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AN EXTENSION OF PLANE STRAIN ANALYSIS

by Kwan Rim and Roger O. Stafford

Prepared by
UNIVERSITY OF IOWA
Iowa City, Iowa

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ABSTRACT

AN EXTENSION OF PLANE STRAIN ANALYSIS

This report reviews the derivation of the governing equations for an isotropic and linear elastic material in a state of plane strain, for the purpose of extending the Kolosov-Muskhelishvili formulation to cover the most general case. Included into the reformulation are axial strain, body forces and thermoelastic effects. Boundary conditions and the single-valuedness of solutions are also discussed for non-dislocation type problems. The present formulation is particularly useful for various types of stress analyses of solid propellent rocket motors.

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AN EXTENSION OF PLANE STRAIN ANALYSIS

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SUMMARY

This report presents a generalization of the equations of linear plane strain elasticity of homogeneous isotropic materials. A set of general equations, corresponding to the Kolosov-Muskhelishvili formulation, is developed by including the effects of body forces, axial and thermal strain. The general formulation would be particularly useful for various types of stress analyses of solid propellant rocket motors.

INTRODUCTION

The main objective of this research project is to perform the elastic and viscoelastic analysis of two-dimensional problems with star-shaped boundaries by the method of complex variables. The derivation of required mapping functions through an application of the Schwarz-Christoffel transformation was presented in the previous report, NASA CR-192 dated March, 1965.

The stress analysis of solid propellent rocket motors with star-shaped perforation has been carried out by other investigators (ref. 1) using the Kolosov-Muskhelishvili formulae of plane elasticity. There now exists a need for more general engineering analyses which includes the axial strain, general body forces and thermoelastic effects. Inclusion of axial strain removes the unnecessary restriction imposed on the axial deformation by the classical formulation of plane strain.

The explicit purpose of this report is to reformulate the general equations of plane strain with these additional considerations, yet in such a way that the existing plane elasticity solutions may be used in analyzing more general problems. This approach offers certain advantages in terms of generality and less expenditure of labor. Also discussed in this report are the single-valuedness of solutions and the boundary conditions.

SYMBOLS

x, y, z	Cartesian coordinates.
$\zeta, \bar{\zeta}$	Complex coordinates, $\zeta = x + iy$.
$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$	Components of stress.
F_x, F_y, F_z	Body forces.
V	Body force potential function.
$\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$	Components of strain.
$\alpha = \varepsilon_z(x,y,z)$	A Superimposed axial strain.
u, v, w	Displacement components.
$D, \bar{D}$	Complex displacements, $D = u + iv$.
E, ν	Young's modulus of elasticity and Poisson's ratio.
$G = E/2(1 + \nu)$	Shear modulus of rigidity.
$T(x,y,z)$	Relative temperature.
μ	Coefficient of thermal expansion.
$\nabla^2 = 4\partial^2/\partial\zeta\partial\bar{\zeta}$	Laplace's operator.

GENERAL EQUATIONS OF ELASTICITY

For future reference, the fifteen equations of linear elasticity are given under the following groupings (ref. 2). The equations of equilibrium:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0, \tag{1a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0, \tag{1b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0. \tag{1c}$$

The linear strain-displacement relations:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \tag{2a}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{2d}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}, \tag{2b}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \tag{2e}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}, \tag{2c}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \tag{2f}$$

The compatibility conditions:

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right), \tag{3a}$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y}\right), \tag{3b}$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right), \tag{3c}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}, \tag{3d}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}, \tag{3e}$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}. \tag{3f}$$

The total strain, that is, the strain produced by deformation gradients plus that of thermal effects is given by

$$\varepsilon_i + \mu T, \quad i = x, y, z.$$

EQUATIONS OF PLANE STRAIN

The simplifying assumption of plane problems is that the out-of-plane shear stresses are zero. This assumption is also taken here, i.e.,

$$\gamma_{xz} \equiv \gamma_{yz} \equiv 0.$$

Hence, equations (2e) and (2f) require the displacement components to satisfy the following equations,

$$0 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad 0 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \tag{4a,b}$$

In the classical theory of plane strain, the normal or axial strain ε_z is also taken to be zero. Herein the axial strain will be considered to be a general function,

$$\varepsilon_z = \alpha(x,y,z).$$

By making use of σ_z expressed in terms of α, T and other stress components,

$$\sigma_z = \nu(\sigma_x + \sigma_y) + E(\alpha - \mu T),$$

one obtains the following stress-strain relations:

$$\varepsilon_x = \frac{1}{2G}\left[\sigma_x - \nu(\sigma_x + \sigma_y)\right] - \left[\nu\alpha - (1 + \nu)\mu T\right],$$

$$\varepsilon_y = \frac{1}{2G}\left[\sigma_y - \nu(\sigma_x + \sigma_y)\right] - \left[\nu\alpha - (1 + \nu)\mu T\right],$$

$$\tau_{xy} = G\gamma_{xy},$$

$$\sigma_x = \frac{2G}{1-2\nu}\left[(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \alpha)\right] - \frac{E\mu}{1-2\nu}T,$$

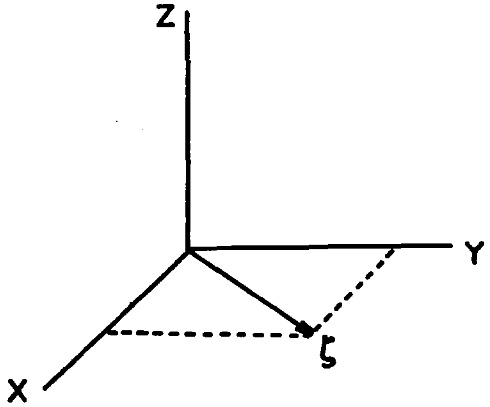
$$\sigma_y = \frac{2G}{1-2\nu}\left[(1 - \nu)\varepsilon_y + \nu(\varepsilon_x + \alpha)\right] - \frac{E\mu}{1-2\nu}T.$$

The symbol T denotes the difference in temperature from some reference level.

The body forces are assumed to be derivable from a scalar potential function V, i.e.,

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}.$$

For simplicity, equations (1) through (4) are converted into corresponding complex representations by using the well-known relations given below:



$$\zeta = x + iy, \qquad i = \sqrt{-1} ,$$

$$x = \frac{\zeta + \bar{\zeta}}{2} , \qquad y = \frac{\zeta - \bar{\zeta}}{2i} .$$

Figure 1. Coordinate system

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}} , \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}}\right) ,$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \zeta^2} + 2\,\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} + \frac{\partial^2}{\partial \bar{\zeta}^2} , \qquad \frac{\partial^2}{\partial y^2} = -\,\frac{\partial^2}{\partial \zeta^2} + 2\,\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} - \frac{\partial^2}{\partial \bar{\zeta}^2} ,$$

$$\frac{\partial^2}{\partial x \partial y} = i\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial \bar{\zeta}^2}\right) , \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} .$$

Complex displacements and body forces in the plane are defined as

$$D = u + iv, \qquad F = F_x + iF_y = -2\,\frac{\partial V}{\partial \bar{\zeta}} .$$

Equilibrium equations (1a) and (1b) combine into

$$\frac{\partial}{\partial \zeta}(\sigma_x - \sigma_y + 2i\tau_{xy}) + \frac{\partial}{\partial \bar{\zeta}}(\sigma_x + \sigma_y) = 2\,\frac{\partial V}{\partial \bar{\zeta}} . \qquad (5)$$

The ζ-plane strain-displacement relations become

$$\epsilon_x = \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}}\right)\left(\frac{D + \bar{D}}{2}\right), \qquad \epsilon_y = \left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}}\right)\left(\frac{D - \bar{D}}{2}\right), \qquad \gamma_{xy} = i\left(\frac{\partial \bar{D}}{\partial \zeta} - \frac{\partial D}{\partial \bar{\zeta}}\right) .$$

The six compatibility conditions reduce to three equations,

$$\frac{\partial^2}{\partial\zeta^2}(\epsilon_x - \epsilon_y + i\gamma_{xy}) + \frac{\partial^2}{\partial\bar{\zeta}^2}(\epsilon_x - \epsilon_y - i\gamma_{xy}) - 2\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}(\epsilon_x + \epsilon_y) = 0, \tag{6a}$$

$$4\frac{\partial^2\alpha}{\partial\zeta^2} + \frac{\partial^2}{\partial z^2}(\epsilon_x - \epsilon_y - i\gamma_{xy}) = 0, \tag{6b}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\zeta}(\epsilon_x + \epsilon_y) - \frac{\partial}{\partial\bar{\zeta}}(\epsilon_x - \epsilon_y + i\gamma_{xy})\right] = 0. \tag{6c}$$

Equation (6a) comes from (3d), equation (6b) is obtained by combining (3c), (3e) and (3f), and equation (6c) results from combining (3a) and (3b). Substitution of the stress-strain relations into the strain-displacement relations yields

$$\epsilon_x - \epsilon_y + i\gamma_{xy} = \frac{1}{2G}(\sigma_x - \sigma_y + 2i\tau_{xy}) = 2\frac{\partial D}{\partial\bar{\zeta}}, \tag{7a}$$

$$\epsilon_x + \epsilon_y = \frac{1 - 2\nu}{2G}(\sigma_x + \sigma_y) - 2\left[\nu\alpha - (1 + \nu)\mu T\right] = \frac{\partial D}{\partial\zeta} + \frac{\partial\bar{D}}{\partial\bar{\zeta}}. \tag{7b}$$

Since the boundary conditions of elasticity problems are described in terms of either stress or displacement components, equations (6) are written in terms of stress components by using equations (7). Also equation (6a) is written in terms of stress components by using equation (5) and its conjugate. These rearrangements result into

$$\frac{\partial^2(\sigma_x + \sigma_y)}{\partial\zeta\partial\bar{\zeta}} = \frac{1}{1 - \nu}\frac{\partial^2 V}{\partial\zeta\partial\bar{\zeta}} + \frac{2G}{1 - \nu}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\left[\nu\alpha - (1 + \nu)\mu T\right], \tag{8}$$

$$4\frac{\partial^2\alpha}{\partial\zeta^2} + \frac{1}{2G}\frac{\partial^2}{\partial z^2}(\sigma_x - \sigma_y - 2i\tau_{xy}) = 0, \tag{9}$$

$$\frac{\partial}{\partial z}\left[(1 - 2\nu)\frac{\partial}{\partial\zeta}(\sigma_x + \sigma_y) - \frac{\partial}{\partial\bar{\zeta}}(\sigma_x - \sigma_y + 2i\tau_{xy})\right]$$

$$= 4G\frac{\partial^2}{\partial z\partial\zeta}\left[\nu\alpha - (1 + \nu)\mu T\right], \tag{10}$$

$$E\frac{\partial\alpha}{\partial z} = \frac{\partial}{\partial z}\left[V + E\mu - \nu(\sigma_x + \sigma_y)\right], \tag{11}$$

$$\frac{\partial w}{\partial\bar{\zeta}} + \frac{1}{2}\frac{\partial D}{\partial z} = 0, \tag{12a}$$

$$\frac{\partial w}{\partial z} = \alpha. \tag{12b}$$

Equation (11) comes from (1c). Equation (12a) is the condition for the vanishing of out-of-plane shear stresses, and (12b) comes from (2c).

Disregarding conjugates, equations (5) and (7) through (12) constitute nine equations which the five unknowns must satisfy. The over-determination is caused by the restriction of the deformation to plane strain. This over-determination does not occur in simple plane strain as $\alpha = 0$ makes (9) through (12) null equations. Note that if the stress components are known, D and w are given by a particular integral of (7) and (12). Hence, the basic problem is to find stress components and strain α which satisfy (5) and (8) through (11), five equations for three unknowns.

The standard procedure by which the number of equations is reduced by one is to define a stress function which automatically satisfies the equilibrium condition. The scalar stress function ϕ is defined by

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} + V, \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V, \qquad \bar{\tau}_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}.$$

In complex notation, the stresses are

$$\sigma_x - \sigma_y + 2i\tau_{xy} = -4\frac{\partial^2\phi}{\partial\bar{\zeta}^2}, \qquad \sigma_x + \sigma_y = 4\frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}} + 2V. \tag{13}$$

Hence the four governing differential equations become

$$4\frac{\partial^4\phi}{\partial\zeta^2\partial\bar{\zeta}^2} = -\left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial^2 V}{\partial\zeta\partial\bar{\zeta}} + \frac{2G}{1-\nu}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\left[\nu\alpha - (1+\nu)\mu T\right], \tag{14}$$

$$2G\frac{\partial^2\alpha}{\partial\zeta^2} = \frac{\partial^4\phi}{\partial z^2\partial\zeta^2}, \tag{15}$$

$$\frac{\partial}{\partial z}\left[(1-2\nu)\frac{\partial^3\phi}{\partial\bar{\zeta}\partial\zeta^2} + \frac{\partial^3\phi}{\partial\zeta^3} + \frac{1-2\nu}{2}\frac{\partial V}{\partial\zeta}\right] = G\frac{\partial^2}{\partial z\partial\zeta}\left[\nu\alpha - (1+\nu)\mu T\right], \tag{16}$$

$$E\frac{\partial\alpha}{\partial z} = \frac{\partial}{\partial z}\left[(1 - 2\nu)V + E\mu T - 4\nu\frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}}\right]. \tag{17}$$

The problem is now reduced to finding functions ϕ and α which satisfy these equations. The boundary conditions are identical to those of the simple plane strain theory with the addition that $\epsilon_z = \alpha$ may be nonzero. The displacements D and W are determined by (7) and (12), respectively.

SOLUTIONS OF PARTICULAR CASES

To illustrate the generality of these equations, the formal solutions of four particular cases will be presented.

I. Simple plane strain.--Assume that all additional effects are zero, i.e.,

$$V \equiv T \equiv \alpha \equiv 0.$$

Equations (15), (16) and (17) are null and equation (14) becomes the governing equation of simple plane strain (ref. 3),

$$\nabla^2(\nabla^2\phi) = 0;$$

the solution of which may be written as

$$\phi = \zeta\bar{\Omega} + \bar{\zeta}\Omega + \omega + \bar{\omega},$$

where Ω and ω are analytic functions of the complex variable ζ.

II. Plane thermoelasticity.--Take the body force potential V and axial strain α to be zero, and assume the temperature T to be independent of z. Again equations (15), (16) and (17) are null and equation (14) becomes the governing equation of plane thermoelasticity (ref. 4),

$$4\frac{\partial^4\phi}{\partial\zeta^2\partial\bar{\zeta}^2} = -\frac{E\mu}{1 - \nu}\frac{\partial^2 T}{\partial\zeta\partial\bar{\zeta}};$$

the formal solution of which may be written as

$$\phi = \zeta\bar{\Omega} + \bar{\zeta}\Omega + \omega + \bar{\omega} + T_0,$$

where the new function T_0 is a particular integral of the following equation.

$$4 \frac{\partial^2 T_0}{\partial \zeta \partial \bar{\zeta}} = - \frac{E\mu T}{1 - \nu} .$$

III. Plane strain including ζ-plane body forces.--Take all additional effects to be zero, except the body forces and assume that V is independent of z. Then equations (15), (16) and (17) are satisfied and (14) becomes

$$4 \frac{\partial^4 \phi}{\partial \zeta^2 \partial \bar{\zeta}^2} = - \left(\frac{1 - 2\nu}{1 - \nu}\right) \frac{\partial^2 V}{\partial \zeta \partial \bar{\zeta}} .$$

This equation is identical in form to that of plane thermoelasticity, as expected, and the same methods of solution will apply (ref. 5).

IV. Axial strain superposed on plane strain with thermal effects and body forces.--This problem is of considerable significance in the analysis of solid propellant rockets, hence the assumptions are tailored to fit the real situation as closely as possible. The simplifying assumptions are:

1. The temperature T is independent of z.
2. The axial strain α is a function of z only. (This will be approximately true for long cylinders.)
3. The body force potential is of the form

$$V = f(\zeta, \bar{\zeta}) + g(z).$$

These restrictions make equations (15) and (16) null, and (14) and (17) become

$$4 \frac{\partial^4 \phi}{\partial \zeta^2 \partial \bar{\zeta}^2} = - \left(\frac{1 - 2\nu}{1 - \nu}\right) \frac{\partial^2 V}{\partial \zeta \partial \bar{\zeta}} - \frac{E\mu}{1 - \nu} \frac{\partial^2 T}{\partial \zeta \partial \bar{\zeta}} ,$$

$$\frac{\partial \alpha}{\partial z} = \left(\frac{1 - 2\nu}{E}\right) \frac{\partial V}{\partial z} .$$

Define V_0 and T_0 to be particular integrals of the equations

$$4 \frac{\partial^2 V_0}{\partial \zeta \partial \bar{\zeta}} = -\left(\frac{1 - 2\nu}{1 - \nu}\right) f , \qquad 4 \frac{\partial^2 T_0}{\partial \zeta \partial \bar{\zeta}} = - \frac{E\mu T}{1 - \nu} ,$$

then the biharmonic equation can be integrated directly and ϕ may be expressed in the following form:

$$\phi = \bar{\zeta}\Omega + \zeta\bar{\Omega} + \omega + \bar{\omega} + V_0 + T_0.$$

Also one obtains

$$\alpha = \left(\frac{1 - 2\nu}{E}\right)g(z) + \varepsilon_0.$$

Note that the axial strain α depends upon the body force in the axial direction only, hence only a constant strain ε_0 can be arbitrarily specified.

When the body forces are constant, such as gravitational type forces, then $\nabla^2 f = 0$. Also, $\nabla^2 T = 0$ when there are no heat sources in the domain. In this case it would follow that V_0 and T_0 would not be included in the stress function ϕ. However, since V_0 and T_0 are harmonic functions, their inclusion in the stress function does not violate (14) and is in fact required by (7a). Harmonic functions V_0 and T_0 vanish from ϕ only when the body forces and the relative temperature are zero.

From (7) and (13) one obtains the following expressions for the stresses:

$$\sigma_x + \sigma_y = \frac{2G}{1 - 2\nu}\left(\frac{\partial D}{\partial \zeta} + \frac{\partial \bar{D}}{\partial \bar{\zeta}} + 2\nu\alpha\right) - \frac{2E\mu}{1 - 2\nu} T = 4\frac{\partial^2 \phi}{\partial \zeta \partial \bar{\zeta}} + 2V,$$

$$\sigma_x - \sigma_y + 2i\tau_{xy} = 4G\frac{\partial D}{\partial \bar{\zeta}} = -4\frac{\partial^2 \phi}{\partial \bar{\zeta}^2}.$$

The displacement D is found by integrating the preceding equations:

$$GD = -\frac{\partial \phi}{\partial \bar{\zeta}} + m(\zeta), \qquad m = 4(1 - \nu)\Omega - \nu G\alpha\zeta,$$

where m is determined from the other stress equation. Hence the expression for displacement is given by

$$GD = (3 - 4\nu)\Omega - \zeta\bar{\Omega}' - \bar{\omega}' - \frac{\partial V_0}{\partial \bar{\zeta}} - \frac{\partial T_0}{\partial \bar{\zeta}} - \nu G\alpha\zeta.$$

The displacement w is found by integrating (12b) and evaluating the integration constant from (12a);

$$w = \int \alpha dz - \frac{\zeta\bar{\zeta}}{4}\left(\frac{1 - 2\nu}{E}\right)\left(\frac{1 - \nu + 2\nu^2}{1 - \nu}\right)g'(z) + w_0 .$$

From (12b) it is seen the $g'(z)$ must be a constant. Hence $g(z)$ can be a linear function at the most; i.e., the body force in the axial direction cannot be more than a constant.

The two most common types of body forces are gravitational and centrifugal forces, and examples are shown below.

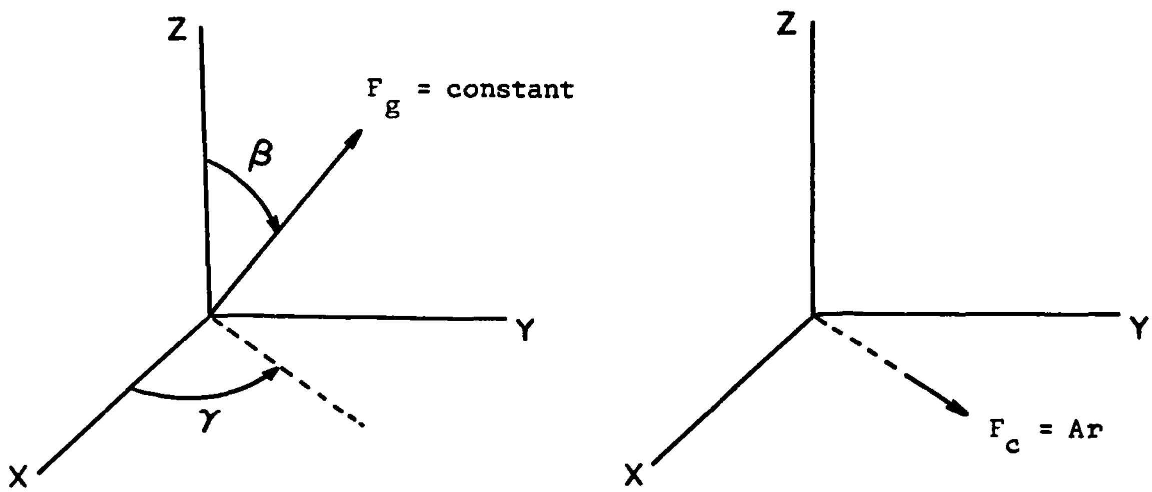


Figure 2. Gravitational and centrifugal body forces.

The scalar potential functions for these two cases are

$$V_g = \frac{-F_g \sin\beta}{2}(\zeta\bar{p} + \bar{\zeta}p) - (F_g \cos\beta)z , \qquad p = e^{i\gamma} ,$$

$$V_c = -A\zeta\bar{\zeta}/2 .$$

SINGLE-VALUEDNESS AND BOUNDARY CONDITIONS

The problems of interest here are non-dislocation type, hence the stress and displacement components must be single-valued at interior points. The condition for single-valuedness may be written as

$$F(\theta) - F(\theta + 2\pi) = [F(\zeta)]_c = 0,$$

where $[\,]_c$ denotes the change in the function during a complete circuit of a contour c lying within the body. Therefore, the displacement D and its first derivative, and the second and higher derivatives of ϕ must satisfy this condition. Since $\partial\phi/\partial\bar{\zeta}$ is related to D, it is assumed to be single-valued. Also V_0 and T_0 and their derivatives up to second order are assumed to be single-valued. Explicitly, the equations to be examined for single-valuedness are

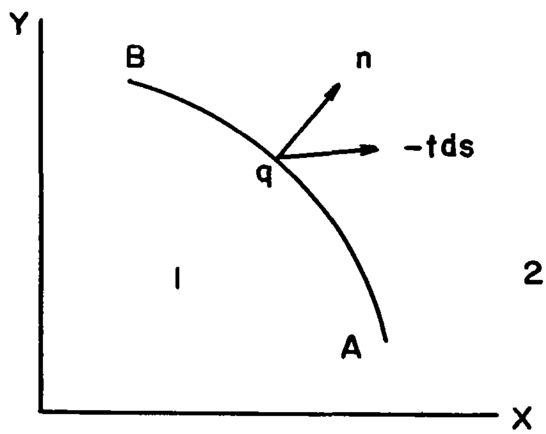
$$D = (3 - 4\nu)\Omega - \zeta\bar{\Omega}' - \bar{\omega}', \qquad \frac{\partial D}{\partial \zeta} = (3 - 4\nu)\Omega - \bar{\Omega}'$$

$$\frac{\partial^2\phi}{\partial\zeta^2} = \bar{\zeta}\Omega'' + \omega'', \qquad \frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}} = \Omega' + \bar{\Omega}', \qquad \frac{\partial^3\phi}{\partial\zeta^2\partial\bar{\zeta}} = \Omega'',$$

where terms assumed to be single-valued are dropped. Since Ω'' must be single-valued, the remaining equations yield

$$[\omega'']_c = 0, \qquad [\Omega']_c = 0, \qquad [(3 - 4\nu)\Omega - \bar{\omega}']_c = 0.$$

The boundary conditions for plane problems are extensively covered in the literature (refs. 3, 4, 5). The sign convention adopted here is shown below.



$$P = \int t\,ds,$$

$$M = \int (r \times t)ds.$$

Figure 3. Boundary conditions.

The total forces P and M represent the force exerted by region 1 on region 2. In complex notation, the general equations are

$$P = \left[2i\frac{\partial\phi}{\partial\bar{\zeta}}\right]_A^B + i\int_A^B V d\zeta ,$$

$$M = \left[\zeta\frac{\partial\phi}{\partial\zeta} + \bar{\zeta}\,\frac{\partial\phi}{\partial\bar{\zeta}}\right]_A^B - [\phi]_A^B + \frac{1}{2}\int_A^B V(\zeta d\bar{\zeta} + \bar{\zeta}d\zeta) .$$

Substitution of the stress function of example IV into the preceding equations yields

$$P = 2i\left[(\Omega + \zeta\bar{\Omega}' + \bar{\omega}' + \frac{\partial V_0}{\partial\bar{\zeta}} + \frac{\partial T_0}{\partial\bar{\zeta}})\right]_A^B + i\int_A^B V d\zeta ,$$

$$M = \left[\zeta\bar{\zeta}(\Omega' + \bar{\Omega}') + \zeta\omega' + \overline{\zeta\omega'} - \omega - \bar{\omega}\right]_A^B + \frac{1}{2}\int_A^B V\,(\zeta d\bar{\zeta} + \bar{\zeta}d\zeta)$$

$$+\left[-V_0 + \zeta\frac{\partial V_0}{\partial\zeta} + \bar{\zeta}\frac{\partial V_0}{\partial\bar{\zeta}}\right]_A^B + \left[-T_0 + \zeta\frac{\partial T_0}{\partial\zeta} + \bar{\zeta}\frac{\partial T_0}{\partial\bar{\zeta}}\right]_A^B ,$$

The evaluation of P and M over a closed contour yields the remarkably simple results,

$$P = 8i(1 - \nu)[\Omega]_c , \qquad M = [\zeta\omega' + \overline{\zeta\omega'} - \omega - \bar{\omega}]_c ,$$

where single-valuedness conditions have been utilized.

CONCLUDING REMARKS

The generality of the plane strain analysis has been extended by removing one of the basic assumptions, i.e., the vanishing of the axial strain; and by including into the re-formulation body forces and thermoelastic effects. These additional considerations are some of the important factors which should be included in the refined analysis of solid propellent rocket motors.

The format of the general equations derived in this report has been carefully designed to agree with those available in the literature. Therefore, the existing solutions of plane strain problems and even their computer programs may be readily modified to include the additional effects discussed in this report.

The general equations derived herein have been checked with the first order approximation equations of the finite elasticity which were derived independently as a special case of the large deformation analysis. They agree with each other, as it is expected. The details of the large deformation analysis will be presented in a future report.

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