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JUNE 1966

This research was sponsored by
the National Aeronautics and Space Administration
under Research Grant NsG 133-61

SUDAAR NO. 279

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Stanford, California

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ABSTRACT

By utilizing results of Hamiltonian theory and the von Zeipel method for treating artificial satellite orbits, error bounds are derived for a general class of orbits with eccentricity less than one. In order to extend the error bounds for the general axisymmetric problem to time intervals of the order $1/J_2$, the known integral of energy is utilized to calibrate the governing differential equations for the rapidly rotating phase. The non-singular rapid phase in this analysis is taken to be the sum of the mean anomaly, argument of periapsis and the right ascension of the ascending node. A corresponding analysis for the general asymmetric problem (including the tesseral harmonics) is also given. From the general error analysis an algorithm is derived for the computation of the correct initial conditions consistent with the expected accuracy of the theory. Numerical results verifying the conclusions of the theory presented in this paper are also given.

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I. INTRODUCTION

The analytical theory of artificial satellite motion has been the subject of very intensive study since the launching of the first artificial satellite in 1957. In fact, many aspects of the problem had been studied before that time in connection with the theories of celestial mechanics. The result of the study has been a very extensive list of papers offering solutions of many differing forms and techniques of achieving them. However, with the exception of the work of Kyner (Ref. 1), no other solution is known to the authors that offers rigorous error bounds on the position and velocity for a general class of orbits, e.g., inclined orbits of any eccentricity less than one. Naturally, the orbits at critical inclination and orbits in resonance with the tesseral harmonics must be excepted from the general class. It is then a matter of general interest to derive such error bounds.

From a fundamental point of view, the problem of artificial satellite motion can be classified as a special case of a general class of non-linear oscillation problems. Non-linear oscillation problems can be treated with varying degrees of success by the general averaging methods developed by Krylov, Bogoliubov and Mitropolskii (i.e., Ref. 2). For these methods of averaging there exists an associated technique for establishing bounds on the error build-up in a specified time between the exact and the approximate solutions (first order or higher order). Now, the method of application of the technique of averaging to the problem of artificial satellite motion depends rather heavily on the particular choice of variables employed. In the case of Kyner's work, averaging could be applied directly; in most other approaches to the problem the use of averaging is more or less disquised.

One of the most widely used perturbation methods in treating artificial satellite orbits has been the method of von Zeipel as adopted by Brouwer (Ref. 3) and Kozai (Ref. 4). This method is one of successive canonical transformations and is necessarily carried out in the variables of Delaunay (L, G, H, ℓ, g, h). With a slight change of variables and a choice of a different intermediary orbit, the same method was applied by Garfinkel (Refs. 5,6). Furthermore, it has been shown (Refs. 7,8) that the von Zeipel method of canonical transformations is a particular form of the method of averaging. Hence, by drawing on the equivalence to averaging, rigorous error bounds could be established for the Delaunay variables directly. Unfortunately, bounds obtainable in this way for the Delaunay variables ℓ and g are unsatisfactory for very small eccentricity (i.e., $e' < J_2$ where J_2 is the oblateness parameter of $O(10^{-3})$) due to a singularity at zero eccentricity in the short period terms. A further drawback is the singularity at zero inclination. Since no singularities exist in the coordinates for zero eccentricity and/or inclination, one would expect that these objections to the bounds would not exist for a suitable choice of variables. The error bounds derived by such direct application of differential equation theory turn out to be unsatisfactory for large time intervals i.e., time intervals of the order $1/J_2$. Since one of the problems of interest in applying closed-form orbit theories is orbit prediction over long periods of time, the error theory must be modified. The modification is a more involved problem and a separate treatment is presented here.

In this report, the problem is analyzed in canonical variables;

the three sets of interest are those due to Delaunay, Hill and Poincaré. Of these variables, the Poincaré set is non-singular for both zero eccentricity and inclination, the Hill set singular for zero inclination and the Delaunay set singular for both zero inclination and eccentricity. The advantages of the Hill set are the simple forms of the in-plane coordinate perturbations which are obtained directly from known generating functions. It was shown by Izsak (Ref. 9) that, to first order in the oblateness coefficient J_2, the in-plane position and velocity components of a satellite are obtainable by converting via Keplerian formulae from Brouwer's averaged Delaunay variables (L', G', H', ℓ', g') to corresponding "averaged" position and velocity and then superimposing the short-period fluctuations. These short-period fluctuations were shown to be obtainable by rewriting Brouwer's short-period generating function S_1 in terms of the Hill variables and taking appropriate partial derivatives. These short-period fluctuations are well-behaved (unlike those in ℓ, g) when eccentricity goes to zero. Recent investigations by Vagners (Ref. 10) have obtained in the same manner first order long-period fluctuations in the Hill variables by rewriting Brouwer's long-period generating function S_1^* relating $(L'', G'', H'', \ell'', g'', h'')$ to $(L', G', H', \ell', g', h')$, including general formulas for the effects of any zonal harmonic. Analogous "medium-period" (i.e., daily) fluctuations in the Hill variables were obtained in a general form for the effects of the tesseral and sectorial harmonics. Since the analysis given by Vagners was applicable to any set of canonical variables, then similar results could readily be obtained for the Poincaré variables.

Utilizing the results of Izsak and Vagners, an analysis is carried out in this report which parallels every canonical tranformation of the Delaunay variables by an appropriate canonical transformation of some general set of canonical variables including the removal of second order short-period terms from the Hamiltonian. In this way, rigorous error bounds on the first-order solution are established which are independent of the eccentricity for Hill variables and independent of eccentricity and inclination for the Poincaré variables (as long as e is not too close to one). As is shown, these bounds are unsatisfactory for long time intervals and another method is offered.

A discussion is presented of the various terms arising in the error bound. Particular attention is focused on the question of initial condition errors; this question is of interest when computing by means of a "closed-form" satellite theory a satellite's ephemeris from some given initial position and velocity vectors. In view of the extensive comparison studies of different orbit theories conducted by Arsenault, Enright and Purcell (Ref. 11), wherein the problem of initialization plays such an important role, this question assumes considerable importance. An energy method is then given for greatly decreasing the primary in-track position error build-up due to initial conditions and some typical results are quoted. The algorithm of computing the correct initial conditions arises directly from the extended error bound theory.

The authors wish to acknowledge the contribution of Small (Ref. 12), who first utilized the energy method in reducing initialization errors in his solution to the problem of satellite motion about an oblate planet.

II. GENERAL BOUNDS ON SATELLITE MOTION

Before proceeding to more specific treatment of the error problem, some general statements concerning the a priori bounds on the motion may be made. First, one can consider the motion of a satellite in a general axi-symmetric gravitational field for which two integrals of the motion are known. If the potential field is represented by

$$V = -\frac{\mu}{r}\left\{1 - \sum_{N=2}^{\infty} J_N \left(\frac{R_\oplus}{r}\right)^N P_N(\sin\beta)\right\} = -\left[U_o(r) + U_1(r,\beta)\right] \quad \text{(II-1)}$$

where β is the latitude, $R_\oplus$ the equatorial radius, μ the gravitational constant, r the radius and J_N numerical coefficients, then it can readily be shown that the total energy and the polar component of the angular momentum are constants of the motion. The two exact integrals may be written in the form

$$\frac{1}{a} + \frac{2}{\mu} U_1(r,\beta) = k_1 \quad \text{(II-2)}$$

and $$H = \sqrt{\mu a(1 - e^2)} \cos i = k_2 \quad \text{(II-3)}$$

where a is the semi-major axis of the orbit, e the eccentricity, i the orbital inclination and k_1 k_2 are constants.

The two integrals (II-2) and (II-3) imply that if k_1 and k_2 are given, then the motion of the satellite is confined to a region bounded by a "zero velocity" surface (Ref. 13). With initial conditions specifying k_1 and k_2 one can write the a priori bounds in the form

$$0 < \delta_1(k_1,k_2,\epsilon) < r < \delta_2(k_1,k_2,\epsilon) \quad \text{(II-4)}$$

where $\epsilon \triangleq J_2$, $k_1 > 0$, $k_2 > 0$ and J_N/J_2^2 are assumed values of $0(1)$.

General bounds of this type are developed by Poritsky (Ref. 14) and given for $\epsilon = 0$ by Kyner (Ref. 1). Here the explicit forms of δ_1 and δ_2 are not of direct interest.

In the more general problem of a longitude dependent potential one no longer has the two integrals (II-2) and (II-3). Such a potential arises when one includes the tesseral harmonics of the Earth's field in the general satellite problem. However, by considering a rotating coordinate system fixed in the primary, one can readily determine the Jacobi integral of the system. In this case one specifies only the upper bound by the zero-velocity surface.

One assumes then that a priori bounds on the state vector x are known, namely, if the initial state vector $x(0)$ is in a set D, then

$$|x| \leq C(x(0),\ \epsilon) \tag{II-5}$$

where the solution depends on a small parameter ϵ. Since for near-earth satellites one is concerned with elliptical orbits, the set D will be specified by the requirement of negative energy and a non-zero initial value of the angular momentum. If the state vector chosen for the description of the motion is some canonical set (q,p), then the equations of motion take the form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \dot{x} = \Phi_o \mathcal{H}_{\tilde{x}} (x,\epsilon) \tag{II-6}$$

where $\mathcal{H}$ is the Hamiltonian of the problem

$\Phi_o = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ the canonical matrix

$\tilde{x}$ denotes the partials of $\mathcal{H}$ with respect to x and the super tilda denotes the transpose of the vector x.

Then, since $\mathcal{H}_{\tilde{x}}$ is continuous and satisfies a Lipschitz condition locally in x in some bounded region $\mathcal{R}$ (then) a solution for all t exists as a consequence of (II-5).

Note that implicit in (II-5) is also a restriction on how close the energy and the angular momentum may be to zero. For the general bounds to hold, these initial values must be sufficiently different from zero so that the perturbations, of order ϵ in the satellite problem, do not cause the state vector x to become arbitrarily large.

III. THE SECOND-ORDER HAMILTONIAN

Inherent in a specific discussion of error bounds is a knowledge of the characteristics of the analytical method used in the fundamental solution and a knowledge of the behavior of various functions arising therein. The method utilized in the following analysis is the von Zeipel method and the system analyzed is a Hamiltonian system. For a brief review of the von Zeipel procedure, the reader is referred to Ref. 10; the specific details of the orbit problem solution may be found in Refs. 3 and 4.

It turns out to be convenient to introduce the three sets of canonical variables due to Delaunay, Hill and Poincaré. (Recall that the original solution of Brouwer was carried out in Delaunay variables.) These sets of variables are defined in the following manner: The Delaunay variables, denoted by y, are given as

$$y = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \ell \\ g \\ h \\ L \\ G \\ H \end{bmatrix} \tag{III-1}$$

where ℓ is the mean anomaly

$g = \omega$ the argument of pericenter

$h = \Omega$ the right ascension of the ascending node

$L = \sqrt{\mu a}$

$G = \sqrt{\mu a (1 - e^2)}$

$H = G \cos i$

Denoting the Hill variables by z, one finds

$$z = \begin{bmatrix} r \\ u \\ h \\ R \\ G \\ H \end{bmatrix} \qquad \text{(III-2)}$$

where u the central angle or argument of latitude

$R = \dot{r}$ the radial velocity

And, finally, the Poincaré variables, denoted by x, are

$$x = \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \lambda \\ \eta_1 \\ \eta_2 \\ L \\ \xi_1 \\ \xi_2 \end{bmatrix} \qquad \text{(III-3)}$$

with

$$\lambda = \ell + g + h \qquad L = L$$

$$\eta_1 = [2(L - G)]^{\frac{1}{2}} \cos (g + h) \qquad \xi_1 = [2(L - G)]^{\frac{1}{2}} \sin (g + h) \qquad \text{(III-4)}$$

$$\eta_2 = [2(G - H)]^{\frac{1}{2}} \cos h \qquad \xi_2 = [2(G - H)]^{\frac{1}{2}} \sin h$$

Note that equations (III-4) give the transformation from Delaunay to Poincaré, and that no singularities are introduced in this transformation. The inverse transformation is given by

$$\ell = \lambda - \tan^{-1} \frac{\xi_1}{\eta_1} \qquad L = L$$

$$g = \tan^{-1} \frac{\xi_1}{\eta_1} - \tan^{-1} \frac{\xi_2}{\eta_2} \qquad G = L - \frac{\xi_1^2 + \eta_1^2}{2} \qquad \text{(III-5)}$$

$$h = \tan^{-1} \frac{\xi_2}{\eta_2} \qquad H = G - \frac{\xi_2^2 + \eta_2^2}{2}$$

In the transformation (III-5), the equations for the momenta L,G,H exhibit no singularities, whereas in the coordinates ℓ,g,h singularities will arise for zero eccentricity ($\eta_1 = 0$) and for zero inclination ($\eta_2 = 0$). This feature of the transformation will be important in later analysis.

If one denotes a general canonical set of variables by w, then the equations of motion take the form (see Eq. (II-6)):

$$\dot{w} = \Phi_o \frac{\partial}{\partial \tilde{w}} \mathcal{H}(w, \epsilon) \qquad \text{(III-6)}$$

with $w = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ α the generalized coordinates, β the associated momenta

where for the artificial Earth satellite problem the Hamiltonian is written as

$$\mathcal{H}(w, \epsilon) = - \frac{\mu^2}{2L^2(w)} + \epsilon \mathcal{H}^{(1)}(w) + \epsilon^2 \mathcal{H}^{(2)}(w) \qquad \text{(III-7)}$$

The oblateness coefficient J_2 has been taken as the small parameter ϵ for convenience. Since all of the higher harmonics in the expansion for the Earth's field are of at least $0(J_2^2)$, one can represent their contribution as an ϵ^2 term (Eq. (II-4)).

Now, apply a stationary canonical transformation to define a new set of variables w':

$$\begin{aligned} \alpha &= \alpha' - \epsilon \mathcal{D}^{(1)}_{\tilde{\beta}'}(\beta', \alpha) - \epsilon^2 \mathcal{D}^{(2)}_{\tilde{\beta}'}(\beta', \alpha) \\ \beta &= \beta' + \epsilon \mathcal{D}^{(1)}_{\tilde{\alpha}}(\beta', \alpha) + \epsilon^2 \mathcal{D}^{(2)}_{\tilde{\alpha}}(\beta', \alpha) \end{aligned} \qquad \text{(III-8)}$$

which has been truncated with the second order terms. The $\mathcal{D}^{(i)}$ are the "generating functions" of the canonical transformation. The new

Hamiltonian is then

$$\mathcal{H}'(w',\epsilon) = \mathcal{H}(w,\epsilon) = -\frac{\mu^2}{2L^2} + \epsilon\left\{\mathcal{H}^{(1)} + \frac{\mu^2}{L^3}\left(L_{\beta'}\mathcal{D}^{(1)}_{\tilde{\alpha}'} - L_{\alpha'}\mathcal{D}^{(1)}_{\tilde{\beta}'}\right)\right\} +$$

$$+ \epsilon^2\left\{\mathcal{H}^{(2)} + \left(\mathcal{H}^{(1)}_{\beta'}\mathcal{D}^{(1)}_{\tilde{\alpha}'} - \mathcal{H}^{(1)}_{\alpha'}\mathcal{D}^{(1)}_{\tilde{\beta}'}\right) + \frac{\mu^2}{L^3}\left(L_{\beta'}\mathcal{D}^{(2)}_{\tilde{\alpha}'} - L_{\alpha'}\mathcal{D}^{(2)}_{\tilde{\beta}'}\right)\right.$$

$$- \frac{3\mu^2}{2L^4}\left(L_{\beta'}\mathcal{D}^{(1)}_{\tilde{\alpha}'} - L_{\alpha'}\mathcal{D}^{(1)}_{\tilde{\beta}'}\right)^2 + \frac{\mu^2}{L^3}\left(-L_{\beta'}\mathcal{D}^{(1)}_{\tilde{\alpha}'\alpha'}\mathcal{D}^{(1)}_{\tilde{\beta}'} + L_{\alpha'}\mathcal{D}^{(1)}_{\tilde{\beta}'\alpha'}\mathcal{D}^{(1)}_{\tilde{\beta}'}\right)$$

$$\left. + \frac{1}{2}\frac{\mu^2}{L^3}\left(\mathcal{D}^{(1)}_{\alpha'}, -\mathcal{D}^{(1)}_{\beta'}\right)\begin{pmatrix} L_{\tilde{\beta}'\beta'} & L_{\tilde{\beta}'\alpha'} \\ L_{\tilde{\alpha}'\beta'} & L_{\tilde{\alpha}'\alpha'} \end{pmatrix}\begin{pmatrix} \mathcal{D}^{(1)}_{\tilde{\alpha}'} \\ -\mathcal{D}^{(1)}_{\tilde{\beta}'} \end{pmatrix}\right\} + \epsilon^3 f(w',\epsilon) \qquad \text{(III-9)}$$

All functions in Eq. (III-9) are to be evaluated at w'. Choose $\mathcal{D}^{(1)}(\alpha,\beta')$ and $\mathcal{D}^{(2)}(\alpha,\beta')$ so that $\mathcal{H}'(w')$ contains no short period terms except in $f(w',\epsilon)$. This requirement is defined by

$$\frac{\partial}{\partial \ell'}\left[\mathcal{H}'(w',\epsilon) - \epsilon^3 f(w',\epsilon)\right] = 0 \qquad \text{(III-10)}$$

with ℓ' the Delaunay variable conjugate to $L' = L(w')$. The Poisson bracket

$$[A,B] = A_{\beta'}B_{\tilde{\alpha}'} - A_{\alpha'}B_{\tilde{\beta}'} = A_{w'}\Phi_o B_{\tilde{w}'} \qquad \text{(III-11)}$$

is easily shown to be invariant under a canonical transformation. In particular

$$[L', \mathcal{D}^{(1)}] = \frac{\partial \mathcal{D}^{(1)}}{\partial \ell'} \qquad \text{(III-12)}$$

then if one writes

$$\mathcal{H}^{(i)}(w') = \overline{\mathcal{H}}^{(i)}(w') + \hat{\mathcal{H}}^{(i)}(w')$$

$$\text{with} \quad \overline{\mathcal{H}}^{(i)}(w') = \underset{\ell'}{\text{av}}\ \mathcal{H}^{(i)}(w')$$

one chooses $\mathcal{D}^{(1)}(w')$ such that

$$\frac{\mu^2}{L^3}\frac{\partial \mathcal{D}^{(1)}}{\partial \ell'} = - \hat{\mathcal{H}}^{(1)} \tag{III-13}$$

This defines $\mathcal{D}^{(1)}(w')$ uniquely up to an additive function of the Delaunay variables other than ℓ'. It is then convenient to choose $\mathcal{D}^{(1)}$ to be identical with Brouwer's $S^{(1)}(L',G',H,\ell,g,--)$ expressed as $S^{(1)}(\alpha,\beta')$. Note that the function $S^{(1)}$ is non-singular for zero eccentricity and/or inclination and is a function (as Brouwer writes it) of both L,G explicitly and implicitly through e and f, the true anomaly. When computing the required partial derivatives for ℓ and g short period variations, the singularity for zero eccentricity, for example, arises in the following way

$$\begin{aligned} \frac{\partial S^{(1)}}{\partial L'} &= \left(\frac{\partial S^{(1)}}{\partial L'}\right)_{expl.} + \left(\frac{G'^2}{e'L'^3}\right)\frac{\partial S^{(1)}}{\partial e'} \\ \frac{\partial S^{(1)}}{\partial G'} &= \left(\frac{\partial S^{(1)}}{\partial G'}\right)_{expl.} - \left(\frac{G'}{e'L'^2}\right)\frac{\partial S^{(1)}}{\partial e'} \end{aligned} \tag{III-14}$$

As shown in Ref. 10, no $\frac{1}{e'}$ terms arise in the case of the Hill variables; however, zero inclination singularities still exist. That no singularities occur for the Poincaré variables can readily be demonstrated. The argument is given for the variable λ; similar arguments apply to the other variables. The function $S^{(1)}$ is given explicitly as $S^{(1)}(e',f',g',G',H')$ so $S^{(1)}$ depends on L' also through e' and f'. According to the von Zeipel procedure the first order short-period variations of λ are given by

$$\delta\lambda_1 = \frac{\partial S^{(1)}}{\partial L'} \qquad \text{(III-15)}$$

which then can be written as (dropping the primes for convenience)

$$\delta\lambda_1 = \frac{\partial S^{(1)}}{\partial L} = \left(\frac{\partial S^{(1)}}{\partial L}\right)_{\text{expl.}} + \frac{\partial e}{\partial L}\frac{\partial S^{(1)}}{\partial e} + \frac{\partial G}{\partial L}\frac{\partial S^{(1)}}{\partial G} + \frac{\partial H}{\partial L}\frac{\partial S^{(1)}}{\partial H} \qquad \text{(III-16)}$$

with

$$\frac{\partial S^{(1)}}{\partial e} = \left(\frac{\partial S^{(1)}}{\partial e}\right)_{\text{expl.}} + \left(\frac{a}{r} - \eta^{-2}\right)\frac{\partial S^{(1)}}{\partial f}\sin f$$

$$\eta = (1 - e^2)^{\frac{1}{2}}$$

So

$$\delta\lambda_1 = \frac{\partial e}{\partial L}\left[\frac{\partial S^{(1)}}{\partial e} + \left(\frac{a}{r} - \eta^{-2}\right)\frac{\partial S^{(1)}}{\partial f}\sin f\right] + \left(\frac{\partial S^{(1)}}{\partial G}\right)_{\text{expl.}} + \left(\frac{\partial S^{(1)}}{\partial H}\right)_{\text{expl.}}$$

where

$$e = \left\{1 - \frac{1}{L^2}\left[L - \frac{\xi_1^2 + \eta_1^2}{2}\right]^2\right\}^{\frac{1}{2}} \qquad \text{(III-17a)}$$

then

$$\frac{\partial e}{\partial L} = \frac{\sqrt{1 - e^2}}{eL}\left[\sqrt{1 - e^2} - 1\right] = \frac{\eta(\eta - 1)}{eL}$$

$$= \frac{\eta(\eta - 1)(\eta + 1)}{eL(\eta + 1)} = -\frac{e\eta}{L(\eta + 1)} \qquad \text{(III-17b)}$$

The derivatives with respect to e and f explicitly introduce no singularities and neither do the last two terms of Eq. (III-17a). Thus $\delta\lambda_1$ is well-behaved as e(and i) $\rightarrow 0$.

Next, one can show that the second order long-period Hamiltonian is independent of the particular canonical variables used and, furthermore, that the second order generating function $\mathcal{D}^{(2)}$ is non-singular for zero eccentricity and/or inclination. Recall that the function $\mathcal{D}^{(2)}$ is chosen so as to cancel all second order short-period terms of the

Hamiltonian. In order to obtain the desired results, note that

$$\mathcal{H}^{(1)}_{\beta'} S^{(1)}_{\tilde{\alpha}'} - \mathcal{H}^{(1)}_{\alpha'} S^{(1)}_{\tilde{\beta}'} = \left[\mathcal{H}^{(1)}, S^{(1)} \right] \tag{III-18}$$

is an invariant for canonical variables. Furthermore,

$$L_{\beta'} \mathcal{D}^{(2)}_{\tilde{\alpha}'} - L_{\alpha'} \mathcal{D}^{(2)}_{\tilde{\beta}'} = \frac{\partial \mathcal{D}^{(2)}}{\partial \ell'} \tag{III-19}$$

and

$$\left(L_{\beta'} S^{(1)}_{\tilde{\alpha}'} - L_{\alpha'} S^{(1)}_{\tilde{\beta}'} \right) = \left(\hat{\mathcal{H}}^{(1)} \right)^2 \tag{III-20}$$

One can rewrite (from Eq. (III-9))

$$- L_{\beta'} S^{(1)}_{\tilde{\alpha}'\alpha'} S^{(1)}_{\tilde{\beta}'} + L_{\alpha'} S^{(1)}_{\tilde{\beta}'\alpha'} S^{(1)}_{\tilde{\beta}'}$$

as follows (where α, β are understood to be the primed variables)

$$\begin{aligned}
& - L_\beta S^{(1)}_{\tilde{\alpha}\alpha} S^{(1)}_{\tilde{\beta}} + L_\alpha S^{(1)}_{\tilde{\beta}\alpha} S^{(1)}_{\tilde{\beta}} = -\frac{1}{2} L_\beta S^{(1)}_{\tilde{\alpha}\alpha} S^{(1)}_{\tilde{\beta}} + \frac{1}{2} L_\alpha S^{(1)}_{\tilde{\beta}\alpha} S^{(1)}_{\tilde{\beta}} - \\
& - \frac{1}{2} \left(L_\beta \frac{\partial}{\partial \tilde{\alpha}} - L_\alpha \frac{\partial}{\partial \tilde{\beta}} \right) \left(S^{(1)}_\alpha S^{(1)}_\beta \right) + \frac{1}{2} L_\beta S^{(1)}_{\tilde{\alpha}\beta} S^{(1)}_{\tilde{\alpha}} - \frac{1}{2} L_\alpha S^{(1)}_{\tilde{\beta}\beta} S^{(1)}_{\tilde{\alpha}} \equiv \\
& = - \frac{1}{2} \left\{ \frac{\partial}{\partial \alpha} \left(L_\beta S^{(1)}_{\tilde{\alpha}} - L_\alpha S^{(1)}_{\tilde{\beta}} \right) \right\} S^{(1)}_{\tilde{\beta}} + \frac{1}{2} \left\{ \frac{\partial}{\partial \beta} \left(L_\beta S^{(1)}_{\tilde{\alpha}} - L_\alpha S^{(1)}_{\tilde{\beta}} \right) \right\} S^{(1)}_{\tilde{\alpha}} + \\
& + \frac{1}{2} \left(S^{(1)}_{\tilde{\alpha}} L_{\tilde{\beta}\alpha} - S^{(1)}_{\beta} L_{\tilde{\alpha}\alpha} \right) S^{(1)}_{\tilde{\beta}} - \frac{1}{2} \left(S^{(1)}_{\tilde{\alpha}} L_{\tilde{\beta}\beta} - S^{(1)}_{\tilde{\beta}} L_{\tilde{\alpha}\beta} \right) S^{(1)}_{\tilde{\alpha}} - \\
& - \frac{1}{2} \frac{\partial}{\partial \ell} \left(S^{(1)}_\alpha S^{(1)}_{\tilde{\beta}} \right) = - \frac{1}{2} \left[\hat{\mathcal{H}}^{(1)}, S^{(1)} \right] - \frac{1}{2} \frac{\partial}{\partial \ell} \left(S^{(1)}_\alpha S^{(1)}_{\tilde{\beta}} \right) + \\
& + \frac{1}{2} \left(S^{(1)}_{\tilde{\alpha}} L_{\tilde{\beta}\alpha} - S^{(1)}_{\beta} L_{\tilde{\alpha}\alpha} \right) S^{(1)}_{\tilde{\beta}} - \frac{1}{2} \left(S^{(1)}_{\tilde{\alpha}} L_{\tilde{\beta}\beta} - S^{(1)}_{\tilde{\beta}} L_{\tilde{\alpha}\beta} \right) S^{(1)}_{\tilde{\alpha}}
\end{aligned} \tag{III-21}$$

then also

$$\tfrac{1}{2}\left(S_{\alpha}^{(1)}, -S_{\beta}^{(1)}\right)\begin{pmatrix} L_{\tilde{\beta}\beta} & L_{\tilde{\beta}\alpha} \\ L_{\tilde{\alpha}\beta} & L_{\tilde{\alpha}\alpha} \end{pmatrix}\begin{pmatrix} S_{\tilde{\alpha}}^{(1)} \\ -S_{\tilde{\beta}}^{(1)} \end{pmatrix} =$$

$$= \tfrac{1}{2}\left(S_{\alpha}^{(1)}, -S_{\beta}^{(1)}\right)\begin{pmatrix} S_{\tilde{\alpha}}^{(1)} L_{\tilde{\beta}\beta} - S_{\tilde{\beta}}^{(1)} L_{\tilde{\beta}\alpha} \\ S_{\tilde{\alpha}}^{(1)} L_{\tilde{\alpha}\beta} - S_{\tilde{\beta}}^{(1)} L_{\tilde{\alpha}\alpha} \end{pmatrix} \qquad \text{(III-22)}$$

Thus it can be seen that Eq. (III-22) cancels the last two terms of Eq. (III-21). The second order part of the Hamiltonian (III-9) consequently is given by

$$\epsilon^2 \left\{ \mathcal{H}^{(2)} + \left[\mathcal{H}^{(1)}, S^{(1)}\right] + \frac{\mu^2}{L'^3} \mathcal{D}^{(2)}_{\ell'} - \frac{3}{2}\frac{L'^2}{\mu^2}\left(\hat{\mathcal{H}}^{(1)}\right)^2 - \frac{\mu^2}{2L'^3}\left[\hat{\mathcal{H}}^{(1)}, S^{(1)}\right] - \tfrac{1}{2}\frac{\partial}{\partial \ell'}\left(S_{\alpha'}^{(1)} \; S_{\tilde{\beta}'}^{(1)}\right) \right\} \qquad \text{(III-23)}$$

The only term of Eq. (III-23) apart from $\frac{\mu^2}{L'^3} \mathcal{D}^{(2)}_{\ell'}$ that depends on the particular canonical variables used is $-\frac{1}{2}\frac{\partial}{\partial \ell'}\left(S_{\alpha'}^{(1)} \; S_{\tilde{\beta}'}^{(1)}\right)$, which is necessarily short period, and $\mathcal{D}^{(2)}$, of course, is chosen so that the term $\frac{\mu^2}{L'^3} \mathcal{D}^{(2)}_{\ell'}$ cancels all short-period terms.

From this invariance property of the terms in Eq. (III-23) one deduces that the difference between the second order generating functions $\mathcal{D}^{(2)}$ of two different sets of canonical variables, such as any arbitrary set w and the Delaunay set y for example, will be given by

$$\mathcal{D}^{(2)} - S^{(2)} = \frac{L'^3}{2\mu^2}\left(S_{\alpha'}^{(1)} \; S_{\tilde{\beta}'}^{(1)}\right) - \frac{L'^3}{2\mu^2}\left(S_{Q'}^{(1)} \; S_{\tilde{P}}^{(1)}\right) + \text{arbitrary long period term} \qquad \text{(III-24)}$$

Equation (III-24) gives a convenient algorithm for computing the generating function $\mathfrak{D}^{(2)}$ for any set of canonical variables. In order to assure that all of the functions arising in the error bound determination remain bounded, one must establish that $\mathfrak{D}^{(2)}$ contains no singularities. This may be done utilizing the known results of Izsak (Ref. 9), Brouwer (Ref. 3) and Kozai (Ref. 4).

In Kozai's paper, the expression given for the J_2^2 component of the function $S^{(2)}$ shows the factor $1/e$ for the trigonometric arguments $\sin f$, $\sin (f + 2g)$, $\sin (3f + 2g)$, $\sin (3f + 4g)$ and $\sin (5f + 4g)$. The appearance of this factor is unnecessary and a suitable rearrangement of terms eliminates it. Such rearrangement will be shown explicitly here for the coefficient of $\sin f$; the other terms can be treated similarly. The coefficient of $\sin f$ as given by Kozai is (omitting a nonsingular multiplying factor):

$$\frac{1}{e}\Big[9(11 - 30\,\theta^2 + 27\,\theta^4) - 8\eta^2(17 - 38\,\theta^2 + 11\,\theta^4) - 4\,\eta^3(1 - 3\,\theta^2)^2 \times (3 + \eta^2) + \eta^4(53 - 130\,\theta^2 - 11\,\theta^4)\Big] \qquad \text{(III-25)}$$

$$\text{where} \quad \theta = \frac{H}{G} = \cos i$$

Equation (III-25) can be rewritten as

$$\frac{1}{e}\Big[99 - 270\,\theta^2 + 243\,\theta^4 - 136 + 304\,\theta^2 - 88\,\theta^4 - 4\eta^3(1 - 3\,\theta^2)^2(3 + \eta^2) + 53 - 130\,\theta^2 - 11\,\theta^4 - 2e^2(121 - 282\,\theta^2 + 33\,\theta^4) + e^4(53 - 13\,\theta^2 - 11\,\theta^4)\Big]$$

or dropping the e^2 and e^4 terms and combining:

$$\frac{4(1 - 3\,\theta^2)^2}{e}\left[4 - 3\eta^3 - \eta^5\right]$$

Now, this can be rewritten as follows:

$$\frac{4 - 3\eta^2 - \eta^5}{e} = \frac{(1 - \eta)}{e} [\eta^4 + \eta^3 + 4\eta^2 + 4\eta + 4]$$

$$= \frac{(1 - \eta)(1 + \eta)}{e(1 + \eta)} [\eta^4 + \eta^3 + 4\eta^2 + 4\eta + 4]$$

$$= \frac{e}{1 + \eta} [\eta^4 + \eta^3 + 4\eta^2 + 4\eta + 4] \qquad \text{(III-26)}$$

which remains bounded as $e \to 0$. In a similar manner the other expressions given by Kozai (for higher order harmonics J_N) may be rearranged and thus it can be shown that $S^{(2)}$ contains no $1/e$ factors. Of the terms on the right-hand side of Eq. (III-24), the first is known to be bounded (III-17 a,b), the second can be shown to be bounded by the above technique of rearranging.

The (new) canonical variables w' satisfy the differential equations

$$\dot{w}' = \Phi_o \, \mathcal{H}'_{\tilde{w}'}(w', \epsilon) \qquad \text{(III-27)}$$

where one can write the Hamiltonian in the form

$$\mathcal{H}'(w', \epsilon) = - \frac{\mu^2}{2L'^2(w')} + \epsilon \, \bar{\mathcal{H}}^{(1)}(w') + \epsilon^2 K^{(2)}(w') + \epsilon^3 s(w', \epsilon) \qquad \text{(III-28)}$$

an analytic function of the variables w' and the small parameter ϵ. Define next a transformation (canonical) to the "secular" variables w'' by the truncated expressions

$$\alpha'' = \alpha' - \epsilon \, \overset{*}{S}_{\tilde{\beta}''}(\beta'', \alpha')$$
$$\beta'' = \beta' + \epsilon \, \overset{*}{S}_{\tilde{\alpha}'}(\beta'', \alpha') \qquad \text{(III-29)}$$

where† S^* is chosen so as to cancel the long-period part of the Hamiltonian (except near critical inclination). Then $S^*_{\tilde{\beta}''}$ and $S^*_{\tilde{\alpha}'}$ give the first order long-period variations of α and β i.e.:

$$\frac{1}{2\pi}\int_0^{2\pi} S^*_{\tilde{\beta}''}\, dg' = \frac{1}{2\pi}\int_0^{2\pi} S^*_{\tilde{\alpha}'}\, dg' = 0 \qquad \text{(III-30)}$$

The governing differential equations for w'' become then

$$\dot{w}'' = \Phi_o \left[\mathcal{H}''_{\tilde{w}''}(w'',\epsilon) + \epsilon^3 \psi_{\tilde{w}''}(w'',\epsilon) \right] \qquad \text{(III-31)}$$

and the solution can be written in the form

$$\alpha = \alpha'' - \epsilon S^*_{\tilde{\beta}''}(\beta'',\alpha') - \epsilon S^{(1)}_{\tilde{\beta}'}(\beta',\alpha) - \epsilon^2 \mathfrak{L}^{(2)}_{\tilde{\beta}'}(\beta',\alpha)$$

$$\beta = \beta'' + \epsilon S^*_{\tilde{\alpha}'}(\beta'',\alpha') + \epsilon S^{(1)}_{\tilde{\alpha}}(\beta',\alpha) + \epsilon^2 \mathfrak{L}^{(2)}_{\tilde{\alpha}}(\beta',\alpha)$$

(III-32)

where

$$\alpha' = \alpha'' - \epsilon S^*_{\tilde{\beta}''}(\beta'',\alpha')$$

$$\beta' = \beta'' + \epsilon S^*_{\tilde{\alpha}'}(\beta'',\alpha')$$

In the definitions of what constituted long-period and/or short-period variations, the Delaunay variables were used explicitly (see Eqs. (III-30) and (III-10)). If the von Zeipel technique is carried out for the Delaunay variables, then it is found that $P''(\equiv \beta'')$ are constants, the "secular" Hamiltonian is a function of P'' only and the coordinates $Q''(\equiv \alpha'')$ have constant rates. If one is dealing in any other canonical variables, for example the Poincaré set x, then x'' are defined to be the same functions of y'' as x are of y.

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† The function S^* can be chosen to be identical to Brouwer's long-period generating function considered as a function of w (see Ref. 10).

In this section it has been established that the generating functions of transformations (III-8) and (III-29) and their partial derivatives are bounded. Note that Eqs. (III-8) constitute transcendental equations for w' which may be written in the form

$$\alpha' = \alpha + \epsilon\, \mathcal{L}_1(\beta',\alpha)$$
$$\beta' = \beta + \epsilon\, \mathcal{L}_2(\beta',\alpha) \qquad \text{(III-33)}$$

From the general bounds on w one has

$$|\beta| \leq B \qquad \text{(III-34)}$$

The functions $\mathcal{L}_i(\beta',\alpha)$ depend on trigonometric functions of α. Suppose now that β is bounded within some region R, and specifically, that β is bounded away from the boundary of R by at least $\epsilon A/1-\epsilon K$. Where A and K are defined by

$$|\mathcal{L}_2(\beta,\alpha)| \leq A$$
$$\text{and} \quad |\mathcal{L}_2(\beta_i,\alpha) - \mathcal{L}_2(\beta_j,\alpha)| \leq K\,|\beta_i - \beta_j| \text{ for all } \beta_i,\beta_j \text{ in } R \qquad \text{(III-35)}$$

Assume further that $\epsilon K < 1$; this in effect imposes a restriction on how close the energy and angular momentum may be to zero. Now assume the following iterative algorithm for computing the primed variables β':

$$\beta'_{n+1} = \beta + \epsilon\, \mathcal{L}_2(\beta'_n,\alpha)$$
$$\text{with} \quad \beta'_o \triangleq \beta \qquad \text{(III-36)}$$

then
$$|\beta'_1 - \beta| \leq \epsilon A$$
$$|\beta'_2 - \beta'_1| \leq \epsilon K|\beta'_1 - \beta|\,, \quad \text{since } \beta'_1 \text{ is also in } R$$
$$|\beta'_3 - \beta'_2| \leq \epsilon K|\beta'_2 - \beta'_1|\,, \quad \text{since } \beta'_2 \text{ is also in } R \qquad \text{(III-37)}$$
$$\vdots$$
$$|\beta'_n - \beta'_{n-1}| \leq \epsilon K|\beta'_{n-1} - \beta'_{n-2}|$$

and $$|\beta_2' - \beta| \leq |\beta_2' - \beta_1'| + |\beta_1' - \beta| \leq |\beta_1' - \beta|(1 + \epsilon K)$$

$$|\beta_3' - \beta| \leq |\beta_3' - \beta_2'| + |\beta_2' - \beta| \leq |\beta_1' - \beta|(1 + \epsilon K + (\epsilon K)^2)$$

$$\vdots$$

$$|\beta_n' - \beta| \leq |\beta_1' - \beta| \sum_{j=0}^{n-1} (\epsilon K)^j \qquad \text{(III-38)}$$

or $$|\beta_n' - \beta| \leq \epsilon A \sum_{j=0}^{n-1} (\epsilon K)^j$$

Taking the limit

$$\lim_{n\to\infty} |\beta_n' - \beta| \leq \epsilon A \lim_{n\to\infty} \sum_{j=0}^{n-1} (\epsilon K)^j$$

$$\lim_{n\to\infty} |\beta_n' - \beta| \leq \epsilon A/1-\epsilon K \qquad \text{(III-39)}$$

A similar argument can be applied to the long-period transformation (III-29) to deduce that β'' will remain (sufficiently) close to β' . As a consequence of the above, one has

$$|\alpha' - \alpha| = \epsilon|\mathcal{L}_1(\beta',\alpha)| \leq \epsilon A_1 \qquad \text{(III-40)}$$

Also, note that the iterative procedure converges, i.e.

$$|\beta_{n+1}' - \beta_n'| \leq \epsilon A(\epsilon K)^n \qquad \text{(III-41)}$$

so that as $n \to \infty$, $|\beta_{n+1}' - \beta_n'| \to 0$ if $\epsilon K < 1$.

IV. ERROR BOUNDS FOR THE AXISYMMETRIC PROBLEM

All of the information necessary to derive formal error bounds has been given in Sections II and III. One can proceed then in a straightforward manner to derive the bounds utilizing known theorems from the theory of differential equations. However, it turns out that because of the nature of the differential equations, the bounds obtainable in this manner prove to be unsatisfactory for time intervals of the order of $1/\epsilon$. This fact is a natural consequence of the existence of a rapidly rotating phase in the governing system of differential equations; however, since only one such phase appears in the case of satellite motion, one can circumvent the difficulty by appealing to a known integral of the motion. In this section the conventional method of error analysis will be presented first and then the extension to the large time intervals will be given for the problem with an axisymmetric potential.

A. BOUNDS FOR SMALL TIME INTERVALS

In order to simplify the following presentation, some new notation will be introduced at this point. If A and B denote n-dimensional vectors then $|A-B|$ will denote the matrix of absolute values of the component differences of A and B i.e.:

$$|A-B| \triangleq \begin{bmatrix} |A_1-B_1| \\ |A_2-B_2| \\ \vdots \\ |A_n-B_n| \end{bmatrix} \qquad \text{(IV-1)}$$

The governing differential equations for the "secular" variables were given as (Eq. (III-31))

$$\dot{w}'' = \Phi_o \frac{\partial}{\partial \tilde{w}'} [\mathcal{H}''(w'', \epsilon) + \epsilon^3 \psi(w'', \epsilon)] \qquad \text{(IV-2)}$$

from which the approximate state vector w''_A is defined by

$$\dot{w}''_A = \Phi_o \frac{\partial}{\partial \tilde{w}''_A} \mathcal{H}''(w''_A, \epsilon)$$

$$w''_A(0) = w''(0) \qquad \text{(IV-3)}$$

For convenience, Eqs. (IV-2) and (IV-3) can be rewritten in the form

$$\dot{w}'' = \Lambda(w'', \epsilon) + \epsilon^3 \Psi(w'', \epsilon)$$

$$\dot{w}''_A = \Lambda(w''_A, \epsilon) \qquad \text{(IV-4)}$$

Since the functions $\frac{\partial}{\partial \tilde{w}''} \mathcal{H}''$ and $\frac{\partial}{\partial \tilde{w}''_A} \mathcal{H}''$ satisfy a Lipschitz condition on the domain of definition of $w(t)$, it follows that

$$| \Lambda(w'', \epsilon) - \Lambda(w''_A, \epsilon) | \leq \underline{k} |w'' - w''_A| \triangleq \underline{k}m'' \qquad \text{(IV-5)}$$

where $\underline{k}$ is an $n \times n$ matrix if m'', the matrix of absolute values of the component differences of $w'' - w''_A$, is $n \times 1$. The particular form of Eq. (IV-5) was chosen since a vector function, say $\Lambda(w'')$, satisfies a Lipschitz condition on w'' if and only if each of its components $\Lambda_i(w'', t)$ does. Since the constants may be different, the use of the matrix of Lipschitz constants $\underline{k}$ can afford a more precise bound than that usually provided by the norm $\|w'' - w''_A\|$.

As a consequence of (IV-5) one can immediately write

$$\frac{d}{dt} m'' \leq \underline{k}m'' + \epsilon^3 \Psi(w'', \epsilon) \qquad \text{(IV-6)}$$

where from a priori bounds on $w(t)$

$$|\Psi(w'', \epsilon)| \leq W(C) \tag{IV-7}$$

Hence

$$\frac{d}{dt} m'' - \underline{k} m'' \leq \epsilon^3 W \tag{IV-8}$$

which is readily integrated to give

$$m'' \leq m''(0) \exp \underline{k}t + \epsilon^3 W \underline{k}^{-1} [\exp \underline{k}t - I]$$
$$0 \leq t \leq T \ , \ t_o \triangleq 0 \tag{IV-9}$$

However, since it was assumed that $w''_A(0) = w''(0)$, the initial error $m''(0) = 0$ and

$$m'' = |w'' - w''_A| \leq \epsilon^3 W \underline{k}^{-1} [\exp \underline{k}t - I] \tag{IV-10}$$

At this point, several difficulties of (IV-10) can be pointed out. The bounds (IV-10) prove to be unsatisfactory for Delaunay variables for small eccentricity and/or inclination since $\underline{k}$ contains the factors $\frac{1}{e}$ and $\frac{1}{\sin i}$. If w is taken to be the Hill set z, the zero eccentricity difficulty is removed. Although the zero inclination singularity remains, for many purposes the Hill variables are a convenient set to use due to the relatively simple expressions for the periodic variations of the in-plane coordinates (see Vagners, Ref. 10). Taking w to be the Poincaré set x, satisfactory behavior is assured for both zero eccentricity and inclination. A much more serious difficulty occurs if one wishes to examine the bounds for time intervals of the order $1/\epsilon$. Expansion of (IV-10) yields, for "small" time intervals

$$\epsilon^3 W \underline{k}^{-1} [\exp \underline{k}t - I] = \epsilon^3 W t + \epsilon^3 W \sum_{j=2}^{\infty} \frac{\underline{k}^{j-1} t^j}{j!} \tag{IV-11}$$

However, for time intervals of order $1/\epsilon$, the bound (IV-10) becomes very large i.e., behaves like $\exp 1/\epsilon$.

Assuming now that m'' is at most $O(\epsilon)$(i.e., bound (IV-10) is satisfactory) then one can complete the analysis by including the periodic terms. If this is done, the total approximate solution of interest here is written as

$$w_c = w''_A + \epsilon\gamma(w''_A) \tag{IV-12}$$

with $\gamma(w''_A)$ giving the first order periodic parts of w as defined by eqs. (III-8) and (III-29) with the generating functions considered as functions of the double primed variables. Equations (III-32) can be written in the form

$$w = w'' + \epsilon\gamma(w'') + \epsilon^2\zeta(w'',\epsilon) \tag{IV-13}$$

then

$$\begin{aligned}|w - w_c| &= |w'' + \epsilon\gamma(w'') + \epsilon^2\zeta(w'',\epsilon) - w''_A - \epsilon\gamma(w''_A)| \le \\ &|w'' - w''_A| + \epsilon|\gamma(w'') - \gamma(w''_A)| + \epsilon^2|\zeta(w'',\epsilon)|\end{aligned} \tag{IV-14}$$

Since $\epsilon|\gamma(w'') - \gamma(w''_A)|$ gives the error in the first order periodic terms of the solution and m'' is $O(\epsilon)$, then the term contributes error of second order. Thus the effect of the last two terms of (IV-14) can be combined into one second order term $\epsilon^2 Z$ to account for all periodic errors of the solution. The error bound for "small" time intervals, assuming exact initial conditions, assumes the form

$$|w - w_c| \le \epsilon^3 W\underline{k}^{-1}\ [\exp \underline{k}t - I] + \epsilon^2 Z \tag{IV-15}$$

or, effectively,

$$\epsilon^3 Wt + \epsilon^2 Z \tag{IV-16}$$

The difficulty of the above bounds for $t \sim 1/\epsilon$ is a direct consequence of the existence of a rapidly rotating phase in the dynamical system. In the treatment of systems with rapidly rotating phases by the method of averaging, the governing equations for these phases are considered separately. The general result obtained then is that the error is $0(\epsilon)$ for $t \sim 1/\epsilon$ rather than $0(\epsilon^2)$ as one would expect from the truncations performed i.e., truncation of $0(\epsilon^2)$ periodic and $0(\epsilon^3)$ secular terms. In the following, such separation will be effected and, by appealing to known integrals, the bounds will be derived for all variables to $0(\epsilon^2)$ for $t \sim 1/\epsilon$.

B. EXTENDED TIME ERROR BOUNDS

The following analysis will be carried out for the Poincaré variables explicitly utilizing known results for the Delaunay variables and their rates. The secular Hamiltonian was defined from the von Zeipel procedure as being a function of the Delaunay momenta P'' only (to second order), hence in the Poincaré variables one writes

$$\bar{\mathcal{H}}''(x'',\epsilon) = -\frac{\mu^2}{2L''^2} + \epsilon\,\bar{\mathcal{H}}^{(1)}(x'') + \epsilon^2 \mathcal{R}^{(2)}(x'') - \epsilon^3\,\varphi(x'',\epsilon) \quad \text{(IV-17)}$$

where $\bar{\mathcal{H}}^{(1)} = \dfrac{\mu^4 R_\oplus^2}{4L''^3 G''^3}\left(3\dfrac{H^2}{G''^2} - 1\right)$, H and G functions of x'' (Eq. (III-5)) and $\mathcal{R}^{(2)}$ is F^{**} of Brouwer considered as a function of x'' . (Explicit expressions for $\mathcal{R}^{(2)}$ in terms of Hill variables for any J_n may be found in Ref. 10, which could then be transformed to Poincaré variables if necessary.) Equation (IV-17) can be rewritten more conveniently as

$$\bar{\mathcal{H}}''(x'',\epsilon) = -\frac{\mu^2}{2L''^2} + \epsilon\,\mathcal{F}(x'',\epsilon) - \epsilon^3\varphi(x'',\epsilon) \qquad \text{(IV-18)}$$

where

$$\mathcal{F}(x'',\epsilon) = \mathcal{F}\left(L'', \eta_1''^2 + \xi_1''^2, \eta_2''^2 + \xi_2''^2, \epsilon\right) \tag{IV-19}$$

The equations for the Poincaré variable rates become

$$\dot{\lambda}'' = \frac{\mu^2}{L''^3} + \epsilon \frac{\partial}{\partial L''} [\mathcal{F}(x'',\epsilon) - \epsilon^2 \varphi(x'',\epsilon)] \tag{IV-20}$$

$\dot{L}'' = \epsilon^3 \frac{\partial \varphi}{\partial \lambda''}(x'',\epsilon)$ since $\overline{\mathcal{K}}^{(1)}$ and $\mathcal{R}^{(2)}$ do not contain λ'' and

$$\dot{x}_R'' = \epsilon \Phi_o \frac{\partial}{\partial \tilde{x}_R''} [\mathcal{F}(x'',\epsilon) - \epsilon^2 \varphi(x'',\epsilon)] \tag{IV-21}$$

with

$$x_R'' = \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \xi_1'' \\ \xi_2'' \end{bmatrix}, \quad \Phi_o \text{ a } 4 \times 4 \text{ matrix.}$$

The approximate variables x_A'' are defined by Eqs. (IV-20) and (IV-21) with $\varphi(x'',\epsilon)$ set equal to zero and $x_A''(0) = x''(0)$. Consider first the differential equations for η_1'' and ξ_1'':

$$\dot{\eta}_1'' = \epsilon \frac{\partial}{\partial \xi_1''} \mathcal{F}(x'',\epsilon) - \epsilon^3 \frac{\partial \varphi}{\partial \xi_1''}$$

$$\dot{\xi}_1'' = -\epsilon \frac{\partial}{\partial \eta_1''} \mathcal{F}(x'',\epsilon) + \epsilon^3 \frac{\partial \varphi}{\partial \eta_1''} \tag{IV-22}$$

Recall that $\mathcal{F}(x'',\epsilon)$ is given by Eq. (IV-19) so that with

$$N_1 \triangleq 2 \frac{\partial \mathcal{F}}{\partial(\eta_1''^2 + \xi_1''^2)}$$

a bounded quantity, one obtains

$$\dot{\eta}_1'' = \epsilon N_1 \xi_1'' - \epsilon^3 \frac{\partial}{\partial \xi_1''} \varphi$$

$$\dot{\xi}_1'' = - \epsilon N_1 \eta_1'' + \epsilon^3 \frac{\partial}{\partial \eta_1''} \varphi \tag{IV-23}$$

Hence

$$\frac{1}{2} \frac{d}{dt} \left[\xi_1''^2 + \eta_1''^2 \right] = \epsilon^3 \left[\xi_1'' \frac{\partial \varphi}{\partial \eta_1''} - \eta_1'' \frac{\partial \varphi}{\partial \xi_1''} \right] \tag{IV-24}$$

The approximate solution x_A'' of interest is given by $\varphi = 0$, thus from the boundedness of φ (and its partial derivatives)

$$\left| \left(\eta_1''^2 - \eta_{1A}''^2 \right) + \left(\xi_1''^2 - \xi_{1A}''^2 \right) \right| = \left| \Delta \left(\xi_1''^2 + \eta_1''^2 \right) \right| \leq \epsilon^3 M_o t \tag{IV-25}$$

Here, as well as in the following discussion, the extended time interval will be taken as $t \sim 1/\epsilon$ so that

$$\left| \Delta \left(\xi_1''^2 + \eta_1''^2 \right) \right| \leq \epsilon^2 M_o \tag{IV-26}$$

The reader may prefer to think of the time interval as defined by $nt \sim 1/\epsilon$ where n is taken to be the (suitable) mean motion. For mathematical convenience, the definition $t \sim 1/\epsilon$ will be used.

Now, rewrite Eqs. (IV-23) as a single complex equation ($j \triangleq \sqrt{-1}$):

$$\dot{\xi}_1'' + j\dot{\eta}_1'' = j\epsilon N_1 \left(\xi_1'' + j\eta_1'' \right) + \epsilon^3 \left[\frac{\partial \varphi}{\partial \eta_1''} - j \frac{\partial \varphi}{\partial \xi_1''} \right] \tag{IV-27}$$

and the approximate equations as

$$\dot{\xi}_{1A}'' + j \dot{\eta}_{1A}'' = j \epsilon N_{1A} \left(\xi_{1A}'' + j\eta_{1A}'' \right) \tag{IV-28}$$

since

$$N_1 = N_1 \left(\xi_1''^2 + \eta_1''^2 , \xi_2''^2 + \eta_2''^2 , L'', \epsilon \right)$$

Difference Eqs. (IV-27) and (IV-28) to get

$$\Delta\dot{\xi}_1'' + j\Delta\dot{\eta}_1'' = j\epsilon\Delta N_1(\xi_1'' + j\eta_1'') + j\epsilon N_{1A}(\Delta\xi_1'' + j\Delta\eta_1'') + \epsilon^3\left[\frac{\partial\varphi}{\partial\eta_1''} - j\frac{\partial\varphi}{\partial\xi_1''}\right] \tag{IV-29}$$

From Eq. (IV-26):

$$\Delta N_1 = |N_1 - N_{1A}| \le M_1\Delta\left(\xi_1''^2 + \eta_1''^2\right) \le \epsilon^2 M_2$$

and (IV-29) thus becomes, with the aid of an integrating factor,

$$\left|\frac{d}{dt}\left[(\Delta\xi_1'' + j\Delta\eta_1'')e^{-j\epsilon N_{1A}t}\right]\right| \le \epsilon^3\left|e^{-j\epsilon N_{1A}t}\left[\frac{\partial\varphi}{\partial\eta_1''} - j\frac{\partial\varphi}{\partial\xi_1''} + jM_2\xi_1'' - M_2\eta_1''\right]\right| \tag{IV-30}$$

Then, since the right hand side of (IV-30) is bounded, it follows that

$$\left|(\Delta\xi_1'' + j\Delta\eta_1'')e^{-j\epsilon N_{1A}t}\right| \le \epsilon^3 M_3 t = \epsilon^2 M_3 \quad t \sim 1/\epsilon \tag{IV-31}$$

but $|e^{-j\epsilon N_{1A}t}| = 1$ so

$$|\Delta\xi_1'' + j\Delta\eta_1''| \le \epsilon^2 M_3 , \tag{IV-32}$$

From similar arguments, it follows that for ξ_2'' and η_2''

$$|\Delta\xi_2'' + j\Delta\eta_2''| \le \epsilon^2 M_4 , \quad t \sim 1/\epsilon \tag{IV-33}$$

Also, from the differential Eq. (IV-20)

$$|L'' - L_A''| \equiv \left|\epsilon^3 \int \frac{\partial\varphi}{\partial\lambda''} dt\right| \le \epsilon^3 M_5 t = \epsilon^2 M_5 \tag{IV-34}$$

The remaining coordinate λ'' causes some difficulty, since with $|L'' - L_A''|$ known to $O(\epsilon^2)$, a straightforward analysis of the $\dot{\lambda}''$

equation gives $|\lambda'' - \lambda''_A|$ only to $0(\epsilon)$ for $t \sim 1/\epsilon$. In order to obtain bounds for λ'' consistent with those of the other coordinates, one must appeal to the knowledge of an exact integral for the axi-symmetric problem. In effect, one can re-define the mean motion as introduced by Brouwer (Ref. 3), who wrote

$$n_o \triangleq \frac{\mu^2}{L'^3} \tag{IV-35}$$

and hence, with ℓ_1 and ℓ_2 functions of L', G'', H only

$$\dot{\ell}'' = n_o[1 + \epsilon\ell_1 + \epsilon^2\ell_2] + 0(\epsilon^3) \tag{IV-36}$$

Recall that $\dot{\lambda}''$ was defined by Eq. (IV-20), which in a more explicit form is given by

$$\dot{\lambda}'' = \frac{\mu^2}{L''^3}\left\{1 - \epsilon\frac{3}{2}k_1L''^{-1} - \epsilon\frac{3\mu^2R_\oplus^2}{4G''^4}\left[5\left(\frac{H''}{G''}\right)^2 - 1 - 2\frac{H''}{G''}\right] + \epsilon^2\delta_2\right\} - \epsilon^3\frac{\partial\varphi}{\partial L''} \tag{IV-37}$$

where

$$k_1 = \frac{\mu^2R_\oplus^2}{2G''^3}\left(3\left(\frac{H''}{G''}\right)^2 - 1\right) \quad \text{with} \quad G'' = G''(x'') \quad \text{and} \quad H'' = H''(x'')$$

$$\delta_2 = \frac{L'^3}{\mu^2}\frac{\partial\mathcal{R}^{(2)}}{\partial L''}, \quad \text{a bounded quantity}$$

Define now a new constant $\hat{a}$ by $\left(\text{with } \mathcal{R}^* = \frac{2L''^2}{\mu^2}\mathcal{R}^{(2)}\right)$

$$\frac{\mu}{2\hat{a}} = -\mathcal{H} = -\mathcal{H}'' = \frac{\mu^2}{2L''^2}[1 - \epsilon k_1L''^{-1} - \epsilon^2\mathcal{R}^* + \epsilon^3\varphi_1] \tag{IV-38}$$

and a new "mean motion" by

$$\hat{n} = \mu^{\frac{1}{2}}\left(\frac{1}{\hat{a}}\right)^{3/2} = \frac{\mu^2}{L''^3}\left[1 - \epsilon k_1L''^{-1} - \epsilon^2\mathcal{R}^* + \epsilon^3\varphi_1\right]^{3/2} \tag{IV-39}$$

or expanding

$$\hat{n} = \frac{\mu^2}{L''^3}\left[1 - \epsilon \frac{3}{2} k_1 L''^{-1} - \epsilon^2 \frac{3}{2} \mathcal{R}^* + \epsilon^2 \frac{3}{8} k_1^2 L''^{-2} + \epsilon^3 \varphi_1^*(x'', \epsilon)\right] \quad \text{(IV-40)}$$

Thus

$$\dot{\lambda}'' = \hat{n} - \epsilon \frac{\mu^2}{L''^3}\left\{ \frac{3\mu^2 R_\oplus}{4G''^4}\left[5\left(\frac{H''}{G''}\right)^2 - 1 - 2\frac{H''}{G''}\right] - \epsilon\left[\delta_2 + \frac{3}{2}\mathcal{R}^* - \frac{3}{8}k_1^2 L''^{-2}\right]\right\} - \epsilon^3\left[\frac{\mu^2}{L''^3}\varphi_1^*(x'', \epsilon) + \frac{\partial \varphi}{\partial L''}\right] \quad \text{(IV-41)}$$

Again, the approximate λ''_A is defined by (IV-41) with $\frac{\partial \varphi}{\partial L''} = 0$ and $\varphi_1^* = 0$. From the exact known integral, $H'' = H$ in Eq. (IV-41) and from the definition of G'':

$$G'' = L'' - \frac{\xi_1''^2 + \eta_1''^2}{2} \quad \text{(IV-42)}$$

then from Eqs. (IV-32) and (IV-34)

$$|G'' - G''_A| \leq \epsilon^2 M_6 \qquad t \sim 1/\epsilon \quad \text{(IV-43)}$$

so that finally

$$|\lambda'' - \lambda''_A| \leq \epsilon^2 M_7 \qquad t \sim 1/\epsilon \quad \text{(IV-44)}$$

If one is interested in orbits with non-zero eccentricity and/or inclination (i.e., $e \gg \epsilon$, $\sin i \gg \epsilon$) then the above analysis can be carried out analogously for the Delaunay and/or Hill variables. In particular, for the Delaunay variables, the rapid phase is ℓ'' and an equation similar to (IV-41) (but somewhat simpler in form) results for $\dot{\ell}''$. Due to choice of g and h as the other two coordinates, the ϵ term of (IV-41) is found to disappear. (Of course, the functions $\delta_2, \mathcal{R}^*$ and φ are different than for the Poincaré variables).

C. THE INITIALIZATION PROBLEM

At this point the relevance of the above results to the so-called initialization problem may be noted. The two primary uses of an analytic (artificial satellite) orbit theory are orbit determination by fitting to observational data and orbit prediction from some initial state vector. In the case of orbit determination, the mean (double primed) variables are obtained to high accuracy by fitting to observational data. This accuracy depends on the number and quality of the data points. In this application, the question of initial value errors does not arise.

The initialization problem may be defined as follows: given some initial radius and velocity vectors, compute a satellite ephemeris for some extended time interval via an analytic theory. The initial radius and velocity, and hence the instantaneous elements, are assumed to be known exactly. Analytic theories are usually formulated so that certain constants of the solution are mean elements, for example L', G'' and H in the Brouwer theory, instead of initial values. Thus from the known set of instantaneous elements, the mean elements must be formed by subtracting out the periodic variations. Since one is considering a first order theory, the mean elements thus defined will be in error by $0(\epsilon^2)$. It can be noted here that a numerical iteration procedure has been applied to the determining equations (Cain Ref. 15, Arsenault, et al Ref. 11) which are written as

$$Q = Q'' - \epsilon\left[S^{(1)}_{\tilde{P}'}(P',Q) + S^{(*)}_{\tilde{P}''}(P'',Q')\right]$$

$$P = P'' + \epsilon\left[S^{(1)}_{\tilde{Q}}(P',Q) + S^{(*)}_{\tilde{Q}'}(P'',Q')\right] \qquad \text{(IV-45)}$$

Such a procedure can, of course, only remove that second order error that arises from considering the S functions to be functions of the instantaneous elements (P,Q), but still cannot account for the truncated second order terms. Thus from an accuracy point of view, such iteration procedures are of dubious value, since as shown by Eq. (IV-37) (with ϵ^3 terms truncated) the error in λ''_A, or equivalently ℓ''_A, will still grow as $\epsilon^2 t$ from the zero order term. The other variables of either x'' or y'' do not present any problem since their rates are either zero or multiples of ϵ, so that an initial value error of $0(\epsilon^2)$ will grow as $\epsilon^3 t$ giving results consistent with the expected accuracy of the truncated theory.

With the algorithm suggested by the analysis of subsection IV-B, the initialization difficulty can be resolved. As noted, for all variables except the rapidly rotating phase, the use of mean elements defined by instantaneous value minus the periodic terms (considered as functions of the instantaneous elements) will lead to no difficulty. The necessary initialization procedure for λ''_A is given by Eqs. (IV-38), (IV-39) and (IV-41). The numerical value of $\hat{n}$ is known exactly from instantaneous $\mathcal{H}$ and the remaining terms of (IV-41) have at least an ϵ multiplier. For the Delaunay variables, one rewrites Eq. (IV-36) with n'' the mean
bb motion defined by

$$n'' \cong \dot{\ell}'' = \frac{\mu^2}{L'^3}\left[1 - \epsilon \frac{3}{2} k_1 L'^{-1} + \epsilon^2 \delta_2\right] + 0\ (\epsilon^3) \qquad \text{(IV-46)}$$

so that, with use of energy†

---

† The explicit expression for $\mathcal{R}^*$ is identical to $\frac{2L'^3}{\mu} F^{**}$. [See Vagners (Ref. 10) where it is given as $\frac{2L'^3}{\mu^2} [\bar{\psi}_2 + \bar{F}_2]$].

$$n'' = \dot{\ell}'' = \hat{n} + \epsilon^2 \frac{\mu^2}{L'^3} \left[\frac{3}{2} \mathcal{R}^* + \delta_2 - \frac{3}{8} k_1^2 L'^{-2} \right]_{t=0} + \epsilon^3 \text{ terms} \qquad \text{(IV-47)}$$

in which in the ϵ^2 terms one replaces the double primed variables with the instantaneous elements. The new mean motion $\hat{n}$ is again given by (IV-38) and (IV-39). All terms appearing the brackets of (IV-47) are functions already known from the general theory.

V. THE ASYMMETRIC POTENTIAL FIELD

If one includes the longitude dependent terms (tesseral harmonics) in the gravitational potential, some modifications to the analysis of Section IV are necessary. The additional terms in the Hamiltonian are

$$\mathcal{H}_T = \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} J_{n,m} \left(\frac{R_\oplus}{r} \right)^n P_n^m(\sin \beta) \cos m(\lambda - \lambda_{n,m}) \qquad \text{(V-1)}$$

where $J_{n,m}$ $\lambda_{n,m}$ are constants with $J_{n,m} \sim 0(J_2^2)$

$$\lambda = h + \tan^{-1}(\cos i \tan u) - \omega_\oplus t$$

$\omega_\oplus$ the angular velocity of the Earth

(Time is measured from an instant when the right ascension of Greenwich is zero.)

In this discussion, $\mathcal{H}_T$ will be considered first as a function of the Delaunay variables $\mathcal{H}_T(L, G, H, \ell, g, h - \omega_\oplus t)$. To remove the explicit time dependence, define a new canonical variable as $h^* = h - \omega_\oplus t$ conjugate to H with the associated Hamiltonian given by

$$K = \mathcal{H} - \omega_\oplus H \qquad \text{(V-2)}$$

where $\mathcal{H}$ now is the original Hamiltonian including both zonal and tesseral harmonic effects. Since time is not present explicitly in K, it is a constant of the motion.

Following the von Zeipel procedure, "remove" all of the periodic parts of the extended Hamiltonian K via a suitable generating function, defined here up to second order (since $\mathcal{H}_T$ is second order in ϵ) so that the new variables become

$$L = L' + \epsilon \frac{\partial S^{(1)}}{\partial \ell} + \epsilon^2 \frac{\partial S^{(2)}}{\partial \ell} + \frac{\partial S_T}{\partial \ell}$$

$$\ell' = \ell + \epsilon \frac{\partial S^{(1)}}{\partial L'} + \epsilon^2 \frac{\partial S^{(2)}}{\partial L'} + \frac{\partial S_T}{\partial L'} \qquad \text{(V-3)}$$

$$H = H' + \frac{\partial S_T}{\partial h^*}$$

Equations similar to those above hold for the other variables. Note that H now contains fluctuations but that these are of second order. As before, $S^{(1)}$ and $S^{(2)}$ are chosen to cancel all zonal short-period terms up to, and including, second order. Thus one is left with (omitting the ϵ^3 function for the time being)

$$K' = -\frac{\mu^2}{2L'^2} + \epsilon\, \bar{\mathcal{H}}_1 + \epsilon^2 \mathcal{R}^{(2)} - \omega_\oplus H' + \frac{\mu^2}{L'^3}\frac{\partial S_T}{\partial \ell} - \omega_\oplus \frac{\partial S_T}{\partial h^*} - \mathcal{H}_T \qquad \text{(V-4)}$$

where

$$\mathcal{H}_T = \tilde{\mathcal{H}} + \bar{\mathcal{H}}_T = \sum_{k_2} \sum_{k_1} \sum_{m} A_{k_2 k_1 m}(a,e,i)\cos(k_1\ell + k_2 g + mh^* + \text{phase}) \qquad \text{(V-5)}$$

in which $\tilde{\mathcal{H}}_T$ includes all terms with $k_1 \neq 0$, the short-period part of $\mathcal{H}_T$, and $\bar{\mathcal{H}}_T$ gives the daily fluctuations from h^*, $k_1 = 0$. Choose S_T so that

$$-\frac{\mu^2}{L'^3}\frac{\partial S_T}{\partial \ell} + \omega_\oplus \frac{\partial S_T}{\partial h^*} = -\mathcal{H}_T \qquad \text{(V-6)}$$

Strictly speaking, (V-6) should be written as

$$-\frac{\mu^2}{L'^3}\frac{\partial S_T}{\partial \ell} + \omega_\oplus \frac{\partial S_T}{\partial h^*} + (\omega_\oplus - \dot{\Omega}'')\frac{\partial S_T^*}{\partial h^{*\prime}} =$$

$$-\tilde{\mathcal{H}}_T(L',G',H',\ell,g,h^*) - \bar{\mathcal{H}}_T(L'',G'',H'',g',h^{*\prime}) \qquad \text{(V-7)}$$

however, (V-6) is accurate to $O(\epsilon^2)$ since $\dot{\Omega}''$ is $O(\epsilon)$. Solving by inspection, with $n' \triangleq \frac{\mu^2}{L'^3}$

$$S_T = \sum_{k_2} \sum_{k_1} \sum_{m} \frac{m\, A_{k_1 k_2 m}}{k_1 n' - m\omega_\oplus} \sin(k_1 \ell' + k_2 g' + m h^{*\prime} + \text{phase}) \tag{V-8}$$

In Eq. (V-8) (as well as (V-6)) one can use the primed variables or instantaneous variables with $O(\epsilon^3)$ error. Clearly, there exist orbits for which n' is commensurable with $\omega_\oplus$ and hence a particular $(k_1 n' - m\omega_\oplus)$ goes to zero. These are the so-called tesseral resonance cases and will not be considered here. Introduction of S_T in the above manner leaves

$$K'' = -\frac{\mu^2}{2L'^2} + \epsilon\, \bar{\mathcal{H}}_1 + \epsilon^2 \mathcal{R}^{(2)} - \omega_\oplus H'' - \epsilon^3 \varphi^{\ddagger} \tag{V-9}$$

where the ϵ^3 function is now included, and is different from the ϵ^3 function of $\mathcal{H}''$ of the axisymmetric problem.

The instantaneous elements may be written as a sum of the "secular" (double-primed) part and the periodic parts. In particular, with H'' a constant,

$$H = H'' + H_{S.P} + H_{M.P} \tag{V-10}$$

From (V-3) and (V-6) it follows that

$$\omega_\oplus H'' = \omega_\oplus H + \mathcal{H}_T - n' \frac{\partial S_T}{\partial \ell} \tag{V-11}$$

thus, from (V-8)

$$n' \frac{\partial S_T}{\partial \ell} = \sum_{k_2} \sum_{k_1} \sum_{m} \frac{k_1 n'}{k_1 n' - m\omega_\oplus} A_{k_1 k_2 m} \cos(k_1 \ell' + k_2 g' + m h^{*\prime} + \text{phase}) \tag{V-12}$$

Combining (V-12) with (V-5) one obtains

$$\mathcal{H}_T - n' \frac{\partial S_T}{\partial \ell} = \sum_{k_2} \sum_{k_1} \sum_{m} \left(1 - \frac{k_1 n'}{k_1 n' - m\omega_\oplus}\right) A_{k_1 k_2 m} \cos(k_1 \ell' + k_2 g' + mh^{*'} + \text{phase})$$

$$= - \sum_{k_2} \sum_{k_1} \sum_{m} \frac{m\omega_\oplus}{k_1 n' - m\omega_\oplus} A_{k_1 k_2 m} \cos(k_1 \ell' + k_2 g' + mh^{*'} + \text{phase}) \tag{V-13}$$

the constancy of K and $\mathcal{H}''$ to 2nd order implies (because of (V-11)) that

$$-\mathcal{H}_T + n' \frac{\partial S_T}{\partial \ell} = \text{tesseral fluctuation of } \mathcal{H} = \int_A \frac{\partial \mathcal{H}}{\partial t} dt \tag{V-14}$$

where $\int_A \frac{\partial \mathcal{H}}{\partial t} dt$ denotes a specific second order approximation to the indefinite integral, considering only ℓ' and h^* as time varying. This may be checked by forming

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}_T}{\partial t} = \sum_{k_2} \sum_{k_1} \sum_{m} m\omega_\oplus A_{k_1 k_2 m} \sin(k_1 \ell' + k_2 g' + mh^{*'} + \text{phase}) \tag{V-15}$$

If one assumes that, to the order necessary here,

$$k_1 \ell' + mh^{*'} \approx [k_1 n' + m\omega_\oplus]t \tag{V-16}$$

then conclusion (V-14) follows.

For discussion of the error bounds, the formalism of Section IV can be retained to a large extent. Due to the absence of secular tesseral terms, only the φ function of the secular Hamiltonian will change and hence the bound on that term will be different. It follows then, that the error bounds on x'', with the exception of λ'', are derived in exactly the same manner as before with different values for the constants M_i. The derivation of an error bound on λ'' is not as simple as before, since for the asymmetrical problem the two separate integrals of energy and polar component of angular momentum no longer exist.

From the extended Hamiltonian (V-14) one finds

$$\dot{\lambda}'' = \frac{\mu^2}{L'^3}\left\{1 - \epsilon\frac{3}{2}k_1 L'^{-1} - \epsilon\frac{3\mu^2 R_\oplus}{4G''^4}\left[5\left(\frac{H''}{G''}\right)^2 - 1 - 2\frac{H''}{G''}\right] + \right.$$

$$\left. + \epsilon^2\delta_2\right\} - \epsilon^3\frac{\partial\varphi^\ddagger}{\partial L'} \tag{V-17}$$

The object again is to obtain an expression for $\frac{\mu^2}{L'^3}\left(1 - \epsilon\frac{3}{2}k_1 L'^{-1}\right)$ accurate up to, and including, second order. Since K is a constant of the motion

$$K = \mathcal{H} - \omega_\oplus H = K'' = \mathcal{H}'' - \omega_\oplus H'' \tag{V-18}$$

so that

$$\mathcal{H}'' = \mathcal{H} - \omega_\oplus(H - H'') \tag{V-19}$$

From general theory for time dependent Hamiltonians

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} \tag{V-20}$$

and

$$\mathcal{H} - \int\frac{\partial\mathcal{H}}{\partial t}\,dt = \text{constant} \tag{V-21}$$

The constant is related to quantity $\mathcal{H}''$, which is also a constant to second order as defined by the von Zeipel procedure. In fact, if one chooses a particular approximate evaluation of the indefinite integral, as in (V-14), then

$$\mathcal{H}'' = \mathcal{H} - \int_A \frac{\partial\mathcal{H}}{\partial t}\,dt \tag{V-22}$$

Note that as defined by (V-19), $\mathcal{H}''$ is known only to second order with third order secular terms (from H''). After times of order $1/\epsilon$ this constitutes an error in $\mathcal{H}''$ of second order secular and thus finally to first order in λ'' if L' is defined through $\mathcal{H}''$. Using the relation (V-22) avoids this difficulty, since it turns out that the third order evaluation of the specific integral $\int_A \partial\mathcal{H}/\partial t \; dt$ yields terms that are still third order after $t \sim 1/\epsilon$, with tesseral resonance situations still ruled out. This may be verified by considering the first order variations of the variables in $\partial\mathcal{H}/\partial t$ and noting that $m \neq 0$ in (V-15).

Next, define as before (with $\mathcal{H}''$ defined by (V-22))

$$\frac{\mu}{2\tilde{a}} = - \mathcal{H}'' \tag{V-23}$$

and the mean motion by

$$\tilde{n} = [\mu/\tilde{a}^3]^{\frac{1}{2}} \tag{V-24}$$

so that

$$\tilde{n} = \frac{\mu^2}{L'^3}\left[1 - \epsilon \frac{3}{2} k_1 L'^{-1} + \epsilon^2 \frac{3}{8} k_1^2 L'^{-2} - \epsilon^2 \frac{3}{2} \mathcal{R}^*\right] + \epsilon^3 \varphi_1^{\ddagger}(x'', \epsilon) \tag{V-25}$$

and finally, the $\dot{\lambda}''$ equation

$$\dot{\lambda}'' = \tilde{n} - \epsilon \frac{\mu^2}{L'^3}\left\{3 \frac{\mu^2 R_\oplus^2}{4G''^4}\left[5\left(\frac{H''}{G''}\right)^2 - 1 - 2\frac{H''}{G''}\right] - \epsilon\left[\delta_2 + \frac{3}{2}\mathcal{R}^* - \frac{3}{8} k_1^2 L'^{-2}\right]\right\} - \epsilon^3\left[\frac{\partial \varphi^{\ddagger}}{\partial L'} + \varphi_1^{\ddagger}\right] \tag{V-26}$$

The approximate solution λ_A'' is again defined by (V-26) with the $\varphi^{\ddagger}$ functions equal to zero. The argument for obtaining the error estimate follows as before.

The algorithm for computing the correct initial value of the mean motion $\tilde{n}$ now involves the evaluation of the integral $\int_A \partial\mathcal{H}/\partial t \; dt$. This may be done by a suitable expansion on eccentricity; one such evaluation is given in the Appendix.

VI. NUMERICAL VERIFICATION

Equation (V-26) then provides an algorithm for computing the correct initial conditions (to the order of accuracy demanded by the general solution) in the case when the total potential of the Earth is taken into account. The algorithm includes the (suitable) evaluation of the indefinite integral $\int_A \frac{\partial \mathcal{H}_T}{\partial t} dt$. It is of interest to obtain numerical verification of the general accuracy theory of Section V. The explicit expression for $\int_A \frac{\partial T_r}{\partial t} dt$ has been derived earlier by Vagners (Ref. 16), and was subsequently incorporated into the Lockheed Closed Form Orbit Determination Program (Ref. 17). This program utilizes a complete first order analytic solution that is equivalent to the extended Brouwer solution. (The extended Brouwer solution is taken to include J_2 short-period, J_2^2 and general J_N long-period, $J_{n,m}$ medium period (daily) effects and all second order secular effects not accounting for tesseral resonances c.f. Giacaglia (Ref. 18), and Garfinkel (Ref. 19).) The Lockheed solution is due to Small, (Ref. 12), and Vagners, (Ref. 16).

Since the error in the mean anomaly (or equivalently λ'') is directly related to in-track position error, the simplest test of overall accuracy is to compare the in-track, cross-track and radial positions as predicted by the analytic solution and numerical integration of coordinates. Since the time intervals of interest are of the order $1/\epsilon$, the comparison was performed over a seven day interval. The test orbit was of 2000 nautical mile altitude and circular. For such an orbit, not including the results of Section V (roughly a 200 foot error in the semi-major axis) resulted in a 200 mile in-track error†. After "tuning" the mean motion with the

† The comparison study was carried out to determine the effects of inclusion (or omission) of tesseral harmonic short period terms in the semi-major axis. The energy had already been incorporated in the formulation of the axisymmetric problem.

energy, the secular error was decreased to 900 feet, which is an order ϵ decrease as demonstrated by the theory of Section V. The comparison is shown in Fig. 1, where it can be seen that the periodic errors and the secular error are now of the same order of magnitude i.e., $O(\epsilon^2)$. The cross-track and radial errors are periodic and have amplitudes of $\pm$ 120 feet and $\pm$ 350 feet, respectively, for the comparison orbit.

VII. CONCLUDING REMARKS

In the previous sections error bounds were derived specifically for the Brouwer procedure using the Poincaré variables. From the general theory, an algorithm was derived for the correct computation of the initial conditions for the Brouwer theory. It is then of interest to note the relevance of the results of this paper to other orbit theories, and also to present the computation of coordinates from the Poincaré elements.

Insofar as the first item is concerned, exactly equivalent errors are to be expected from any complete first order theory provided that care is taken in establishing the correct mean elements for that theory. A complete first order theory is defined as one that includes the first order periodic and second order secular influences of any harmonic. This distinction is necessary if one wishes to compare theories for prediction of orbits from a fit to observational data or for prediction from an initial state vector, i.e. the initial value problem. For example, the theories of Kyner (Ref. 1), Petty and Breakwell (Ref. 21), including a time equation carried only to first order secular terms, would give satisfactory results if applied to orbit prediction from a fit to data. However, for the initial value problem, these theories would prove unsatisfactory (giving ϵ errors for time $t \sim 1/\epsilon$). The latter difficulty could be remedied if the time equations (or its equivalent) would be carried out to include second order secular effects and an energy algorithm used to calibrate the mean motion. The theory of Small (Ref. 12, Ref. 16) is a complete first order theory and includes the correct algorithm for computation of initial conditions.

With more or less difficulty, any theory appearing in the literature may be analyzed in a manner analogous to that given in this paper and equivalent results obtained. In each case, the energy will have to be used to establish the mean motion (or the constant rate of the fast variable) to second order, unless complete second order periodic expressions for the semi-major axis are available. The questions of error bounds become more difficult if one admits orbits at critical inclination and/or orbits at resonance with the tesseral harmonics. Such orbits are excluded from the general class investigated in this report and remain the topic of future investigations.

The last point to consider is the computation of the coordinates from the Poincaré elements in which most of the theory of this paper was developed. In terms of conventional orbit elements,

$$\lambda = M + \omega + \Omega \qquad L = \sqrt{\mu a}$$

$$\eta_1 = [2\sqrt{\mu a}\,(1 - \sqrt{1-e^2})\,]^{\frac{1}{2}}\cos(\omega + \Omega) \qquad \xi_1 = [2\sqrt{\mu a}\,(1 - \sqrt{1-e^2})\,]^{\frac{1}{2}}\sin(\omega + \Omega)$$

$$\eta_2 = [2\sqrt{\mu p}\,(1 - \cos i)]^{\frac{1}{2}}\cos\Omega \qquad \xi_2 = [2\sqrt{\mu p}\,(1 - \cos i)]^{\frac{1}{2}}\sin\Omega$$

where M is the mean anomaly and $p = a(1 - e^2)$. The remaining elements were defined in Section III, Eq. (III-1). Known the time t one can find $\lambda, \xi_1, \eta_1, \xi_2, \eta_2$ and L, then compute

$$e\cos(\omega + \Omega) = \frac{\eta_1}{L^{\frac{1}{2}}}\left[1 - \frac{\xi_1^2 + \eta_1^2}{4L}\right]^{\frac{1}{2}}$$

$$e\sin(\omega + \Omega) = \frac{\xi_1}{L^{\frac{1}{2}}}\left[1 - \frac{\xi_1^2 + \eta_1^2}{4L}\right]^{\frac{1}{2}} \qquad \text{(VII-2)}$$

An iterative procedure yields Δ, $e\cos f$, $e\sin f$ defined by

$$\Delta_n = 2\,\frac{\tan^{-1}\,(e\sin f)_n}{1+\sqrt{1-e^2}+(e\cos f)_n} + \frac{\sqrt{1-e^2}\,(e\sin f)_n}{1+(e\cos f)_n} \qquad \text{(VII-3)}$$

$$(e\cos f)_{n+1} = (e\cos(\omega+\Omega))\cos(\lambda+\Delta_n) + (e\sin(\omega+\Omega))\sin(\lambda+\Delta_n)$$

$$(e\sin f)_{n+1} = (e\cos(\omega+\Omega))\sin(\lambda+\Delta_n) - (e\sin(\omega+\Omega))\cos(\lambda+\Delta_n)$$

where $\sqrt{1-e^2} = 1 - \dfrac{\xi_1^2+\eta_1^2}{2L}$

So that the radius is given by

$$r = \frac{L^{5/2}(1-e^2)}{D} \qquad \text{(VII-4)}$$

where

$$D = \mu\left\{ L^{\frac{1}{2}} + \left[1 - \frac{\xi_1^2+\eta_1^2}{4L}\right]^{\frac{1}{2}} [\eta_1\cos(\lambda+\Delta) + \xi_1\sin(\lambda+\Delta)]\right\}$$

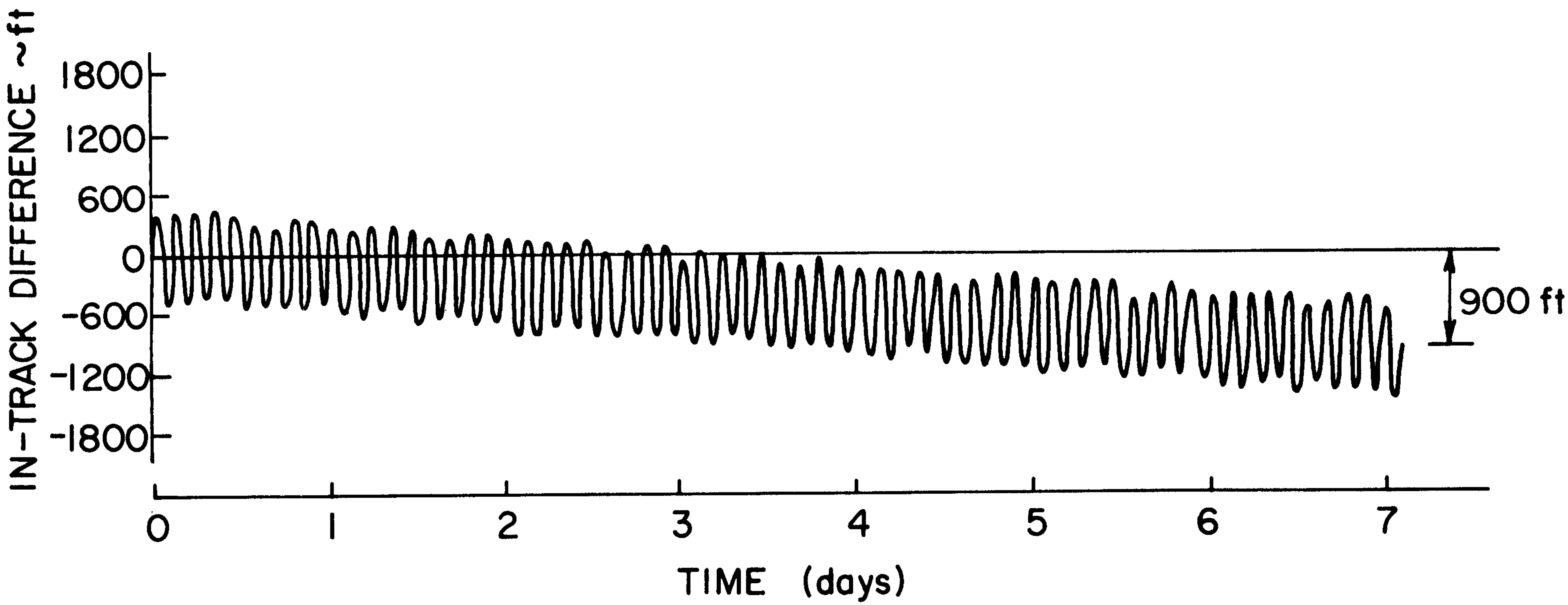
and the cartesian coordinates x,y,z by

$$x = \frac{L^{3/2}(1-e^2)^{\frac{1}{2}}}{2D}\,[(2L-\xi_1^2-\eta_1^2-\xi_2^2)\cos(\lambda+\Delta) + \eta_2\xi_2\sin(\lambda+\Delta)]$$

$$y = \frac{L^{3/2}(1-e^2)^{\frac{1}{2}}}{2D}\,[\xi_2\eta_2\cos(\lambda+\Delta) + (2L-\xi_1^2-\eta_1^2-\eta_2^2)\sin(\lambda+\Delta)]$$

$$z = \frac{L^{3/2}(1-e^2)^{\frac{1}{2}}}{2D}\,(4L-2\xi_1^2-2\eta_1^2-\xi_2^2-\eta_2^2)^{\frac{1}{2}}\,[\eta_2\sin(\lambda+\Delta) - \xi_2\cos(\lambda+\Delta)]$$

FIGURE 1



COMPARISON OF NUMERICAL INTEGRATION
AND ANALYTIC THEORY ORBIT PREDICTION

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APPENDIX

Explicit Evaluation of $\int_A \frac{\partial \mathcal{H}_T}{\partial t}\, dt$

For the evaluation of the initial value problem, the indefinite integral $\int_A \frac{\partial \mathcal{H}_T}{\partial t}\, dt$ must be evaluated or, equivalently, the generating function S_T must be found. It will be assumed here that the integrand is given by (V-15) and the integration will be carried out in conventional variables.

The following expressions prove useful:

$$P_n^m(\sin\beta) = \frac{\cos^m\beta}{2^n n!} \sum_{\zeta=0}^{\nu} \frac{(2n-2\zeta)!}{(n-m-2\zeta)!} \binom{n}{\zeta} (-1)^{\zeta} \sin^{n-m-2\zeta}_{i} \sin^{n-m-2\zeta}_{u} \quad \text{(A-1)}$$

where

$$\nu = \begin{cases} \frac{n-m}{2} & \text{for } n-m \text{ even} \\ \frac{n-m-1}{2} & \text{for } n-m \text{ odd} \end{cases}$$

$$\cos m(\lambda - \lambda_{n,m}) = \sum_{s=0}^{m} \binom{m}{s} \frac{\cos^{m-s} u \sin^{s} u \cos^{s} i}{\cos^{m}\beta} (-1)^{\xi} \times$$

$$[\gamma_1 \cos m(h^* - \lambda^*) + \gamma_2 \sin m(h^* - \lambda^*)] \quad \text{(A-2)}$$

where

$$\xi = \begin{cases} s/2 & \text{if } s \text{ is even;} \quad \gamma_1 = 1, \quad \gamma_2 = 0 \\ \frac{s+1}{2} & \text{if } s \text{ is odd;} \quad \gamma_1 = 0, \quad \gamma_2 = 1 \end{cases}$$

$\lambda^* = \alpha_o + \lambda_{n,m}$ with α_o the right ascension of Greenwich at a base time t_o.

$$\frac{1}{r^{n+1}} = \sum_{p=0}^{n+1} \binom{n+1}{p} \frac{e^p}{2^p(1 - e^2)^{p+1}} \frac{1}{a^{n+1}} \sum_{q=0}^{p} \binom{p}{q} \cos(p - 2q)(u - \omega) \quad \text{(A-3)}$$

and

$$\sin^j u \cos^k u = \sum_{c=0}^{j} \sum_{d=0}^{k} \binom{j}{c} \binom{k}{d} \frac{(-1)^{\ell}}{2^{j+k}} [\delta_1 \cos(j + k - 2c - 2d)u$$

$$+ \delta_2 \sin(j + k - 2c - 2d)u] \quad \text{(A-4)}$$

where

$$\ell = \begin{cases} j + c + j/2 & \text{if } j \text{ is even; } \delta_1 = 1, \ \delta_2 = 0 \\ j + c + \dfrac{j+1}{2} & \text{if } j \text{ is odd; } \delta_1 = 0, \ \delta_2 = 1 \end{cases}$$

At this point, the assumptions under which $\int_A \frac{\partial \mathcal{H}}{\partial t} dt$ will be integrated may be stated. The inclination angle i and the eccentricity e will be taken as constants. Since no appreciable difficulty is incurred thereby, the following will be adopted

$$\omega = \omega_o + \omega' u$$

$$h^* = \Omega_o + \Omega' u - \omega_{\oplus} t$$

with

$$\omega' = \frac{3}{2} \epsilon \left(\frac{R_{\oplus}}{p}\right)^2 (2 - 5/2 \sin^2 i) \quad \text{(A-5)}$$

$$\Omega' = -\frac{3}{2} \epsilon \left(\frac{R_{\oplus}}{p}\right)^2 \cos i$$

The last item is the central angle-time relationship. Since the integrand is (essentially) now a function of u, one would prefer to integrate with respect to u. To the first approximation

$$du = \tilde{n} \, dt + O(e\epsilon^2) \quad \text{(A-6)}$$

so that the contribution of the $J_{n,m}$ term is given by

$$E_{n,m} \int \sin^{n-m-2\zeta+s} u \cos^{m-s} u \, [\gamma_1 \sin m(h^* - \lambda^*) - \gamma_2 \cos m(h^* - \lambda^*)] \times$$

$$\cos(p - 2q)(u - \omega) \, du$$

with‡

$$E_{n,m} = \frac{m\omega_\oplus}{\tilde{n}} \mu J_{n,m} \frac{R_\oplus^n}{a^{n+1}} \frac{1}{2^n n!} \sum_{\zeta=0}^{\nu} \sum_{p=0}^{n+1} \sum_{q=0}^{p} \sum_{s=0}^{m} \frac{(2n-2\zeta)!}{(n-m-2\zeta)!} \binom{n}{s} \binom{n+1}{p} \times \tag{A-7}$$

$$\binom{p}{q} \binom{m}{s} (-1)^{\zeta+\xi} \frac{e^p}{2^p (1 - e^2)^{p+1}} \times \cos^s i \sin^{n-m-2\zeta} i$$

or with $\tilde{h} = h^* - \lambda^*$,

$$E^*_{n,m} \int \cos(p - 2q)(u - \omega)[\gamma_1 \sin m\tilde{h} - \gamma_2 \cos m\tilde{h}][\delta_1 \cos(j + k - 2c - 2d)u$$

$$+ \delta_2 \sin(j + k - 2c - 2d)u] du$$

with

$$E^*_{n,m} = E_{n,m} \sum_{c=0}^{j} \sum_{d=0}^{k} \binom{j}{c} \binom{k}{d} \frac{(-1)^\ell}{2^{j+k}}$$

then let

$$B_o = - \omega_o (p - 2q)$$

$$B_1 = (1 - \omega')(p - 2q)$$

$$B_2 = j + k - 2c - 2d$$

$$B_3 = m(\Omega_o - \lambda^*)$$

$$B_4 = - m\left(- \Omega' + \frac{\omega_\oplus}{\tilde{n}}\right)$$

so that the integrand becomes

‡ If one prefers, the F and G functions of Kaula, Ref. 22, may be used instead.

$$\int \cos(B_o + B_1 u)[\delta_1 \cos B_2 u + \delta_2 \sin B_2 u][\gamma_1 \sin(B_3 + B_4 u) - \gamma_2 \cos(B_3 + B_4 u)]du \qquad \text{(A-8)}$$

The following non-zero combinations arise in the above integral:

$$I_1 = \int \cos(B_o + B_1 u) \cos B_2 u \sin(B_3 + B_4 u)du$$

$$I_2 = - \int \cos(B_o + B_1 u) \cos B_2 u \cos(B_3 + B_4 u)du$$

$$I_3 = \int \cos(B_o + B_1 u) \sin B_2 u \sin(B_3 + B_4 u)du \qquad \text{(A-9)}$$

$$I_4 = - \int \cos(B_o + B_1 u) \sin B_2 u \cos(B_2 + B_4 u)du$$

which can all be evaluated explicitly to give

$$I_1 = \frac{1}{4}\left\{ \frac{1}{B_1-B_4-B_2} \cos[B_3-B_o+(B_2+B_4-B_1)u] + \frac{1}{B_1-B_4+B_2} \cos[B_o-B_3+(B_2+B_1-B_4)u] \right.$$

$$\left. + \frac{1}{B_2-B_4-B_1} \cos[B_3+B_0+(B_1+B_4-B_2)u] - \frac{1}{B_4+B_1+B_2} \cos[B_3+B_o+(B_2+B_4+B_1)u] \right\}$$

$$I_2 = -\frac{1}{4}\left\{ \frac{1}{B_2-B_4+B_1} \sin[B_o-B_3+(B_2-B_4+B_1)u] - \frac{1}{B_1-B_4-B_2} \sin[B_3-B_o+(B_2+B_4-B_1)u] \right.$$

$$\left. - \frac{1}{B_2-B_4-B_1} \sin[B_3+B_o+(B_1+B_4-B_2)u] - \frac{1}{B_4+B_1+B_2} \sin[B_3+B_o+(B_1+B_2+B_4)u] \right\}$$

$$I_3 = \frac{1}{4}\left\{ \frac{1}{B_1-B_4-B_2} \sin[B_3-B_o+(B_2+B_4-B_1)u] + \frac{1}{B_2-B_4+B_1} \sin[B_o-B_3+(B_2+B_1-B_4)u] \right.$$

$$\left. - \frac{1}{B_2-B_4-B_1} \sin[B_o+B_3+(B_1+B_4-B_2)u] - \frac{1}{B_4+B_1+B_2} \sin[B_3+B_o+(B_2+B_4+B_1)u] \right\}$$

$$I_4 = -\frac{1}{4}\left\{ \frac{1}{B_4-B_1-B_2} \cos[B_o-B_3+(B_2+B_1-B_4)u] + \frac{1}{B_4-B_2+B_1} \cos[B_o+B_3+(B_1+B_4-B_2)u] + \right.$$

$$+ \frac{1}{B_1-B_4-B_2} \cos[B_3-B_o+(B_2+B_4-B_1)u] - \frac{1}{B_4+B_1+B_2} \cos[B_3+B_o+(B_2+B_4+B_1)u] \Bigg\}$$

In the expressions of Ref. 16, it was assumed that i, ω, Ω and r were constants and in the test case the orbit was circular. The extensions of the above development cause no difficulty other than increasing the number of terms. However, any improvement of accuracy for non-zero eccentricity orbits is difficult to assess due to the approximation of Eq. (A-6). In order to define the error remaining as of order $e^2\epsilon^2$ one must include the e terms in (A-6).