LMSC/HREC A783146

LOCKHEED MISSILES & SPACE COMPANY
HUNTSVILLE RESEARCH & ENGINEERING CENTER
HUNTSVILLE RESEARCH PARK
4800 BRADFORD DRIVE, HUNTSVILLE, ALABAMA

RESEARCH IN STABILITY OF PERIODIC MOTIONS

QUARTERLY PROGRESS

5 May to 4 August 1966

Contract NAS8-20323

Prepared for George C. Marshall Space Flight Center
by

D. L. Hitzl and John White

APPROVED BY:
K. L. Remmler, Supvr.,
Vehicle Dynamics

G. E. Christopher, Mgr.,
Dynamics & Guidance

J. S. Farrior, Resident Manager
Huntsville R & E Center

FOREWORD

The principal objective of Contract NAS8-20323, Research in Stability of Periodic Motions, is to derive exact analytical results concerning the degree of instability of certain periodic orbits in the restricted and reduced three body problems. However, before this problem can be formulated, considerable background material must be assimilated. This report summarizes the background material studied during this period of contract performance.

This investigation is being performed by personnel of Lockheed Missiles & Space Company, Huntsville Research & Engineering Center for the Aero-Astrodynamics Laboratory of the George C. Marshall Space Flight Center.

SUMMARY

During this reporting period, Chapters 20, 21, 22, 26 and 27 of Dr. Carl Ludwig Siegel's Vorlesungen über Himmelsmechanik were read. In Chapter 22, area-preserving transformations are discussed in view of application to proving Birkoff's fixed point theorem. The normal form is established and its physical interpretation discussed. In particular, the idea of invariant curves is introduced in their simplest form. The importance of these invariant curves in studying the stability of periodic motions is pointed out.

The Liapunov and Dirichlet theorems on the stability of equilibrium solutions of a set of differential equations are discussed.

CONTENTS

INTRODUCTION

Two methods of establishing the existence of periodic solutions to the problem of three bodies has met with distinguished success. The first method, Poincaré's continuity method, was described briefly in the first quarterly progress report. The second, the fixed point method (also credited to Poincaré) has been a major topic for study during the succeeding quarter, reported herein. The establishing of the fixed point method as a useful tool is largely due to Birkhoff. The basic concept behind this method is that the consecutive intersections of the solution with, in general, a hyperplane in phase space constitutes a mapping of that hyperplane into itself. Periodic solutions, after one or more revolutions if necessary, return to intersect the hyperplane in the same point. These periodic solutions are represented by fixed points in the mapping, and also, obviously, the converse is true. The task then reduces to defining a mapping and then finding or proving the existence of fixed points. In both Poincaré's and Birkhoff's work a known periodic solution, which is not an equilibrium solution, is required; and the mapping is of one neighborhood onto another neighborhood of this fixed point. The key to the existence of periodic solutions is that the mapping is area-preserving, that is, the areas of the neighborhood of the fixed point and its image neighborhood are equal.

During the later part of the reporting period the stability of equilibrium solutions was studied, in particular the theorems of Liapunov and Dirichlet. Liapunov gives a sufficient condition for instability and a necessary condition for stability for the equilibrium solution of a set of first order linear differential equations. Dirichlet, on the other hand, gives a sufficient condition for stability provided that a time-dependent integral of the equations exists.

TECHNICAL DISCUSSION

The solutions

$$x_k = x_k(t, \xi_1, \ldots, \xi_m) \quad (k = 1, \ldots, m) \tag{1}$$

of the differential equations

$$\dot{x}_k = f_k(x_1, \ldots, x_m) \quad (k = 1, \ldots, m), \tag{2}$$

which satisfy the conditions of Cauchy's existence theorem, define, at a time t, a mapping of the initial conditions, $\xi_k = x_k(t_o, \xi_1, \ldots, \xi_m)$ $(k = 1, \ldots, m)$, onto $x_k = x_k(t, \xi_1, \ldots, \xi_m)$. A suitably chosen domain U of m-volume > 0, but finite, is mapped onto a domain U'. The m-volume is preserved if the functional determinant

$$\Delta = \left| x_{k\xi_\ell} \right| = \pm 1 . \tag{3}$$

Since at time t = 0 the matrix $(x_{k\xi_\ell}) = I$, the unit matrix the condition (3) becomes

$$\dot{\Delta} = 0 .$$

By some careful manipulations this reduces to

$$\sum_{k=1}^{m} f_{kx_k} = 0 . \tag{4}$$

This condition is obviously satisfied for a Hamiltonian system

$$\left.\begin{aligned} \dot{x}_k &= E_{y_k} \\ \dot{y}_k &= -E_{x_k} \end{aligned}\right\} (k = 1, \ldots n) \tag{5}$$

since the condition becomes

$$\sum_{k=1}^{n} \left(E_{y_k x_k} - E_{x_k y_k}\right) = 0 \ .$$

Now with the eventual application to motion near a non-equilibrium periodic solution, we consider a solution with initial conditions $\xi_1, \ldots, \xi_{m-1}, \xi_m^*$, ξ_m^* = const., at time t_o, which after a time $t = t'$ again intersects the hyperplane ξ_m^* = const., at $\xi_1', \ldots, \xi_{m-1}', \xi_m^*$. From continuity considerations, a neighborhood, B, of $\xi_1, \ldots, \xi_{m-1}, \xi_m^*$ contained in the hyperplane ξ_m^* = const. is mapped onto a neighborhood, B', of $\xi_1', \ldots, \xi_{m-1}', \xi_m^*$. Now consider the set of all points of the solution connecting B to B' as the domain U of initial conditions at time $t = t_o$. After a time t', U maps onto U', i.e., the stream tube U is moved on, following the trajectories, so that volume is preserved. This situation is depicted, in three dimensions, in Figure 1.

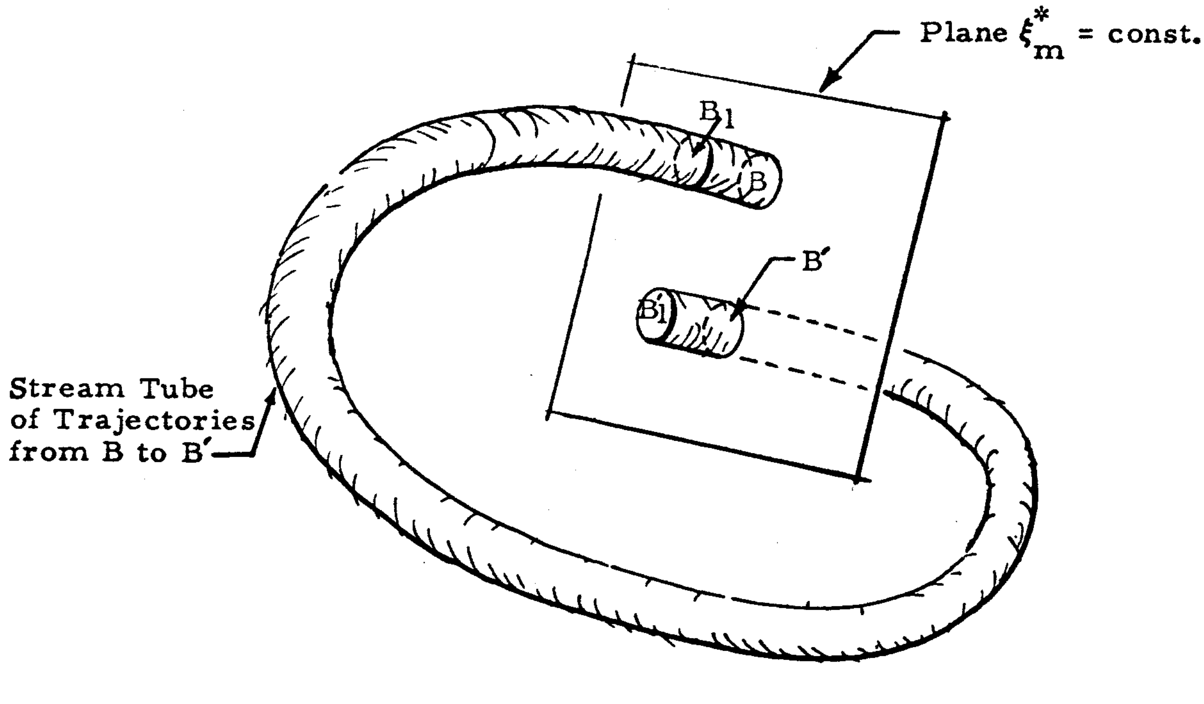

Figure 1 - Diagram of Mapping Defined by the Solutions of a Set of Differential Equations

Let the points of the surface $B \rightarrow B_1$ and $B' \rightarrow B_1'$ and R and R′ be the m-volumes of the stream tubes between B and B_1, and B′ and B_1' respectively. Then since the mapping is m-volume preserving

$$U = U' = U - R + R'$$

$$\therefore R = R' \ .$$

In general the (m-1) volume of B $\neq$ (m-1) volume of B′, but we can establish that

$$\int_B f_m(\xi)\, d\xi = \int_{B1} f_m(\xi)\, d\xi \qquad (d\xi = d\xi_1, \ldots, d\xi_{m-1}) \ . \qquad (6)$$

The time t has been eliminated by letting $t' \rightarrow 0$. In order to reduce the dimension of the set of initial conditions even further it is assumed that a time independent integral $\psi(x)$ of the system (2) exists and that $\psi_{x_{m-1}} \neq 0$. Under this assumption the invariant integral (6) becomes

$$\int_F \frac{f_m(\xi)}{\psi_{x_{m-1}}(\xi)}\, dv = \int_{F_1} \frac{f_m(\xi)}{\psi_{x_{m-1}}(\xi)}\, dv \qquad (dv = d\xi_1 \ldots d\xi_{m-2};\ \xi_m = \xi_m^*) \quad (7)$$

where F and F_1 are the appropriate subsets of B and B_1 respectively.

For the Hamiltonian system (4) E itself satisfies the requirements for ψ and so

$$\frac{f_m(\xi)}{\psi_{x_{m-1}}(\xi)} = \frac{-E_{x_n}}{E_{x_n}} = -1$$

on a suitable ordering of the variables, and so

$$\int_F dv = \int_{F_1} dv \qquad (8)$$

i.e., the mapping preserves (m-2) volumes.

In the case of the restricted problem of three bodies, this means that the mapping near a periodic solution preserves area.

To take full advantage of area-preserving mappings, we must choose our coordinate system so that the mapping takes on some recognizable form. Siegel accomplishes this task by a coordinate transformation written as an infinite power series which may or may not converge. The area-preserving mapping itself is also written as an infinite power series, which, to be of use, must converge. We let our area-preserving mapping be

$$\begin{aligned} x_1 &= ax + by + p(x, y) \\ y_1 &= cx + dy + q(x, y). \end{aligned} \tag{9}$$

where $p(x, y)$ and $q(x, y)$ start with terms of at least degree two, and the origin (0, 0) is the fixed point of the mapping. Three cases are recognized, namely the hyperbolic, parabolic and elliptic according to whether $(a-d)^2 - 4bc$ is >, =, < 0 respectively. The matrix of coefficients $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the linear terms is reduced to diagonal form $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ by a linear transformation. The normal form in the hyperbolic case, is

$$\xi_1 = \pm e^{w}\xi, \quad \eta_1 = \pm e^{-w}\eta, \quad \lambda = \pm e^{\gamma_o}$$

where

$$w = \sum_{k=0}^{\infty} \gamma_k (\xi\eta)^k$$

and in the elliptic case

$$r_1 = r \cos w - s \sin w, \quad s_1 = r \sin w + s \cos w$$

where

$$w = \sum_{k=0}^{\infty} \gamma_k (r^2 + s^2)^k$$

and

$$\xi = r + is, \quad \eta = r - is, \quad \xi_1 = r_1 + is_1, \quad \eta_1 = r_1 - is_1$$

and, in each case, formally

$$\begin{aligned} x &= \phi(\xi, \eta), & y &= \psi(\xi, \eta) \\ x_1 &= \phi(\xi_1, \eta_1), & y_1 &= \psi(\xi_1, \eta_1) \end{aligned}$$

where ϕ, ψ are formal power series in the stated variables.

We now consider the interpretation of these forms. In the hyperbolic case

$$\xi_1 \eta_1 = \xi \eta$$

i.e., if ξ, η are interpreted as rectangular coordinates of the point $P \neq (0, 0)$, then P is mapped onto P_1 which lies on the rectangular hyperbola passing through P. Further we may find some disc $S = x^2 + y^2 \leq \rho^2$ $(\rho > 0)$ in the domain of convergence of (9) such that all the image points of $P \epsilon s$ for successive applications of the transformation (9) will not lie within the disc for any point $P \neq (0, 0)$. Now in the elliptic case, again r, s are interpreted as rectangular coordinates, the equations

$$r_1 = r \cos w - s \sin w, \quad s_1 = r \sin w + s \cos w$$

means that the radius vector to the point $P = (r, s)$ is of invariant magnitude and is rotated through an angle θ dependent upon $\rho = (r^2 + s^2)^{1/2}$. We see, then, that every circle of radius ρ, ρ sufficiently small, is invariant if ϕ and ψ are convergent. Now from continuity of the solutions (1) we conclude that the invariant circles and the points on the trajectories, compare to Figure 1, $B = B'$, define a closed surface and so motion is restricted to within or without this surface. A motion initially within this surface is therefore bounded, and, from our elementary concepts of stability, a motion contained within a finite invariant curve is stable. Since in the elliptic case, convergent ϕ, ψ implies all circles of sufficiently small radius are invariant we are not surprised to learn that, in the elliptic case divergence of ϕ, ψ, is, in fact, general. However, this divergence of ϕ, ψ does not preclude invariant curves, as shown by Moser, nor stability. Applying our intuitive concept of stability to the hyperbolic case, we concluded that the hyperbolic case will give instability since we can always find a circle such that the image of P lies outside this circle.

In the final paragraphs, a convergent ϕ', ψ' are generated, which reduces the area-preserving transformation to a form which is identical to the normal form to an arbitrary power in ξ, η plus a remainder term which, of course, is convergent if the original area-preserving transformation is convergent.

The form considered is

$$\xi_1 = p(\xi,\eta) = u\xi + P, \quad \eta_1 = q(\xi,\eta) = v\eta + Q, \quad uv = 1$$

$$u = e^{i(\alpha + r^{2\ell})}, \quad r^2 = \xi\eta, \quad \overline{p}(\xi,\eta) = q(\eta,\xi)$$

where P, Q are infinite series in ξ, η commencing with terms of degrees $2\ell + 2$. This is the starting point for the proof of the Birkhoff fixed point theorem which states:

There exist fixed points $Z \neq 0$ of S^n such that $S^k Z \epsilon G$ $(k = 0, \ldots, n-1)$ exist, in any neighborhood G, which may be as small as we please, of the origin in the (x, y)-plane, and for all sufficiently large integers $n > n_o$ (G).

This theorem is proven by showing that for n sufficiently large, each point of a continuous curve surrounding a fixed point, Z = 0, is mapped along the radius vector from the origin. The mapping is area-preserving, so that the image curve cannot lie entirely inside nor entirely outside the selected curve; therefore, it has at least two fixed points. (See Figure 2.)

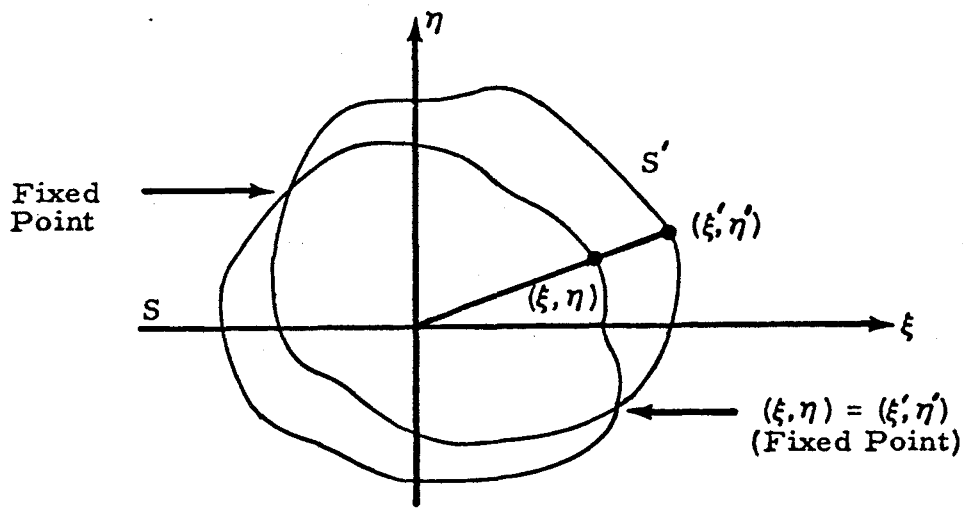

Figure 2 - Diagram of Radial Mapping under an Area-Preserving Mapping

Stability of the equilibrium solutions of a system (1) is the topic of the Liapunov and Dirichlet theorems. Periodic solutions may be investigated by these theorems provided these solutions are in fact, equilibrium solutions in some suitably chosen coordinate system, for example, the libration points of the restricted problem of three bodies in rotating coordinates.

If the eigenvalues of the matrix A of the linear terms of $f_1(x), \ldots, f_m(x)$ are denoted by $\lambda_1, \ldots, \lambda_m$, then the Liapunov Theorem tells us that the equilibrium solution is unstable if the real parts of $\lambda_1, \ldots, \lambda_m$ are all nonzero and that if the equilibrium solution is stable, then the real parts of $\lambda_1, \ldots, \lambda_m$ are all zero.

Dirichlet gives a sufficient condition for the stability of the equilibrium solution of the system (1) provided it possesses a time independent integral $g(x)$. This theorem states that:

If the system (1) possesses a time independent integral $g(x)$ which has a relative extremum in the strict sense at $x = 0$, then the equilibrium solution $x = 0$ is stable.

FUTURE WORK

Planned for the future is a continued study of the normalization of the Hamiltonian and existence of invariant curves under area-preserving mappings with particular emphasis on applications to the stability of periodic orbits. This will include portions of Siegel as well as appropriate publications by Dr. Moser and Dr. Arenstorf.