
20°	,113347289/+01 ,833436165/+00	,632857185/+00	(-24)	(-22)
30°	,123284138/+01 ,906501253/+00	,410750603/+00	(-18)	(-16)
40°	,123796996/+01 ,910272267/+00	-,471084203/+00	(-13)	(-12)
50°	,103168567/+01 ,758592604/+00	-,196454071/+00	(-40)	(-38)
60°	,549803003/+00 ,404267041/+00	-,348158071/+01	(-25)	(-23)
70°	-,124341227/+00 -,914273966/-01	-,398760686/+01	(-18)	(-16)
80°	-,736175131/+00 -,541305384/+00	,27133[illegible]8/+01	(-13)	(-11)
90°	-,985303635/+00 -,724488154/+00	-,372529029/-07	(-09)	(-07)

(6.32) **Remarks on the results:**

We had everywhere the expressions $(x-x_0)^\rho . Y_\rho/\rho!$ printed out. Thus, it turned out that they rapidly decreased with increasing ρ, in our examples. It seems, therefore, to be obvious that in those domains of the (q,λ)-plane in which this decrease is rapid enough the error caused by breaking-off the sum (VI,39) is of about the same order of magnitude as the smallest (or last) term used if no error of higher order of magnitude is superimposed, e.g., by repeated analytical continuation. Since we expanded in the neighborhood of 0° and 45°,

the values of columns 4 and 5 should give useful hints to an estimate between 0° and 45° and in the neighborhood of 90°.

Unfortunately, no tables with more than 5 digits were available so that we could merely estimate the accuracy of the remaining digits. (Ref. 42 contents only single values $\bar{y}(x;q,\lambda)$ to certain pairs of (q,λ)).

Consequently, the results at the boundary of the interval were specially significant for judging the efficiency of the method. From (VI,60) follows:

$$ce_2'(\pi/2) = 0$$

$$se_1'(\pi/2) = 0$$

$$ce_3(\pi/2) = 0$$

whereas our values are:

$ce_2'(\pi/2)$	= -,372.../-07	(-07)	
$se_1'(\pi/2)$	= -,178.../-06	(-07)	
$ce_3(\pi/2)$	= ,372.../-08	(-09)	N = 30
	= -,347.../-04	(-04)	N = 17
	= ,375.../-02	(-01)	N = 10

(N means the number of terms used to calculate the sum (VI,39)).
The agreement is rather satisfactory, above all, if we take into account the fact that $x-x_o$ is already rather great, namely 0,8. In the case of repeated analytical continuation the result should still be better whereas the computer times remain comparable. Besides, the values of the 4. and 5. column yield surprisingly good estimates in the case considered.

(6.4) Discussion of the codes; Possibilities of Improvement and Extension

(6.41) The program MNVM-AD:

The programs for calculating Mathieu functions by means of Lie series were intended to become acquainted with the possibilities and limitations of this method (accuracy, domains of rapid convergence, necessary number of terms) rather than to yield computer times as short as possible. Accordingly, AD was arranged so that the run of the calculation could be influenced to a wide extent (e.g., the input of a new number of terms with "GO TO: (GZAHL)") while some clumsy actions, as, e.g., the complete calculation of the $\Psi_\rho(x=0)$ instead of using those already calculated, were not eliminated, partly by lack of time, partly since only the computer time was increased, by that. In an improved program, analytical continuation would have to be possible in the neighborhood of arbitrary points. For this purpose, a reasonable, i.e., theoretically well-founded and numerically useful estimate of errors would be needed if one does not want to give the step size completely arbitrarily.

Another promising way of further improvement is a method which has been elaborated here, at Innsbruck, by H. Knapp (Ref. 10 to Ref. 13), i.e. using rearrangement of the series and iteration processes.

(6.42) The other programs:

The programs AA, AB are preliminary stages of AC and AD; in the same way, BA, BB are preliminary stages of BC.

The comparison program CA for the representation by means of Fourier series was, again by lack of time, only sketched and is not yet ready.

DA, however, was completely worked out. As it was to be expected, the results were satisfactory only for rather small q. Consequently, this program could not be referred to for purposes of comparison.

Conclusion:

In the exact calculation of Mathieu functions also the use of Lie series is accompanied by certain difficulties. However, investigations carried out at Innsbruck in the last few years have shown that these difficulties can be avoided to a greater extend, e.g., by rearrangement, iteration processes, etc. For less accurate calculations the method is thoroughly appropriate in the form described above.

The fact that the Lie series method yields satisfactory results, also in the comparatively unfavorable case of Mathieu functions suggests that it might be appropriate also for calculating the other special functions of mathematical physics (see chapter VII).

Chapter VII

The Numerical Calculation of Weber Parabolic Cylinder Functions,

by A.SCHETT and J.WEIL.

From a separation of the Helmholtz equation

$$\Delta\Phi(\mu,\nu,z) \pm \varkappa^2\Phi(\mu,\nu,z) = 0 \qquad \text{(VII,1)}$$

in parabolic-cylinder coordinates, we obtain the following three equations:

$$\begin{aligned}
&\frac{d^2M}{d\mu^2} - (\alpha_2 + \alpha_3\mu^2)M = 0\\
&\frac{d^2N}{d\nu^2} + (\alpha_2 - \alpha_3\nu^2)N = 0 \qquad \text{(VII,2)}\\
&\frac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)Z = 0
\end{aligned}$$

the first two of which are solved by Weber parabolic-cylinder functions.

As it is well known these equations have already been investigated extensively: /33, and references given there/, /34/, /35/, /36/, /37/, /38/, /39/.

We are going to present a short review of all three formulas representing the solution of the general homogeneous linear differential equation derived so far - all of them based on Lie series formalism.

These equations are of the general type:

$$Y''(t) - f_1(t)Y'(t) - f_2(t)Y(t) = 0 \qquad \text{(VII,3)}$$

which is converted into a first-order system in the usual way and solved

$$Y(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu y$$

The first formula representing the solution - on which the numerical calculations are based - reads:

$$Y(t) = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \sum_{\varrho=0}^{\nu-2} \binom{\nu-2}{\varrho} \left(f_1^{(\varrho)}(y_0) D^{\nu-1-\varrho} y_1 + \right.$$

$$\left. + f_2^{(\varrho)}(y_0) D^{\nu-2-\varrho} y_1\right) + y_1 + t y_2 \qquad \text{(VII,4)}$$

Evidently, this formula is a recurrence formula for the powers of the D-operator; following a way proposed by Prof.Cap, we gave an alternative formula obtained by splitting off given functions (trigonometric and hyperbolic) from the total solution, e.g.:

$$Z(t) = z_2 \cdot \frac{1}{f_1}\left[\cosh(tf_1) - 1 - \frac{(tf_1)^2}{2}\right] + z_1 \left[\cosh(t\sqrt{f_2}) - 1 - \frac{t^2 f_1}{2}\right] +$$

$$+ z_2 \frac{1}{f_1}\left[\sinh(tf_1) - tf_1 - \frac{(tf_1)^3}{3!}\right] + z_1 \frac{f_2}{2}\left[\sinh(tf_1) - \right.$$

$$\left. - tf_1 - \frac{(tf_1)^3}{3!}\right] + \frac{z_2}{\sqrt{f_2}}\left[\sinh(t\sqrt{f_2}) - t\sqrt{f_2} - \frac{(t\sqrt{f_2})^3}{3!}\right] +$$

$$+ \sum_{\varrho=2}^{\infty} \frac{t^\varrho}{\varrho!} s_\varrho(z_0, z_1, z_2, f_1, f_2) + z_1 + t z_2$$

The third formula (see chapter III, Eq.(III,6)) consists of a main term and a perturbation integral which can be evaluated by iterations:

$$\begin{pmatrix} Y_2(t) \\ Y_1(t) \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_1^\nu \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} + \sum_{\alpha=0}^{\infty} \int_0^t \frac{(t-\tau)^\alpha}{\alpha!} \left[D_2 D^\alpha \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}\right]_{\bar{a}} d\tau$$

In our case, Eq.(VII,4) for which the code was written reduces to a simpler form as $f_1 = 0$ and $f_2 = (a + bt^2)$; consequently, Eq.(VII,4) becomes:

$$Y(t) = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} \Big[(-a-by_o^2)D^{\nu-2}y_1 + (\nu-2)(-2by_o) D^{\nu-3}y_1 + \frac{(\nu-2)(\nu-3)}{2}(-2b)D^{\nu-4}y_1\Big] + y_1 + ty_2 \qquad \text{(VII,5)}$$

According to /33/, we write the two equations leading to Weber' functions in the following form:

$$Y''(t) + (\frac{1}{4}t^2 - a)Y(t) = 0 \qquad \text{(VII,6)}$$

$$Y''(t) - (\frac{1}{4}t^2 + a)Y(t) = 0 \qquad \text{(VII,7)}$$

i.e., the conventional ways of solutions requi e a disjunction of cases which are simultaneously contained in the Lie formalism; this is one of the advantages of the latter method. To start with, we will treat Eq.(VII,6):

a) The solution of Eq.(VII,6) is given by:

$$Y = AU + BV \qquad \text{(VII,8)}$$

where U, V are two independent solutions and A, B constants.

α) In the well-known series representation /33/ we have:

$$U(a,t) = D_{-a-\frac{1}{2}}(t) = \cos\pi\,(\frac{1}{4} + \frac{1}{2}a)Y_1 - \sin\pi\,(\frac{1}{4} + \frac{1}{2}a)Y_2$$

and

$$V(a,t) = \frac{1}{\Gamma(\frac{1}{2} - a)}\Big\{\sin\pi\,(\frac{1}{4} + \frac{1}{2}a)Y_1 + \cos\pi\,(\frac{1}{4} + \frac{1}{2}a)Y_2\Big\}$$

with

$$Y_1 = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4} - \frac{1}{2}a)}{2^{\frac{1}{2}a + \frac{1}{2}}} y_1$$

and

$$Y_2 = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{3}{4} - \frac{1}{4}a)}{2^{\frac{1}{2}a - \frac{1}{2}}} y_2$$

y_1, y_2 are given by the following series:

$$y_1 = e^{\frac{1}{4}t^2} \left\{1 + (a - \tfrac{1}{2}) \frac{t^2}{2!} + (a - \tfrac{1}{2}) (a - \tfrac{5}{2}) \frac{t^4}{4!} + \dots\right\}$$

and

$$y_2 = e^{\frac{1}{4}t^2} \left\{t + (a - \tfrac{3}{2}) \frac{t^3}{3!} + (a - \tfrac{3}{2}) (a - \tfrac{7}{2}) \frac{t^5}{5!} + \dots\right\}$$

which are convergent for all values of t.

β) On the other hand, U and V are represented by integrals in the following way /33/:

$$U(a,z) = \frac{e^{-\frac{1}{4}z^2}}{2\pi i} z^{-a-\frac{1}{2}} \int_{-\infty i}^{-\infty i} \frac{\Gamma(s)\Gamma(\frac{1}{2} + a - 2s)}{\Gamma(\frac{1}{2} + a)} (\sqrt{2}z)^{2s} ds$$

$$(|\arg z| < \tfrac{3}{4}\pi)$$

where the contour separates the zeros of $\Gamma(s)$ from those of $\Gamma(a + \frac{1}{2} - 2s)$.

Similarly we have for V:

$$V(a,z) = \sqrt{\frac{2}{\pi}} \frac{e^{\frac{1}{4}z^2}}{2\pi i} z^{a-\frac{1}{2}} \int_{-\infty i}^{-\infty i} \frac{\Gamma(s)\Gamma(\frac{1}{2} - a - 2s)}{\Gamma(\frac{1}{2} - a)} \cdot (\sqrt{2}z)^{2s} \cos s\pi ds$$

$$(|\arg z| < \tfrac{1}{4}\pi)$$

b) Equation (VII,7) is solved by:

$$Y(a, \pm t) = \frac{(\cosh \pi a)^{\frac{1}{4}}}{2\sqrt{\pi}} (G_1 y_1 \mp \sqrt{2}\, G_3 y_2)$$

where

$$G_1 = \left| \Gamma(\tfrac{1}{4} + \tfrac{1}{2} ia) \right|$$

and

$$G_3 = \left| \Gamma(\tfrac{3}{4} + \tfrac{1}{2} ia) \right|$$

while

$$y_1 = 1 + a\frac{t^2}{2!} + (a^2 - \tfrac{1}{2}) \frac{t^4}{4!} + (a^3 - \tfrac{7}{2} a) \frac{t^6}{6!} + (a^4 - 11a^2 + \tfrac{15}{4}) \frac{t^8}{8!} +$$

$$+ (a^5 - 25a^3 + \frac{211}{4} a) \frac{t^{10}}{10!} + \ldots\ldots$$

and

$$y_2 = t + a\frac{t^3}{3!} + (a^2 - \tfrac{3}{2}) \frac{t^5}{5!} + (a^3 - \tfrac{13}{2} a) \frac{t^7}{7!} + (a^4 - 17a^2 + \tfrac{63}{4}) \frac{t^9}{9!} +$$

$$+ (a^5 - 35a^3 + \frac{531}{4} a) \frac{t^{11}}{11!} + \ldots\ldots.$$

in which non-zero coefficients a_n of $\frac{t^n}{n!}$ are connected by

$$a_{n+2} = a \cdot a_n - \frac{1}{4} n (n-1) a_{n-2}$$

Numerical calculation:

Evidently, we have for the solution and its derivative:

$$Y(a,t) = AU(a,t) + BV(a,t) = Y_1(a,t)$$

$$Y'(a,t) = AU'(a,t) + BV'(a,t) = Y_2(a,t)$$

For $t = 0$, the independent solutions U and V and their derivatives reduce to /33/:

$$U(a,o) = \frac{\sqrt{\pi}}{2^{\frac{1}{2}a + \frac{1}{4}}\,\Gamma(\frac{3}{4} + \frac{1}{2}a)}$$

$$U'(a,o) = -\frac{\sqrt{\pi}}{2^{\frac{1}{2}a - \frac{1}{4}}\,\Gamma(\frac{1}{4} + \frac{1}{2}a)}$$

$$V(a,o) = \frac{2^{\frac{1}{2}a + \frac{1}{4}}\sin\pi(\frac{3}{4} - \frac{1}{2}a)}{\Gamma(\frac{3}{4} - \frac{1}{2}a)} \qquad \text{(VII,9)}$$

$$V'(a,o) = \frac{2^{\frac{1}{2}a + \frac{3}{4}}\sin\pi(\frac{1}{4} - \frac{1}{2}a)}{\Gamma(\frac{1}{4} - \frac{1}{2}a)}$$

From Eq.(VII,9) $U(a,o)$, $U'(a,o)$, $V(a,o)$ and $V'(a,o)$ can be calculated for different values of the parameter a.

Without restriction of generality, we may choose:

$$Y(a,o) = U(a,o) = y(a,o) = y_1(a,o) \qquad \text{(VII,10)}$$

$$Y'(a,o) = U'(a,o) = y'_1(a,o) = y_2(a,o) \qquad \text{(VII,11)}$$

i.e., we have put $A = 1$ and $B = 0$; the small letters are initial values.

With the help of (VII,10) and (VII,11) we may write the solution:

$$Y(a,t) = \sum_{\nu=2}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu}y + y_1 + y_2 t = U(a,t)$$

$$Y'(a,t) = \sum_{\nu=1}^{\infty} \frac{t^{\nu}}{\nu!} D^{\nu}y + y_2 = U'(a,t)$$

Furthermore we choose:

$$Y(a, o) = V(a,o) = y(a,o) = y_1(a,o)$$

$$Y'(a,o) = V'(a,o) = y'(a,o) = y_2(a,o)$$

i.e. we have $A = 0$ and $B = 1$.

Again the solution is expressed by:

$$Y(A,t) = \sum_{\nu=2}^{\infty} \frac{t^\nu}{\nu!} D^\nu y + y_1 + y_2 t = V(a,t) \qquad (VII,12)$$

$$Y'(a,t) = \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} D^\nu y + y_2 = V'(a,t) \qquad (VII,13)$$

the philosophy of this choice being the construction of solutions which are tabulated.

We now expand $Y(a,t)$ and $Y'(a,t)$ in the neighborhood of $t = 0$ and choose a step size of $\Delta t = 0{,}1$. As t increases more terms $\frac{t^\nu D^\nu}{\nu!} y(a,t = 0)$ have to be calculated if the accuracy is prescribed. Since the computers have a limited numerical range, only a limited number of terms $\frac{t^\nu D^\nu}{\nu!}$ $y(a,t = 0)$ can be calculated ("overflow"). Consequently, we expand the functions $Y(a,t)$ and $Y'(a,t)$ at $t = 0$ and, using a step size of $\Delta t = 0{,}1$, we calculate until $Y(a,t_1)$ and $Y'(a,t_1)$ are reached. $Y(a,t_1)$ and $Y'(a,t_1)$ are, then expanded at t_1 where t_1 was chosen to be 2, in this calculation. Continuing this method, we can compute $Y(a,t)$ and $Y'(a,t)$ for arbitrary a and t, with the accuracy wanted.

In the following appendix, the values calculated on the basis of the Eqs.(VII,5), (VII,12) and (VII,13) are compared to the five digit values extracted from /33/. 23 terms of the Lie series were used for this calculation (it is worth mentioning that also 16 terms which were used in a preliminary test yielded very accurate results). The calculation time for one value of the functions $U(a,t)$, $V(a,t)$, $U'(a,t)$, $V'(a,t)$ was about 3 sec.

Tables: In the following tables, the values of U, U', V, V' are given for several values of the parameter a; the column to the left and right are Lie series results, the central one presenting results taken from /33/.

t	U(-5.0/t) Lie Series	U(-5.0/t) Table /33/	U'(-5.0/t) Lie Series
0.0	,30522 (+01)	,30522 (+01)	,68418 (+01)
0.1	,36547 (+01)	,36547 (+01)	,51583 (+01)
0.2	,40753 (+01)	,40753 (+01)	,32200 (+01)
0.3	,42935 (+01)	,42935 (+01)	,11257 (+01)
0.4	,42988 (+01)	,42988 (+01)	-,10179 (+01)
0.5	,40918 (+01)	,40918 (+01)	-,31029 (+01)
0.6	,36836 (+01)	,36836 (+01)	-,50253 (+01)
0.7	,30953 (+01)	,30953 (+01)	-,66912 (+01)
0.8	,23566 (+01)	,23566 (+01)	-,80214 (+01)
0.9	,15042 (+01)	,15042 (+01)	-,89558 (+01)
1.0	,57999 (+00)	,5799 (+00)	-,94557 (+01)
1.1	-,37182 (+00)	-,3719 (+00)	-,95055 (+01)
1.2	-,13063 (+01)	-,13064 (+01)	-,91126 (+01)
1.3	-,21806 (+01)	-,21806 (+01)	-,83064 (+01)
1.4	-,29554 (+01)	-,29554 (+01)	-,71355 (+01)
1.5	-,35976 (+01)	-,35976 (+01)	-,56644 (+01)
1.6	-,40808 (+01)	-,40808 (+01)	-,39695 (+01)
1.7	-,43868 (+01)	-,43868 (+01)	-,21345 (+01)
1.8	-,45059 (+01)	-,45059 (+01)	-,24521 (+00)
1.9	-,44369 (+01)	-,44368 (+01)	,16143 (+01)
2.0	-,41866 (+01)	-,41866 (+01)	,33664 (+01)

t	U(-5-0/t) Lie Series	U(-5.0/t) Table /33/	U'(-5.0/t) Lie Series
2.1	-,37694 (+01)	-,37694 (+01)	,49429 / +01
2.2	-,32057 (+01)	-,32057 (+01)	,62884 / +01
2.3	-,25208 (+01)	-,25208 (+01)	,73616 / +01
2.4	-,17434 (+01)	-,17434 (+01)	,81364 / +01
2.5	-,90387 (+00)	-,9039 (+00)	,86017 / +01
2.6	-,33225 (-01)	-,332 (-01)	,87612 / +01
2.7	,83870 (+00)	,8387 (+00)	,86312 / +01
2.8	,16842 (+01)	,16842 (+01)	,82389 / +01
2.9	,24789 (+01)	,24789 (+01)	,76202 / +01
3.0	,32021 (+01)	,32021 (+01)	,68170 / +01
3.1	,38377 (+01)	,38377 (+01)	,58744 / +01
3.2	,43739 (+01)	,43739 (+01)	,48386 / +01
3.3	,48038 (+01)	,48038 (+01)	,37545 / +01
3.4	,51246 (+01)	,51246 (+01)	,26637 / +01
3.5	,53376 (+01)	,53376 (+01)	,16034 / +01
3.6	,54473 (+01)	,54473 (+01)	,60485 / +00
3.7	,54614 (+01)	,54614 (+01)	-,30683 / +00
3.8	,53895 (+01)	,53895 (+01)	-,11132 / +01
3.9	,52427 (+01)	,52427 (+01)	-,18021 / +01
4.0	,50332 (+01)	,50332 (+01)	-,23677 / +01
4.1	,47733 (+01)	,47733 (+01)	-,28094 / +01
4.2	,44753 (+01)	,44753 (+01)	-,31311 / +01

t	U(-4.5/t) Lie Series	U(-4,5/t) Table /33/	U!(-4,5/t) Lie Series
o.o	,30000 (+o1)	,30000 (+o1)	,00000 (oo)
o.1	,29328 (+o1)	,29328 (+o1)	-,13396 (+o1)
o.2	,27341 (+o1)	,27341 (+o1)	-,26178 (+o1)
o.3	,24132 (+o1)	,24132 (+o1)	-,37763 (+o1)
o.4	,19846 (+o1)	,19846 (+o1)	-,47627 (+o1)
o.5	,14678 (+o1)	,14678 (+o1)	-,55337 (+o1)
o.6	,88615 (+oo)	,88615 (+oo)	-,60565 (+o1)
o.7	,26550 (+oo)	,26550 (+oo)	-,63106 (+o1)
o.8	-,36676 (+oo)	-,36676 (+oo)	-,62886 (+o1)
o.9	-,98321 (+oo)	-,98321 (+oo)	-,59963 (+o1)
1.o	-,15576 (+o1)	-,15576 (+o1)	-,54516 (+o1)
1.1	-,20661 (+o1)	-,20661 (+o1)	-,46837 (+o1)
1.2	-,24882 (+o1)	-,24882 (+o1)	-,37312 (+o1)
1.3	-,28077 (+o1)	-,28077 (+o1)	-,26396 (+o1)
1.4	-,30131 (+o1)	-,30131 (+o1)	-,14587 (+o1)
1.5	-,30982 (+o1)	-,30982 (+o1)	-,24-38 (+oo)
1.6	-,30617 (+o1)	-,30617 (+o1)	,96448 (+oo)
1.7	-,29073 (+o1)	-,29073 (+o1)	,21081 (+o1)
1.8	-,26435 (+o1)	-,26435 (+o1)	,31479 (+o1)
1.9	-,22824 (+o1)	-,22824 (+o1)	,40484 (+o1)
2.o	-,18394 (+o1)	-,18394 (+o1)	,47824 (+o1)

t	V(3.5/t) Lie Series	V(3.5/t) Table /33/	V'(3.5/t) Lie Series
0.0	,00000 (+00)	,00000 (+00)	0,23937 (+01)
0.1	,24076 (+00)	,24076 (+00)	,24357 (+01)
0.2	,48999 (+00)	,48999 (+00)	,25634 (+01)
0.3	,75648 (+00)	,75647 (+00)	,27820 (+01)
0.4	,10497 (+01)	,10497 (+01)	,30999 (+01)
0.5	,13802 (+01)	,13802 (+01)	,35301 (+01)
0.6	,17600 (+01)	,17600 (+01)	,40900 (+01)
0.7	,22033 (+01)	,22033 (+01)	,48025 (+01)
0.8	,27266 (+01)	,27266 (+01)	,56974 (+01)
0.9	,33501 (+01)	,33501 (+01)	,68126 (+01)
1.0	,40980 (+01)	,40980 (+01)	,81962 (+01)
1.1	,50003 (+01)	,50002 (+01)	,99089 (+01)
1.2	,60933 (+01)	,60933 (+01)	,12027 (+02)
1.3	,74225 (+01)	,74224 (+01)	,14649 (+02)
1.4	,90440 (+01)	,90439 (+01)	,17896 (+02)
1.5	,11028 (+02)	,11028 (+02)	,21924 (+02)
1.6	,13461 (+02)	,13461 (+02)	,26930 (+02)
1.7	,16454 (+02)	,16454 (+02)	,33164 (+02)
1.8	,20145 (+02)	,20145 (+02)	,40945 (+02)
1.9	,24708 (+02)	,24708 (+02)	,50683 (+02)
2.0	,30364 (+02)	,30364 (+02)	,62898 (+02)
2.1	,37393 (+02)	,37393 (+02)	,78264 (+02)

t	V(2.o/t) Lie Series	V(2.o/t) Table /33/	V'(2.o/t) Lie Series
o.o	,34311 (+oo)	,34311 (+oo)	,492oo (+oo)
o.1	,39591 (+oo)	,39591 (+oo)	,56581 (+oo)
o.2	,45665 (+oo)	,45665 (+oo)	,65118 (+oo)
o.3	,5266o (+oo)	,5266o (+oo)	,75o12 (+oo)
o.4	,6o721 (+oo)	,6o721 (+oo)	,865o7 (+oo)
o.5	,7oo24 (+oo)	,7oo24 (+oo)	,99893 (+oo)
o.6	,8o774 (+oo)	,8o77 (+oo)	,11552 (+o1)
o.7	,93217 (+oo)	,93217 (+oo)	,13381 (+o1)
o.8	,1o764 (+o1)	,1o764 (+o1)	,15528 (+o1)
o.9	,1244o (+o1)	,1244o (+o1)	,18o54 (+o1)
1.o	,1439o (+o1)	,1439o (+o1)	,21o35 (+o1)
1.1	,16665 (+o1)	,16665 (+o1)	,24563 (+o1)
1.2	,19325 (+o1)	,19325 (+o1)	,28751 (+o1)
1.3	,22442 (+o1)	,22442 (+o1)	,33735 (+o1)
1.4	,261o4 (+o1)	,261o4 (+o1)	,39686 (+o1)
1.5	,3o418 (+o1)	,3o418 (+o1)	,46812 (+o1)
1.6	,35514 (+o1)	,35514 (+o1)	,55371 (+o1)
1.7	,41551 (+o1)	,41551 (+o1)	,65682 (+o1)
1.8	,48722 (+o1)	,48722 (+o1)	,78142 (+o1)
1.9	,57267 (+o1)	,57267 (+o1)	,93247 (+o1)
2.o	,6748o (+o1)	,6748o (+o1)	,11162 (+o2)
2.1	,79725 (+o1)	,79725 (+o1)	,134o2 (+o2)

t	U(-1.5/t) Lie Series	U(-1.5/t) Table /33/	U'(-1.5/t) Lie Series
0.0	,00000 (+00)	,00000 (+00)	,10000 (+01)
0.1	,99750 (-01)	,9975 (-01)	,99251 (+00)
0.2	,19801 (+00)	,19801 (+00)	,97025 (+00)
0.3	,29333 (+00)	,29333 (+00)	,93375 (+00)
0.4	,38432 (+00)	,38432 (+00)	,88393 (+00)
0.5	,46971 (+00)	,46971 (+00)	,82198 (+00)
0.6	,54836 (+00)	,54836 (+00)	,74942 (+00)
0.7	,61929 (+00)	,61929 (+00)	,66795 (+00)
0.8	,68171 (+00)	,68171 (+00)	,57945 (+00)
0.9	,73502 (+00)	,73502 (+00)	,48593 (+00)
1.0	,77880 (+00)	,77880 (+00)	,38940 (+00)
1.1	,81287 (+00)	,81287 (+00)	,29189 (+00)
1.2	,83721 (+00)	,83721 (+00)	,19535 (+00)
1.3	,85203 (+00)	,85203 (+00)	,10159 (+00)
1.4	,85768 (+00)	,85768 (+00)	,12253 (-01)
1.5	,85467 (+00)	,85467 (+00)	-,71223 (-01)
1.6	,84367 (+00)	,84367 (+00)	-,14764 (+00)
1.7	,82541 (+00)	,82541 (+00)	-,21606 (+00)
1.8	,80074 (+00)	,80074 (+00)	-,27581 (+00)
1.9	,77055 (+00)	,77055 (+00)	-,32647 (+00)
2.0	,73576 (+00)	,73576 (+00)	-,36788 (+00)
2.1	,69728 (+00)	,69728 (+00)	-,40011 (+00)

t	V(-2.o/t) Lie Series	V(-2.o/t) Table /33/	V'(-2.o/t) Lie Series
o.o	-,45748 (+oo)	-,45748 (+oo)	-,65600 (+oo)
o.1	-,51829 (+oo)	-,51829 (+oo)	-,55830 (+oo)
o.2	-,56878 (+oo)	-,56877 (+oo)	-,44973 (+oo)
o.3	-,60796 (+oo)	-,60796 (+oo)	-,33280 (+oo)
o.4	-,63515 (+oo)	-,63515 (+oo)	-,21021 (+oo)
o.5	-,64991 (+oo)	-,64991 (+oo)	-,84769 (-o1)
o.6	-,65210 (+oo)	-,65210 (+oo)	,40696 (-o1)
o.7	-,64186 (+oo)	-,64186 (+oo)	,16344 (+oo)
o.8	-,61959 (+oo)	-,61959 (+oo)	,28089 (+oo)
o.9	-,58594 (+oo)	-,58594 (+oo)	,39072 (+oo)
1.o	-,54177 (+oo)	-,54177 (+oo)	,49093 (+oo)
1.1	-,48813 (+oo)	-,48813 (+oo)	,57986 (+oo)
1.2	-,42621 (+oo)	-,42621 (+oo)	,65631 (+oo)
1.3	-,35731 (+oo)	-,35731 (+oo)	,71947 (+oo)
1.4	-,28277 (+oo)	-,28278 (+oo)	,76899 (+oo)
1.5	-,20396 (+oo)	-,20396 (+oo)	,80496 (+oo)
1.6	-,12222 (+oo)	-,12222 (+oo)	,82786 (+oo)
1.7	-,38797 (-o1)	-,3880 (-o1)	,83855 (+oo)
1.8	,45127 (-o1)	,45127 (-o1)	,83822 (+oo)
1.9	,12853 (+oo)	,12852 (+oo)	,82834 (+oo)
2.o	,21053 (+oo)	,21053 (+oo)	,81060 (+oo)

Appendix

Solving Eq.(IV,33) by Means of Laplace Transformation by A.SCHETT.

We start from Eq.(IV,33), having the form:

$$F(t) = Z''(t) + \frac{1}{t_o+t} Z'(t) + \varkappa^2 Z(t) = 0 \qquad (1)$$

where $t_o > 0$ is the point at which the solution function is expanded.
Applying Laplace transformation

$$L\{F\} = 0 \qquad (2)$$

to this equation, one obtains:

$$\xi'(s) + \xi(s)\,\frac{\varkappa^2 t_o + s^2 t_o - s}{-\varkappa^2 - s^2} - \frac{s t_o Z(0)}{-\varkappa^2 - s^2} - \frac{t_o Z'(0)}{-\varkappa^2 - s^2} = 0 \qquad (3)$$

which is solved by:

$$\xi(s) = e^{t_o s - \frac{1}{2}\ln(\varkappa^2+s^2)} \int_{\infty}^{s} \left\{ e^{-r_o s + \frac{1}{2}\ln(\varkappa^2+s^2)} \cdot \frac{-sZ(0)t_o - t_o Z'(0)}{\varkappa^2+s^2} \right\} ds \qquad (4)$$

taking into account that $\xi(s=\infty) = 0$.
After some amount of work, one obtains for $\xi(s)$:

$$\xi(s) = \frac{sZ(o)}{\varkappa^2+s^2} + \frac{Z'(o)}{\varkappa^2+s^2} + \frac{1}{\varkappa^2+s^2} \sum_{\nu=1}^{\infty} \frac{1}{t_o^{\nu}} \frac{d^{(\nu)}}{ds^{(\nu)}} \frac{sZ(o)+Z'(o)}{\varkappa^2+s^2} \qquad (5)$$

The solution function Z(t) is given by:

$$Z(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{st} f(s)\,ds \qquad (6)$$

Making use of (34) and (35), (30) is solved by:

$$Z(t) = (\cos \varkappa t - \delta_1(t))Z(t=o) + \left(\frac{\sin \varkappa t}{\varkappa} - \delta_2(t)\right)Z'(t=o) \qquad (7)$$

with

$$-\delta_1(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\left\{e^{st}\frac{1}{\varkappa^2+s^2}\sum_{\nu=1}^{\infty}\frac{1}{t_o^{\nu}}\frac{d^{(\nu)}}{ds^{(\nu)}}\left[\frac{s}{\varkappa^2+s^2}\right]\right\} ds \qquad (8)$$

and

$$-\delta_2(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\left\{e^{st}\frac{1}{\varkappa^2+s^2}\sum_{\nu=1}^{\infty}\frac{1}{t_o^{\nu}}\frac{d^{(\nu)}}{ds^{(\nu)}}\left[\frac{1}{\varkappa^2+s^2}\right]\right\} ds \qquad (9)$$

Making use of the convolution and multiplications theorems of Laplace transformation theory, δ_1 and δ_2 can be evaluated:

$$\delta_1(t) = \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{t_o^{\nu}}\int_0^t \tau^{\nu}\, I_1(\varkappa\tau)I_o(\varkappa[t-\tau])d\tau \qquad (10)$$

$$\delta_2(t) = \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{t_o^{\nu}}\int_0^t \tau^{\nu} I_o(\varkappa\tau)I_o(\varkappa[t-\tau])d\tau \qquad (11)$$

where I_o is the zero order Bessel function and I_1 the first order Bessel function.

These integrals are tabulated (e.g., Tables of Integral Transforms, Vol.2, Erdélyi, Magnus and Oberhettinger, Tricomi, Mc Graw-Hill Book Company Inc.1954, S 354).

Remark: The recurrence formulas obtained by using Lie series (Eq.(IV,34)) do not allow the terms occurring in δ_1 and δ_2 to be ordered according to powers of $1/t_o$. This is advantageous for numerical calculations, as this factor produces a good convergence. The method of Laplace transformations yields δ_1 and δ_2 in a closed form, ordered according to powers of $1/t_o$. These are two advantages compared to the representation by means of recurrence formulas.

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