Aerospace Research Center

Final Report

STUDY OF REDUCED PROBLEMS OF THREE AND FOUR BODIES IN CONNECTION WITH CISLUNAR TRAJECTORIES

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SUMMARY OF WORK ACCOMPLISHED DURING FIRST THREE QUARTERS OF PROJECT

Initially, the equations of motion of the Reduced Problem of Four Bodies were formulated. These equations contain, as special cases, the equations which describe the reduced or elliptical three-body problem, the restricted (circular) four-body problem, and the restricted (circular) three-body problem. Following this, conditions under which the forcing function of the reduced four-body problem is periodic, were determined. Specifically, a necessary and sufficient condition for the forcing function to be periodic is that the periods of the two fundamental elliptical motions - Moon about the Earth, and Earth - Moon barycenter about the Sun - be commensurable. Finally, the existence of periodic solutions in the elliptical four-body problem about the libration points of the restricted three-body problem was proven for sufficiently small values of the parameter μ. Furthermore, symmetry arguments were used to prove the existence of a symmetric periodic solution about the collinear libration points of the restricted four-body model, and, under added assumptions, of the elliptical four-body model.

Details of these results are described in the First, Second and Third Quarterly Reports, Contract NAS 5-9169, Goddard Space Flight Center, 1965-1966.

During the fourth quarter of the project, work was begun to apply the Birkhoff rate-of-growth concept to study motion near periodic trajectories of time-dependent Hamiltonian systems. This will enable us to obtain new qualitative information about motion in the reduced or elliptical three-body problem as well as in more general Hamiltonian systems. The preliminary analysis is discussed in detail in the following pages.

FOURTH QUARTERLY REPORT

The equations of motion of the planar elliptic three-body problem may be written, [1],

$$\ddot{x} - 2\dot{y} = \frac{1}{1+e\cos t}\,\frac{\partial U(x,y)}{\partial x},$$
$$\ddot{y} + 2\dot{x} = \frac{1}{1+e\cos t}\,\frac{\partial U(x,y)}{\partial y}, \qquad (1)$$

where
$$U = \frac{1-m}{\sigma_{o1}} + \frac{m}{\sigma_{o2}} + \tfrac{1}{2}(x^2+y^2) .$$

These equations can be put into Hamiltonian form by introducing additional variables defined by

$$u = \dot{x} - y ,$$
$$v = \dot{y} - x .$$

We now may write (1) as

$$\dot{x} = H_u(x,y,u,v)$$
$$\dot{y} = H_v(x,y,u,v)$$
$$\dot{u} = -H_x(x,y,u,v)$$
$$\dot{v} = -H_y(x,y,u,v) ,$$

where

$$H = \tfrac{1}{2}(u^2+v^2) + \tfrac{1}{2}(x^2+y^2) + (uy - xv) - \frac{1}{1+e\cos t}\,U(x,y) .$$

Expanding H about the L_4 point, which without loss of generality, we will assume to be the origin of coordinates, we obtain

$$H = H_o(t) + H_2(x,y,u,v,t) + \ldots + H_n(x,y,u,v,t) + \ldots , \qquad (2)$$

where H_n is a homogeneous polynomial of degree n in the variables x,y,u,v, with coefficients having period 2π in t. As $H_o(t)$ plays no role in the analysis, we will not consider it in the sequel. An application of the following lemma allows us to eliminate the time dependence from H_2.

Lemma

Let us represent a linear Hamiltonian system of dimension $2n$ with a periodic time dependence by

$$\dot{w} = A(t)w, \quad A(t+2\pi) = A(t) . \tag{3}$$

Then there exists a canonical change of variables

$$w = P(t)z, \quad P(t+2\pi) = P(t) , \tag{4}$$

such that

$$\dot{z} = Bz , \tag{5}$$

where B is a constant matrix, if the characteristic exponents associated with (3) are distinct.

Proof

By Floquet theory, [2], the fundamental matrix $\Phi(t)$ of (3) with $\Phi(o) = I$, can be represented as

$$\Phi(t) = P(t) \exp(Bt) , \tag{6}$$

where

$$B = \frac{1}{2\pi} \Phi(2\pi) ,$$

and

$$P(t+2\pi) = P(t) .$$

It can then be verified that the change of variables (4) gives (5). We now demonstrate that (4) is a canonical change of variables. A sufficient condition for this is that P satisfy

$$X'JX = J , \tag{7}$$

where $'$ denotes the transpose and where

$$J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} .$$

As the set of matrices that satisfy (7) form a group, it follows from (6) that it is sufficient to show that $\Phi(t)$ and $\exp(Bt)$ satisfy (7).

From (3) we obtain

$$\frac{d}{dt}(\Phi' J \Phi) = \dot{\Phi}' J \Phi + \Phi' J \dot{\Phi} = \Phi' A' J \Phi + \Phi' J A \Phi \tag{8}$$
$$= \Phi' [A' J + J A] \Phi .$$

Now as $A(t)$ is derived from a Hamiltonian function, we have that

$$A(t) = J S(t) \quad , \tag{9}$$

where S is a symmetric matrix. Substituting (9) in (8) we obtain

$$\frac{d}{dt}(\Phi' J \Phi) = O ,$$

and as $\Phi(o) = I$, $\quad \Phi' J \Phi = J$.

To show that $\exp(Bt)$ satisfies (7), we note that for any non-singular matrix S,

$$\exp(Bt) = S[\exp(S^{-1} B S t)] S^{-1} .$$

As the characteristic exponents of our system are distinct, it can be shown, [3], that there exists such an S which satisfies (7) and is such that

$$S^{-1} B S = D,$$

where D is diagonal. Furthermore, as we have a Hamiltonian system, the eigenvalues of D are $\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n$. Thus, by the group property of canonical matrices, the proof has been reduced to showing that $\exp(Dt)$ satisfies (7). This follows readily from the form of the eigenvalues of D.

By applying the linear, canonical change of variables

$$w = P(t) S r ,$$

where

$$r' = (p_1, p_2, q_1, q_2) \qquad \text{and} \qquad w' = (x, y, u, v)$$

it follows readily from the above lemma that the equations defined by H take the form

$$\dot{r} = D r + \ldots \quad . \tag{10}$$

Thus in the new variables we have

$$\hat{H} = \lambda_1 p_1 q_1 + \lambda_2 p_2 q_2 + \ldots \quad .$$

Equations (10) are now integrable if terms of degree two and higher are neglected. This linear approximation is good over some fixed finite time interval if we restrict ourselves to a sufficiently small neighborhood of the L_4 point. To improve on this approximation, we will apply a theorem of Birkhoff, [4], which will enable us to obtain better estimates by considering the nonlinear terms.

Theorem

Let the Hamiltonian $H(x,y,t)$ of a dynamical system with an equilibrium point at the origin, be analytic in x and y, periodic in t of period 2π, and represented by

$$H(x,y,t) = \sum_{i=1}^{n} \lambda_i x_i y_i + \sum_{v_1 + v_2 + \dots + v_{2n} = 3}^{\infty} a_{v_1, v_2, v_3, v_4}(t)\, x_1^{v_1} x_2^{v_2} \dots x_n^{v_n} y_1^{v_1} y_2^{v_2} \dots y_n^{v_n}$$

$$= H_2(x,y) + H_3(x,y,t) + \dots + H_k(x,y,t) + \dots \quad .$$

Let the $2n$ characteristic exponents associated with H_2 be distinct and purely imaginary. As the system is Hamiltonian, they maybe represented as

$$\lambda_1, \dots, \lambda_n, -\lambda_1, \dots -\lambda_n \quad .$$

Furthermore, let the exponents satisfy

$$m_1 \lambda_1 + m_2 \lambda_2 + \dots + m_n \lambda_n + m_{n+1}\, i \neq o$$

for all integers m_i such that

$$o < |m| = \sum_{i=1}^{n} |m_i| \leq N \leq 3.$$

Then there exists a canonical change of variables

$$x_v = f_v(\xi, \eta, t) = \xi_v + \ldots$$
$$v = 1, \ldots, n,$$
$$y_v = g_v(\xi, \eta, t) = \eta_v + \ldots$$

where f_v and g_v are convergent power series in the components of ξ and η with coefficients having period 2π in t, such that the Hamiltonian in the new variables has the form

$$\widetilde{H}(\xi, \eta, t) = \widetilde{H}_1(\xi_1 \eta_1, \xi_2 \eta_2, \ldots, \xi_n \eta_n) + \widetilde{H}_2(\xi, \eta, t) ,$$

where $\widetilde{H}_1$ is a polynomial with constant coefficients of degree N if N is even and degree $N-1$ if N is odd in the variables $z_v = \xi_v \eta_v$, and where $\widetilde{H}_2(\xi, \eta, t)$ is a power series in ξ_v, η_v, beginning with terms of degree $N+1$. With $\widetilde{H}$ in this form, the Hamiltonian is said to be normalized up to order N.

The usefulness of this theorem is the following. If we are studying solutions near the equilibrium point $\xi = \eta = 0$, then $\widetilde{H}_2$ is of higher order than $\widetilde{H}_1$ and is discarded for the moment. The equations then have the form

$$\dot{\xi}_v = (\widetilde{H}_1)_{\eta_v}(z) = \xi_v (\widetilde{H}_1)_{z_v}$$
$$v = 1, \ldots, n. \qquad (11)$$
$$\dot{\eta}_v = -(\widetilde{H}_1)_{\xi_v}(z) = -\eta_v (\widetilde{H}_1)_{z_v}$$

If we multiply the first equation by η_v, the second by ξ_v, and add, it follows that

$$\frac{d}{dt}(\xi_v \eta_v) = 0, \quad v = 1, \ldots, n.$$

Thus, $\xi_v \eta_v = c_v$ (constant) so that (4) becomes integrable yielding

$$\xi_v = \xi_v(o) e^{i(\widetilde{H}_1)_{z_v}(c)t}$$
$$v = 1, \ldots, n. \qquad (12)$$
$$\eta_v = \eta_v(o) e^{-i(\widetilde{H}_1)_{z_v}(c)t}$$

If we restrict ourselves to a large finite time interval and a suitable region in phase space it can be shown that the higher order terms previously truncated can be made small so that (5) is a close representation to the actual solution in this region. By use of (3) approximate solutions to the original problem may be obtained. For precise statements along these lines see [4] and [5].

We will consider the case of $n = 2$ for notational convenience, and prove the theorem by successively introducing canonical changes of variables that normalize terms of degree $3,4,\ldots,N$ respectively. Thus let $\widetilde{H}$ be normalized up to order $S - 1$. (We note that $\widetilde{H}_2$ is already in normal form).

$$\widetilde{H}(\xi,\eta,t) = \lambda_1\xi_1\eta_1 + \lambda_2\xi_2\eta_2 + \widetilde{H}_1(\xi_1\eta_1, \xi_2\eta_2) \tag{13}$$

$$+ \sum_{v_1+v_2+v_3+v_4=S} h_{v_1,v_2,v_3,v_4}(t)\, \xi_1^{v_1} \xi_2^{v_2} \eta_1^{v_3} \eta_2^{v_4} + \ldots$$

We introduce a canonical change of variables defined by the contact transformation, [4],

$$\widetilde{\xi}_k = \xi_k + \frac{\partial V}{\partial \widetilde{\eta}_k} \qquad k = 1,2, \tag{14}$$

$$\eta_k = \widetilde{\eta}_k + \frac{\partial V}{\partial \xi_k}$$

where $V(\xi_k, \widetilde{\eta}_k)$ has the form

$$V = \sum_{v_1+v_2+v_3+v_4=S} c_{v_1,v_2,v_3,v_4}(t)\, \xi_1^{v_1} \xi_2^{v_2} \widetilde{\eta}_1^{v_3} \widetilde{\eta}_2^{v_4} \tag{15}$$

We attempt to choose $c_{v_1,v_2,v_3,v_4}(t)$ so as to eliminate as many $S^{\underline{th}}$ order terms as possible from (13). The Hamiltonian, under change of variables (14), takes the form in the new variables

$$\widetilde{\widetilde{H}} = \lambda_1 \widetilde{\xi}_1 \widetilde{\eta}_1 + \lambda_2 \widetilde{\xi}_2 \widetilde{\eta}_2 + \lambda_1 (\widetilde{\xi}_1 \frac{\partial V}{\partial \widetilde{\xi}_1} - \widetilde{\eta}_1 \frac{\partial V}{\partial \widetilde{\eta}_1}) + \lambda_2 (\widetilde{\xi}_2 \frac{\partial V}{\partial \widetilde{\xi}_2} - \widetilde{\eta}_2 \frac{\partial V}{\partial \widetilde{\eta}_2}) \tag{16}$$

$$+ \frac{\partial V}{\partial t} + \widetilde{H}_1 (\widetilde{\xi}_1 \widetilde{\eta}_1 , \widetilde{\xi}_2 \widetilde{\eta}_2) + \sum_{v_1 + v_2 + v_3 + v_4 = S} g_{v_1 , v_2 , v_3 , v_4} (t) \widetilde{\xi}_1^{v_1} \widetilde{\xi}_2^{v_2} \widetilde{\eta}_1^{v_3} \widetilde{\eta}_2^{v_4}$$

$$+ \ldots \; .$$

Collecting S^{th} order terms in $\widetilde{\xi}_1^{v_1} \widetilde{\xi}_2^{v_2} \widetilde{\eta}_1^{v_3} \widetilde{\eta}_2^{v_4}$ in (16) and using (15) we obtain for a typical coefficient ($c_{v_1 , v_2 , v_3 , v_4} = c_v$)

$$\frac{d c_v}{d t} + c_v [\lambda_1 (v_1 - v_3) + \lambda_2 (v_2 - v_4)] + g_v \; . \tag{17}$$

We wish to make (17) as simple as possible by choosing c_v appropriately. From the hypothesis of the theorem $\lambda_1 (v_1 - v_3) + \lambda_2 (v_2 - v_4)$ can vanish only if $v_1 = v_3$, $v_2 = v_4$. If this is the case, we let

$$c_v (t) = - \int_0^t g_v (s) \, ds + [\frac{1}{2\pi} \int_0^{2\pi} g_v (s) \, ds] t \; .$$

It follows that $c_v (t)$ is periodic and (17) reduces to the constant

$$\frac{1}{2\pi} \int_0^{2\pi} g_v (s) \, ds .$$

We note that such terms are of the form of products of $\tilde{\xi}_1 \tilde{\eta}_1$ and $\tilde{\xi}_2 \tilde{\eta}_2$ which are in the desired normal form.

If $\lambda_1 (v_1 - v_3) + \lambda_2 (v_2 - v_4)$ doesn't vanish, it follows from the hypothesis of the theorem that there exists a unique periodic $c_v(t)$ such that (17) vanishes. Thus, such terms can be completely eliminated from the new Hamiltonian and the desired normal form is obtained.

Summing up, we see that by a combination of algebraic manipulation involving polynomials whose coefficients are periodic and by integrating linear equations, we can obtain useful information about solutions near critical points of nonlinear Hamiltonian systems. To perform the operations required to obtain the normal form, a computer program will be written.

References

[1] First Quarterly Report, Contract NAS 5-9169, General Precision, Inc. Little Falls, New Jersey August 1965

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[4] Birkhoff, G. D., "Dynamical Systems", New York 1927

[5] Siegel, C. L., "Vorlesungen über Himmelsmechanik", Springer, Berlin, 1956