Propagation of Spherical Waves Through an Inhomogeneous Medium Containing Anisotropic Irregularities

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Chao-han Liu

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Abstract

Amplitude and phase fluctuations of a spherical wave and amplitude and phase correlation functions of two spherical waves propagating in an inhomogeneous medium containing anisotropic irregularities are calculated in this paper. With ionospheric propagation application in mind (e.g., speced-receiver experiments), the medium is assumed to be characterized by a dielectric permittivity which consists of two parts: an average permittivity which varies as a function of height and a part which is a random function of position. For radio frequency waves and weakly random irregularities, methods of WKB approximation and small perturbation are used to derive general expressions for fluctuations and correlations. Calculations are carried out for a specific case where the background medium in a parabolic plasma layer with small anisotropic irregularities imbedded in it. Results are compared with the expressions derived for a homogeneous medium containing anisotropic irregularities. It is found that the corrections due to the regular inhomogeneities of the parabolic layer depend on various things: the position of the irregularity slab, the thickness of the parabolic layer, the space between the observation points, the ratio of the maximum plasma frequency for the layer to the applied frequency, and the random properties of the irregular region. The correction may be as large as 15% of the values for the homogeneous case for some experimental situations. In general, the

(one)

correction for the fluctuations is maximum when the irregularity slab is at the peak of the electron density profile.

1. Introduction

The study of wave propagation in a medium with inhomogeneities has long been a problem of theoretical as well as practical interests in many branches of physics. In general, one assumes that the medium can be characterized macroscopically by some parameters, such as the refractive index, the density of the medium, etc., which are functions of space and time. Many authors have studied this problem in connection with tropospherical as well as ionospherical electromagnetic wave propagation, underwater sound wave propagation, or earth seismic wave propagation [Pekeris, 1946; Budden, 1961]. One aspect of the problem is to study the wave equation when the medium is characterized by a smoothly varying parameter, such as the permittivity $\epsilon(r)$ in the case of electromagnetic wave propagation. Special attention has been given to the so-called stratified medium, a medium in which properties vary only in one particular direction. Investigations of this problem have led to many important results such as normal mode theory of wave propagation, surface waves, etc. [Brekhovskikh, 1960; Wait, 1962]. Another aspect of the problem is to investigate the scattering of waves by randomly spaced small inhomogeneities, or irregularities. This sometimes is called wave propagation in a random medium. This kind of medium is in general characterized by a parameter of the form $1 + \epsilon_1(\vec{r})$ where $\epsilon_1(\vec{r})$ is a random variable of position. For weakly random media, perturbation methods have been used successfully to calculate fluctuations and correlations of the waves [Chernov, 1960; Tatarski, 1961; Keller, 1964]. It is noticed that in this problem, the "background" medium is assumed to be homogeneous. These two aspects of the problem can be categorized as wave propagation in a medium with regular inhomogeneities and wave propagation in a medium with inhomogeneities of a random character. Investigations in both aspects are far from complete. But, in general, the medium in reality is one which contains inhomogeneities of both types. The problem is then an even more complex one. For electromagnetic case, the medium can be characterized by a permittivity $\epsilon(\vec{r}) + \epsilon_1(\vec{r})$, where $\epsilon(\vec{r})$ is just a function of space and $\epsilon_1(\vec{r})$ is a random variable of position. Very little work has been done along this line, mainly because of its enormous complexity [Chen, 1964]. Yet, in order to investigate wave propagation in, as well as the properties of, a true medium more closely, this problem is of very basic importance.

In this report, an attempt is made to study this problem with the application to wave propagation through ionospheric irregularities in mind. The problem is formulated in the usual way so that method of small perturbation can be applied for weakly random inhomogeneities. The average permittivity of the medium is assumed to vary only as a function of z. Superimposed on this regular inhomogeneous "background" is a weakly random inhomogeneity characterized by $\epsilon_1(\vec{r})$. A Green's function for a point source in the regular inhomogeneous medium will be derived and used in the calculation of fluctuations and correlations. In the derivation, WKB approximation will be applied to calculate the explicit form of the Green's function and small perturbation method will be used to derive the statistical expressions for the fields. A special example relevant to the ionospherical propagation case will be studied and the results will be compared with those derived for the case of homogeneous background [Yeh, 1962; Liu, 1966].

2. Formulation

To start with, an infinite, unbounded medium is assumed to be characterized by a macroscopic relative dielectric permittivity in the most general sense:

$$\underset{\approx}{\epsilon}(\vec{x}) = \underset{\approx}{\epsilon}_0(\vec{x}) + \underset{\approx}{\epsilon}_1(\vec{x}) \tag{1}$$

where both the inhomogeneous and anisotropic properties of the background medium are characterized by the tensor $\underset{\approx}{\epsilon}_0$, while $\underset{\approx}{\epsilon}_1$ represents the randomness of the medium. Each component of the tensor $\underset{\approx}{\epsilon}_1$ is a random function of position. For weakly random media, $\epsilon_{0ij} \gg \epsilon_{1ij}$. Furthermore, it is assumed that $\langle \epsilon_{1ij} \rangle = 0$, where $\langle\,\rangle$ denotes the average. With harmonic time variation $e^{-i\omega t}$ in mind, the electric field is governed by the equation

$$L\,\vec{E}(\vec{x}) = [\nabla^2 \underset{\approx}{I} - \nabla\nabla + k_0^2\,\underset{\approx}{\epsilon}_0(\vec{x})] \cdot \vec{E}(\vec{x}) = -k_0^2\,\underset{\approx}{\epsilon}_1 \cdot \vec{E}(\vec{x}) \qquad (2)$$

and the radiation condition, where $\underset{\approx}{I}$ is the unit dyad and k_0 is the free space wave number.

Define $$\vec{E}_1(\vec{x}) = \vec{E}(\vec{x}) - \vec{E}_0(\vec{x}) \qquad (3)$$

where $$\vec{E}_0(\vec{x}) = \langle \vec{E}(\vec{x}) \rangle \qquad (4)$$

For weakly random medium, $|E_1| \ll |E_0|$. When (3) is substituted into (2) and terms of the same order of magnitude are collected, the following can be obtained:

Zeroth order: $$L\,\vec{E}_0(\vec{x}) = 0 \qquad (5)$$

First order: $$L\,\vec{E}_1(\vec{x}) = -k_0^2\,\underset{\approx}{\epsilon}_1(\vec{x}) \cdot \vec{E}_0(\vec{x}) \qquad (6)$$

In general, eq. (5) and the proper radiation condition define the average field $\vec{E}_0$ uniquely. To find the fluctuation $\vec{E}_1$ to the first order, operate on (6) by L^{-1}, the inverse operation of L (assume its existence), hence, from (6),

$$\vec{E}_1(\vec{x}) = -k_0^2\,L^{-1}\,\underset{\approx}{\epsilon}_1(\vec{x}) \cdot \vec{E}_0(\vec{x}) \qquad (7)$$

In terms of the dyadic Green's function, eq. (7) can be written as

$$\vec{E}_1(\vec{x}) = -k_0^2 \int_{V'} \underset{\approx}{G}(\vec{x},\vec{x}') \cdot \vec{E}_0(\vec{x}')\, d^3\vec{x}' \tag{7a}$$

where $\underset{\approx}{G}$ satisfies

$$L\,\underset{\approx}{G} = \underset{\approx}{I}\,\delta(\vec{x}-\vec{x}') \tag{8}$$

The mean square values of fluctuation and correlation functions of the field can be calculated once $\vec{E}_1$ is obtained from (7a). For an isotropic, homogeneous background medium such that $\epsilon_0 = 1$, this problem has been treated by many authors [Chen, 1964; Yeh, 1962; Liu, 1966]. In the following, the case of an isotropic, inhomogeneous medium will be considered. In particular, the permittivity of the medium is assumed to be given by

$$\epsilon(\vec{x}) = \epsilon_0(\vec{z}) + \epsilon_1(\vec{x}) \tag{1a}$$

where $\epsilon_0(z)$ is a function of z only, corresponding to a stratified medium. For this case, the operator L in (2) can be written as

$$L = [\nabla^2 + k_0^2\,\epsilon_0(z)]\,\underset{\approx}{I} + \nabla\,\frac{\nabla\,\epsilon}{\epsilon} \tag{9}$$

The relation $\nabla \cdot \vec{D} = \nabla \cdot (\epsilon\vec{E}) = 0$ has been used in deriving this expression.

If (1) the spatial variation of $\epsilon_0(z)$ is small in one wavelength, (2) the correlation length of $\epsilon_1(\vec{x})$ is large compared to the wavelength and (3) $\epsilon(\vec{x})$ does not vanish in the region of interest, then, the last term of eq. (9) can be neglected. Therefore, in the following, instead of the vector wave equation, a scalar wave equation will be studied:

$$L_0\,\psi(\vec{x}) = [\nabla^2 + k_0^2\,\epsilon_0(z)]\,\psi(\vec{x}) = -k_0^2\,\epsilon_1(\vec{x})\,\psi(\vec{x}) \tag{10}$$

where $\psi(\vec{x})$ may be any component of the electric field.

Following the usual procedure [Chen, 1964; Yeh, 1962; Liu, 1966], define:

$$\psi(\vec{x}) = \psi_0(\vec{x})\, e^{-i\phi_1(\vec{x})} = \psi_0(\vec{x})\, e^{-iU(\vec{x})/\psi_0(\vec{x})} \qquad (11)$$

where $\psi_0(\vec{x})$ is the solution of $L_0\psi_0(\vec{x}) = 0$.

Substituting (11) into (10), the following equation for $U(\vec{x})$ is obtained:

$$L_0 U(\vec{x}) = -ik_0^2\, \epsilon_1(\vec{x})\, \psi_0(\vec{x}) \qquad (12)$$

Eqs. (11) and (12) constitute the basic equations for the following analysis.

3. Green's Function

Consider a medium in which $\epsilon_0(z)$ is a positive continuous function of z and reaches constant values as z approaches + ∞ and − ∞. The boundary condition for the wave is the usual "outgoing wave" at ± ∞. The Green's function of the operator L_0 satisfies:

$$L_0\, G(\vec{x},\vec{x}') = -4\pi\, \delta(\vec{x}-\vec{x}') \qquad (13)$$

where the factor -4π is added for convenience. (Not consistent with G defined by (8).) Take the Fourier transform of (13)

$$\left[\frac{d^2}{dz^2} + k_0^2\, \epsilon_0(z) - k^2\right] g(k_x,k_y,z,\vec{x}') = -\frac{1}{\pi}\, e^{-ik_x x' - ik_y y'}\, \delta(z-z') \qquad (14)$$

where

$$g(k_x,k_y,z,\vec{x}') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} e^{-ik_x x - ik_y y}\, G(\vec{x},\vec{x}')\, dxdy \qquad (15)$$

$$G(\vec{x},\vec{x}') = \iint_{-\infty}^{+\infty} e^{ik_x x + ik_y y}\, g(k_x,k_y,z,\vec{x}')\, dk_x dk_y \qquad (16)$$

$$k^2 = k_x^2 + k_y^2$$

Let $u_1(z)$ and $u_2(z)$ be the two independent solutions of the homogeneous equation:

$$\frac{d^2u}{dz^2} + [k_0^2 \epsilon_0(z) - k^2]\, u = 0 \tag{18}$$

where $u_1(z)$ and $u_2(z)$ satisfy the boundary conditions at $z \to -\infty$ and $z \to +\infty$ respectively.

Then the solutions of equation (14) can be written as [Friedman, 1956]

$$g(k_x, k_y, z, z') = \frac{e^{-ik_x x' - ik_y y'}}{\pi\, J(u_1 u_2)} [u_1(k,z)\, u_2(k,z')\, H(z'-z) + u_2(k,z) u_1(k,z) H(z-z')] \tag{19}$$

where H(z) is the Heaviside step function and J is the Jocobian of u_1 and u_2. From eq. (16), the Green's function is:

$$G(\vec{x},\vec{x}') = \iint_{-\infty}^{+\infty} \frac{e^{ik_x(x-x')+ik_y(y-y')}}{\pi\, J(u_1 u_2)} [u_1(k,z)u_2(k,z')H(z'-z) +$$

$$+ u_1(k,z')u_2(k,z')\, H(z-z')] \cdot dk_x dk_y \tag{20}$$

Change variables to:

$$k_x = k_0 \sin\Theta \cos\varphi, \qquad k_y = k_0 \sin\Theta \sin\varphi, \qquad k_z = k_0 \cos\Theta \tag{21}$$

$$x-x' = r\cos\varphi_1\,, \qquad y-y' = r\sin\varphi_1 \tag{22}$$

φ varies from 0 to 2π, Θ varies from 0 to $\frac{\pi}{2} - i\infty$. Eq. (20) becomes:

$$G(\vec{x},\vec{x}') = \int_0^{\frac{\pi}{2}-i\infty} d\Theta \int_0^{2\pi} d\varphi \frac{u_1(z)u_2(z')H(z'-z)+u_1(z')u_2(z)H(z-z')}{\pi\, J(u_1,u_2)} \cdot$$

$$\cdot\, e^{ik_0 r \sin\Theta \cos(\varphi-\varphi_1)} \cdot k_0^2 \cos\Theta \sin\Theta \tag{23}$$

The φ-integration yields [Watson, 1958]:

$$G(\vec{x},\vec{x}') = k_0^2 \int_{\Gamma_1} H_0^{(1)}(v) \frac{[u_1(\theta,z)u_2(\theta,z')H(z'-z)+u_1(\theta,z')u_2(\theta,z)H(z-z')]}{J(u_1,u_2)} \sin\theta\cos\theta d\theta \tag{24}$$

where $H_0^{(1)}(v)$ is the Hankel function of first kind and zeroth order, $v = k_0 r \sin\theta$ and Γ_1 is the path of integration shown in Figure 1.

This Green's function represents the field generated by a point source at (x',y',z') in an unbounded inhomogeneous medium. When the medium becomes homogeneous, it reduces to the well-known expression for spherical wave.

Since the original assumption for this problem is that $\epsilon_0(z)$ varies slowly in one wave length, the solution u_1 and u_2 of eq. (18) can be represented by its WKB approximations:

$$u_1(z) \cong \frac{1}{k_0^{1/2} q^{1/2}} e^{-ik_0 \int^z q(\tau)d\tau} \tag{25}$$

$$u_2(z) \cong \frac{1}{k_0^{1/2} q^{1/2}} e^{+ik_0 \int^z q(\tau)d\tau} \tag{26}$$

where $q^2(z) = \epsilon_0(z) - \sin^2\theta.$ (27)

In the neighborhood of the turning points, points where q vanishes, the WKB solutions are no longer valid. If in the integration of eq. (24), we change the path Γ_1 to Γ_2 (Figure 2) such that on Γ_2 q does not approach zero, then on this new path of integration, we can approximate u_1 and u_2 by eqs. (25) and (26) all the time. The original integration along Γ_1 is now equal to:

$$\int_{\Gamma_1} = \int_{\Gamma_2} + \int_C \tag{28}$$

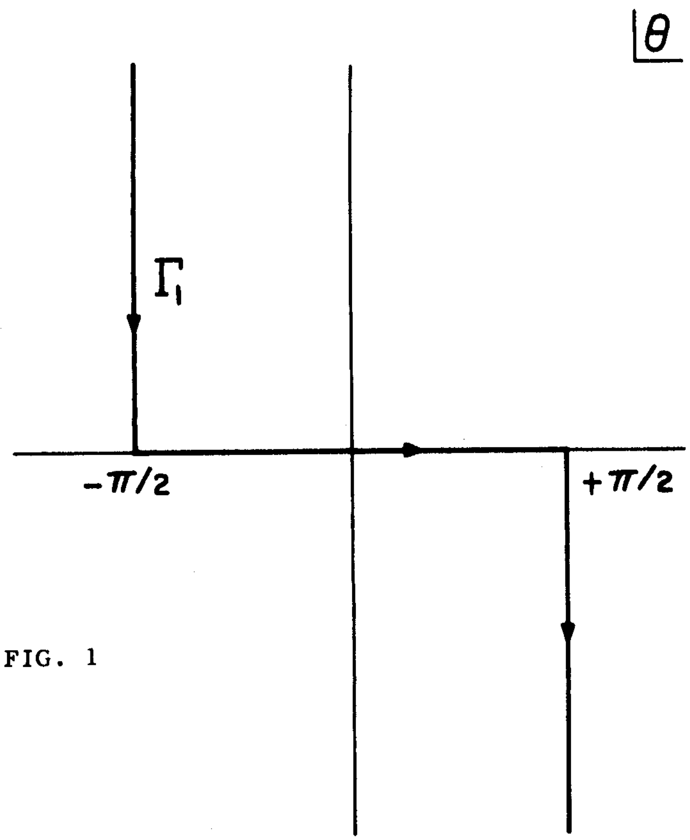

FIG. 1

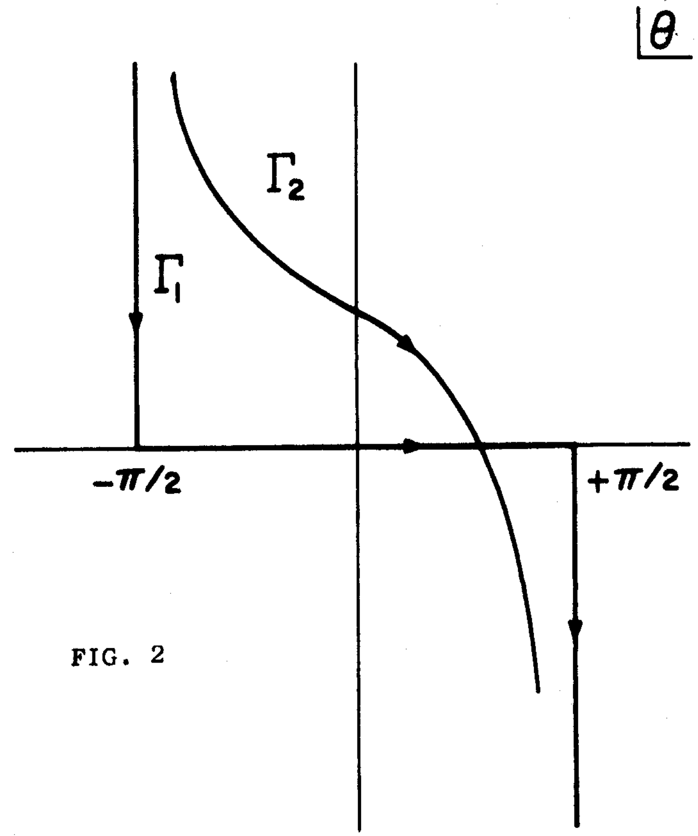

FIG. 2

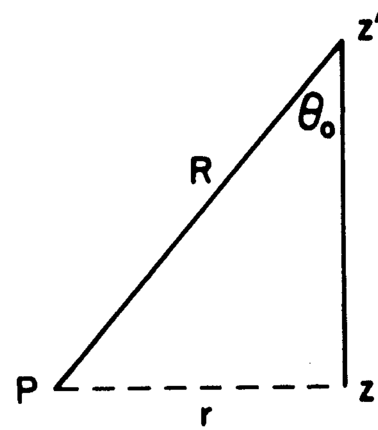

FIG. 3

where $\int_C$ represents the contributions from the poles or branch points of the integrand of eq. (24) between the two paths Γ_1 and Γ_2 (Figure 2). These contributions correspond to the normal modes and lateral modes. If one only considers problems with frequency high enough so that most of the energy from the source is transmitted outside of the inhomogeneous region, only a very small number of rays, coming out of the source, with directions almost parallel to the x-y plane, are "trapped" in the inhomogeneous medium. The Green's function can then be approximated by the contribution from Γ_2 alone:

$$G(\vec{x},\vec{x}') \cong \frac{ik_0}{2} \int_{\Gamma_2} \frac{H_0^{(1)}(v)}{[q(z)q(z')]^{1/2}} \left[e^{-ik_0 \int_{z'}^{z} q(\tau)d\tau} H(z'-z) + e^{-ik_0 \int_{z'}^{z} q(\tau)d\tau} H(z-z') \right] \cdot$$

$$\cdot \sin\theta \cos\theta \, d\theta \tag{29}$$

For far field outside the region where those "trapped" modes are important, $k_0 r = k_0 R \sin\theta_0$ is large, (Figure 3), asymptotic expression for $H_0^{(1)}(v)$ can be used. Eq. (29) is then reduced to:

$$G(\vec{x},\vec{x}') \cong \begin{cases} i\sqrt{\dfrac{k_0}{2\pi R\sin\theta_0}}\, e^{-i\frac{\pi}{4}} \displaystyle\int_{\Gamma_2} e^{ik_0[R\sin\theta_0\sin\theta - w(\theta)]} \sin\theta \dfrac{\cos\theta d\theta}{[q(z)q(z')]^{1/2}} & \text{for } -\infty < z < z' \\ -i\sqrt{\dfrac{k_0}{2\pi R\sin\theta_0}}\, e^{-i\frac{\pi}{4}} \displaystyle\int_{\Gamma_2} e^{ik_0[R\sin\theta_0\sin\theta + w(\theta)]} \sin\theta \dfrac{\cos\theta d\theta}{[q(z)q(z')]^{1/2}} & \text{for } z' < z < +\infty \end{cases} \tag{30}$$

where

$$w = \int_{z'}^{z} q(\tau)d\tau \tag{31}$$

The geometry of the source and field points is shown in Figure 3. Note that in this choice of R and θ_0

$$z - z' = R\cos\Theta_0 \qquad z > z'$$

$$z - z' = -R\cos\Theta_0 \qquad z < z' \tag{32}$$

$$r = R\sin\Theta_0 \qquad z < z'$$

Θ_0 varies from 0 to $\pi/2$.

As in the case for homogeneous medium, method of steepest descent can be used to evaluate the integral (30). Once the Green's function is obtained, the solution of eq. (12) can be expressed by:

$$U(\vec{x}) = \frac{ik_0^2}{4\pi} \int_{V_1} \epsilon_1(\vec{x}')\,\psi_0(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3\vec{x}' \tag{33}$$

The total field is then:

$$\psi(\vec{x}) = \psi_0(\vec{x})\,e^{-i\phi_1(\vec{x})} \tag{9}$$

where

$$\phi_1(\vec{x}) = \frac{ik_0^2}{4\pi} \int_{V'} \epsilon_1(\vec{x}')\,\frac{\psi_0(\vec{x}')}{\psi_0(\vec{x})}\,G(\vec{x},\vec{x}')\,d^3\vec{x}' \tag{34}$$

This field $\psi(\vec{x})$ is a random variable of position. The mean of $\psi(\vec{x})$ will be the unperturbed field $\psi_0(\vec{x})$, but there will be mean square fluctuations both in amplitude and phase of the field. The explicit form of the field depends on the spatial variation of the medium. In the next section, an example will be considered for waves in the ionosphere.

4. An Example

Consider a medium for which $\epsilon_0(z)$ varies as

$$\epsilon_0(z) = \begin{cases} 1 - X_m[1 - (z-z_0)^2/z_0^2] & 0 < z < 2z_0 \\ 1 & \text{otherwise} \end{cases} \tag{35}$$

where z_0 and X_m are positive constants. Eq. (35) corresponds to a plasma medium in which the electron density varies parabolically. For this medium,

$$X_m = \omega_{Nm}^2/\omega^2 \tag{36}$$

where ω_{Nm} is the plasma frequency at the peak of the electron density profile and ω is the frequency of the wave. If in some region of this plasma, the electron density has a random variation, then the dielectric constant will have a random part $\epsilon_1(\vec{x})$ in it. Figure 4 shows ϵ_0 as a function of z. For this medium, eq. (31) becomes:

$$w = \int_{z'}^{z} \left\{\cos^2\theta - X_m[1-(\tau-z_0)^2/z_0^2]\right\}^{1/2} d\tau \quad \text{for} \quad \begin{matrix} 0 < z < 2z_0 \\ 0 < z' < 2z_0 \end{matrix} \tag{37}$$

$$= \int_{z'}^{2z_0} \left\{\cos^2\theta - X_m[1-(\tau-z_0)^2/z_0^2]\right\}^{1/2} d\tau + \int_{2z_0}^{z} \cos^2\theta \, d\tau \quad \text{for} \quad \begin{matrix} 2z_0 < z \\ 0 < z' < 2z_0 \end{matrix} \tag{38}$$

If the frequency of the wave is high so that $X_m << 1$, then to the order of X_m, eqs. (37), (38) can be expressed as:

$$w(\theta) \cong z\cos\theta + X_m(z^3 - 3z^2 z_0)/6z_0^2\cos\theta \quad \text{for} \quad \begin{matrix} 0 < z < 2z_0 \\ z' = 0 \end{matrix} \tag{39}$$

$$w(\theta) \cong (z-z')\cos\theta - X_m(z'^3 - 3z'^2 z_0 + 4z_0^3)/6z_0^2\cos\theta \quad \text{for} \quad \begin{matrix} 2z_0 < z \\ 0 < z' < 2z_0 \end{matrix} \tag{40}$$

The field generated by a point source at the origin can then be evaluated by substituting eq. (39) into eq. (30). The field for $z > 0$ is

$$\psi_0(\vec{x}) \cong i\sqrt{\frac{k_0}{2\pi R\sin\theta_0}}\, e^{-i\frac{\pi}{4}} \int_{\Gamma_2} e^{k_0 R f(\theta)} \sqrt{\sin\theta}\, d\theta \qquad z > 0 \tag{41}$$

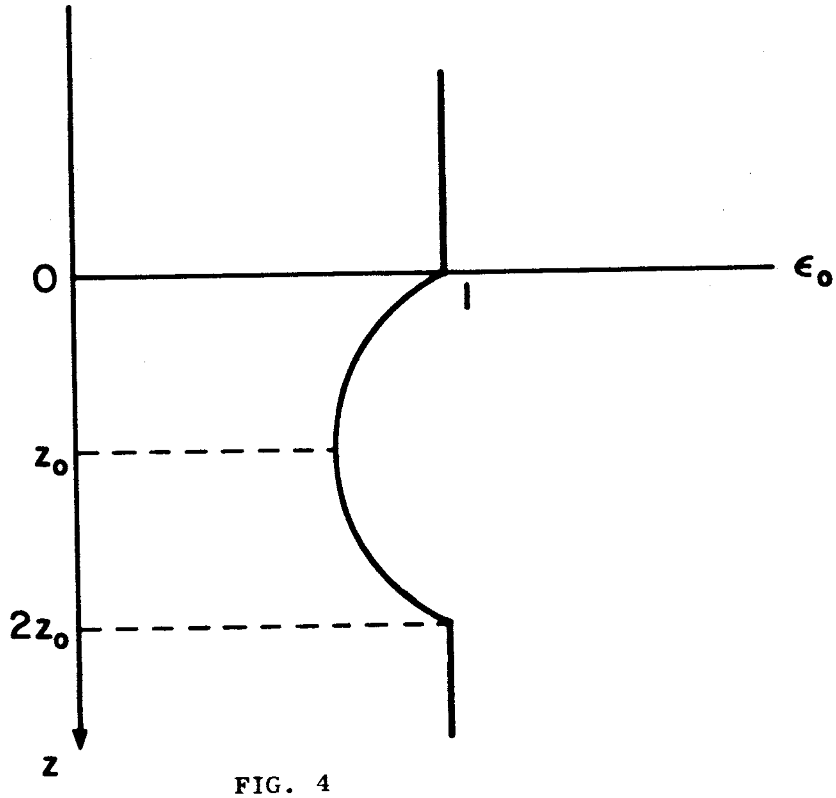

FIG. 4

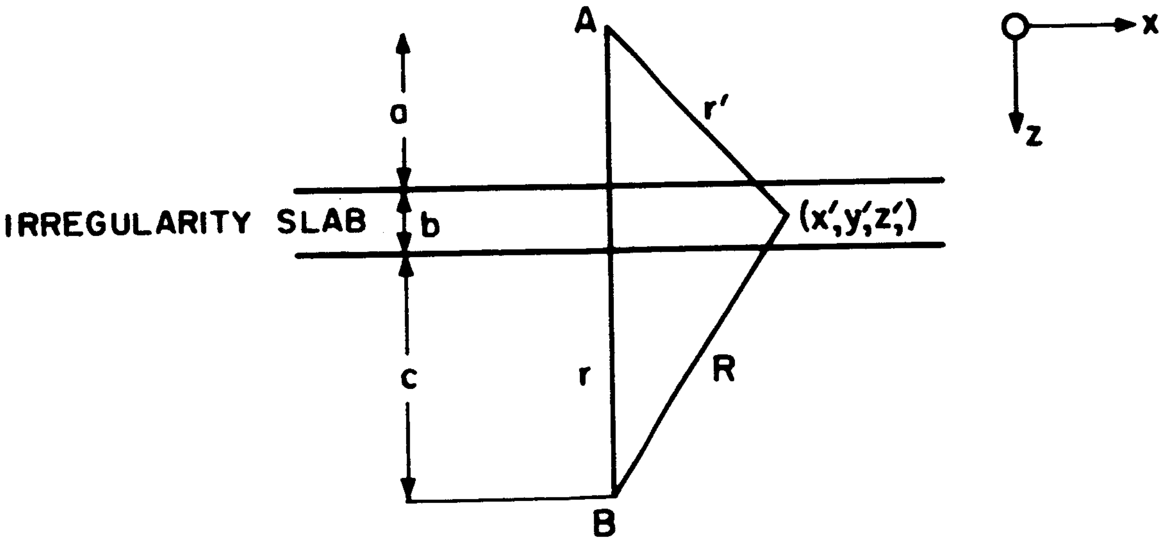

FIG. 5

where

$$f(\theta) = i[\sin\theta_0 \sin\theta + \frac{z}{R}\cos\theta + z^2 X_m(z - 3z_0)/6z_0^2 R \cos\theta] \quad (42)$$

and the approximation $[q(z)\, q(z')]^{1/2} \cong \cos\theta$ has been used for small X_m.

Method of steepest descent will be used for the calculation of eq. (41) for $k_0 R \gg 1$. From eq. (42), the saddle point corresponds to

$$\theta_s \cong \theta_0 + X_m(z^2 - 3zz_0)\sin\theta_0/6z_0^2 \cos\theta_0 \quad (43)$$

to the first order of X_m for $\theta_0 \ll \pi/2$. Expand $f(\theta)$ in the neighborhood of θ_s,

$$f(\theta) \cong f(\theta_s) + f''(\theta_s)(\theta - \theta_s)^2/2 + \ldots \quad (44)$$

where

$$f''(\theta_s) \cong -i[1 - X_m(z^2 - 3zz_0)(1 + 2\tan^2\theta_0)\sin\theta_0/6z_0^2 \cos\theta_0] \quad (45)$$

The usual method of steepest descent yields:

$$\psi_0(x) = \frac{1}{R}\, e^{ik_0 R[1 + X_m z(z - 3z_0)/6z_0^2]} \quad (46)$$

Similarly, if the point source is located at $\vec{x}'(x', y', z')$, the field is obtained by substituting eq. (40) into eq. (30). Carrying out the integration by steepest descent method, the Green's function has the form:

$$G(x, x') \cong \frac{1}{R}\, e^{ik_0 R[1 - X_m(z'^3 - 3z'^2 z_0 + 4z_0^3)/6(z - z')z_0^2]} \qquad z' < z < \infty \quad (47)$$

where $R = |\vec{x} - \vec{x}'|$, $z - z' = R\cos\theta_0$.

In the following, a specific configuration will be considered. The geometry of the problem is shown in Figure 5. The slab of irregularities is in the region $a \leq z \leq a + b$. The background electron density varies parabolically to give $\epsilon_0(x)$ as in eq. (35). The observation point is at a level $z > 2z_0$. If the dimension of the irregularities is much larger than the wavelength, then the contribution for the scattered field comes predominately from the scatterer in the neighborhood of the straight line connecting the source point and field point [Tatarski, 1961]. Therefore, the condition $\theta_0 << \pi/2$ is satisfied, the field at $B(\vec{x})$ can then be expressed:

$$\psi(\vec{x}) = \psi_0(\vec{x})\, e^{-i\phi_1(\vec{x})} \tag{48}$$

where

$$\psi_0(\vec{x}) = \frac{A_0}{r}\, e^{ik_0 r[1 - 2z_0 X_m/3z]} \tag{49}$$

$$\phi_1(\vec{x}) = \frac{ik_0^2}{4\pi} \int_{v'} \epsilon_1(\vec{x}') \frac{\psi_0(\vec{x}')}{\psi_0(\vec{x})} \frac{1}{R}\, e^{ik_0 R[1 - (z'^3 - 3z'^2 z_0 + 4z_0^3) X_m / 6z_0^2 (z - z_0)]}\, d^3\vec{x}' \tag{50}$$

$$\psi_0(\vec{x}') = \frac{A_0}{r}\, e^{ik_0 r[1 + X_m(z^2 - 3zz_0)/6z_0^2]} \tag{51}$$

Define the logarithmic amplitude $S(x)$ and phase departure $Q(x)$ by:

$$S(\vec{x}) = \ln \frac{A}{A_0} = \operatorname{Im} \phi_1 = \frac{r}{4\pi} \int_{v'} \frac{\epsilon_1(\vec{x}')}{r'R} \cos \Delta \, d^3\vec{x}' \tag{52}$$

$$Q(\vec{x}) = -\operatorname{Re} \phi_1 = \frac{r}{4\pi} \int_{v'} \frac{\epsilon_1(\vec{x}')}{r'R} \sin \Delta \, d^3\vec{x}' \tag{53}$$

where

$$\Delta = r' - r + R + X_m[r'(z'^2 - 3z'z_0)/6z_0^2 + 2rz_0/3z - R(z'^3 - 3z'^2 z_0 + 4z_0^3)/6z_0^2(z - z')] \tag{54}$$

In eqs. (52), (53), and (54) all distances are normalized with respect to the wavelength.

The correlation functions of the field at two points $\vec{x}_1$ and $\vec{x}_2$ are defined by:

$$\langle Q(\vec{x}_1)Q(\vec{x}_2)\rangle = \frac{r_1 r_2 \langle\epsilon_1^2\rangle}{(4\pi)^2} \int_{V_1}\int_{V_2} \frac{\sin\Delta_1}{r_1' R_1} \frac{\sin\Delta_2}{r_2' R_2} \rho_\epsilon(\vec{x}') \, d^3\vec{x}_1' \, d^3\vec{x}_2' \qquad (55)$$

$$\langle S(\vec{x}_1)S(\vec{x}_2)\rangle = \frac{r_1 r_2 \langle\epsilon_1^2\rangle}{(4\pi)^2} \int_{V_1}\int_{V_2} \frac{\cos\Delta_1}{r_1' R_1} \frac{\cos\Delta_2}{r_2' R_2} \rho_\epsilon(\vec{x}') \, d^3\vec{x}_1' \, d^3\vec{x}_2' \qquad (56)$$

where the normalized correlation function of the medium is defined by

$$\rho_\epsilon(\vec{x}) = \langle\epsilon_1(\vec{x}_1)\,\epsilon_1(\vec{x}_2)\rangle / \langle\epsilon_1^2\rangle \qquad (57)$$

and

$$\vec{x} = \vec{x}_2 - \vec{x}_1 \qquad (58)$$

Here the assumption is made that the irregularity region in the medium is statistically homogeneous so that the correlation function depends only on x.

5. Mean Square Fluctuations

To calculate the mean square values of the amplitude and phase fluctuations, let $\vec{x}_1 = \vec{x}_2 = (0,0,z)$ in eqs. (55) and (56) and define:

$$I_1 = 4\pi(\langle Q^2\rangle + \langle S^2\rangle)/\langle\epsilon_1^2\rangle \qquad (59)$$

$$I_2 = 4\pi(\langle Q^2\rangle - \langle S^2\rangle)/\langle\epsilon_1^2\rangle \qquad (60)$$

The following approximations are made in the integrand of eqs. (55) and (56), the phase factor Δ is approximated by:

$$\Delta = (1/2\zeta' + \xi')(x'^2 + y'^2) = \eta'(x'^2 + y^2) \tag{61}$$

where

$$\zeta' = z'(z-z')/z \tag{62}$$

$$\xi' = x_m[(z'^2-3z'z_0)z(z-2z')-4z_0^3z']/12z_0^2z'(z-z')^2 \tag{63}$$

$$\eta' = 1/2\zeta' + \xi' \tag{64}$$

and when they appear in the denominator:

$$r' \cong z', \qquad R \cong z-z' \tag{65}$$

Then from eqs. (55), (56), (59), and (60), the following can be derived:

$$I_1 = \frac{1}{\pi} \int_{v_1} \int_{v_2} \frac{\rho_\epsilon(\vec{x}')}{4\zeta_1'\zeta_2'} \cos[\eta_1'(x_1'^2 + y_1'^2) - \eta_2'(x_2'^2 + y_2'^2)]\, d\vec{x}_1'^3\, d\vec{x}_2'^3 \tag{66}$$

$$I_2 = \frac{-1}{\pi} \int_{v_1} \int_{v_2} \frac{\rho_\epsilon(\vec{x}')}{4\zeta_1'\zeta_2'} \cos[\eta_1'(x_1'^2 + y_1'^2) + \eta_2'(x_2'^2 + y_2'^2)]\, d\vec{x}_1'^3\, d\vec{x}_2'^3 \tag{67}$$

Change variables to:

$$x' = x_2' - x_1', \qquad y' = y_2' - y_1', \qquad z' = z_2' - z_1'$$

$$\alpha' = (x_2' + x_1')/2, \qquad \beta' = (y_1' + y_2')/2, \qquad \gamma' = (z_2' + z_1')/2 \tag{68}$$

The α', β' integration will yield:

$$I_1 = \int \frac{1}{4\zeta_1'\zeta_2'} \frac{\rho_\epsilon(\vec{x}')}{(\eta_2'-\eta_1')} \sin\left[\frac{\eta_2'\eta_1'}{\eta_2'-\eta_1'} (x'^2 + y'^2)\right] dx'\, dy'\, dz'\, d\gamma' \tag{69}$$

$$I_2 = \int \frac{1}{4\zeta_1'\zeta_2'} \frac{\rho_\epsilon(\vec{x}')}{(\eta_2'+\eta_1')} \sin\left[\frac{\eta_2'\eta_1'}{\eta_2'-\eta_1'} (x'^2 + y'^2)\right] dx'\, dy'\, dz'\, d\gamma' \tag{70}$$

Further integration depends on the explicit form of $\rho_\epsilon(\vec{x})$. If

$$\rho_\epsilon(\vec{x}) = e^{-x'^2/\ell_x^2 - y'^2/\ell_y^2 - z'^2/\ell_z^2} \tag{71}$$

Integrations over x' and y' can be carried out.

$$I_1 = \pi \text{ Im} \int_{\gamma'=a}^{\gamma'=a+b} d\gamma' \int_{z'=-b}^{+b} dz' \frac{1}{4\zeta_1'\zeta_2'\eta_1'\eta_2'} \frac{e^{-z'^2/\ell_z^2}}{(C_1^2/\ell_y^2 - i)^{1/2}(C_1^2/\ell_x^2 - i)^{1/2}} \tag{72}$$

$$I_2 = \pi \text{ Im} \int_{\gamma'=a}^{a+b} \int_{z'=-b}^{+b} dz' \frac{1}{4\zeta_1'\zeta_2'\eta_1'\eta_2'} \frac{e^{-z'^2/\ell_z^2}}{(C_2^2/\ell_y^2 - i)^{1/2}(C_2^2/\ell_x^2 - i)^{1/2}} \tag{73}$$

where

$$C_1^2 = (\eta_2' - \eta_1')/\eta_2'\eta_1' \tag{74}$$

$$C_2^2 = (\eta_2' + \eta_1')/\eta_2'\eta_1' \tag{75}$$

To the order of X_m, the parameters are:

$$(4\zeta_1'\zeta_2'\eta_1'\eta_2')^{-1} \cong 1 - 2(\zeta_1'\zeta_1' + \zeta_2'\zeta_2') \tag{76}$$

$$C_1^2 \cong 2(\zeta_1' - \zeta_2') + 4(\zeta_2'^2\zeta_2' - \zeta_1'^2\zeta_1') \tag{77}$$

$$C_2^2 \cong 2(\zeta_1' + \zeta_2') - 4(\zeta_2'^2\zeta_2' + \zeta_1'^2\zeta_1') \tag{78}$$

Because of the exponential dependence in the integrand, the contributions to I_1 and I_2 come predominately from the region $z' \leq \ell z$. Since the thickness of the irregularity slab is much larger than the correlation distance of the irregularities, the z' integration can be extended to $-\infty$ and $+\infty$ without introducing appreciable errors. Furthermore, in the integrand, the approximations $z \gg z'$ and $\gamma' \gg z'$ can be made since $z \gg \ell$ and $\gamma' \gg \ell$. And finally, terms of the order $1/\ell$ can be neglected compared to unity, since $\ell \gg 1$ from the starting assumptions. Introducing these approximations in the integrals I_1 and I_2, the z' integration can be carried out and yield:

$$I_1 = \pi^{3/2}\ell_z[b + 2X_m \int_a^{a+b} f_1(\gamma')\, d\gamma'] \tag{79}$$

$$I_2 = \pi^{3/2}\ell_z \int_a^{a+b} \beta(\gamma')\, [1 + 2X_m\, f_1(\gamma')]\, d\gamma'$$

$$+ 2X_m\, \pi^{3/2}\ell_z \int_a^{a+b} [\frac{\alpha+\beta D_y}{\ell_y^2(1+D_y^2)} + \frac{\alpha + \beta D_x}{\ell_x^2(1+D_x^2)}]\, f_2(\gamma')\, d\gamma' \tag{80}$$

where

$$D_y = 4\gamma'(z-\gamma')/\ell_y^2 z\ , \qquad D_x = 4\gamma'(z-\gamma')/\ell_x^2 z \tag{81}$$

$$\alpha(\gamma') = \sqrt{\frac{(1+D_x^2)^{1/2}\,(1+D_y^2)^{1/2} - (1 - D_x D_y)}{2(1+D_x^2)\,(1+D_y^2)}} \tag{82}$$

$$\beta(\gamma') = \sqrt{\frac{(1+D_x^2)^{1/2}\,(1+D_y^2)^{1/2} + (1 - D_x D_y)}{2(1+D_x^2)\,(1+D_y^2)}} \tag{83}$$

$$f_1(\gamma') = -\gamma'^2/3z_0^2 + (6z_0-z)\gamma'/6z_0^2 + (z/2z_0-z^2/6z_0^2-2z_0/3z)[1+z/(\gamma'-z)] \tag{84}$$

$$f_2(\gamma') = -\gamma'^4/3zz_0^2 + (z+6z_0)\,\gamma'^3/6zz_0^2 - (3z^2 + 4z_0^2)\,\gamma'^2/6z^2 z_0 \tag{85}$$

The γ'-integration for I_1 is elementary and yields:

$$I_1 = \pi^{3/2}\ell_z b[1 + 2X_m\, F_1] \tag{86}$$

where

$$F_1 = [a^3 - (z-c)^3]/9z_0^2 + (6z_0-z)[(z-c)^2 - a^2]/12z_0^2$$

$$+ (z/2z_0 - z^2/6z_0^2 - 2z_0/3z)[b + z\ln[(a+b-z)/(a-z)]]/b \qquad (87)$$

The integration for I_2 cannot be carried out in general. However, if the slab is thin or under conditions that D_x and D_y are some constant average values $\bar{D}_x$ and $\bar{D}_y$, then α and β can also be represented by their average values, $\bar{\alpha}$ and $\bar{\beta}$ respectively. I_2 then becomes:

$$I_2 = \pi^{3/2} \ell_z b\bar{\beta}(1 + 2X_m F_1) + 2\pi^{3/2} \ell_x X_m \left[\frac{\bar{\alpha}+\bar{\beta}\bar{D}_y}{\ell_y^2(1+\bar{D}_y^2)} + \frac{\bar{\alpha}+\bar{\beta}\bar{D}_x}{\ell_x^2(1+\bar{D}_x^2)}\right] F_2 \qquad (88)$$

where

$$F_2 = -[(a+b)^5 - a^5]/5zz_0^2 + (z+6z_0)[(a+b)^4 - a^4]/24zz_0^2$$

$$- (3z^2 + 4z_0^2)[(a+b)^3 - a^3]/18z^2z_0 \qquad (89)$$

From eqs. (59) and (60), the mean square values of fluctuations are given by,

$$\langle Q^2 \rangle = \langle \epsilon_1^2 \rangle (I_1 + I_2)/8\pi \qquad (90)$$

$$\langle S^2 \rangle = \langle \epsilon_1^2 \rangle (I_1 - I_2)/8\pi \qquad (91)$$

where I_1 and I_2 are given by eqs. (86) and (88).

In the following, some special cases will be considered for a thin slab.

(1) $D \gg 1$. For this case, the mean square values of phase and the logarithmic amplitude are found to be equal.

$$\langle Q^2\rangle = \langle S^2\rangle = \langle \epsilon_1^2\rangle \pi^{1/2} b\, \ell_z[1 + \delta_1]/8 \tag{92}$$

where δ_1 is the contribution from the regular inhomogeneities of the medium.

$$\delta_1 = 2X_m[-a^2/3z_0^2 + a(6z_0-z)/6z_0^2 + (z/2z_0 - z^2/6z_0^2 - 2z_0/3z)(1 - z/c)] \tag{93}$$

(2) D << 1.

$$\langle Q^2\rangle = \langle \epsilon_1^2\rangle \pi^{1/2} \ell_z b\, [1 + \delta_1]/4 \tag{95}$$

$$\langle S^2\rangle = \langle \epsilon_1^2\rangle \pi^{1/2} \ell_z b\, (3\bar{D}_x^2 + 2\bar{D}_x\bar{D}_y + 3\bar{D}_y^2)\, [1 + \delta_1]/64 \tag{95}$$

where δ_1 is given in (93).

6. Correlation Functions

The general expressions for correlation functions are given by eqs. (55) and (56). If the two receivers are placed at (-x,0,z) and x,0,z) respectively, the correlation functions can be calculated explicitly. The following approximations are made in the calculation

$$r_1' + R_1 - r_1 \cong \frac{1}{2}\frac{1}{\zeta_1'}[y_1'^2 + (x_1' - xz_1'/z)^2] \tag{96}$$

$$r_2' + R_2 - r_2 \cong \frac{1}{2}\frac{1}{\zeta_2'}[y_2'^2 + (x_2' + xz_2'/z)^2] \tag{97}$$

$$\Delta_1 \cong \eta_1'(y_1'^2 + x_1'^2) - \xi_1''^2/4\eta_1' \tag{98}$$

$$\Delta_2 \cong \eta_2'(y_2'^2 + x_2'^2) - \xi_2''^2/4\eta_2' \tag{99}$$

where

$$X_1' = x_1' - xz_1'/z + \xi_1''/2\eta_1' \quad (100)$$

$$X_2' = x_2' + xz_2'/z - \xi_2''/2\eta_2' \quad (101)$$

$$\xi_1'' = X_m \, xz_1'[4z_0^2 + z(z_1' - 3z_0)]/6z_0^2(z-z_1')z \quad (102)$$

$$\xi_2'' = X_m \, xz_2'[4z_0^2 + z(z_2' - 3z_0)]/6z_0^2(z-z_2')z \quad (103)$$

and ζ' and η' are given in (62) and (64) respectively. Define,

$$I_3 = 4\pi(\langle Q_1Q_2\rangle + \langle S_1S_2\rangle)/\langle\epsilon_1^2\rangle \quad (104)$$

$$I_4 = 4\pi(\langle Q_1Q_2\rangle - \langle S_1S_2\rangle)/\langle\epsilon_1^2\rangle \quad (105)$$

Putting the above approximations into integrals (55) and (56), eqs. (104) and (105) can be written

$$I_3 = \frac{1}{\pi}\int_{V_1}\int_{V_2} \frac{\rho_\epsilon(\vec{x}')}{4\zeta_1'\zeta_2'} \cos[\eta_1'(X_1'^2+y_1'^2)-\eta_2'(X_2'^2+y_2'^2)-\tfrac{1}{4}(\xi_1''^2/\eta_1'-\xi_2''^2/\eta_2')]d^3\vec{x}_1'd^3\vec{x}_2' \quad (106)$$

$$I_4 = \frac{-1}{\pi}\int_{V_1}\int_{V_2} \frac{\rho_\epsilon(\vec{x}')}{4\zeta_1'\zeta_2'} \cos[\eta_1'(X_1'^2+y_1'^2)+\eta_2'(X_2'^2+y_2'^2)-\tfrac{1}{4}(\xi_1''^2/\eta_1'+\xi_2''^2/\eta_2')]d^3\vec{x}_1'd^3\vec{x}_2' \quad (107)$$

Following the usual procedure of transforming into a relative and center of mass coordinate system, the α', β' integrations can be carried out immediately to give

$$I_3 = \iiiint \frac{\rho_\epsilon(\vec{x}')}{4\zeta_1'\zeta_2'(\eta_1'-\eta_2')} \sin \frac{\eta_2'\eta_1'}{\eta_1'-\eta_2'} [(x'+\bar{A})^2+y'^2]-\tfrac{1}{4}(\xi_1''^2/\eta_1'-\xi_2''^2/\eta_2')] \, dx'dy'dz'd\gamma' \quad (108)$$

$$I_4 = \iiiint \frac{\rho_\epsilon(\vec{x}')}{4\zeta_1'\zeta_2'(\eta_2'+\eta_1')} \sin \frac{\eta_2'\eta_1'}{\eta'+\eta_2'} [(x'+\bar{A})^2+y'^2] - \frac{1}{4}(\xi_1''^2/\eta_1' + \xi_2''^2/\eta_2')] \, dx'dy'dz'd\gamma' \tag{109}$$

where

$$\bar{A} = 2x\gamma'/z - (\xi_1''/\eta_1' + \xi_2''/\eta_2')/2$$

When eq. (71) for $\rho_\epsilon(\vec{x}')$ is substituted into (108) and (109), the x' and y' integration can be performed in the similar manner as in the last section. Furthermore, if the same assumptions are followed as before, the integrals I_3 and I_4 can be written approximately as:

$$I_3 = \frac{\pi^2 \ell_x \ell_z z}{4x} \left[\mathrm{erf}\left(\frac{2x(a+b)}{\ell_x z}\right) - \mathrm{erf}\left(\frac{2xa}{\ell_x z}\right)\right] + X_m \pi^{3/2} \ell_z F_3 \tag{110}$$

$$I_4 = \frac{\pi^2 \ell_x \ell_z z}{4x} \mathrm{Im}\left\{ i(1+i\bar{D}_y)^{-1/2} \left[\mathrm{erf}\left(\frac{2x(a+b)}{\ell_x z(1+i\bar{D}_x)^{1/2}}\right) - \mathrm{erf}\left(\frac{2xa}{\ell_x z(1+i\bar{D}_x)^{1/2}}\right)\right]\right\}$$

$$+ X_m \pi^{3/2} \ell_z \,\mathrm{Im}\, F_4 \tag{111}$$

where

$$F_3 = \frac{\ell_x^2 z^2}{6z_0^2 z\cdot(4x^2)} (4z_0 + 3z_0 z - z^2) [e^{-p_3^2 a^2} - e^{-p_3^2(a+b)^2}]$$

$$+ \frac{(4z_0 - 3z)}{6zz_0} [a^2 e^{-p_3^2 a^2} - (a+b)^2 e^{-p_3^2(a+b)^2}] + \frac{2}{9z_0^2}[a^3 e^{-p_3^2 a^2} - (a+b)^3 \cdot$$

$$\cdot e^{-p_3^2(a+b)^2}] + \frac{4x^2}{9z_0^2 \ell_x^2 z^2} \int_a^{a+b} \gamma'^4 e^{-p_3^2\gamma'^2} d\gamma'$$

$$+ (z/z_0 - z^2/3z_0^2 - 4z_0/3z) \int_a^{a+b} \frac{\gamma'}{\gamma'-z} e^{-p_3^2\gamma'^2} d\gamma' \tag{112}$$

$$F_4 = \frac{1}{(\bar{D}_x - i)^{1/2}(\bar{D}_y - i)^{1/2}} \quad \frac{(\bar{D}_x - i)\ell_x^2 z^2}{(-i4x^2)6zz_0^2} [4z_0^2 + 3zz_0 - z^2 - i(z+6z_0)z^2/x^2] \cdot$$

$$[e^{-p_4^2 a^2} - e^{-p_4^a (a+b)^2}]$$

$$+ \frac{z_0(4z_0 - 3z) - i(z+6z_0)z^2/x^2}{6zz_0^2} [a^2 e^{-p_4^2 a^2} - (a+b)^2 e^{-p_4^2 (a+b)^2}]$$

$$+ \frac{2}{9}\frac{1}{z_0^2} [a^3 e^{-p_4^2 a^2} - (a+b)^3 e^{-p_4^2 (a+b)^2}]$$

$$- \frac{z(z+6z_0)}{3\ell_x^2 z^2 (\bar{D}_x - i) z_0^2} [a^4 e^{-p_4^2 a^2} - (a+b)^4 e^{-p_4^2 (a+b)^2}]$$

$$+ (z/z_0 - z^2/3z_0^2 - 4z_0/3z) \int_a^{a+b} \frac{\gamma'}{\gamma' - z} e^{-p_4^2 \gamma'^2} d\gamma'$$

$$+ \frac{-i\,4x^2}{\ell_x^2 z^2 (\bar{D}_x - i) z_0^2} \left[\frac{1}{9} + \frac{2z(3z^2 + 4z_0^2)}{3\ell_x^2 z^2 (\bar{D}_x - i)}\right] \int_a^{a+b} \gamma'^4 e^{-p_4^2 \gamma'^2} d\gamma'$$

$$+ \frac{-i16x^2 z}{3\ell_x^4 z^4 z_0^2 (\bar{D}_x - i)^2} \int_a^{a+b} \gamma'^6 e^{-p_4^2 \gamma'^2} d\gamma' \qquad (113)$$

$$p_3^2 = 4x^2/\ell_x^2 z^2 \qquad (114)$$

$$p_4^2 = -i4x^2/\ell_x^2 z^2 (\bar{D}_x - i) \qquad (115)$$

For a very thin slab, the following approximations are obtained:

$$F_3 \cong [ab(4z_0^2 + 3z_0z - z^2)/3z_0^2z + (4z_0 - 3z)ab(4x^2a^2/\ell_x^2z^2 - 1)/3z_0z$$

$$+ 2a^2b(8a^2x^2/\ell_x^2z^2 - 3)/9z_0^2 + 4x^3a^46/9z_0^2\ell_x^2z^2$$

$$+ ab(z/z_0 - z^2/3z_0^2 - 4z_0/3z)/(a-z)]\, e^{-4x^2a^2/\ell_x^2z^2} \quad (116)$$

$$F_4 \cong \frac{ab}{3z_0^2z}(4z_0^2+3z_0z-z^2) + \frac{(4z_0-3z)ab}{3zz_0}\left[\frac{-i4x^2a^2}{\ell_x^2z^2(\bar{D}_x-i)} - 1\right]$$

$$+ \frac{2a^2b}{9z_0^2}\left[\frac{-i8a^2x^2}{\ell_x^2z^2(\bar{D}_x-i)} - 3\right] - \frac{2z(z+6z_0)a^3}{3\ell_x^2z^2z_0^2(\bar{D}_x-i)}\left[\frac{-i4x^2a^2}{\ell_x^2z^2(\bar{D}_x-i)} - 2\right]$$

$$+ \left(\frac{z}{z_0} - \frac{z_0^2}{3z_0^2} - \frac{4z_0}{3z}\right)\frac{ab}{a-z} + \frac{-i4x^2a^4b}{9\ell_x^2z^2z_0^2(\bar{D}_x-i)} \quad \frac{e^{+i4x^2a^2/\ell_x^2z^2(\bar{D}_x-i)}}{(\bar{D}_x-i)^{1/2}(\bar{D}_y-i)^{1/2}}$$

(117)

$$I_3 \cong \pi^{3/2}\, b\, \ell_z[e^{-4x^2a^2/\ell_z^2z^2} + X_mF_3/b] \quad (118)$$

$$I_4 \cong \pi^{3/2}b\ell_z\ \mathrm{Im}[i(i+i\bar{D}_y)^{-1/2}(1+\bar{D}_x)^{-1/2}e^{i4x^2a^2/\ell_z^2z^2(\bar{D}_x-i)} + X_mF_4/b] \quad (119)$$

From the definitions (104) and (105), the correlation functions can be expressed by

$$\langle Q_1Q_2\rangle = \langle\epsilon_1^2\rangle\,(I_3 + I_4)/8\pi \quad (120)$$

$$\langle S_1S_2\rangle = \langle\epsilon_1^2\rangle\,(I_3 - I_4)/8\pi \quad (121)$$

where I_3 and I_4 are given by eqs. (110) and (111) in general and eqs. (118) and (119) for a very thin slab.

Again, as in the last section, two special cases will be considered for a very thin slab.

(1) $\bar{D} \gg 1$.

Eqs. (117) to (119) yield for this case

$$\langle Q_1 Q_2 \rangle = \langle S_1 S_2 \rangle \cong \pi^{1/2} b \ell_z \langle \epsilon_1^2 \rangle e^{-4x^2a^2/\ell_x^2 z^2} (1 + \delta_2)/8 \tag{122}$$

where

$$\delta_2 = X_m F_3 e^{4x^2a^2/\ell_x^2 z^2} /b \tag{123}$$

The normalized correlation functions are defined by

$$\rho_Q(x) = \langle Q_1 Q_2 \rangle / \langle Q^2 \rangle \quad \text{and} \quad \rho_S(x) = \langle S_1 S_2 \rangle / \langle S^2 \rangle \tag{124}$$

Using the results derived in the last section, the expressions for normalized phase and amplitude correlation functions can then be written as

$$\rho_Q(x) = \rho_S(x) = (1 + \delta) e^{-4x^2a^2/\ell_x^2 z^2} \tag{125}$$

where

$$\delta = \delta_2 - \delta_1$$

$$= 4a^2x^2 X_m [3az_0(4z_0 - 3z) + 5a^2 z]/4z_0^2 \ell_x^2 z^3 \tag{126}$$

is the correction in the correlation functions to the first order of X_m.

(2) $\bar{D} \ll 1$

For this case, the normalized correlation function for the phase is expressed approximately by:

$$\rho_Q(x) = (1 + \delta)\, e^{-4x^2a^2/\ell_x^2 z^2} \qquad (127)$$

And the normalized amplitude correlation function is:

$$\rho_S(x) = \left[1 - \frac{16a^2x^2\bar{D}_x(3\bar{D}_x + \bar{D}_y - 4a^2x^2\bar{D}_x/z^2\ell_x^2)}{z^2\ell_x^2(3\bar{D}_x^2 + 3\bar{D}_y^2 + 2\bar{D}_x\bar{D}_y)}\right] e^{-4x^2a^2/\ell_x^2 z^2}\,(1+\delta) \qquad (128)$$

All these expressions reduce to those derived by Yeh for the case $X_m = 0$.

7. Results

It is noticed from eqs. (93) and (126) that the corrections due to the regular inhomogeneous background to the mean square fluctuations and correlation functions depend on various things: the space between two observation points, x; the position of the satellite, z; the position at which the maximum of the electron density profile is, z_0; the position of the irregular slab, a; as well as the transverse dimension of the irregularities, ℓ_x. In the following, these corrections are plotted as functions of the ratio a/z_0 for three different cases: $z = 2z_0$, $z = 5z_0/2$, $z = 3z_0$. Several points of interest are observed:

(1) The correction to the fluctuation, δ_1, is positive for all the cases which are consistent with the assumptions. This is to be expected physically. Since for the medium considered, $\epsilon_0 < 1$, and the ratio ϵ_1 / ϵ_0 is higher for this medium than for a homogeneous one. Therefore, the fluctuation due to scattering is higher.

(2) δ_1 has a maximum about the point $a/z_0 = 1$, corresponding to the case when the irregular slab is near the peak of the electron density profile. Physically, this corresponds to the case when ϵ_1 / ϵ_0 is largest, hence, maximum fluctuation.

(3) The correction to the normalized correlation functions, δ, is shown in Figure 7. For some values of a/z_0 less than one, δ is small and negative. It becomes increasingly positive when the irregularity slab is below the peak.

(4) In Figure 8, the normalized correlation function is plotted against the distance between the two observation points. It is compared with the case for homogeneous background. The correction is more appreciable for greater separation when the magnitude of the correlation is very small.

(5) A numerical example using $z = 700$ km, $z_0 = 350$ km shows that the correction to the fluctuation is about 17%.

8. Conclusion

In this paper, the problem of wave propagation through an inhomogeneous medium containing anisotropic irregularities has been treated by various approximation methods. The mean square values and autocorrelations for both amplitude and phase of the wave were found to be affected by the regular inhomogeneous background. With ionospheric application in mind, the case of a plasma medium with parabolic electron density profile and small irregularities imbedded in it was studied. The corrections to the mean square values and correlation functions due to the regular inhomogeneities were calculated to the first order of X_m, the square of the ratio of maximum plasma frequency to the frequency of the wave. It was found that these corrections depend on the height of the electron density maximum, the height of the transmitter, the distance between the observation point and the height of the irregular slab. The corrections are found to be appreciable for some experimental situations. Other electron density profiles which may represent more closely the true ionosphere can be treated in a similar manner. It is hoped that the results will be of some practical use to the interpretations of earth satellite scintillation data. Due to the nature of the approximations made, the results are applicable only to high frequency waves and media with weakly random inhomogeneities. Further work should be done along this line for a more complex medium where both inhomogeneities and anisotropies exist. The medium in reality, the ionosphere for instance, is exactly one of this nature.

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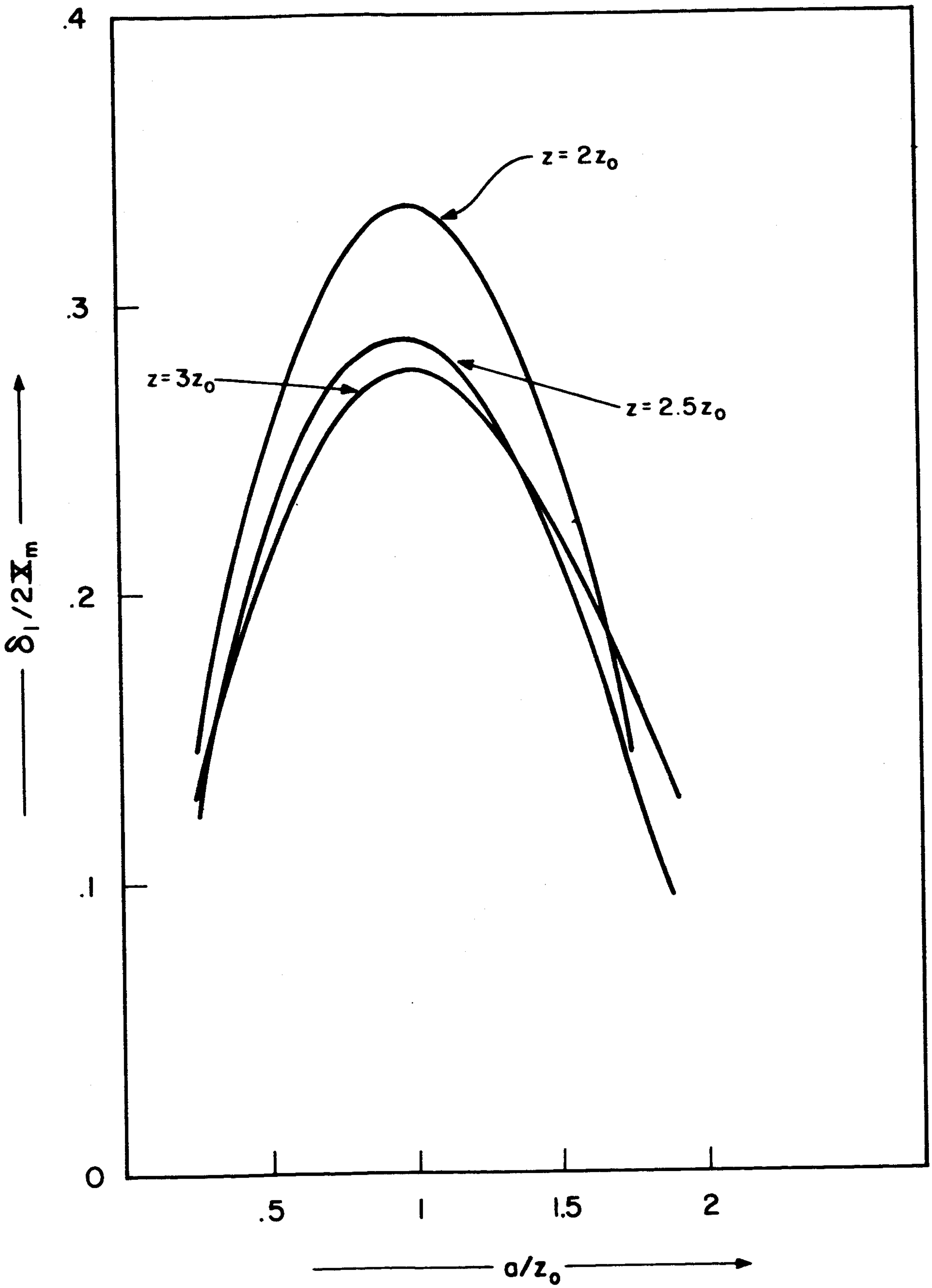

FIG. 6

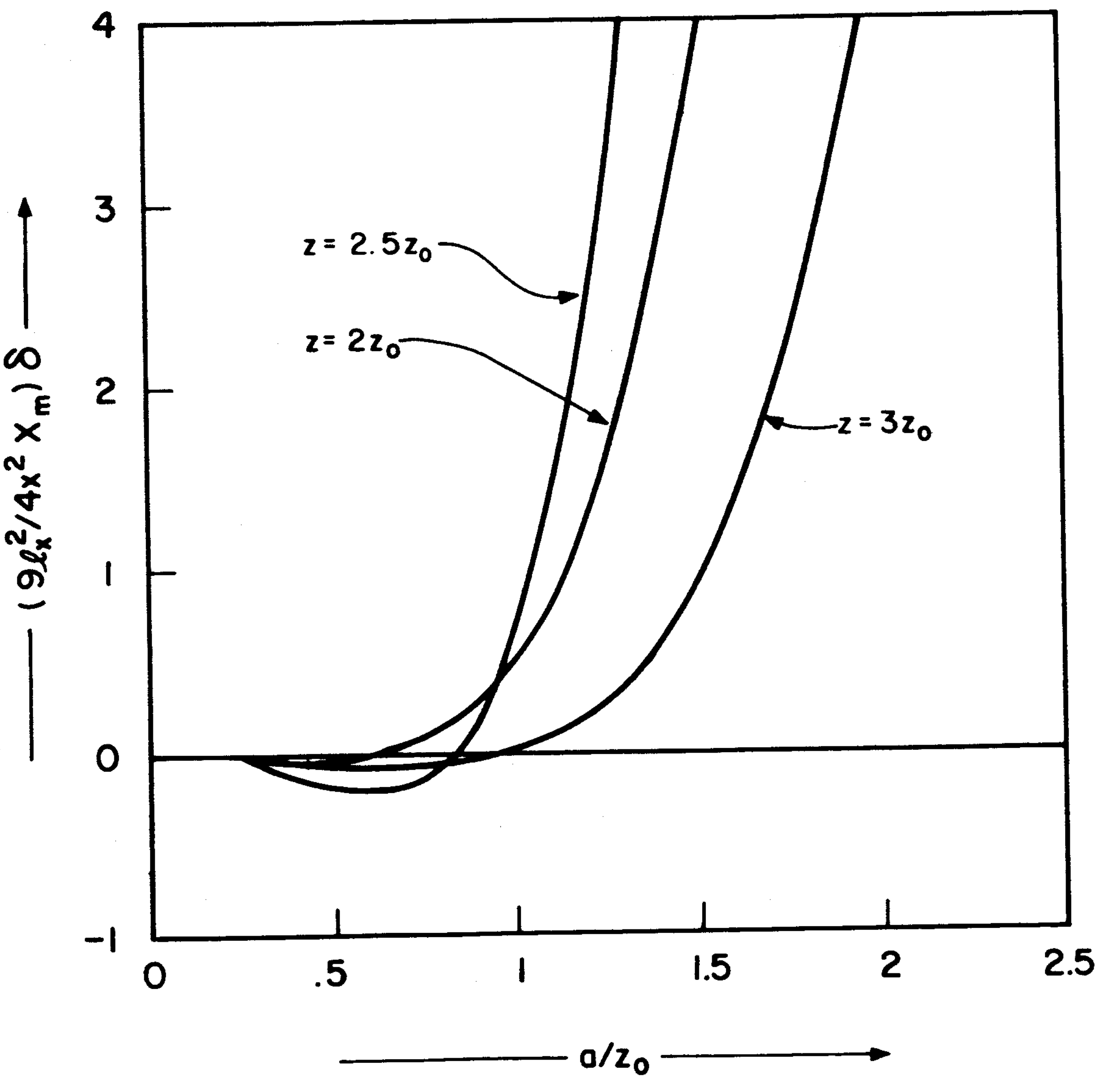

FIG. 7

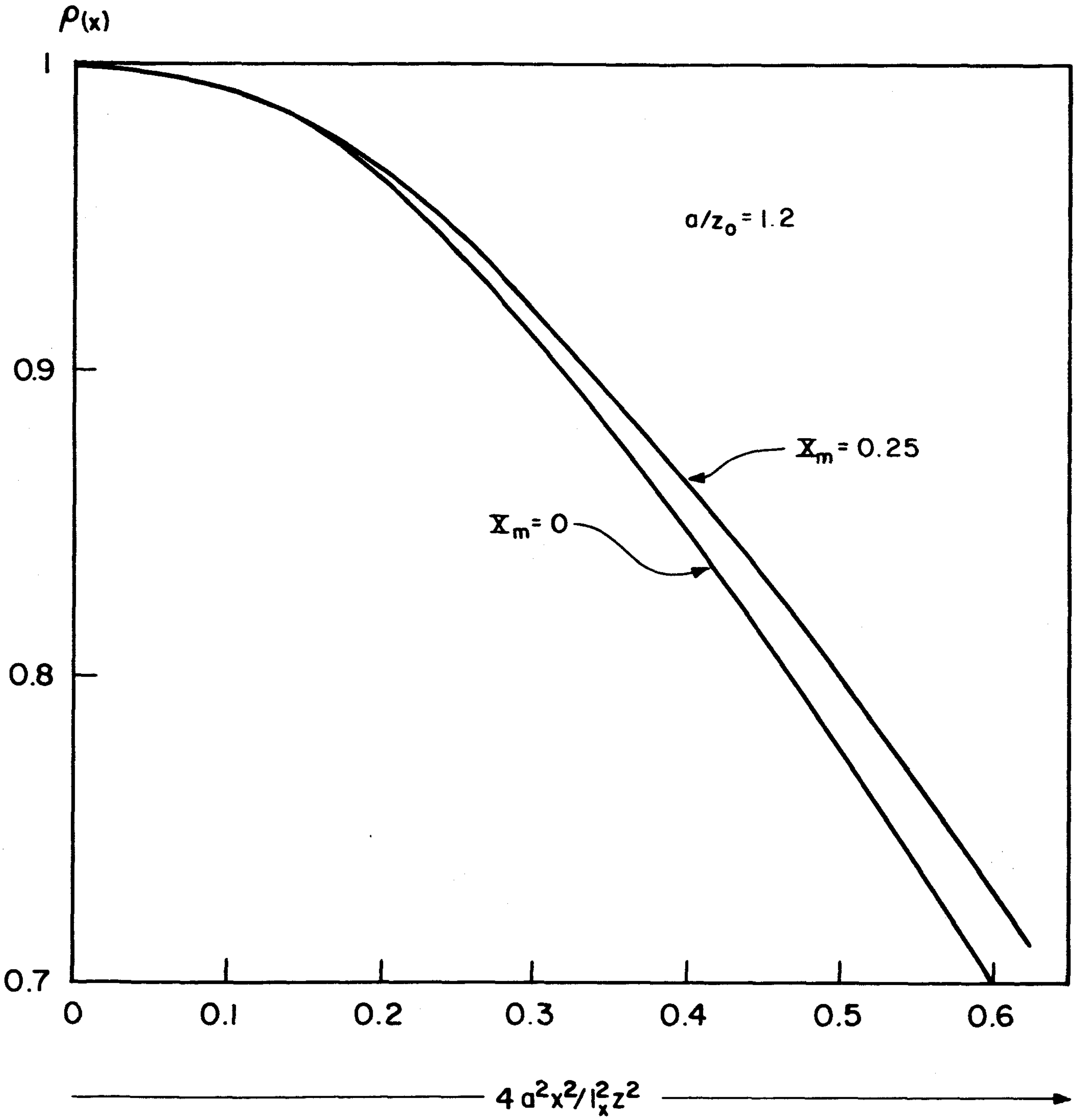

FIG. 8