TECHNICAL NOTE R-91

COMPARISON OF SPECTRA OF SINGLE SIDEBAND AND PHASE MODULATED SIGNALS

Prepared By

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February, 1964

BROWN ENGINEERING COMPANY INC.
HUNTSVILLE, ALABAMA

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February, 1964

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INSTRUMENTATION BRANCH
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ABSTRACT

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Mathematical expressions for a phase modulated signal and a single sideband modulated signal are derived and compared. The effects of nonlinearities in the transponder transmitters are noted, and equations are derived to show the cross-modulation products that are generated by small nonlinearities in the airborne tracking receiver. The conclusion is reached that the nonlinearities in the transponder transmitter and in the airborne receiver cause a single-sideband signal to have sidebands that occupy a major portion of the frequency band occupied by the sidebands of a phase modulated signal.

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INTRODUCTION

The airborne tracking transceiver of the AROD system receives signals from each of four transponders located on the earth. The transponders operate at different carrier frequencies; however, the spectrums of signals from different transponders may overlap if up-link signal channels are not properly spaced. Two factors increase the frequency band that must be allocated to each channel, sidebands due to the carrier modulation and doppler shift due to motion of the vehicle.

The type of carrier modulation determines the location and total number of sidebands. An ideal single sideband signal has energy only at the carrier frequency and, if the upper (lower) sideband is used, at frequencies equal to the carrier frequency plus (minus) frequencies of the modulating tones. On the other hand, a phase modulated carrier has sidebands at the carrier frequency plus and minus all possible combinations of multiples of the modulating tones. One can compare the differences in the sidebands of a phase modulated signal and a single-sideband modulated signal by examining equation A-5 of Appendix A (single sideband) and equation B-1 of Appendix B (phase modulated). These two modulation techniques would seem to represent the extremes in bandwidth requirement for frequency allocation. The purpose of the investigation was to establish practical criteria for comparing the spectra of single-sideband modulated transponders with the spectra of phase modulated transponders.

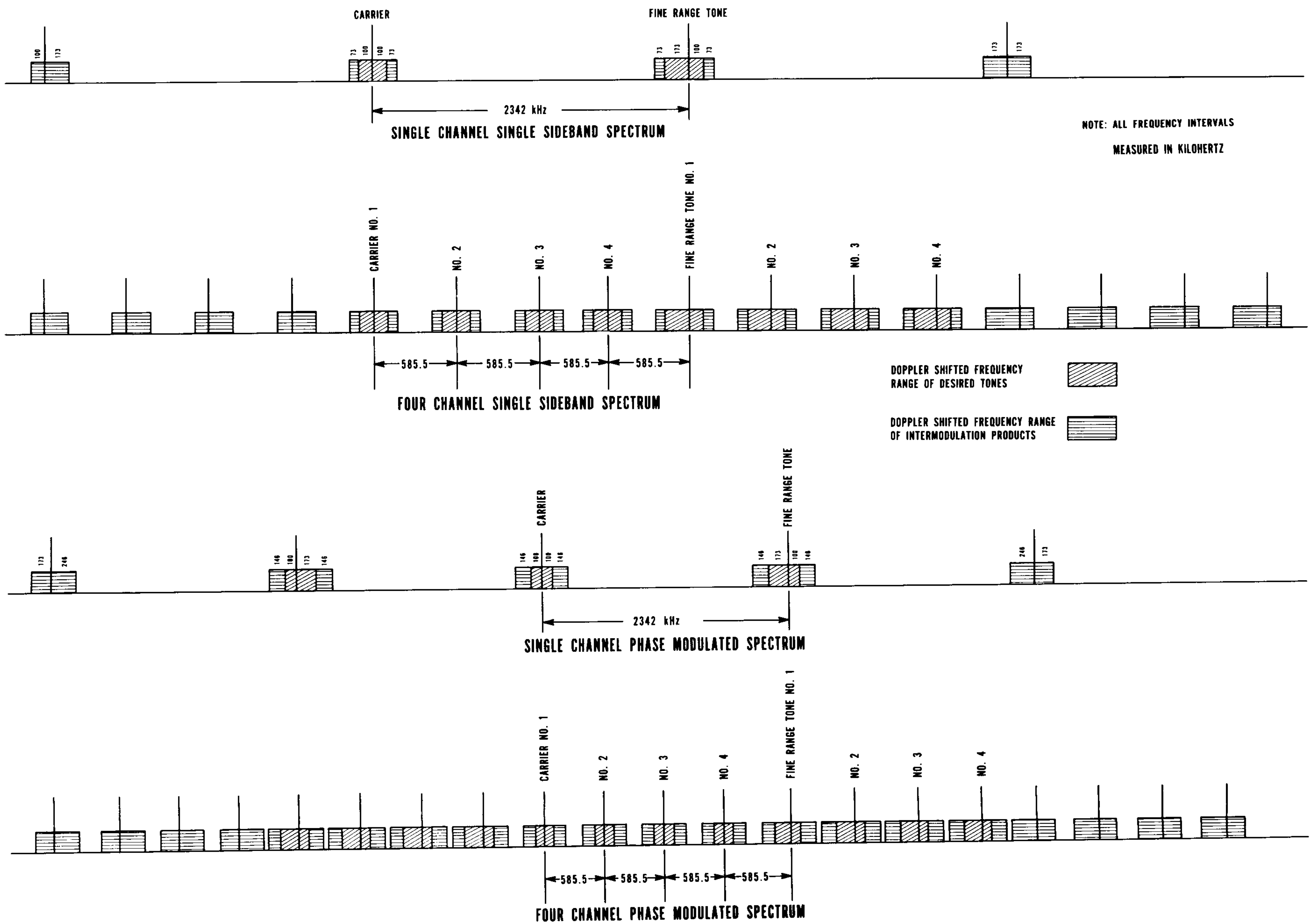
SIGNAL TRANSMITTED BY A TRANSPONDER

Equation A-1 (Appendix A) is not a suitable representation for the signal that would actually be transmitted by a single-sideband transmitter. The effects of incomplete cancellation of the undesired sidebands and of nonlinearities in the transmitter (both likely to be significant for unattended stations) are not included. Equation A-2 is a better representation for a signal that might be obtained in practice; a number of sidebands appear that are not present in the representation of an ideal single-sideband signal. Table A-1 of Appendix A lists the sidebands that would be present in the signal from an operational system.

Because the space vehicle is moving relative to the transponder sites, the spectrum of the transmitted signal is translated in frequency by the doppler shift. The expected range of doppler shift exceeds 70 kHz; therefore, the frequency intervals between sidebands separated by 70 kHz cannot be allotted to another channel. By assigning the region between sidebands separated by 70 kHz to the spectrum of the transmitted signal, one can sketch an interesting diagram of the frequency bands occupied by the spectrum. Figure 1 shows the frequency regions in which the spectrum of a single sideband signal would be located. The cross hatched regions show where the carrier and principal sidebands must be if the doppler shift does not exceed 100 kHz (velocity of approximately 6800 m/sec). The regions with horizontal lines contain unused sidebands. For comparison, Figure 1 also presents the same information for a phase modulated signal (including only the sidebands that have power which is greater than the carrier power less 50 db). The doppler shift of 100 kHz was chosen for convenience to facilitate drawing Figure 1.

FIGURE 1

DOPPLER SHIFTED SPECTRA FOR AROD TRANSPONDER TRANSMISSION



RECEIVER NONLINEARITIES

The above discussion was concerned with the spectrum of the signal transmitted by a single transponder. Actually signals from four transponders are received simultaneously at the vehicle; and before the combined signals are filtered into separate channels (one for each transponder), they are amplified at RF, converted to IF and amplified at IF. Nonlinearities in the amplifiers and mixer generate new sidebands that will, in general, lie close to the carrier and principal sidebands of desired signals.

Analyses that assume small nonlinearities in the receiver front end appear in Appendix C (single-sideband modulation) and in Appendix D (phase modulation). As can be ascertained from equations C-6 and D-19, even a small nonlinearity generates many new sidebands, whether the transponder signals be amplitude modulated (single sideband) or phase modulated. The intermodulation products of the phase modulated signals and the amplitude modulated signals do not differ significantly. Only sidebands at the third and fourth harmonic of the fine range tone cause the phase modulated signal spectrum to be qualitatively different from the spectrum of the single sideband modulated signal.

ASSIGNMENT OF CARRIER FREQUENCIES

If one ignores nonlinearities in the front end of the vehicle tracking receiver, a number of suitable carrier assignments can be made. Figure 1 shows a possible assignment of carrier frequencies that should be satisfactory for either type of modulation being considered here. Doppler shifts up to 220 kHz would not produce interchannel interference for transponders using single-sideband modulation (Figure 1) and doppler shifts up to 185 kHz would be acceptable for phase modulated carriers (Figure 1). Interference is possible if the unused sidebands of one transponder (regions shaded with horizontal lines) overlap a region (cross hatched) that could be occupied by the doppler-shifted useful components of another channel. No fifth order intermodulation products were considered in drawing the single-sideband spectrum in Figure 1. If they had been included, the two upper diagrams on the figure would have been even more similar to the two lower diagrams. Again, only the presence in the phase modulated signal of sidebands near large multiples of the fine range tone frequency significantly distinguish the two types of modulation.

Until a two tone intermodulation test can be performed using the tunnel diode amplifier, the tunnel diode mixer and the first IF amplifier in tandem, no definite conclusion can be drawn about the amplitude of receiver generated intermodulation components. With limited information about the amplitude of intermodulation components, one has difficulty establishing the degree to which receiver nonlinearities should influence the assigning of carrier frequencies to the four transponder channels.

A close examination of equations C-6 and Table D-1 shows certain intermodulation terms which are unaffected by the spacing between carriers (i. e. if $\omega_{c_{\nu_2}} = \omega_{c_{\nu_3}}$ in equation C-6). These terms correspond to the interaction of one carrier, e. g., carrier #1, with another carrier, e. g., carrier #2, and with the sidebands of carrier #2. This interaction produces about carrier #1 sidebands that are identical in frequency and phase to the sidebands about the carrier #2. No manipulation of the transponders, except reduction in transmitted power, changes the amplitude of the intermodulation sidebands. The ratio of power in the intermodulation sidebands to power in carrier #1 is not a function of the amplitude of carrier #1. At the output of the first IF amplifier the carrier for each transponder signal will have superimposed upon it the sidebands of the carriers for the other three transponder signals.

The carriers and sidebands of three distinct transponder signals interact to produce sidebands that could possibly lie in the passband of the fourth transponder signal. Location in frequency of these intermodulation components is a function of the spacing of the carriers of the three signals. In normal operation these intermodulation products probably will affect the operation of the system less than those that are mentioned in the previous paragraph and that are unaffected by the spacing of the four carriers. Only the weakest of the four channels will be more affected by the intermodulation products due to intermixing of all three of the other channels. If the weakest signal is significantly weaker than the other three, it probably is not being used for measuring range and velocity. The tentative conclusion is that the intermodulation products arising from the interaction of three different transponder signals should not influence the assignment of carrier frequencies.

Finally, certain of the intermodulation components are due to an interaction between two signals (i. e. $\omega_{c_{\nu_1}} = \omega_{c_{\nu_2}}$ in equation C-6 and Table D-1). If the signals from two transponders are much stronger than the signals from the other two, the intermodulation products from the mixing of these two signals could be amplitudes that are sizable fractions of the amplitudes of the other two transponder signals. Because large differences in signal level could occur during launch of the vehicle; either the carrier spacing must be such that two signal intermodulation products do not lie in the passband of the other two channels, or provisions must be made to reduce the power radiated by transponders when the vehicle is near to the site.

At present no general technique exists for selecting an optimum carrier spacing to assure that two signal intermodulation products do not lie in the passband of other channels. Channel assignment is made even more difficult if transponder transmitter intermodulation products are included in the consideration. Hopefully, a two-tone intermodulation test of the receiver will show the two-signal intermodulation products are no problem if, during launch, the transmitted power is reduced at transponders located close to the launch site.

Whether or not receiver nonlinearities are a problem, the type of modulation used for the transponder transmitters is not likely to influence the assignment of carrier frequencies to any significant degree.

APPENDIX A

SPECTRUM OF TRANSPONDER SIGNAL - SINGLE SIDEBAND

If the signal transmitted by a transponder were an ideal single sideband signal, the frequency spectrum could be easily obtained. Ideally the signal S(t) has the form

$$S(t) = a_0 \cos \omega_C t + a_1 \cos (\omega_C - \omega_1) t + a_2 \cos (\omega_C - \omega_2) t + a_3 \cos (\omega_C - \omega_3) t + a_4 \cos (\omega_C - \omega_4) t \qquad \text{(A-1)}$$

where

$\omega_C = 2\pi f_C$, f_C is the carrier frequency

$\omega_1 = 2\pi f_1$, $f_1 = 2.2688125 \times 10^6$ Hz

$\omega_2 = 2\pi f_2$, $f_2 = 2.340820765 \times 10^6$ Hz

$\omega_3 = 2\pi f_3$, $f_3 = 2.340892235 \times 10^6$ Hz

$\omega_4 = 2\pi f_4$, $f_4 = 2.342 \times 10^6$ Hz (fine range tone)

The sideband amplitudes are related to the carrier amplitude by the equations

$$\frac{a_1}{a_0} = \frac{a_2}{a_0} = \frac{a_3}{a_0} = \frac{1}{\sqrt{2}}$$

$$\frac{a_4}{a_0} = 1$$

The spectrum could be passed by a channel with pass bands centered at $f = f_C$, $f = f_C - f_1$, $f = f_C - f_2$, $f = f_C - f_3$, and $f = f_C - f_4$, each with bandwidths of 320 kHz. The 320 kHz bandwidths would be sufficient to pass doppler shifts of ± 160 kHz (corresponding to about 11,000 m/sec).

Unfortunately, equation (A-1) does not adequately represent the signal from an actual transmitter. Nonlinearities in the transmitter generate intermodulation components, and incomplete cancellation of the upper sidebands results in spurious signals at the frequencies $f = f_c + f_1$, $f = f_c + f_2$, $f = f_c + f_3$, and $f = f_c + f_4$. The output $v_o(t)$ of the transmitter can more correctly be written as

$$v_o(t) = \alpha_1 \, S(t) + \alpha_1 \, S_u(t) + \alpha_3 \, S^3(t) \tag{A-2}$$

where

$S(t) \equiv S(t)$ in equation (A-1)

$$S_u(t) = b_1 \cos(\omega_c + \omega_1)t + b_2 \cos(\omega_c + \omega_2)t + b_3 \cos(\omega_c + \omega_3)t + b_4 \cos(\omega_c + \omega_4)t$$

α_1 - linear voltage gain

α_3 - voltage gain for cubic terms

The $S_u^{\,3}(t)$ term which could be included is assumed to be negligible.

An expression for $S^3(t)$ is readily derived by writing

$$S^3(t) = \sum_{k_1=0}^{4} \sum_{k_2=0}^{4} \sum_{k_3=0}^{4} a_{k_1} \cdot a_{k_2} \cdot a_{k_3} \cdot \cos(\omega_c - \omega_{k_1})t \cdot \cos(\omega_c - \omega_{k_2})t \cdot \cos(\omega_c - \omega_{k_3})t \tag{A-3}$$

with $\omega_0 = 0$. Using a trigometric identity for the expansion of the product of three cosine terms, we obtain

$$S^3(t) = \frac{1}{4} \sum_{k_1} \sum_{k_2} \sum_{k_3} a_{k_1} \cdot a_{k_2} \cdot a_{k_3} \{\cos(\omega_c + \omega_{k_1} + \omega_{k_2} - \omega_{k_3})t + \cos(\omega_c + \omega_{k_2} + \omega_{k_3} - \omega_{k_1})t + \cos(\omega_c + \omega_{k_3} + \omega_{k_1} - \omega_{k_2})t + \cos(3\omega_c t + \omega_{k_1} + \omega_{k_2} + \omega_{k_3})t\} \tag{A-4}$$

The term containing $3\omega_c$ is of no interest, and the others can be combined to give

$$S^3(t) = \frac{3}{4} \sum_{k_1=0}^{4} \sum_{k_2=0}^{4} \sum_{k_3=0}^{4} a_{k_1} \cdot a_{k_2} \cdot a_{k_3} \cos \left(\omega_c + \omega_{k_1} + \omega_{k_2} - \omega_{k_3}\right) t \qquad \text{(A-5)}$$

Figure A-1 shows the relative magnitudes of certain of the terms in S^3 (t). A list of the frequencies at which the spectrum of S^3 (t) is nonzero is given in Table A-1 together with the relative amplitude of the spectrum at each frequency.

The coefficients a_0, a_1, a_3, b_1, . . . , b_4 can be determined only by making appropriate feed-through and intermodulation tests on the transmitter. However, because the transponder sites will be unattended, a reasonable assumption is that spurious responses due to both uncancelled sidebands and intermodulation terms will have considerably greater power than -50 db below the carrier power. With this assumption, all of the frequencies listed in Table A-1 should be included in the spectrum that is used for investigating problems of interchannel interference.

The spectrum of the transponder signal will translate up or down in frequency by as much as 160 kHz due to the velocity of the target. The carrier and the sidebands must, therefore, be considered to occupy 320 kHz frequency intervals that are centered at the frequencies listed in Table A-1. Superimposing all of the frequency intervals in which the carrier or some sideband (desired and undesired) could be located, one obtains a collection of intervals in which the carrier or principal sidebands of another channel should not appear. Table A-2 is a list of the

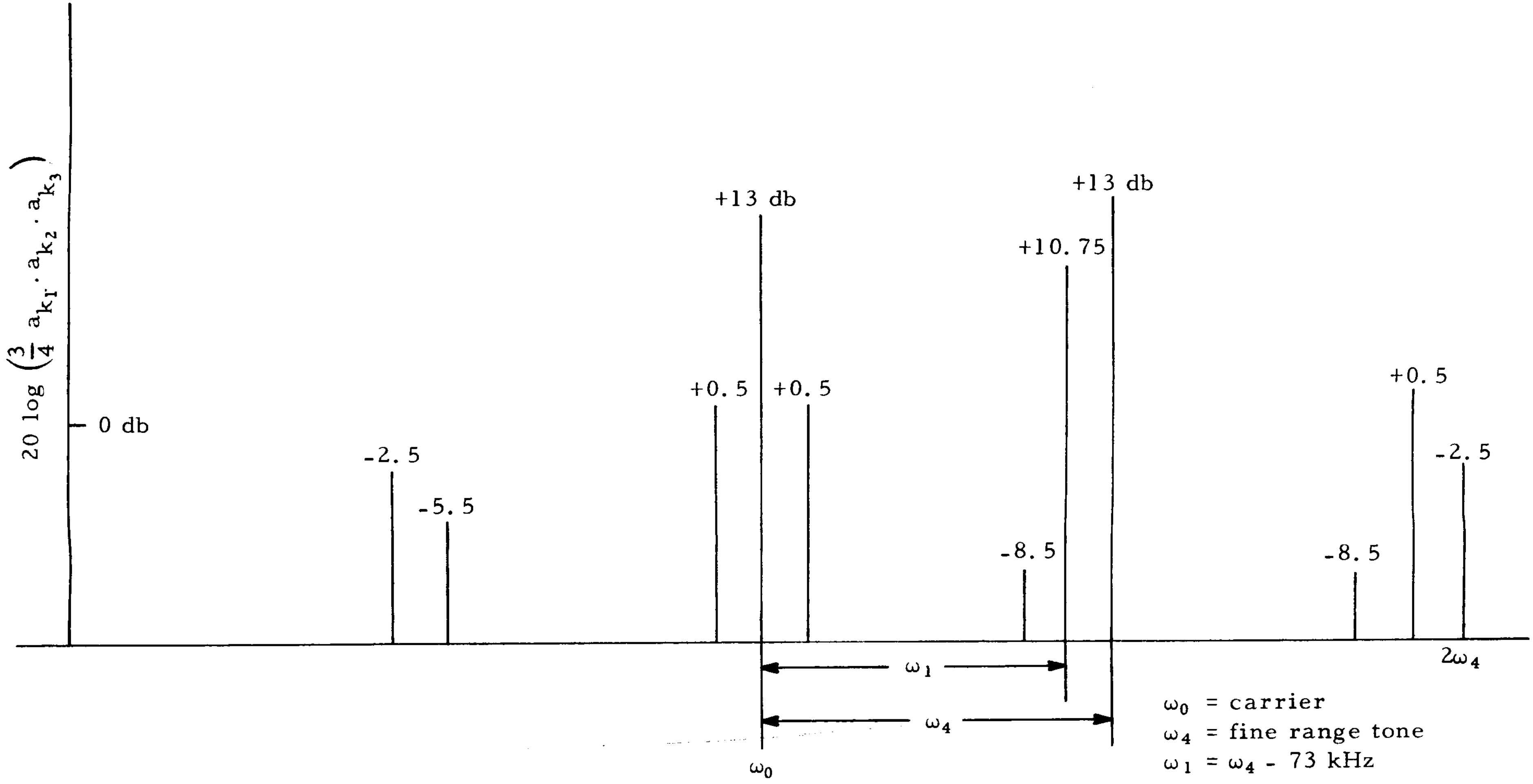


FIGURE A-1

RELATIVE AMPLITUDE OF TERMS IN $S^3(t)$

TABLE A-1

RELATIVE AMPLITUDES OF TERMS IN S^3 (t)

Frequency	$\left(\frac{\text{Power}_{IM}}{\text{Power}_{reference}}\right)$ in db
$f_p = (f_3 - f_2) = 70$ Hz	
$f_c \pm f_p$	-9
$f_c + f_1 \pm f_p$	-12
$f_c + f_2 - f_p$	-12
$f_c + f_3 + f_p$	-12
$f_c + f_4 \pm f_p$	-9

TABLE A-2

FREQUENCY INTERVALS FOR A GIVEN SINGLE SIDEBAND CHANNEL IN WHICH INTERFERENCE MAY BE EXPECTED

Frequencies listed should be superimposed on an appropriate carrier coinciding with zero frequency of the spectrum below.

-2.502 MHz	to	-2.109 MHz
-0.233 MHz	to	+0.233 MHz
+2.109 MHz	to	+2.502 MHz
+4.378 MHz	to	+4.844 MHz

intervals. (The frequency values given in Table A-2 are specified relative to the carrier frequency.) Figure A-2 shows the intervals that include the carrier and the principal sidebands.

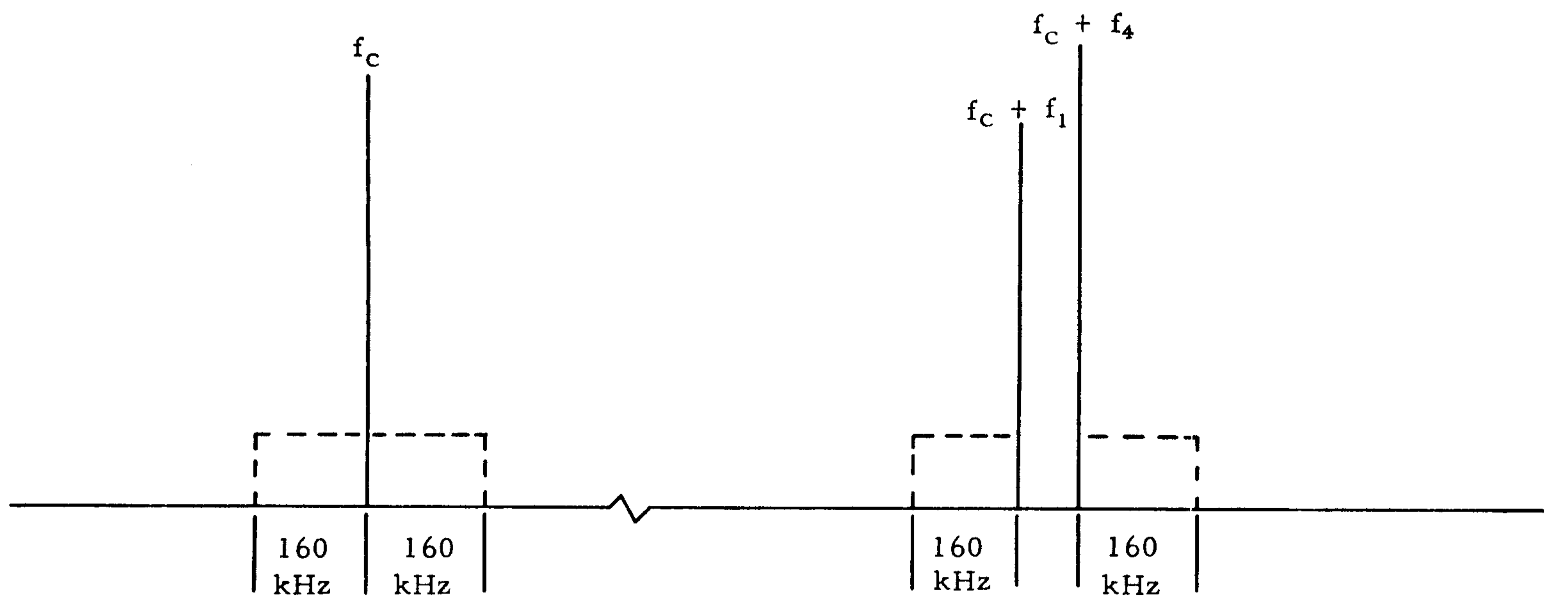


FIGURE A-2

DOPPLER FREQUENCY BANDS ASSOCIATED WITH CARRIER AND SIDEBANDS OF ONE SINGLE SIDEBAND AROD CHANNEL

APPENDIX B

SPECTRUM OF TRANSPONDER SIGNAL - PHASE MODULATED

The brassboard model of the AROD system will have single sideband transmitters at each transponder site. Because distortion is often present in signals generated by single sideband transmitters, study of alternate modulation techniques is advisable. Nonlinearities in the amplifiers of a transmitter do not distort an ideal phase modulated signal; therefore, phase modulation of the up-link (transponder to vehicle link) carrier might be preferable to single sideband modulation. The spectrum of a phase modulated signal has characteristics which may, however, limit the applicability of this type signal.

A phase modulated signal will have a spectrum that is several times wider than the modulating waveform, unless the modulation index is much less than unity. For instance, a carrier (frequency f_c) modulated by a single tone (frequency f_m) has a line spectrum with the lines located at the frequencies $f = f_c \pm n f_m$ ($n = 1, 2, 3, \ldots.$). If several phase modulated signals are to occupy transmission channels with closely spaced center frequencies, interchannel interference can occur from the higher order (than the first) sidebands of a signal in one channel appearing in the passband of another. The AROD vehicle tracking receiver has four parallel channels to simultaneously receive signals from four transponders.

The following equation (Reference 1) is especially convenient for exhibiting the properties of the spectrum of a signal S(t) that is a carrier phase modulated with four superimposed modulating tones:

$$S(t) = E \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} \Pi_{n_1, n_2, n_3, n_4} \cos\left(2\pi f_c t + \phi_{n_1, n_2, n_3, n_4}\right) \qquad \text{(B-1)}$$

where

$$\phi_{n_1, n_2, n_3, n_4} = 2\pi\,(n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4)t + (n_1 + n_2 + n_3 + n_4)\frac{\pi}{2}$$

$$\Pi_{n_1, n_2, n_3, n_4} = J_{n_1}(\delta_1) \cdot J_{n_2}(\delta_2) \cdot J_{n_3}(\delta_3) \cdot J_{n_4}(\delta_4)$$

f_c = carrier frequency

f_1 = 2.2688125 (10^6) Hz

f_2 = 2.340820765 (10^6) Hz

f_3 = 2.340892235 (10^6) Hz

f_4 = 2.342 (10^6) Hz (fine range tone)

$J_n(x)$ = Bessel function (first kind) of order n

$\delta_1, \delta_2, \delta_3, \delta_4$ = modulation indices for the four tones

Each time dependent term in the series is a sinusoid; therefore, the spectrum is zero everywhere except at the frequences $f = \pm (f_c + n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4)$. The power at the frequency $f = f_c + n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4$ (in a resistor of one ohm) is $\frac{1}{2} E^2 \Pi^2_{n_1, n_2, n_3, n_4}$. To determine the frequency band which this signal occupies, one must specify the power level below which the contribution from a sideband may be considered negligible and must compute the frequency intervals over which the spectrum is translated due to doppler effects.

In order to facilitate making a direct comparison with the single sideband signal presently being considered for the up link, the relative power ratios $\left(\frac{\text{Power in a Sideband}}{\text{Power in Carrier}} = \frac{\Pi^2_{n_1, n_2, n_3, n_4}}{\Pi_{0,0,0,0}}\right)$ have been computed for a number of sidebands in the phase modulated signal. Modulation indices were chosen so that the principle sideband to carrier power ratios were as follows:

$$\frac{\Pi^2_{1,0,0,0}}{\Pi^2_{0,0,0,0}} = \frac{1}{4}$$

$$\frac{\Pi^2_{0,1,0,0}}{\Pi^2_{0,0,0,0}} = \frac{1}{4}$$

$$\frac{\Pi^2_{0,0,1,0}}{\Pi^2_{0,0,0,0}} = \frac{1}{4}$$

$$\frac{\Pi^2_{0,0,0,1}}{\Pi^2_{0,0,0,0}} = \frac{1}{2}$$

Table B-1 is a list of values of $\frac{\Pi^2_{n_1, n_2, n_3, n_4}}{\Pi^2_{0,0,0,0}}$ for the sidebands that are particularly important for determining the frequency intervals in which the signal has significant power (relative to the carrier power).

In Table B-1, the sideband frequencies are written in the form $f = f_c + m\, f_4 + m_3\,(f_4 - f_3) + m_2\,(f_4 - f_2) + m_1\,(f_4 - f_1)$, which is more convenient than the form $f = f_c + n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4$, and better illustrates the relationships of various sidebands to the harmonics of the

TABLE B-1

POWER SPECTRUM REFERRED TO CARRIER

Frequency	$\left(\dfrac{\text{Power}_{IM}}{\text{Power}_{carrier}}\right)$ in db
$f_p = (f_3 - f_2) = 70$ Hz	
f_c	
$f_c \pm f_p$	-11
$f_c \pm 2f_p$	-37
$f_c \pm 3f_p$	-71
$f_c \pm 4f_p$	-108
$f_c + f_1 \pm f_p$	-17
$f_c + f_1 \pm 2f_p$	-43
$f_c + f_1 \pm 3f_p$	-76
$f_c + f_1 \pm 4f_p$	-114
$f_c + f_2 - f_p$	-24
$f_c + f_2 - 2f_p$	-53
$f_c + f_2 - 3f_p$	-91
$f_c + f_3 + f_p$	-24
$f_c + f_3 + 2f_p$	-53
$f_c + f_3 + 3f_p$	-91
$f_c + f_4 \pm f_p$	-14
$f_c + f_4 \pm 2f_p$	-40
$f_c + f_4 \pm 3f_p$	-75

fine range tone frequency (f_4). The relative power at frequency

$f = f_c + m_4 f_4 + m_3 (f_4 - f_3) + m_2 (f_4 - f_2) + m_1 (f_4 - f_1)$ is

$$\frac{\Pi^2_{-m_1, -m_2, -m_3, (m_1+m_2+m_3+m_4)}}{\Pi^2_{0,0,0,0}} .$$

As an example, the relative power at several frequencies $f = f_c + m f_4$ is shown in Figure B-1. One observes that at least the third order sidebands (at $f_c \pm 3 \omega_4$) and perhaps the fourth order ones should be included in any considerations of interchannel interference.

In the intervals between harmonics of the fine range tone some contribution to the spectrum is present at all multiples of $f_3 - f_2 = 70$ Hz; however, only the sidebands located close to harmonics of the fine range tone have appreciable power. Figure B-2 shows the relative power at frequencies $f = f_c \pm m_1(f_4 - f_1)$, $f = f_c + f_1 \pm m_1(f_4 - f_1)$ and $f = f_c - f_1 \pm m_1 (f_4 - f_1)$. Table B-1, which was mentioned previously, contains values of the power in sidebands located close to multiples of the fine range tone. Because $f_4 - f_3 << f_4 - f_1$ and $f_4 - f_2 << f_4 - f_1$, the gross characteristics of the spectrum are determined by the terms at frequencies $f = f_c + m_4 f_4 + m_1 (f_4 - f_1)$. The effect of including the multiples of $f_4 - f_3$ and $f_4 - f_2$ is to show the power in sidebands located within several kilohertz of the frequencies $f = f_3 + m_4 f_4 + m_1 (f_4 - f_1)$.

For maximum vehicle velocities of 11,000 m/sec, the doppler shift in the spectrum will be from -160 kHz to 160 kHz. The carrier and each sideband must, therefore, be considered to occupy

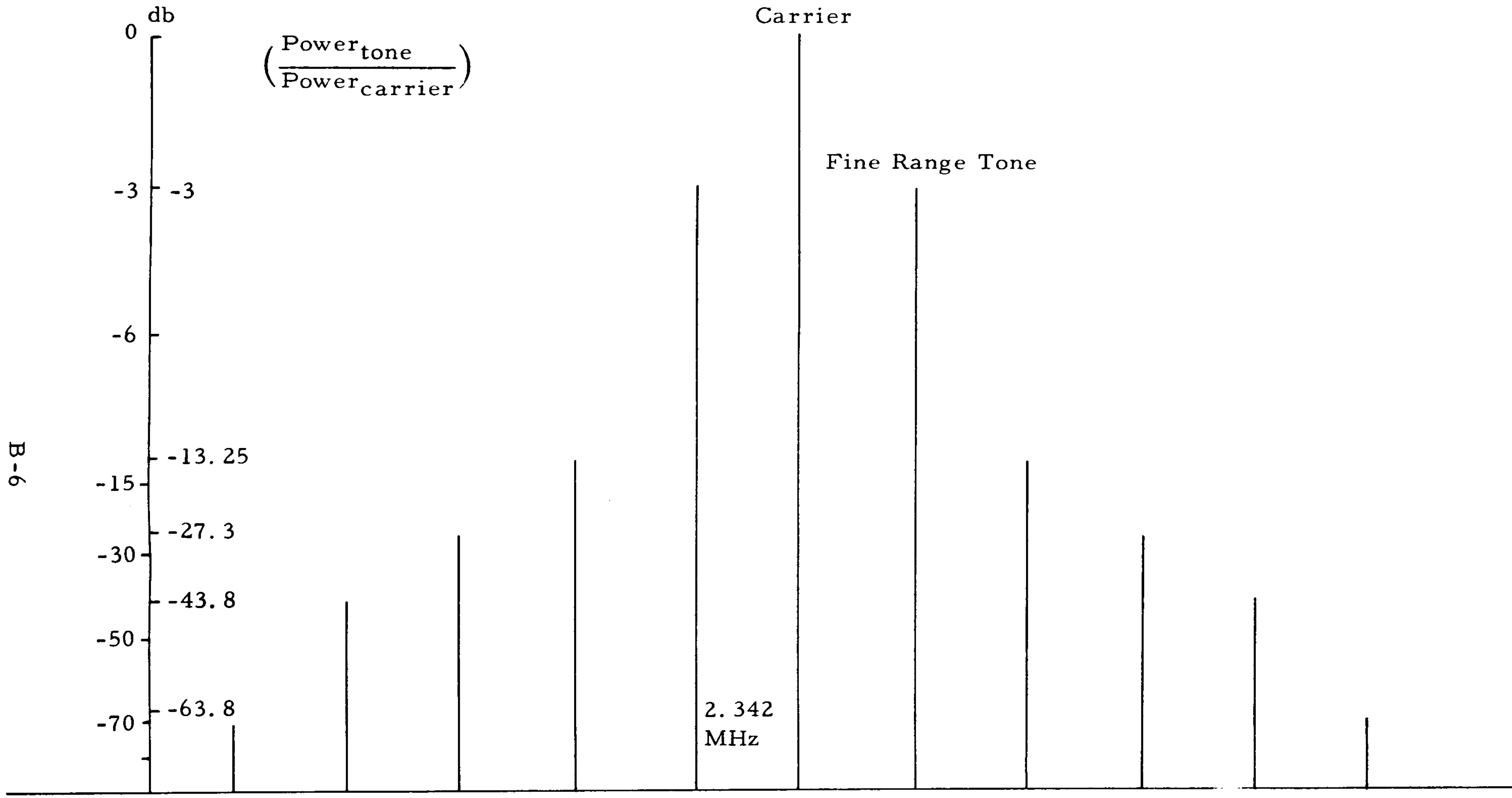
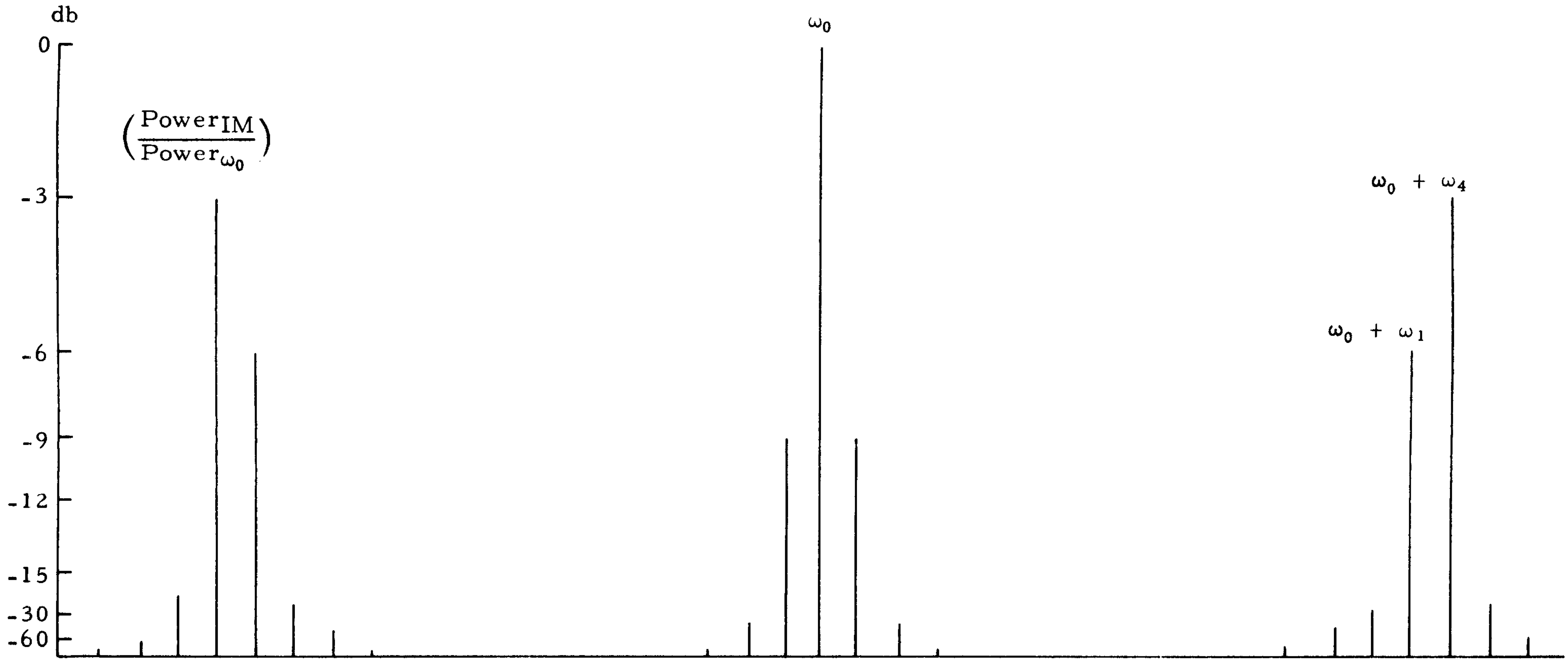


FIGURE B-1

POWER LEVEL OF
FINE RANGE TONE SIDE BANDS



ω_0 = carrier
ω_1 = ω_4 - 73 kHz
ω_4 = fine range tone = 2.342 MHz

FIGURE B-2

AROD INTERMODULATION SPECTRUM
FOR ONE CHANNEL

320 kHz frequency intervals (Figure B-3). Using the frequencies listed in Table B-1, one can construct the frequency regions which will contain some appreciable power from the phase modulated signal if the range of doppler shift given above is taken into account. Table B-2 lists the frequency regions obtained from just such a construction, using frequencies with power greater than the carrier power less 50 db. If one accepts the more or less arbitrary decision to include all terms not more than 50 db down from the carrier, Table B-2 establishes the frequency regions which should not contain the carrier or principal sidebands of any other RF channel of the vehicle tracking receiver. The frequency values in Table B-2 are measured relative to the carrier.

TABLE B-2

FREQUENCY INTERVALS FOR A GIVEN CHANNEL IN WHICH INTERFERENCE MAY BE EXPECTED

The frequencies listed should be superimposed on an appropriate carrier coinciding with zero frequency of the spectrum below.

-11.797 MHz	to	-11.404 MHz
-9.528 MHz	to	-8.989 MHz
-7.259 MHz	to	-6.647 MHz
-4.917 MHz	to	-4.305 MHz
-2.648 MHz	to	-1.953 MHz
-0.306 MHz	to	+0.306 MHz
+1.953 MHz	to	+2.648 MHz
+4.305 MHz	to	+4.917 MHz
+6.647 MHz	to	+7.759 MHz
+8.989 MHz	to	+9.528 MHz
+11.404 MHz	to	+11.797 MHz

NOTE: Doppler shift figures on maximum velocity of 11,000 m/sec^2

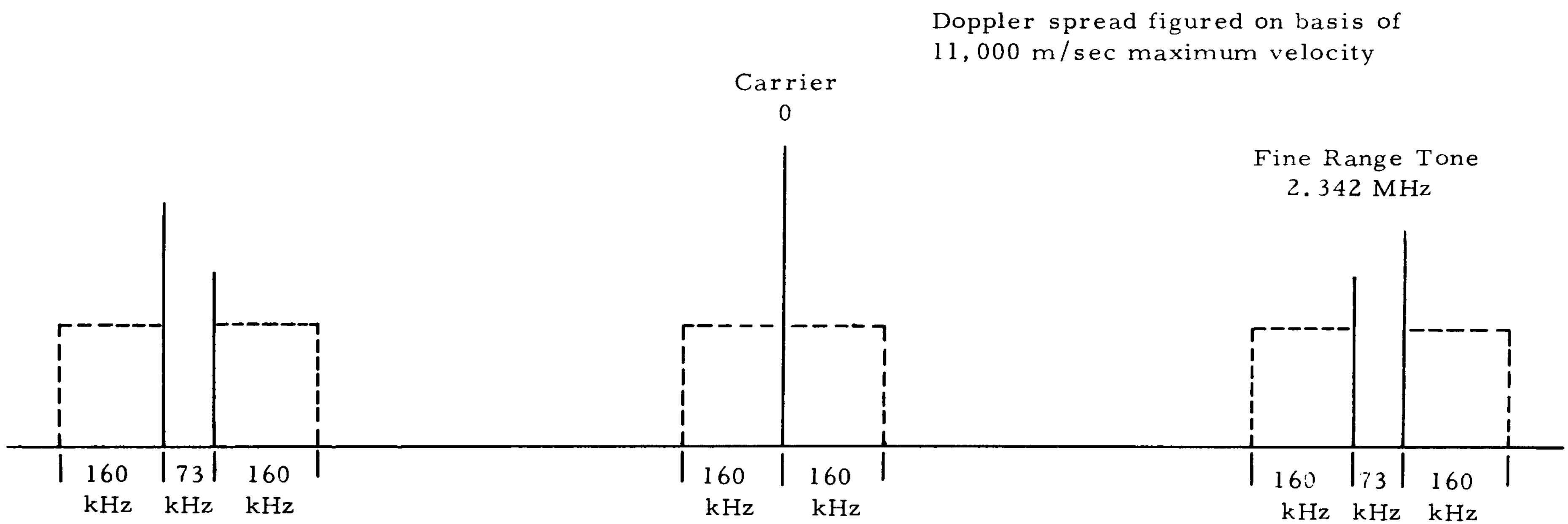


FIGURE B-3

DOPPLER FREQUENCY BANDS ASSOCIATED WITH CARRIER AND PRINCIPAL SIDEBANDS OF ONE PHASE MODULATED AROD CHANNEL

APPENDIX C

OUTPUT FROM RECEIVER FRONT-END - SINGLE SIDEBAND MODULATED SIGNAL

The signal received at the vehicle receiver from the transponder number ν ($\nu = 1, 2, 3, 4$) can be represented by the expression

$$S_\nu(t) = E_\nu \sum_{\gamma=0}^{4} a_\gamma \cos\left(\omega_{c_\nu} t + \omega_\gamma t + \theta_{\nu,\gamma}\right), \tag{C-1}$$

where

ω_{c_ν} - carrier frequency (including doppler shift of signal from ν^{th} transponder),

ω_γ - frequency of γ^{th} modulating tone ($\omega_o = 0$),

$\theta_{\nu,\gamma}$ - phase shift in signal due to transit time from vehicle to transponder and return, and

E_ν - amplitude of signal from ν^{th} transponder.

At the input to the vehicle receiver, the signals from the four transponders are superimposed; therefore, the input is

$$S(t) = \sum_{\nu=1}^{4} \sum_{\gamma=0}^{4} E_\nu a_\gamma \cdot \cos\left(\omega_{c_\nu} t + \omega_\gamma t + \theta_{\nu,\gamma}\right). \tag{C-2}$$

A reasonable representation of the output voltage of the receiver front-end is given by

$$V_o(t) = \alpha_1 S(t) + \alpha_3 S^3(t). \tag{C-3}$$

Directing interest to the $S^3(t)$ factor, we observe that

$$S^3(t) = \sum_{\nu_1, \nu_2, \nu_3 = 1}^{4} \; \sum_{\gamma_1, \gamma_2, \gamma_3 = 0}^{4} \Big\{ E_{\nu_1} \cdot E_{\nu_2} \cdot E_{\nu_3} \cdot a_{\gamma_1} \cdot a_{\gamma_2} \cdot a_{\gamma_3}$$

$$\cdot \cos(\omega_{c\nu_1} t + \omega_{\gamma_1} t + \theta_{\nu_1, \gamma_1})$$

$$\cdot \cos(\omega_{c\nu_2} t + \omega_{\gamma_2} t + \theta_{\nu_2, \gamma_2}) \qquad \text{(C-4)}$$

$$\cdot \cos(\omega_{c\nu_3} t + \omega_{\gamma_3} t + \theta_{\nu_3, \gamma_3}) \Big\}$$

If one neglects terms with the carrier frequency $\omega_{c\nu_1} + \omega_{c\nu_2} + \omega_{c\nu_3}$, the expression for $S^3(t)$ can be written as

$$S^3(t) = \frac{1}{4} \sum_{\nu_1, \nu_2, \nu_3 = 1}^{4} E_{\nu_1} \cdot E_{\nu_2} \cdot E_{\nu_3} \cdot \Big\{ \sum_{\gamma_1, \gamma_2, \gamma_3 = 0}^{4} a_{\gamma_1} \cdot a_{\gamma_2} \cdot a_{\gamma_3}$$

$$\cdot \Big[\cos \Big((\omega_{c\nu_1} + \omega_{c\nu_2} - \omega_{c\nu_3}) t + (\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3}) t$$

$$+ \theta_{\nu_1, \gamma_1} + \theta_{\nu_2, \gamma_2} - \theta_{\nu_3, \gamma_3} \Big) \qquad \text{(C-5)}$$

$$+ \cos \Big((\omega_{c\nu_2} + \omega_{c\nu_3} - \omega_{c\nu_1}) t + (\omega_{\gamma_2} + \omega_{\gamma_3} - \omega_{\gamma_1}) t$$

$$+ \theta_{\nu_2, \gamma_2} + \theta_{\nu_3, \gamma_3} - \theta_{\nu_1, \gamma_1} \Big)$$

$$+ \cos \Big((\omega_{c\nu_3} + \omega_{c\nu_1} - \omega_{c\nu_2}) t + (\omega_{\gamma_3} + \omega_{\gamma_1} - \omega_{\gamma_2}) t$$

$$+ \theta_{\nu_3, \gamma_3} + \theta_{\nu_1, \gamma_1} - \theta_{\nu_2, \gamma_2} \Big) \Big] \Big\}$$

By collecting terms, one obtains

$$S^3(t) = \frac{3}{4} \sum_{\nu_1, \nu_2, \nu_3 = 1}^{4} E_{\nu_1} \cdot E_{\nu_2} \cdot E_{\nu_3} \left\{ \sum_{\gamma_1, \gamma_2, \gamma_3 = 0}^{4} a_{\gamma_1} \cdot a_{\gamma_2} \cdot a_{\gamma_3} \right.$$

$$\left. \cdot \cos \left[\left(\omega_{c_{\nu_1}} + \omega_{c_{\nu_2}} - \omega_{c_{\nu_3}} \right) t + \left(\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3} \right) t + \theta_{\nu_1, \gamma_1} + \theta_{\nu_2, \gamma_2} - \theta_{\nu_3, \gamma_3} \right] \right\} . \tag{C-6}$$

Equation (C-6) clearly exhibits the spectrum of $S^3(t)$. Spectral lines are located at the frequencies

$$\omega = \omega_{c_{\nu_1}} + \omega_{c_{\nu_2}} - \omega_{c_{\nu_3}} + (\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3}) , \tag{C-7}$$

where ν_1, ν_2, ν_3 assume all possible combinations of the values $\nu = 1, 2, 3, 4$ and $\gamma_1, \gamma_2, \gamma_3$ assume all possible combinations of the values $\gamma = 0, 1, 2, 3, 4$. Present in the signal are the components of all combinations of the sum of two carrier frequencies minus the frequency of another carrier together with all combinations of the sum of the frequencies of two modulating tones minus the frequency of another.

A better appreciation of the significance of the term $S^3(t)$ can be obtained by examining the cube of the only one single-sideband modulated signal.

It can be shown (Appendix A) that

$$S_\nu^3(t) = \frac{3}{4} \sum_{\gamma_1, \gamma_2, \gamma_3 = 0}^{4} a_{\gamma_1} \cdot a_{\gamma_2} \cdot a_{\gamma_3}$$

$$\cdot \cos \left[\omega_{c_\nu} t + (\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3}) t + \theta_{\nu, \gamma_1} + \theta_{\nu, \gamma_2} - \theta_{\nu, \gamma_3}\right] . \tag{C-8}$$

By examining equations (C-6) and (C-7), one observes that the equation for $S^3(t)$ is simply a superposition of terms like $S_\nu^3(t)$, but with the carrier frequency translated to

$$\omega_{c_\nu} = \omega_{c_{\nu_1}} + \omega_{c_{\nu_2}} - \omega_{c_{\nu_3}} . \tag{C-9}$$

APPENDIX D

MATHEMATICAL ANALYSIS FOR DETERMINING EFFECT OF NONLINEARITY ON AROD AIRBORNE RECEIVER USING PHASE MODULATION

The voltage output of the IF amplifier of the AROD airborne receiver, for the case of phase modulation, is given to a close approximation by the equation

$$V_o(t) = \alpha_1 S_R(t) + \alpha_3 S_R^{3}(t) \quad , \tag{D-1}$$

where

$$S_R(t) = E_1 \cos \phi_1(t) + E_2 \cos \phi_2(t) + E_3 \cos \phi_3(t) + E_4 \cos \phi_4(t) , \tag{D-2}$$

$$\phi_1(t) = \omega_{c_1} t + \lambda \theta_1(t) \quad , \tag{D-3}$$

$$\phi_2(t) = \omega_{c_2} t + \lambda \theta_2(t) \quad , \tag{D-4}$$

$$\phi_3(t) = \omega_{c_3} t + \lambda \theta_3(t) \quad , \tag{D-5}$$

$$\phi_4(t) = \omega_{c_4} t + \lambda \theta_4(t) \quad , \tag{D-6}$$

$$\theta_1(t) = \theta\left(t - \frac{2R_1(t)}{c}\right) \quad , \tag{D-7}$$

$$\theta_2(t) = \theta\left(t - \frac{2R_2(t)}{c}\right) \quad , \tag{D-8}$$

$$\theta_3(t) = \theta\left(t - \frac{2R_3(t)}{c}\right) \quad , \tag{D-9}$$

$$\theta_4(t) = \theta\left(t - \frac{2\,R_4(t)}{c}\right) \text{ , and} \tag{D-10}$$

$$\theta(t) = \varepsilon_1 \cos \omega_1 t + \varepsilon_2 \cos \omega_2 t + \varepsilon_3 \cos \omega_3 t + \varepsilon_4 \cos \omega_4 t \quad , \tag{D-11}$$

where

α_1 and α_3 are constants,

E_m - the amplitude of the signal from the m^{th} transponder (for m = 1, 2, 3, and 4),

$V_o(t)$ - the voltage output of the IF amplifier of the AROD airborne receiver,

$\lambda=32$ - the multiplication factor,

R_1 - the distance from the first transponder to the space vehicle,

R_2 - the distance from the second transponder to the space vehicle,

R_3 - the distance from the third transponder to the space vehicle,

R_4 - the distance from the fourth transponder to the space vehicle,

$S_R(t)$ - the signal received at the space vehicle receiver from the four transponders,

ω_{c_m} - the angular frequencies of the carrier waves in radians per second,

ω_m - the angular frequencies of the modulating waves in radians per second, and

ε_m - the modulation indices for m = 1, 2, 3, and 4.

The modulation index is the ratio of the frequency deviation to the modulating frequency.

Let

$$b_1 = E_1 \cos \phi_1(t) \quad , \tag{D-12}$$

$$b_2 = E_2 \cos \phi_2(t) \quad , \tag{D-13}$$

$$b_3 = E_3 \cos \phi_3(t) \quad , \tag{D-14}$$

and

$$b_4 = E_4 \cos \phi_4(t) \quad , \tag{D-15}$$

From equations (D-2), (D-12), (D-13), (D-14), and (D-15), one obtains

$$S_R(t) = b_1 + b_2 + b_3 + b_4 \tag{D-16}$$

and

$$S_R^3(t) = (b_1 + b_2 + b_3 + b_4)^3 \quad . \tag{D-17}$$

By expanding equation (D-17), one obtains

$$\begin{aligned} S_R^3(t) = \; & b_1^3 + b_2^3 + b_3^3 + b_4^3 \\ & + 3b_1 b_2^2 + 3b_1^2 b_2 + 3b_3 b_4^2 + 3b_3^2 b_4 \\ & + 3b_1^2 b_2 + 3b_1^2 b_4 + 3b_2^2 b_3 + 3b_2^2 b_4 \\ & + 3b_1 b_3^2 + 3b_1 b_4^2 + 3b_2 b_3^2 + 3b_2 b_4^2 \\ & + 6b_1 b_2 b_3 + 6b_1 b_2 b_4 + 6b_1 b_3 b_4 + 6b_2 b_3 b_4 \; . \end{aligned} \tag{D-18}$$

From equations (D-12), (D-13), (D-14), (D-15), and (D-18), it may be shown that

$$\begin{aligned} S_R^3(t) = \; & d_1 \cos \phi_1 + d_2 \cos \phi_2 + d_3 \cos \phi_3 + d_4 \cos \phi_4 \\ & + c_1 \cos 3\phi_1 + c_2 \cos 3\phi_2 + c_3 \cos 3\phi_3 + c_4 \cos 3\phi_4 \\ & + d_5 [\cos(\phi_1 - 2\phi_2) + \cos(\phi_1 + 2\phi_2)] \\ & + d_6 [\cos(\phi_2 - 2\phi_1) + \cos(\phi_2 + 2\phi_1] \\ & + d_7 [\cos(\phi_3 - 2\phi_4) + \cos(\phi_3 + 2\phi_4] \\ & + d_8 [\cos(\phi_4 - 2\phi_3) + \cos(\phi_4 + 2\phi_3] \\ & + d_9 [\cos(\phi_3 - 2\phi_1) + \cos(\phi_3 + 2\phi_1] \\ & + d_{10} [\cos(\phi_4 - 2\phi_1) + \cos(\phi_4 + 2\phi_1] \\ & + d_{11} [\cos(\phi_3 - 2\phi_2) + \cos(\phi_3 + 2\phi_2)] \\ & + d_{12} [\cos(\phi_4 - 2\phi_2) + \cos(\phi_4 + 2\phi_2)] \end{aligned}$$

$$
\begin{aligned}
&+ d_{13}[\cos(\phi_1 - 2\phi_3) + \cos(\phi_1 + 2\phi_3)] \\
&+ d_{14}[\cos(\phi_1 - 2\phi_4) + \cos(\phi_1 + 2\phi_4)] \\
&+ d_{15}[\cos(\phi_2 - 2\phi_3) + \cos(\phi_2 + 2\phi_3)] \\
&+ d_{16}[\cos(\phi_2 - 2\phi_4) + \cos(\phi_2 + 2\phi_4)] \\
&+ c_{17}[\cos(\phi_1 + \phi_2 + \phi_3) + \cos(\phi_1 + \phi_2 - \phi_3) + \cos(\phi_1 - \phi_2 + \phi_3) \\
&\qquad + \cos(\phi_1 - \phi_2 - \phi_3)] \\
&+ c_{18}[\cos(\phi_1 + \phi_2 + \phi_4) + \cos(\phi_1 + \phi_2 - \phi_4) + \cos(\phi_1 - \phi_2 + \phi_4) \qquad \text{(D-19)} \\
&\qquad + \cos(\phi_1 - \phi_2 - \phi_4] \\
&+ c_{19}[\cos(\phi_1 + \phi_3 + \phi_4) + \cos(\phi_1 + \phi_3 - \phi_4) + \cos(\phi_1 - \phi_3 + \phi_4) \\
&\qquad + \cos(\phi_1 - \phi_3 - \phi_4)] \\
&+ c_{20}[\cos(\phi_2 + \phi_3 + \phi_4) + \cos(\phi_2 + \phi_3 - \phi_4) + \cos(\phi_2 - \phi_3 + \phi_4) \\
&\qquad + \cos(\phi_2 - \phi_3 - \phi_4)]
\end{aligned}
$$

where

$$c_1 = \frac{E_1^{\,3}}{4} , \qquad \text{(D-20)}$$

$$c_2 = \frac{E_2^{\,3}}{4} , \qquad \text{(D-21)}$$

$$c_3 = \frac{E_3^{\,3}}{4} , \qquad \text{(D-22)}$$

$$c_4 = \frac{E_4^{\,3}}{4} , \qquad \text{(D-23)}$$

$$c_5 = \frac{3\, E_1 E_2^{\,2}}{2} , \qquad \text{(D-24)}$$

$$c_6 = \frac{3\, E_1^{\,2} E_2}{2} , \qquad \text{(D-25)}$$

$$c_7 = \frac{3 E_3 E_4^2}{2} , \quad (D\text{-}26)$$

$$c_8 = \frac{3 E_3^2 E_4}{2} , \quad (D\text{-}27)$$

$$c_9 = \frac{3 E_1^2 E_3}{2} , \quad (D\text{-}28)$$

$$c_{10} = \frac{3 E_1^2 E_4}{2} , \quad (D\text{-}29)$$

$$c_{11} = \frac{3 E_2^2 E_3}{2} , \quad (D\text{-}30)$$

$$c_{12} = \frac{3 E_2^2 E_4}{2} , \quad (D\text{-}31)$$

$$c_{13} = \frac{3 E_1 E_3^2}{2} , \quad (D\text{-}32)$$

$$c_{14} = \frac{3 E_1 E_4^2}{2} , \quad (D\text{-}33)$$

$$c_{15} = \frac{3 E_2 E_3^2}{2} , \quad (D\text{-}34)$$

$$c_{16} = \frac{3 E_2 E_4^2}{2} , \quad (D\text{-}35)$$

$$c_{17} = \frac{3 E_1 E_2 E_3}{2} , \quad (D\text{-}36)$$

$$c_{18} = \frac{3 E_1 E_2 E_4}{2} , \quad (D\text{-}37)$$

$$c_{19} = \frac{3 E_1 E_3 E_4}{2} , \quad (D\text{-}38)$$

$$c_{20} = \frac{3 E_2 E_3 E_4}{2} , \quad (D\text{-}39)$$

$$d_1 = 3 c_1 + c_5 + c_{13} + c_{14} , \quad (D\text{-}40)$$

$$d_2 = 3 c_2 + c_6 + c_{15} + c_{16} , \quad (D\text{-}41)$$

$$d_3 = 3c_3 + c_7 + c_9 + c_{11} \quad , \tag{D-42}$$

$$d_4 = 3c_4 + c_8 + c_{10} + c_{12} \quad , \tag{D-43}$$

$$d_5 = \tfrac{1}{2} c_5 = \frac{3 E_1 E_2^{\,2}}{4} \quad , \tag{D-44}$$

$$d_6 = \tfrac{1}{2} c_6 = \frac{3 E_1^{\,2} E_2}{4} \quad , \tag{D-45}$$

$$d_7 = \tfrac{1}{2} c_7 = \frac{3 E_3 E_4^{\,2}}{4} \quad , \tag{D-46}$$

$$d_8 = \tfrac{1}{2} c_8 = \frac{3 E_3^{\,2} E_4}{4} \quad , \tag{D-47}$$

$$d_9 = \tfrac{1}{2} c_9 = \frac{3 E_1^{\,2} E_3}{4} \quad , \tag{D-48}$$

$$d_{10} = \tfrac{1}{2} c_{10} = \frac{3 E_1^{\,2} E_4}{4} \quad , \tag{D-49}$$

$$d_{11} = \tfrac{1}{2} c_{11} = \frac{3 E_2^{\,2} E_3}{4} \quad , \tag{D-50}$$

$$d_{12} = \tfrac{1}{2} c_{12} = \frac{3 E_2^{\,2} E_4}{4} \quad , \tag{D-51}$$

$$d_{13} = \tfrac{1}{2} c_{13} = \frac{3 E_1 E_3^{\,2}}{4} \quad , \tag{D-52}$$

$$d_{14} = \tfrac{1}{2} c_{14} = \frac{3 E_1 E_4^{\,2}}{4} \quad , \tag{D-53}$$

$$d_{15} = \tfrac{1}{2} c_{15} = \frac{3 E_2 E_3^{\,2}}{4} \quad , \quad \text{and} \tag{D-54}$$

$$d_{16} = \tfrac{1}{2} c_{16} = \frac{3 E_2 E_4^{\,2}}{4} \quad . \tag{D-55}$$

By disregarding all terms far removed from the carrier frequency and those terms exactly at the carrier frequency, equation (D-19) reduces to

$$\begin{aligned}
S_R{}^3(t) = {} & \frac{3E_1E_2{}^2}{4}\cos(\phi_1 - 2\phi_2) + \frac{3E_1{}^2E_2}{4}\cos(\phi_2 - 2\phi_1) \\
& + \frac{3E_3E_4{}^2}{4}\cos(\phi_3 - 2\phi_4) + \frac{3E_3{}^2E_4}{4}\cos(\phi_4 - 2\phi_3) \\
& + \frac{3E_1{}^2E_3}{4}\cos(\phi_3 - 2\phi_1) + \frac{3E_1{}^2E_4}{4}\cos(\phi_4 - 2\phi_1) \\
& + \frac{3E_2{}^2E_3}{4}\cos(\phi_3 - 2\phi_2) + \frac{3E_2{}^2E_4}{4}\cos(\phi_4 - 2\phi_2) \\
& + \frac{3E_1E_3{}^2}{4}\cos(\phi_1 - 2\phi_3) + \frac{3E_1E_4{}^2}{4}\cos(\phi_1 - 2\phi_4) \\
& + \frac{3E_2E_3{}^2}{4}\cos(\phi_2 - 2\phi_3) + \frac{3E_2E_4{}^2}{4}\cos(\phi_2 - 2\phi_4) \\
& + \frac{3E_1E_2E_3}{2}\Big[\cos(\phi_1 + \phi_2 - \phi_3) + \cos(\phi_1 - \phi_2 + \phi_3) \\
& \qquad + \cos(\phi_1 - \phi_2 - \phi_3)\Big] \\
& + \frac{3E_1E_2E_4}{2}\Big[\cos(\phi_1 + \phi_2 - \phi_4) + \cos(\phi_1 - \phi_2 + \phi_4) \\
& \qquad + \cos(\phi_1 - \phi_2 - \phi_4)\Big] \\
& + \frac{3E_1E_3E_4}{2}\Big[\cos(\phi_1 + \phi_3 - \phi_4) + \cos(\phi_1 - \phi_3 + \phi_4) \\
& \qquad + \cos(\phi_1 - \phi_3 - \phi_4)\Big] \\
& + \frac{3E_2E_3E_4}{2}\Big[\cos(\phi_2 + \phi_3 - \phi_4) + \cos(\phi_2 - \phi_3 + \phi_4) \\
& \qquad + \cos(\phi_2 - \phi_3 - \phi_4)\Big]
\end{aligned} \tag{D-56}$$

From equations (D-3) - (D-6), inclusive, it may be shown that equations (D-57) - (D-62) are valid.

$$2\phi_i - \phi_j = (\omega_{a_k})t + \lambda\psi_k(t) \tag{D-57}$$

where

$$\omega_{a_k} = 2\omega_{c_i} - \omega_{c_j} \tag{D-58}$$

and

$$\psi_k(t) = 2\theta_i(t) - \theta_j(t) \quad , \tag{D-59}$$

where i, j, and k are integers.

Similarly,

$$\phi_i \pm \phi_j \pm \phi_m = (\omega_{a_k})t + \psi_k(t) \quad , \tag{D-60}$$

where

$$\omega_{a_k} = \omega_{c_i} \pm \omega_{c_j} \pm \omega_{c_m} \tag{D-61}$$

and

$$\psi_k(t) = \theta_i(t) \pm \theta_j(t) \pm \theta_m(t) \quad , \tag{D-62}$$

where i, j, k, and m are integers.

From equations (D-56) - (D-62), inclusive, the amplitudes, carrier frequencies, and modulating frequencies for each of the terms in equation (D-56) were determined. The frequencies and amplitudes of the intermodulation components are listed in Table D-1.

Equations (D-7) - (D-10), inclusive, may be written in the form

$$\theta_i(t) = \theta\left(t - \frac{2R_i(t)}{c}\right) \quad , \tag{D-63}$$

where i = 1, 2, 3, and 4.

TABLE D-1

AMPLITUDES, CARRIER FREQUENCIES, AND MODULATING FREQUENCIES OF IM COMPONENTS DUE TO NONLINEARITY

Amplitudes of IM Components	Carrier Frequencies of IM Components	IM Modulating Frequencies
$\frac{3 E_1 E_2^2}{4} \alpha_3$	$\omega a_1 = 2\omega c_2 - \omega c_1$	$\psi_1(t) = 2\theta_2(t) - \theta_1(t)$
$\frac{3 E_1^2 E_2}{4} \alpha_3$	$\omega a_2 = 2\omega c_1 - \omega c_2$	$\psi_2(t) = 2\theta_1(t) - \theta_2(t)$
$\frac{3 E_3 E_4^2}{4} \alpha_3$	$\omega a_3 = 2\omega c_4 - \omega c_3$	$\psi_3(t) = 2\theta_4(t) - \theta_3(t)$
$\frac{3 E_3^2 E_4}{4} \alpha_3$	$\omega a_4 = 2\omega c_3 - \omega c_4$	$\psi_4(t) = 2\theta_3(t) - \theta_4(t)$
$\frac{3 E_1^2 E_3}{4} \alpha_3$	$\omega a_5 = 2\omega c_1 - \omega c_3$	$\psi_5(t) = 2\theta_1(t) - \theta_3(t)$
$\frac{3 E_1^2 E_4}{4} \alpha_3$	$\omega a_6 = 2\omega c_1 - \omega c_4$	$\psi_6(t) = 2\theta_1(t) - \theta_4(t)$
$\frac{3 E_2^2 E_3}{4} \alpha_3$	$\omega a_7 = 2\omega c_2 - \omega c_3$	$\psi_7(t) = 2\theta_2(t) - \theta_3(t)$
$\frac{3 E_2^2 E_4}{4} \alpha_3$	$\omega a_8 = 2\omega c_2 - \omega c_4$	$\psi_8(t) = 2\theta_2(t) - \theta_4(t)$
$\frac{3 E_1 E_3^2}{4} \alpha_3$	$\omega a_9 = 2\omega c_3 - \omega c_1$	$\psi_9(t) = 2\theta_3(t) - \theta_1(t)$
$\frac{3 E_1 E_4^2}{4} \alpha_3$	$\omega a_{10} = 2\omega c_4 - \omega c_1$	$\psi_{10}(t) = 2\theta_4(t) - \theta_1(t)$
$\frac{3 E_2 E_3^2}{4} \alpha_3$	$\omega a_{11} = 2\omega c_3 - \omega c_2$	$\psi_{11}(t) = 2\theta_3(t) - \theta_2(t)$

TABLE D-1 (Continued)

Amplitudes of IM Components	Carrier Frequencies of IM Components	IM Modulating Frequencies
$\frac{3 E_2 E_4^2}{4} a_3$	$\omega_{a_{12}} = 2\omega_{c_4} - \omega_{c_2}$	$\psi_{12}(t) = 2\theta_4(t) - \theta_2(t)$
$\frac{3 E_1 E_2 E_3}{2} a_3$	$\omega_{a_{13}} = \omega_{c_1} + \omega_{c_2} - \omega_{c_3}$	$\psi_{13}(t) = \theta_1(t) + \theta_2(t) - \theta_3(t)$
$\frac{3 E_1 E_2 E_3}{2} a_3$	$\omega_{a_{14}} = \omega_{c_1} + \omega_{c_3} - \omega_{c_2}$	$\psi_{14}(t) = \theta_1(t) + \theta_3(t) - \theta_2(t)$
$\frac{3 E_1 E_2 E_3}{2} a_3$	$\omega_{a_{15}} = \omega_{c_2} + \omega_{c_3} - \omega_{c_1}$	$\psi_{15}(t) = \theta_2(t) + \theta_3(t) - \theta_1(t)$
$\frac{3 E_1 E_2 E_4}{2} a_3$	$\omega_{a_{16}} = \omega_{c_1} + \omega_{c_2} - \omega_{c_4}$	$\psi_{16}(t) = \theta_1(t) + \theta_2(t) - \theta_4(t)$
$\frac{3 E_1 E_2 E_4}{2} a_3$	$\omega_{a_{17}} = \omega_{c_1} + \omega_{c_4} - \omega_{c_2}$	$\psi_{17}(t) = \theta_1(t) + \theta_4(t) - \theta_2(t)$
$\frac{3 E_1 E_2 E_4}{2} a_3$	$\omega_{a_{18}} = \omega_{c_2} + \omega_{c_4} - \omega_{c_1}$	$\psi_{18}(t) = \theta_2(t) + \theta_4(t) - \theta_1(t)$
$\frac{3 E_1 E_3 E_4}{2} a_3$	$\omega_{a_{19}} = \omega_{c_1} + \omega_{c_3} - \omega_{c_4}$	$\psi_{19}(t) = \theta_1(t) + \theta_3(t) - \theta_4(t)$
$\frac{3 E_1 E_3 E_4}{2} a_3$	$\omega_{a_{20}} = \omega_{c_1} + \omega_{c_4} - \omega_{c_3}$	$\psi_{20}(t) = \theta_1(t) + \theta_4(t) - \theta_3(t)$
$\frac{3 E_1 E_3 E_4}{2} a_3$	$\omega_{a_{21}} = \omega_{c_3} + \omega_{c_4} - \omega_{c_1}$	$\psi_{21}(t) = \theta_3(t) + \theta_4(t) - \theta_1(t)$
$\frac{3 E_2 E_3 E_4}{2} a_3$	$\omega_{a_{22}} = \omega_{c_2} + \omega_{c_3} - \omega_{c_4}$	$\psi_{22}(t) = \theta_2(t) + \theta_3(t) - \theta_4(t)$
$\frac{3 E_2 E_3 E_4}{2} a_3$	$\omega_{a_{23}} = \omega_{c_2} + \omega_{c_4} - \omega_{c_3}$	$\psi_{23}(t) = \theta_2(t) + \theta_4(t) - \theta_3(t)$
$\frac{3 E_2 E_3 E_4}{2} a_3$	$\omega_{a_{24}} = \omega_{c_3} + \omega_{c_4} - \omega_{c_2}$	$\psi_{24}(t) = \theta_3(t) + \theta_4(t) - \theta_2(t)$

It may then be shown that for the case of the range tones, equations (D-64), (D-65), and (D-66) are valid.

$$\theta_i(t) = \epsilon_1 \cos(\omega_1 t + \delta_{i,1}) + \epsilon_2 \cos(\omega_2 t + \delta_{i,2}) + \epsilon_3 \cos(\omega_3 t + \delta_{i,3}) + \epsilon_4 \cos(\omega_4 t + \delta_{i,4}) , \quad \text{(D-64)}$$

where i = 1, 2, 3, and 4 ,

$$\theta_j(t) = \epsilon_1 \cos(\omega_1 t + \delta_{j,1}) + \epsilon_2 \cos(\omega_2 t + \delta_{j,2}) + \epsilon_3 \cos(\omega_3 t + \delta_{j,3}) + \epsilon_4 \cos(\omega_4 t + \delta_{j,4}) , \quad \text{(D-65)}$$

where j = 1, 2, 3, and 4 , and

$$\theta_k(t) = \epsilon_1 \cos(\omega_1 t + \delta_{k,1}) + \epsilon_2 \cos(\omega_2 t + \delta_{k,2}) + \epsilon_3 \cos(\omega_3 t + \delta_{k,3}) + \epsilon_4 \cos(\omega_4 t + \delta_{k,4}) , \quad \text{(D-66)}$$

where

$\delta_{m,n}$ - phase angles of the range tones which depend on the transponder range $\left(\text{i.e.}, \delta_{m,n} = 2\,\omega_m \cdot \frac{R_n}{c}\right)$,

ω_m - angular modulating frequency in radians per second, and

ϵ_m - modulation index for m = 1, 2, 3, and 4.

An examination of the angular modulating frequencies listed in Table D-l indicates that all these frequencies can be represented by an equation of the type

$$\psi_m(t) = \theta_i(t) + \theta_j(t) - \theta_k(t) , \quad \text{(D-67)}$$

where $\theta_i(t)$, $\theta_j(t)$, and $\theta_k(t)$ are given by equations (D-64), (D-65), and (D-66). respectively.

From equations (D-64) - (D-67), inclusive, it may be shown that

$$\begin{aligned}\psi_m(t) = \varepsilon_1 &[\cos(\omega_1 t + \delta_{i,1}) + \cos(\omega_1 t + \delta_{j,1}) - \cos(\omega_1 t + \delta_{k,1})]\\ &+ \varepsilon_2 [\cos(\omega_2 t + \delta_{i,2}) + \cos(\omega_2 t + \delta_{j,2}) - \cos(\omega_2 t + \delta_{k,2})]\\ &+ \varepsilon_3 [\cos(\omega_3 t + \delta_{i,3}) + \cos(\omega_3 t + \delta_{j,3}) - \cos(\omega_3 t + \delta_{k,3})]\\ &+ \varepsilon_4 [\cos(\omega_4 t + \delta_{i,4}) + \cos(\omega_4 t + \delta_{j,4}) - \cos(\omega_4 t + \delta_{k,4})]\end{aligned} \tag{D-68}$$

From equation (D-68), it is evident that

$$\psi_m(t) = P_1{}^{i,j,k}(t) + P_2{}^{i,j,k}(t) + P_3{}^{i,j,k}(t) + P_4{}^{i,j,k}(t) \tag{D-69}$$

where

$$P_1{}^{i,j,k}(t) = \varepsilon_1 [\cos(\omega_1 t + \delta_{i,1}) + \cos(\omega_1 t + \delta_{j,1}) - \cos(\omega_1 t + \delta_{k,1})] , \tag{D-70}$$

$$P_2{}^{i,j,k}(t) = \varepsilon_2 [\cos(\omega_2 t + \delta_{i,2}) + \cos(\omega_2 t + \delta_{j,2}) - \cos(\omega_2 t + \delta_{k,2})] , \tag{D-71}$$

$$P_3{}^{i,j,k}(t) = \varepsilon_3 [\cos(\omega_3 t + \delta_{i,3}) + \cos(\omega_3 t + \delta_{j,3}) - \cos(\omega_3 t + \delta_{k,3})], \text{ and} \tag{D-72}$$

$$P_4{}^{i,j,k}(t) = \varepsilon_4 [\cos(\omega_4 t + \delta_{i,4}) + \cos(\omega_4 t + \delta_{j,4}) - \cos(\omega_4 t + \delta_{k,4})] . \tag{D-73}$$

By application of Euler's formula and elementary trigonometry, it may be shown that

$$\begin{aligned}\psi_m(t) = \varepsilon_1 F_1{}^{i,j,k} &\cos(\omega_1 t + d_1{}^{i,j,k})\\ + \varepsilon_2 F_2{}^{i,j,k} &\cos(\omega_2 t + d_2{}^{i,j,k})\\ + \varepsilon_3 F_3{}^{i,j,k} &\cos(\omega_3 t + d_3{}^{i,j,k})\\ + \varepsilon_4 F_4{}^{i,j,k} &\cos(\omega_4 t + d_4{}^{i,j,k}) ,\end{aligned} \tag{D-74}$$

where

$$A_m = \cos(\delta_{i,m}) + \cos(\delta_{j,m}) - \cos(\delta_{k,m}) , \tag{D-75}$$

$$B_m = \sin(\delta_{i,m}) + \sin(\delta_{j,m}) - \cos(\delta_{k,m}) , \tag{D-76}$$

$$F_m^{i,j,k} = \left| \sqrt{A_m^2 + B_m^2} \right|, \tag{D-77}$$

and

$$d_m^{i,j,k} = \text{arc tan}\left(\frac{B_m}{A_m}\right). \tag{D-78}$$

From equations (D-75), (D-76), and (D-77), it can be shown that

$$F_m^{i,j,k} = \left| \sqrt{3 + 2\,H_m^{i,j,k}} \right|, \tag{D-79}$$

where

$$H_m^{i,j,k} = \cos(\delta_{i,m} - \delta_{j,m}) - \cos(\delta_{i,m} - \delta_{k,m}) - \cos(\delta_{j,m} - \delta_{k,m}). \tag{D-80}$$

It is thus evident from the previous analysis, that the modulating frequencies listed in Table (D-1) reduce to expressions of the form given by equation (D-74).

In accordance with equation (D-7), Reference (1), the components of the signal at the output of the first IF amplifer of the AROD airborne receiver due to nonlinearity, are given by the equation

$$S_{i,j,k}(t) = E_{i,j,k} \sum_{n_1, n_2, n_3, n_4 = -\infty}^{+\infty} \Pi_{n_1, n_2, n_3, n_4} \cdot \cos\left[A(t) + (n_1 + n_2 + n_3 + n_4)\frac{\pi}{2}\right], \tag{D-81}$$

where

$$\Pi_{n_1, n_2, n_3, n_4} = J_{n_1}(\lambda\varepsilon_1 F_1) \cdot J_{n_2}(\lambda\varepsilon_2 F_2) \cdot J_{n_3}(\lambda\varepsilon_3 F_3) \cdot J_{n_4}(\lambda\varepsilon_4 F_4), \tag{D-82}$$

$$\begin{aligned} A(t) = [\,&(\omega_{c_i} + \omega_{c_j} - \omega_{c_k})\,t + n_1(\omega_1 t + d_1^{i,j,k}) + n_2(\omega_2 t + d_2^{i,j,k}) \\ &+ n_3(\omega_3 t + d_3^{i,j,k}) + n_4(\omega_4 t + d_4^{i,j,k})] \quad , \end{aligned} \tag{D-83}$$

corresponding to the modulating frequencies listed in Table D-1, and where

J_{n_m} is a Bessel function of the first kind and $n_m{}^{th}$ order with argument $\lambda \varepsilon_m F_m$, for m = 1, 2, 3, and 4, and λ = 32.

REFERENCES

1. R. R. Parker, "AROD Vehicle Transmitted Signal", Technical Memorandum R-12-63-3, A-12-63-17, Brown Engineering Company, Inc., December 12, 1963.