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Summary Report

BOOSTER ATTITUDE STABILIZATION NETWORK SYNTHESIS

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PROPRIETARY INFORMATION

FOREWORD

This is the summary report on the Booster Attitude Stabilization Network Synthesis. This report was prepared by Republic Aviation Corporation under NASA Contract NAS 8-5016 for the National Aeronautics and Space Administration-Marshall Space Flight Center. The work was administered by Nicholas C. Szuchy of Republic Aviation Corporation, and Mr. Mario H. Rheinfurth and Dr. Helmut F. Bauer of the Dynamic Analysis Branch, Aeroballistics Division, NASA-MSFC.

ABSTRACT

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This report presents the impedance synthesis techniques and transfer function factoring methods developed for realization of complex compensation networks for the Saturn Booster. Two major divisions, 1) approximation, and 2) realization, result from the systematic approach to the synthesis of the shaping network meeting NASA requirements. Author

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NOMENCLATURE

C	Capacitor, farads or microfarads
F(s)	General voltage transfer function
$F_D^1(s)$	Desired over-all compensation transfer function with unity DC gain
$F_D(s)$	Desired over-all constant resistance ladder network transfer function with non-unity DC gain
$F^{(i)}(s)$	Transfer function of the i^{th} ladder network with non-unity DC gain; i = 1, 2, 3.
$F_i(s)$	Transfer function of i^{th} stage of ladder network
F_i^*, F_i^{**}	Modified transfer functions of the i^{th} stage of the 2^{nd} and 3^{rd} ladder networks, respectively
j	$\sqrt{-1}$
K_T	Over-all compensation gain for ladder network to give it unity DC gain
K_i	Realizability gain for the i^{th} transfer function, such that $K_i \left[\mathrm{Re}\frac{1}{F_i(j\omega)}\right]_{min} = 1.0$
K_{T_i}	Compensation gain for i^{th} stage of ladder network to give it a unity DC gain
L	Inductor, henries
$L_{N,D}(\tau)$	First order transfer function (note: N, D subscript refers to numerator or denominator respectively)
$m_{1,2}(s)$	Even part of numerator and denominator, respectively
$n_{1,2}(s)$	Odd part of numerator and denominator, respectively
PR	Positive real
PRNM	Positive real, non-minimum
PRM	Positive real, minimum
PRMR	Positive real, minimum resistance

$Q_{N,D}$ $\dfrac{\zeta_{N,D}}{\omega_{N,D}}$	Second order transfer function
Re	Real part of a complex function
R(s)	Richards function used in Bott-Duffin synthesis procedure
R	Resistor, ohms
s	Laplace variable
$\underline{\bar{X}}_1$	Absolute value of the imaginary part of Z at $\omega = \omega_1$, $\lvert \mathrm{Im}\ Z(j\omega_1) \rvert$
Y(s)	General admittance function, Y(s) = Z(s)
Z	General complex impedance function
Z_{a_i}, Z_{b_i}	Branch impedances of the i^{th} stage of the ladder network for a unity load resistance
Z_{a_i}, Z_{b_i}	Branch impedances of the i^{th} stage of the ladder network for the actual load resistance of 800 ohms
τ	Time constant
$\zeta_{N,D}$	Damping factor
$\omega_{N,D}$	Undamped natural frequency
ω	Angular frequency
ω_1	Angular frequency at which Re $\dfrac{1}{F(j\omega)}$ is a minimum

SECTION I

INTRODUCTION

Attitude control of space vehicles is generally required to enforce a predetermined flight path. The guidance system provides information about necessary maneuvering and defines a required attitude for the vehicle. With the evolution of large liquid-fueled rocket booster vehicles, additional complexities in the problem of attitude stabilization have resulted primarily from the effects of fuel sloshing and body bending.[1,2] Minimizing these effects by solely mechanical means normally results in excessive penalties in weight and system complexity. However, by introducing phase-shaping networks [3] in the stabilization loop, stability problems are resolved efficiently while avoiding the penalties previously mentioned.

Theoretical analysis of the system transfer functions permits the determination of stability regions and points of optimum stability required for the phase-shaping network. Two major divisions, 1) approximation, and 2) realization, result from the systematic approach to the synthesis of the shaping network. The approximation area is fundamentally concerned with the methods for determining rational functions approximating the required performance characteristics of the desired network within the constraints of the appropriate realizability conditions. Realization techniques are then used to find explicit networks that are described by physically realizable rational functions. Individually then, the divisions each form an essential step, which collectively correlated form a technique for the total synthesis of the Saturn Booster Attitude Stabilization Network.

The performance characteristics of the phase-shaping network are defined in Reference (2). Through the judicious use of the flexible character of the approximation problem, the given attenuation curve is approximated by the addition of a finite number of semi-infinite slopes, each of which in turn is closely approximated by the attenuation curve of a Butterworth or Tschebyscheff function. The interrelationships between the attenuation and phase requirements, tolerances, physical realizability, ease of construction and alignment are considered individually and collectively in establishing the appropriate rational transfer function.

The realization problem of the shaping network is concerned with the purpose of defining a suitable optimum combination of linear, passive, lumped networks in order to realize the prescribed rational transfer function. Three general synthesis methods are known that will lead to a network configuration, namely:

1) Brune procedure
2) Darlington procedure
3) Bott-Duffin procedure.

The fourth procedure, that of Miyata, is restricted in the sense that it cannot realize every realizable impedance; however, it is often useful as an alternative to the Bott-Duffin procedure for obtaining networks without transformers.

The non-uniqueness aspect of circuit-synthesis allows for an infinite number of circuits which may have the same response or function at specified points of access, while still satisfying the requirements at the terminals. The synthesis technique used for the Saturn Booster Attitude Stabilization Network considers the makeup of the network as a tandem connected sequence of constant resistance sections, each one of which imposes constraints on the prescribed rational transfer function. Then the required network is synthesized by any one of the general procedures, and the network configurations containing fewer elements are chosen. Figure I-1 represents conceptually the over-all procedures used for the network synthesis operation.

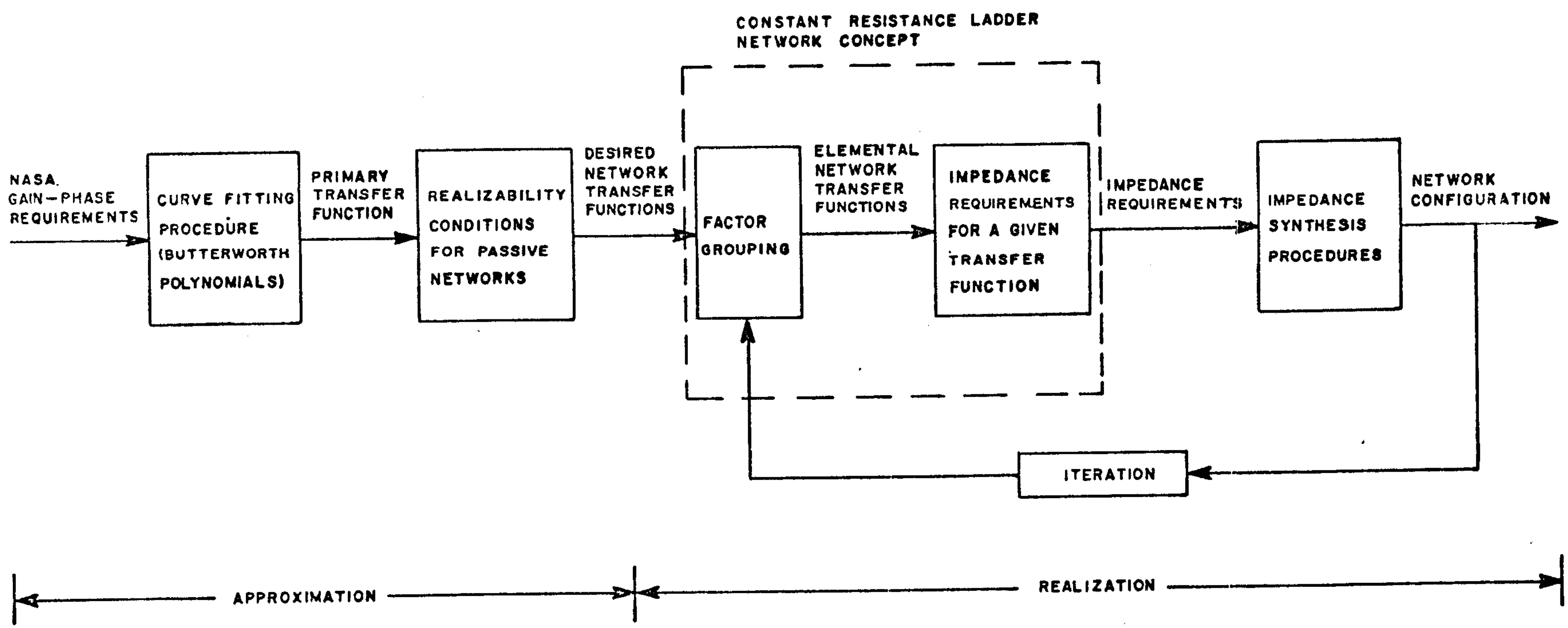


Figure I-1. Block Diagram of Over-All Network Synthesis Operation

SECTION II

APPROXIMATION

The following items will be discussed in this section:

- Piecewise linear approximation
- Butterworth Polynomials
- Tschebyscheff Polynomials
- Approximation of NASA specification

A. PIECEWISE LINEAR APPROXIMATION

By considering an impedance function Z(s), it may be noted that the amplitude (magnitude) function $|Z(j\omega)|$ is an even power of the frequency, i.e.

$$Z(s) = \frac{\sum_{l=0}^{\ell} a_{\ell} s^{\ell}}{\sum_{m=0}^{m} a_m s^m} = \frac{\prod_{i=0}^{i} L_N(\tau_i) \prod_{j=0}^{j} Q_N\left(\frac{\zeta_{Nj}}{\omega_{Nj}}\right)}{\prod_{k=0}^{k} L_D(\tau_k) \prod_{h=0}^{h} Q_D \frac{\zeta_{Dh}}{\omega_{D_h}}} \tag{II-1}$$

or
$$Z(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} \,. \tag{II-2}$$

The m's and n's represent the even and odd part, respectively, of the numerator and denominator. When $s = j\omega$, the m's will be real, while the n's will be imaginary. Separating Z(s) into its even and odd part results in

$$Z(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} \bullet \frac{m_2(s) - n_2(s)}{m_2(s) - n_2(s)}$$

$$= \frac{m_1 m_2(s) - n_1 n_2(s)}{m_1^{\,2}(s) - n_2^{\,2}(s)} + \frac{m_2 n_1(s) - m_1 n_2(s)}{m_2^{\,2}(s) - n_2^{\,2}(s)} \,. \tag{II-3}$$

or semi-infinite slopes. Since the transfer function is characterized as a quotient of two finite degree polynomials, the mathematical expression for the semi-infinite slope approximation must be a rational function. The system or transfer function is constructed from the amplitude specification, then checked to see that both amplitude and phase specifications are satisfied. Adjustments to the transfer function are made in two ways, 1) by modifying semi-infinite slope approximations, and 2) by adding particular gain and phase characteristics to the derived system function outside the system frequency response requirement. Normally both methods are judiciously used to satisfy the amplitude and phase specifications.

The approximation problem then is the determination of a system function that on one hand approximates the given requirement within the specified tolerances, and on the other, is realizable by a network of the desired form. In other words, one has to fit a realizable rational function to the specified data, that is, determine the coefficient of the two polynomials, or equivalently, determine the zeros, poles, and constant multiplier of the rational function. It is also desired that the function be of the lowest possible order so that a small number of elements will be required for its realization. The Butterworth polynomials and the Tschebyscheff polynomials are known to possess the desired properties and consequently are used to approximate the magnitude of the transfer functions.

B. BUTTERWORTH POLYNOMIALS

The magnitude requirement is empirically approximated by semi-infinite slopes for which a mathematical expression has to be derived. The choice of Butterworth functions is a logical suggestion, because:

1) for large values of ω the function approaches the semi-infinite slope as its asymptote;
2) from the theory of filter design their roots are known and tabulated;
3) they actually fit semi-infinite slopes very smoothly by a margin of not more than 3 db.

The substitution of $s = j\omega$ into an even polynomial gives a real number, while the substitution into an odd polynomial gives an imaginary number, i.e.,

$$\text{Even part of } Z(s) \Big|_{s=j\omega} = \text{Ev}\, Z(s) \Big|_{s=j\omega} = \text{Re}\, Z(j\omega) = U(\omega^2) \tag{II-4}$$

$$\text{Odd part of } Z(s) \Big|_{s=j\omega} = \text{Od}\, Z(s) \Big|_{s=j\omega} = j\,\text{Im}\, Z(j\omega) = j\, V(\omega). \tag{II-5}$$

Therefore the amplitude function of Z(s) is described by an expression of the form

$$\left| Z(j\omega) \right| = \left[\frac{\sum_{\ell=0}^{\ell} a_{2\ell}\,\omega^{2\ell}}{\sum_{m=0}^{m} b_{2m}\,\omega^{2m}} \right]^{\frac{1}{2}} . \tag{II-6}$$

It follows from an examination of the above expression that the asymptotic behavior of any physically realizable impedance is characterized by an even power of frequency, or

$$\lim_{\omega \to \infty} |Z(j\omega)|^2 = \text{constant x } \omega^{2n} \tag{II-7}$$

where n is positive or negative integer. To this corresponds a gain (or attenuation) measured in decibels of

$$A(\omega^2) = 10 \log_{10} |Z(j\omega)|^2 = \text{constant} + 20\, n \log_{10} \omega . \tag{II-8}$$

If one plots $A(\omega^2)$ vs $\log_{10} \omega$ - which is conveniently done on semilog paper (Bode Plot) - the curve will be a straight line. This straight line, or semi-infinite slope, is the asymptote to the curve of the amplitude function, and has a slope of $6\,n \frac{\text{db}}{\text{octave}} = 20\,n \frac{\text{db}}{\text{decade}}$. Any other slope cannot be approximated with a finite number of terms, or physically realized by a network with a finite number of components. The above considerations therefore limit the available semi-infinite slopes only to those including an angle of $6\,n \frac{\text{db}}{\text{oct}}$ with the abscissa.

The Bode plot of the NASA specification for the phase stabilization network is first approximated as a finite number of piecewise linear lines

The Butterworth function of order 2n is given by

$$B_{2n}^{2}(\omega^{2}) = 1 + \omega^{2n}. \tag{II-9}$$

For large values of ω

$$B_{2n}^{2}(\omega^{2}) \approx \omega^{2n}. \tag{II-10}$$

A semi-infinite slope of 6n db/octave, but with a cutoff frequency different from unity (1), obviously is approximated by

$$B_{2n}(\omega^{2}) = 1 + \left(\frac{\omega}{\omega_0}\right)^{2n}. \tag{II-11}$$

The roots of the Butterworth function of $B_{2n}(\omega^{2})$ are the 2n roots of (-1). This follows directly from noting that the amplitude response $B_{2n}(\omega^{2})$ and the complex system function $T(j\omega)$ are related by

$$B_{2n}^{2}(\omega^{2}) = T(j\omega)\,T(-j\omega). \tag{II-12}$$

Defining a new function $h(s^{2})$ such that

$$h(s^{2}) = T(s)\,T(-s), \tag{II-13}$$

it may be noted that

$$B_{2n}(\omega^{2}) = h(-\omega^{2}). \tag{II-14}$$

From $h(-\omega^{2})$ all that has to be done is to substitute $s^{2} = -\omega^{2}$ to get $h(s^{2})$. Then $h(s^{2})$ is factored into the product $T(s)\,T(-s)$. Since the poles and zeros of $T(s)$ are the mirror images of the poles and zeros of $T(-s)$, i.e., they form symmetrical patterns on the unit circle about the origin in the s-plane, one simply chooses the Hurwitz factors of $h(s^{2})$ as $T(s)$.

An example will be used to clarify the above discussion. Consider the third order (n = 3) Butterworth response given by

$$B_{2n}^{2}(\omega^{2}) = B_{6}^{2}(\omega^{2}) = \frac{1}{1+\omega^{6}} = \frac{1}{1-(-\omega^{2})^{3}} \tag{II-15}$$

$$\therefore\ h(s^2) = \frac{1}{1-(s^2)^3} \quad . \tag{II-16}$$

Factoring $h(s^2)$, one obtains

$$h(s^2) = \left(\frac{1}{1+2s+2s^2+s^3}\right)\left(\frac{1}{1-2s+2s^2-s^3}\right) \tag{II-17}$$

$$= T(s)\ T(-s), \tag{II-18}$$

therefore

$$T(s) = \frac{1}{s^3+2s^2+2s+1} \tag{II-19}$$

$$= \frac{1}{(s+1)\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} \ . \tag{II-20}$$

The poles of $T(s)$ and $T(-s)$ are shown in Figure II-1. Note that the poles of $T(-s)$ are the mirror images of the poles of $T(s)$.

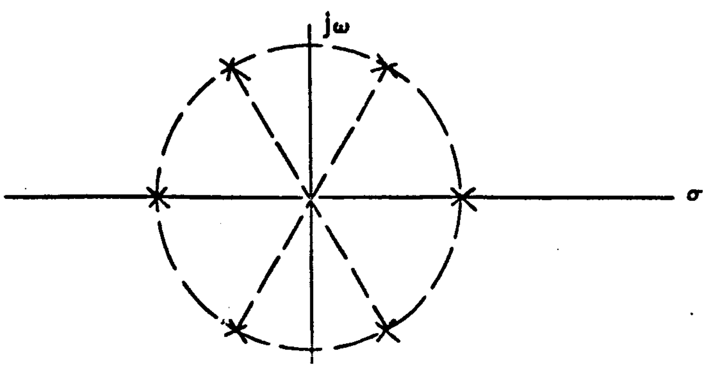


Figure II-1. Poles For the Butterworth Function, n = 3

For a Butterworth response, the poles $T(s)\ T(-s)$ are the roots of

$$(-1)^n s^{2n} = -1 \tag{II-21}$$

$$= \epsilon^{j\Pi(2K-1)} \qquad K = 0,\ 1,\ 2,\ \ldots,\ 2n \tag{II-22}$$

The distinct s values are then given by

$$s_K = \epsilon^{j\pi\left[\frac{2K-1}{2n}\right]} \qquad \text{for n even} \tag{II-23}$$

$$s_K = \epsilon^{j\pi\left(\frac{K}{n}\right)} \qquad \text{for n odd} \tag{II-24}$$

or in general,

$$s_K = \epsilon^{j\pi\left[\frac{2K+n-1}{2n}\right]} \qquad \text{for } K = 0,\ 1,\ 2,\ \ldots,\ 2n. \tag{II-25}$$

Expressing s_K as $s_K = \sigma_K + j\omega_K$, the real and imaginary parts are given by

$$T_K = \cos \frac{(2K+n-1)}{2n}\, \pi = \sin\left(\frac{2K-1}{n}\right)\frac{\pi}{2}, \tag{II-26}$$

$$\omega_K = \sin \frac{(2K+n-1)}{2n}\, \pi = \cos\left(\frac{2K-1}{n}\right)\frac{\pi}{2}, \tag{II-27}$$

It may be noted from the above that all the poles of T(s) T(-s) are located on the unit circle in the s-plane, and are symmetrical about both the σ and the $j\omega$ axes. To satisfy the realizability condition, one associates the poles in the right-half plane with T(-s) and the poles in the left-half plane with T(s). To simplify the use of the Butterworth functions, T(s) is given in Tables II-1 and II-2 for $n = 1$ to $n = 8$, in factored form or in polynomial form.

Table II-1. Coefficients of Butterworth Polynomials

n =	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1							
2	1.414	1						
3	2	2	1					
4	2.613	3.414	2.613	1				
5	3.236	5.236	5.236	3.236	1			
6	3.864	7.464	9.141	7.464	3.864	1		
7	4.494	10.103	14.606	14.606	10.103	4.494	1	
8	5.126	13.138	21.848	25.691	21.848	13.138	5.126	1

Table II-2. Factors of Butterworth Polynomials

n =	$P_n(\lambda)$
1	$(1+\lambda)$
2	$(1+1.4142\lambda+\lambda^2)$
3	$(1+\lambda)\ (1+\lambda+\lambda^2)$
4	$(1+0.7653\lambda+\lambda^2)\ (1+1.8477\lambda+\lambda^2)$
5	$(1+\lambda)\ (1+0.6180\lambda+\lambda^2)\ (1+1.6180\lambda+\lambda^2)$
6	$(1+0.5176\lambda+\lambda^2)\ (1+1.4142\lambda+\lambda^2)\ (1+1.9318\lambda+\lambda^2)$
7	$(1+\lambda)\ (1+0.4449\lambda+\lambda^2)\ (1+1.2465\lambda+\lambda^2)\ (1+1.8022\lambda+\lambda^2)$
8	$(1+0.3896\lambda+\lambda^2)\ (1+1.1110\lambda+\lambda^2)\ (1+1.6630\lambda+\lambda^2)\ (1+1.9622\lambda+\lambda^2)$

C. TSCHEBYSCHEFF POLYNOMIALS

A rational-function approximation of the desired finite frequency specification can often be found by using a particular set of orthogonal polynomials known as Tschebyscheff Polynomials of the first kind, defined as:

$$C_n(\omega) = \cos(n \cos^{-1} \omega) \qquad |\omega| \leq 1 \tag{II-28}$$

$$= \cosh(n \cosh^{-1} \omega) \qquad |\omega| > 1 \tag{II-29}$$

and possessing the orthogonality relation

$$\int_{-1}^{1} C_n(\omega)\, C_m(\omega) \frac{d\omega}{\sqrt{1-\omega^2}} = \begin{cases} 0 & m \neq n \\ \frac{\Pi}{2} & m = n \neq 0 \\ \Pi & m = n = 0 \,. \end{cases} \tag{II-30}$$

For $n = 0$

$$C_0(\omega) = 1 \,. \tag{II-31}$$

For $n = 1$

$$C_1(\omega) = \cos(\cos^{-1} \omega) = \omega \,. \tag{II-32}$$

Higher order Tschebyscheff polynomials are obtained through the recursion formula

$$C_n(\omega) = 2\, \omega\, C_{n-1}(\omega) - C_{n-2}(\omega) \,. \tag{II-33}$$

Thus for $n = 2$

$$C_2(\omega) = 2\, \omega C_1(\omega) - C_0(\omega) \tag{II-34}$$

$$= 2\, \omega^2 - 1 \,. \tag{II-35}$$

To simplify the use of Tschebyscheff functions, $C_n(\omega)$ is given in Table II-3 for $n = 1$ to $n = 10$.

Table II-3. Tschebyscheff Polynomials of Order 1 to 10

n	$T_n(\omega)$
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1,280\omega^8 + 1,120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

It may be noted from Table II-3 that the Tschebyscheff function $C_{2n}(\omega^2)$ of order 2n is a polynomial in ω^2 of highest power 2n, in which the coefficients that are chosen to make the function oscillate between plus and minus one within the interval $-1 < \omega < +1$. For $|\omega| > 1$, the function assumes rapidly increasing values.

Applying Tschebyscheff polynomials to the approximation problem results from a consideration of the function $\epsilon^2 C_n^2(\omega)$, where ϵ is real and small compared to 1. It may be noted that $\epsilon^2 C_n^2(\omega)$ will vary between 0 and ϵ^2 in the interval $|\omega| \leq 1$. The function

$$C_n^2(\omega^2)' = 1 + \epsilon^2 C_n^2(\omega) \qquad \text{(II-36)}$$

where n is a positive integer, obviously oscillates between 1 and $1 + \epsilon^2$, within the same interval $-1 < \omega < +1$. The cutoff of function of this type is much steeper than that of the Butterworth functions.

The roots of the functions $C_n^2(\omega^2)'$ (derived from Tschebyscheff functions) are known[4], and may be obtained graphically from the root star of Butterworth function of the same order, as shown in Figure II-2. Each root vector is prolonged to its intersection with circles of radii a and b, and the points of intersection are projected horizontally and vertically on the ellipse with the long axis 2b, and the short axis 2a, where

$$a = \cosh \left[\frac{\cosh^{-1} (1/\epsilon)}{2n} \right]$$

$$b = \sinh \left[\frac{\cosh^{-1} (1/\epsilon)}{2n} \right] . \qquad \text{(II-37)}$$

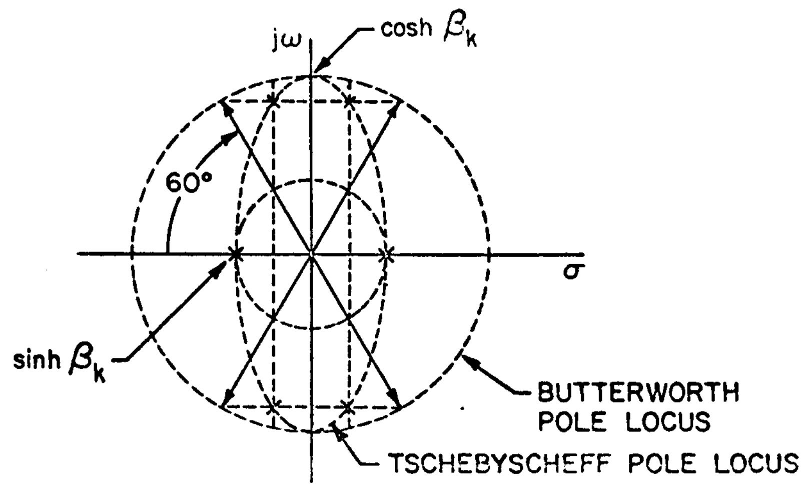


Figure II-2. Poles for the Tschebyscheff Function, n = 3

From the above figure it is apparent that the Butterworth approximation is a degenerate form of the Tschebyscheff approximation in which the ellipse becomes a circle.

D. APPROXIMATION OF NASA SPECIFICATIONS

1. General

This section describes in detail the procedure that was used to select a rational function, identifiable as the response function of a realizable network which approximates the NASA specified magnitude and phase characteristics. The very nature of the way the requirements were defined immediately suggested a graphical or semi-graphical technique. The approximation problem is solved in a systematic manner, in essentially four steps, yet retains the flexibility needed for modifications to do the interrelationship between the attenuation and phase requirement.

First, the gain and phase specifications are plotted on a decibel versus logarithmic frequency scale and angle versus logarithmic frequency scale curves. Second, a curve satisfying the attenuation requirement is approximated by a succession of straight lines. Third, the corresponding mathematical expression is developed. Fourth, the continuous curves resulting from the mathematical expressions are plotted and checked against the attenuation and phase requirements. Then any corrections necessary to meet either attenuation or phase requirements are used to modify the approximating expression and it is again checked against the specification. The iterative process converges rapidly to a satisfactory mathematical expression. It may be noted that by the use of sufficiently large number of straight-line approximations, the approximation to any continuous curve could be made as close as required.

2. NASA Specifications

Figure II-3 shows the required NASA specification; the boxes represent the maximum phase stability requirement while the attenuation is indicated by dotted lines, on a "Bode plot." Consider the succession of semi-infinite slopes S_1, S_2, S_3 in Figure II-4 limited only to those including an angle of 6n db/octave with the horizontal axis. It is evident that, by a simple addition of these slopes, the broken line curve A is obtained, which may be considered as an approximating curve satisfying the requirements.

A semi-infinite slope of 6n db/octave, with a cut-off frequency different from unity, can be approximated by a Butterworth function of order 2n of the form

$$B_{2n} = 1 + \left(\frac{\omega}{\omega_0}\right)^{2n} \qquad \text{(II-38)}$$

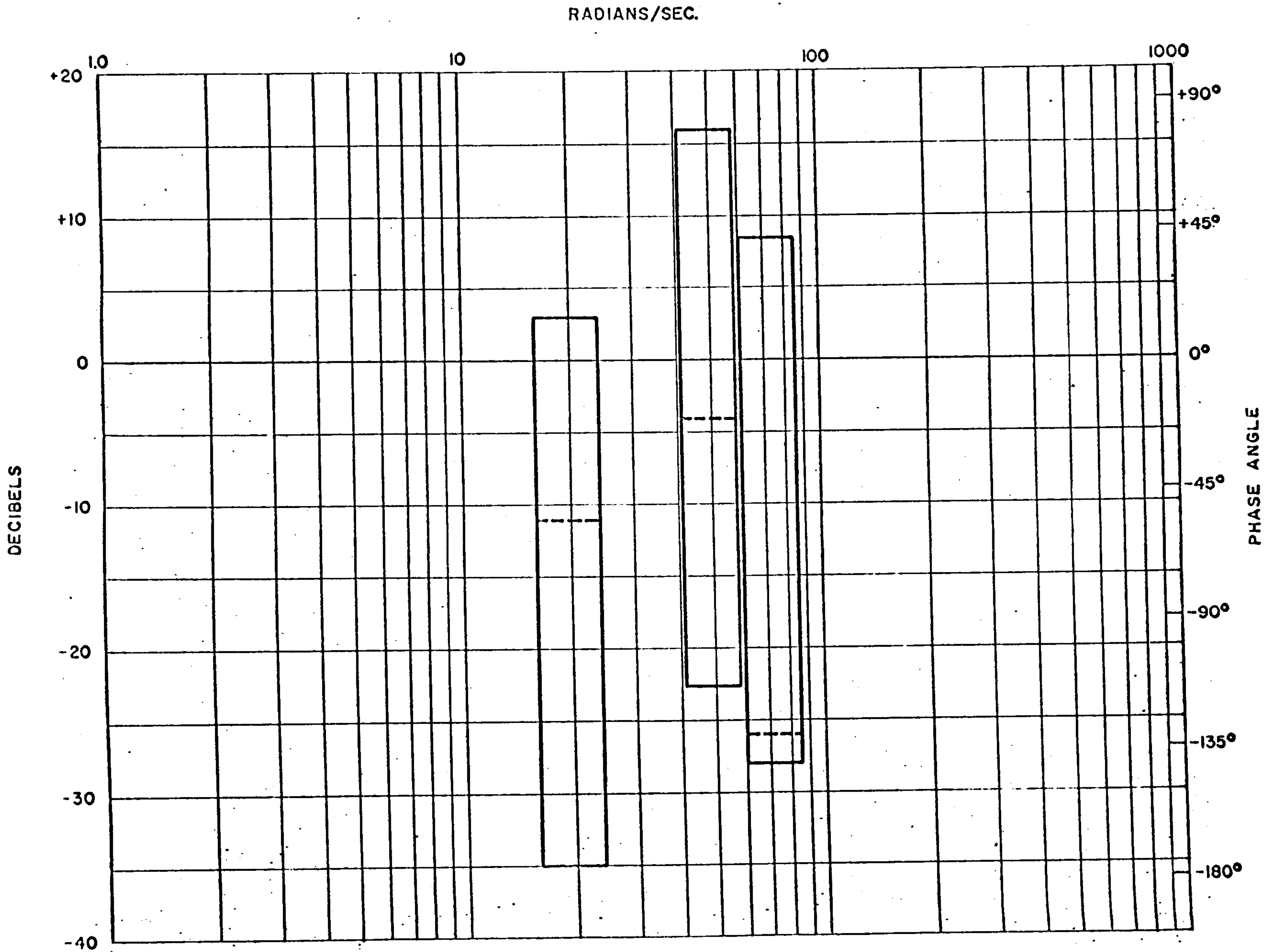


Figure II-3. NASA Specifications

where

B_{2n} = Butterworth function of order 2n

ω = frequency in radians

ω_0 = break frequency.

The three slopes of Figure II-4 have angles of -6, 18, and -36 db, and cut-off points of 4, 20 and 45 radians, respectively. The corresponding attenuation function is

$$|F(j\omega)|^2 = \frac{\left[1+\left(\frac{\omega}{20}\right)^6\right]}{\left[1+\left(\frac{\omega}{4}\right)^2\right]\left[1+\left(\frac{\omega}{45}\right)^{12}\right]} \qquad \text{(II-39)}$$

The roots of the Butterworth functions are the 2n roots of minus one (-1). Table II-1 tabulates all coefficients for Butterworth functions up to the eighth order. For synthesis problems it is more convenient to group only conjugate complex roots together; they are presented in this fashion in Table II-2. Using Table II-2, the transfer function corresponding to $|F(j\omega)|^2$ can be written as

$$F_1(s) = \frac{\left[\frac{s}{20}+1\right]\ \frac{S^2}{20^2}+\frac{2(0.5)}{20}S+1}{\left[\frac{s}{4}+1\ \ \frac{s^2}{45^2}+\frac{2(0.26)}{45}sH\ \ \frac{s^2}{45^2}+\frac{2(0.71)}{45}s+1\ \ \frac{s^2}{45^2}+\frac{2(0.965)}{45}s+1\right.} \qquad \text{(II-40)}$$

Since in the present application the phase angle is of importance, the phase angle of the above function is plotted and compared with the specification as shown in Figure II-5.

By applying lead, lag and quadradic correction factors outside the response of the system, a mathematical representation of the transfer function is developed

$$F_2(s) = \frac{Q\left(\frac{0.5}{20}\right)}{Q\left(\frac{0.7}{45}\right)} \times \frac{Q\left(\frac{0.4}{126}\right)}{Q\left(\frac{0.96}{45}\right)} \times \frac{Q\left(\frac{0.4}{126}\right)}{L(4)\ L(2000)} \times \frac{L(20)}{Q\left(\frac{0.26}{45}\right)} \times \frac{Q\left(\frac{0.4}{126}\right)}{L(2000)} \qquad \text{(II-41)}$$

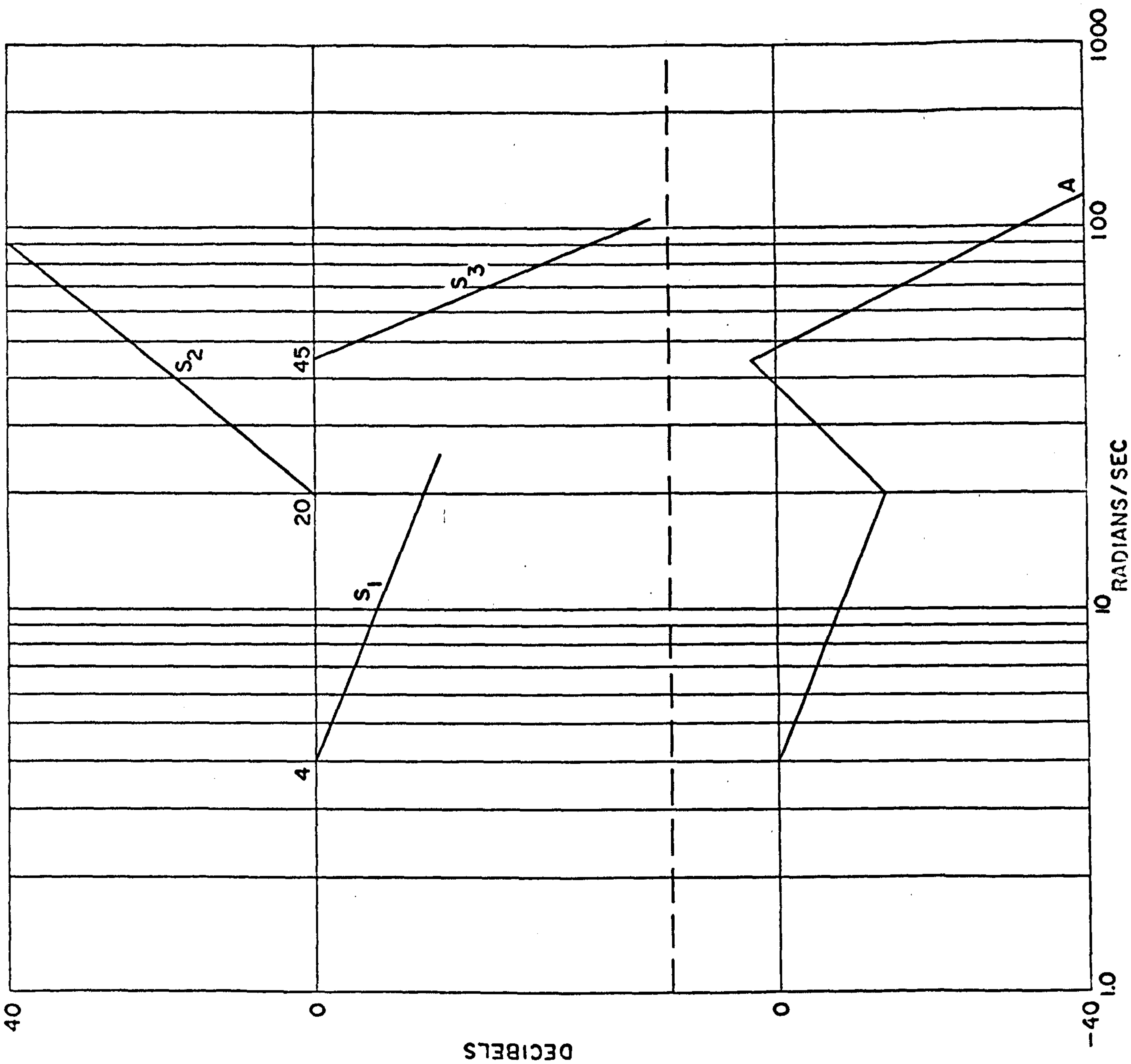


Figure II-4. Graphical Approximation

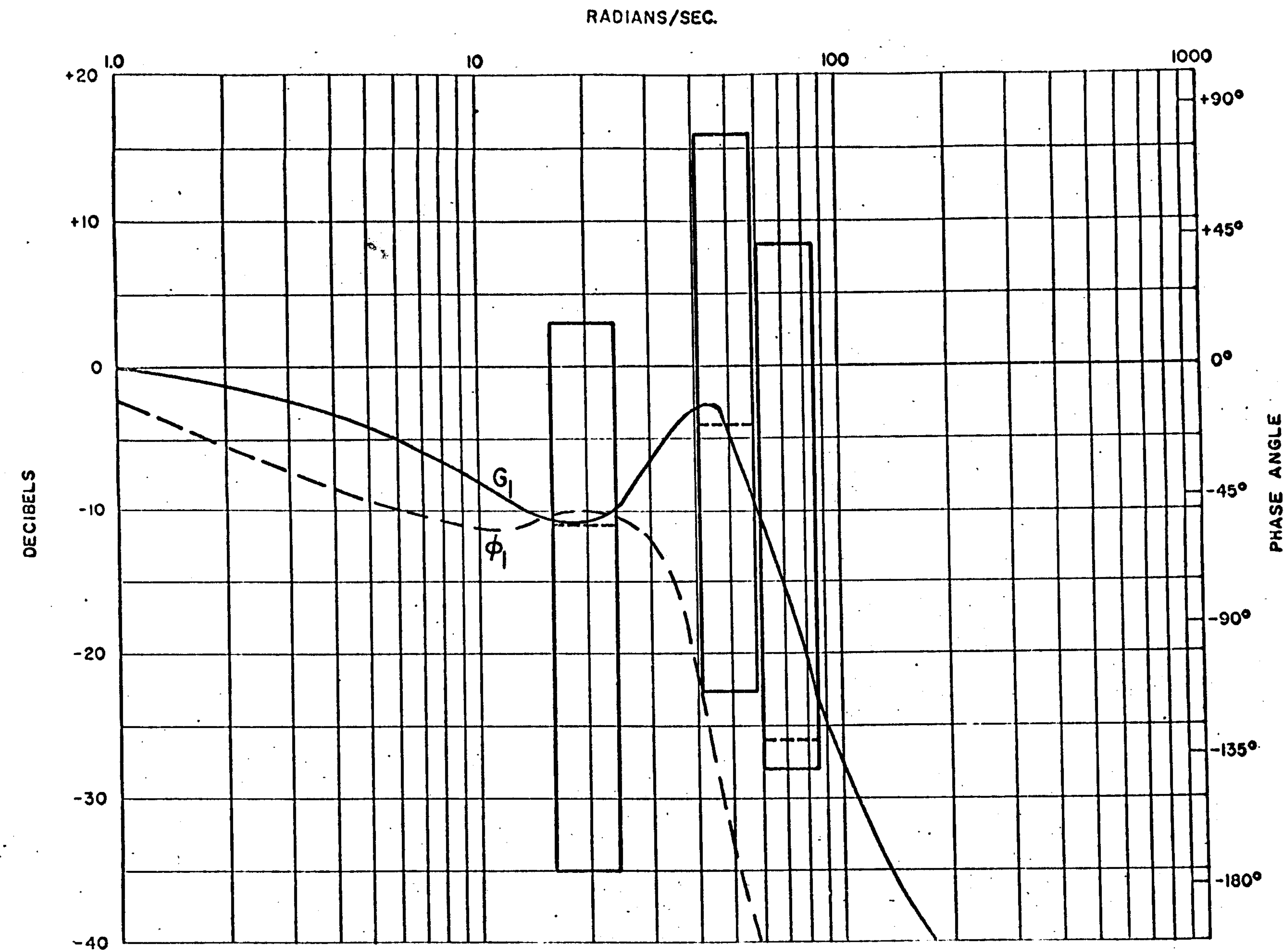


Figure II-5. First Continuous Function Approximation of NASA Specifications that satisfies the NASA specifications, as shown in Figure II-6.

The curve-fitting procedure outlined above gives rise to a transfer function which possesses the desired gain-phase characteristics. However, it must now be ensured that this function is realizable by a passive network. The requirements for this assurance are: 1) the transfer function must be stable; 2) the degree of the numerator of the transfer function must be less than or equal to the degree of the denominator. The first requirement is ensured by the nature of the curve-fitting procedure. The second requirement, however, is not usually met by the function resulting from the curve-fitting procedure. The method of meeting this requirement consists of adding simple lag-type factors to the transfer function. These factors are chosen so that they do not appreciably alter the gain-phase characteristics in the region of interest. The transfer function consisting of the terms obtained from the curve fitting with the appropriate factors for realizability by a passive network appended, is termed the desired network transfer function.

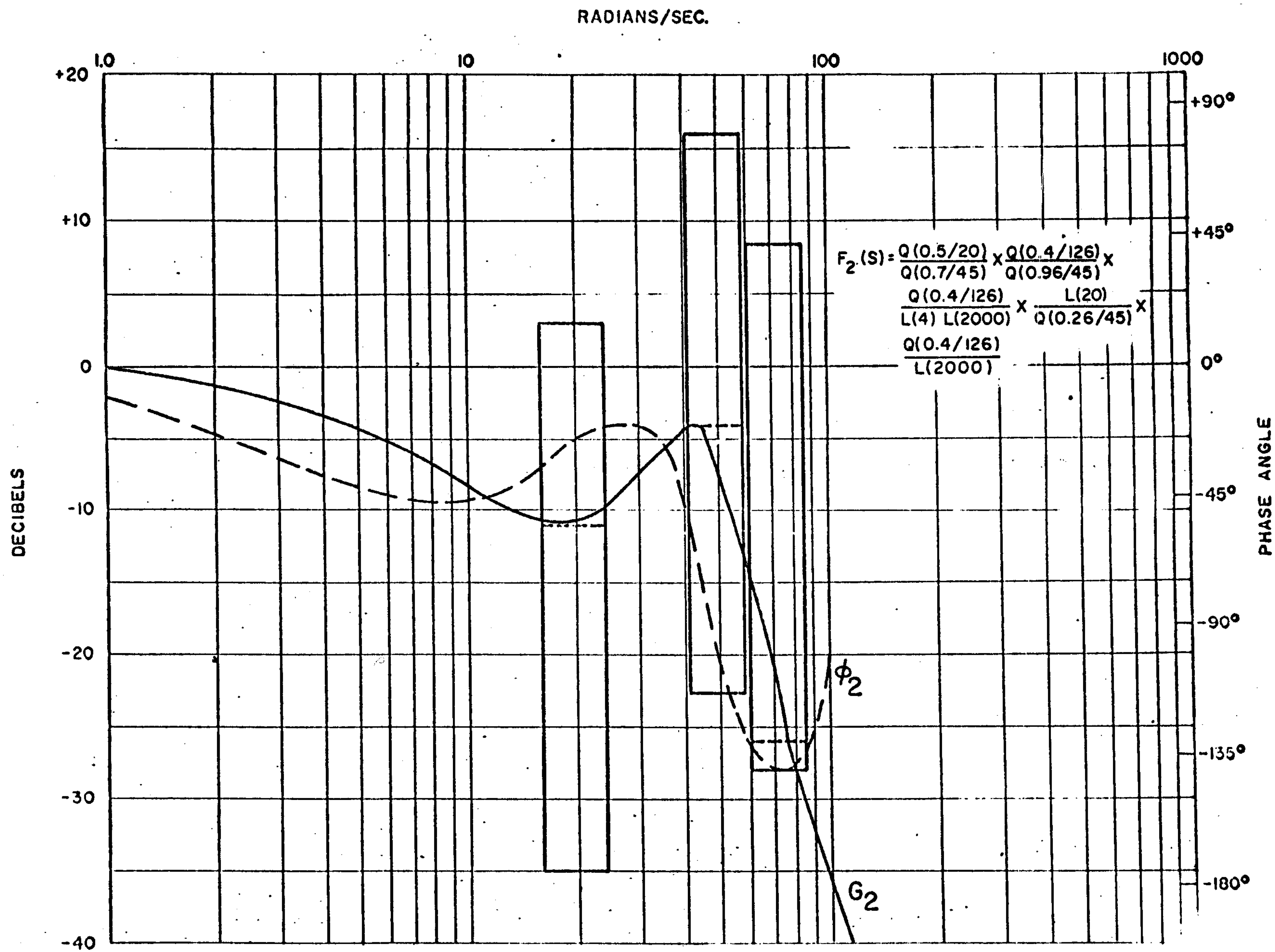


Figure II-6. Curve Satisfying NASA Specifications

SECTION III

REALIZATION

It is observed that the character of the NASA requirements requires that the desired network transfer function be of high order, i.e., about ninth order. The synthesis of a network realizing a transfer function of this complexity becomes quite unwieldy by conventional techniques. An even more difficult problem is the determination of the effect of component tolerances on the network transfer function. For these reasons the constant resistance network approach was chosen. These networks have the property that one stage does not load another. Thus, the stages may be separately synthesized with component transfer functions that are the desired factors of $F_D'(s)$. The most complex form of these factors is a quadratic over a quadratic, termed a biquadratic.

Of the impedance topologies which may be made to exhibit this constant resistance behavior, the ladder configuration was chosen because of the following advantages:

1) relatively low sensitivity of the particular transfer function to component tolerances;

2) desirability of a common ground.

In order to be realizable in the form of a number of cascaded stages of a constant resistance ladder network, the transfer function must be broken down into factors which are realizable as passive networks and which possess in addition the following properties:

1) they must be minimum phase (no zeros in the right half plane);
2) they can have no poles on the imaginary axis;
3) they must be positive real, non-minimum (PRNM).

As described in Appendix A, the necessary and sufficient conditions for a rational function ($F(s) = N(s)/D(s)$) with real coefficients to be positive real non-minimum are that F(s) has no right half plane poles, that F(s) has only simple poles on the imaginary axis with positive and real residues, and that $\mathrm{Re}F(j\omega) > 0$ for all ω. Appendix A gives requirements on the coefficients of the realizable linear and quadratic forms of N(s) and D(s) for F(s) to be PRNM.

The first two conditions are ensured by the method of curve fitting (Butterworth Polynomials). The third condition must be achieved either by judicious grouping of the factors of $F'_D(s)$ or by the introduction of realizability factors as explained in Appendix A and Section IV. It is to be noted that, while each of the factors to be realized by one stage of the ladder network must be PRNM, the over-all transfer function does not have to satisfy this requirement.

One other item of great importance in the factor grouping operation is the DC attenuation of the associated ladder network. First, as will be described, a gain factor must be applied to the component transfer function in order that the branch impedances of the associated ladder network stage are PR MR functions,* as required by the impedance synthesis procedures. Amplification must be supplied which compensates for this factor and makes the ladder network DC gain unity. Unless care is exercised in the factoring process, the amplifier gain required will become excessive. A final item is that care must be exercised in order to generally minimize the number of passive elements and keep their values reasonable. The considerations presented above are utilized in the iteration process as shown in Figure I-1.

---

* A PRMR function denotes a positive real, minimum resistance function, $Z(j\omega)$, whose real part vanishes at some ω_1, i.e., $\mathrm{Re}Z(j\omega_1) = 0$.

The synthesis procedures used for the branch impedances of the various ladder network stages are summarized in Table III-1. The types of component transfer functions that will be encountered and the corresponding methods of branch impedance synthesis to be employed are given in this table. This subject will be discussed in detail in the following section.

Table III-1. Synthesis Procedures

Component Transfer Function*	Method of Branch Impedance Synthesis
$\frac{1}{s+a}$, $\frac{s+a}{s+b}$	Inspection
$\frac{s+\omega_L}{s^2+bs+c}$, $\frac{s^2+bs+c}{(s+d)(s+c)}$	Continued fraction expansion
$\frac{s^2+bs+c}{s^2+bs+d}$, $\frac{(s+a)(s+b)}{s^2+cs+d)}$	Bott-Duffin
* a, b, c, d, and e real and positive	

It is noted that, while the factors of the desired transfer function $F_D'(s)$ that are obtained from the approximation process have a unity DC gain, component transfer functions of the form shown in Table III-1 with a non-unity DC gain were realized in the network synthesis operation. Allowance for this difference is provided in the compensation gain (K_T) for the over-all ladder network.

A. CONSTANT RESISTANCE LADDER NETWORKS

1. General

The following items will be discussed:

1) functional relationships between the stage transfer functions and the associated branch impedances of a constant resistance ladder network;
2) cascading of such stages in a ladder network;
3) assumed network load characteristics;
4) realizability conditions on the component or stage transfer functions;

5) determination of the realizability gain factor for these component transfer functions;

6) determination of the required compensation gain for the over-all ladder network.

2. Transfer Function and Branch Impedance Relationships

The structure of the 2-port* network representing one stage of a constant resistance ladder network is presented in Figure III-1.

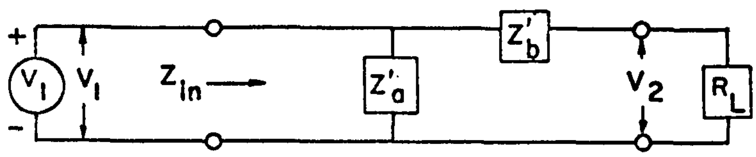


Figure III-1.) Ladder Network

The input impedance of this network, Z_{in}, is readily found to be:

$$Z_{in} = \frac{Z_a'(Z_b' + R_L}{Z_a' + Z_b' + R_L} \quad . \tag{III-1}$$

To satisfy the constant resistance condition, it is required that $Z_{in} = R_L$. The network impedance functions are then related by the following expression:

$$Z_a' = R_L (1 + R_L/Z_b'). \tag{III-2}$$

In order to simplify the initial synthesis considerations, let R_L be normalized to unity so that:

$$Z_a = 1 + Y_b \tag{III-2a}$$

where the admittance $Y_b = Z_b^{-1}$. This, then is the condition for the ladder

* The term "2-port" will mean a network which has two accessible pairs of terminals, i.e., an input and an output pair. A "1-port," will mean a network with one accessible terminal pair, such as Z_a' and Z_b'.

network structure to exhibit a constant resistance property. The network transfer function is:

$$F(s) = \frac{V_2}{V_1}(s) = \frac{R_L}{R_L + Z_b(s)} = \frac{1}{1 + Z_b(s)} . \qquad \text{(III-3)}$$

Again, in the third equality R_L has been normalized to unity.

Equations (III-2) or (III-2a) and (III-3) are the design equations for the constant resistance ladder networks. Given the desired component transfer function, Equation (III-3) is used to obtain the required branch impedance function Z_b. Equation (III-2) is then utilized to obtain the corresponding Z_a.

3. Cascading Stages of a Constant Resistance Ladder Network

Suppose that η ladder stages are cascaded together as shown in Figure III-2.

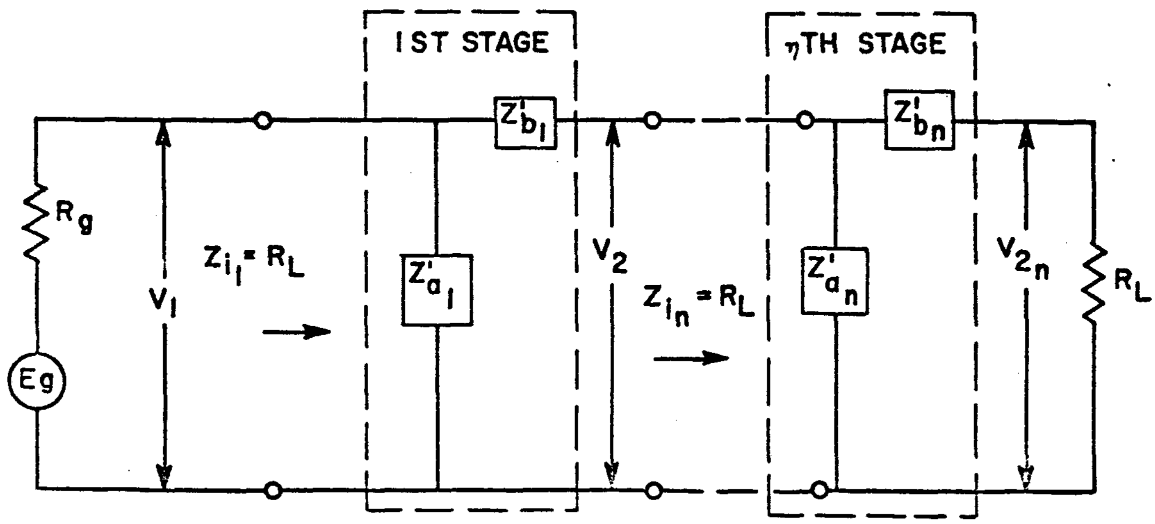


Figure III-2. Cascaded Constant Resistance Ladder Stages

The input impedance looking into any stage is R_L. Hence each stage is loaded by the same resistive load R_L, but each stage has been designed to yield the desired transfer function when loaded by R_L so that, if the network is excited by an ideal generator ($R_g = 0$) and the load resistance is taken as unity, the network transfer function is:

$$F = \frac{V2_{\eta}}{V_1} = \prod_{\lambda=1}^{\eta} F_{\lambda} - F_1 F_2 \cdots F_{\eta}. \tag{III-4}$$

In the general case:

$$F = \frac{V2_{\eta}}{V_1} = \frac{1}{1 + Rg/R_L} \prod_{\lambda=1}^{\eta} F_{\lambda} \tag{III-5}$$

The form of this equation shows the usefulness of using the constant resistance approach in the synthesis of high order transfer functions, i.e., the transfer function may be broken down into a product of simple transfer functions and synthesized "piece by piece." The transfer function must be broken down so that each component transfer function fulfills the realizability conditions of the previous section. This is relatively simple to do, utilizing the methods presented in the appendix.

It is important to note that this ability to build up the network stage by stage is practically very useful. Each stage may be tested and "tuned up" before cascading it so that the effects of component tolerances and non-ideal elements may be taken into account much more simply than if the total transfer function is realized in a stage.

4. Network Load Characteristics

It is seen from the foregoing material that, in order for the method presented to be valid, the network load must be resistive. A more precise statement would be: The load must "look" resistive over the frequency region of interest, i.e., if the network is being synthesized to produce a desired transfer function over a frequency range specified as $0 < \omega \leq \omega_1$, then the load must look resistive over this range. By this is meant that the frequency response of the load must behave as shown in Figure III-3, i.e., the load impedance must not have a significant phase shift in the frequency region of interest. This is the real criterion for applicability of constant-resistance transfer function synthesis.

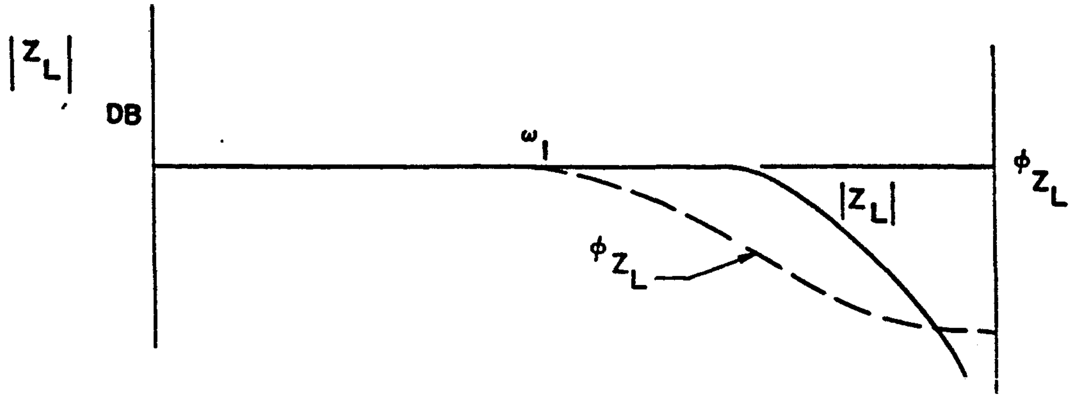


Figure III-3. Load Impedance Frequency Response

For the network synthesis work conducted under this contract, the load impedance has been taken as resistive with $R_L = 800$ ohms.

5. Realizability Conditions on the Stage Transfer Functions

The necessary and sufficient conditions which a component transfer function must satisfy in order that it can be realized as a voltage transfer function for one stage of a constant resistance ladder network are next discussed.

a) Necessary Conditions

$$|F(j\omega)| = \left|\frac{V_2}{V_1}(j\omega)\right| \leq 1 \text{ for all } \omega.$$

This can be seen from Equation (III-3) since $F(j\omega) = \frac{1}{1+Z_b(j\omega)}$ while $Z_b = \alpha + j\beta$ with $\alpha \geq 0$ for all ω by virtue of the fact Z_b must be PR. Hence, $|F| = \frac{1}{\left[(1+\alpha)^2 + \beta^2\right]^{\frac{1}{2}}} \leq 1$ for all ω. Note that this condition implies that F can have no poles, finite or infinite, on the imaginary axis.

b) Sufficient Conditions

From Equation (III-3):

$$Z_b = \frac{1}{F} - 1 \qquad \text{(III-6)}$$

To meet nominal realizability standards for a general ladder network with assured stability of operation, it is first required that F be a minimum phase function with no poles in the right half plane (unstable poles). Now if Z_b is to be PR, then Re $Z_b\,(j\,\omega) \geq 0$ for all ω; hence, Equation (III-3) requires that

$$\mathrm{Re}\left[\frac{1}{F}\right] = \mathrm{Re}\left[\frac{F\,(j\,\omega)}{\left|F\,(j\,\omega)\right|^2}\right] \geq 1\,. \tag{III-7}$$

Equation (III-7) can be satisfied only if:

1) $\mathrm{Re}\left[F\,(j\,\omega)\right] > 0$;

2) a constant K is chosen such that

$$\mathrm{Re}\left[\frac{K}{F\,(j\,\omega)}\right] \geq 1 \text{ for all } \omega\,. \tag{III-7a}$$

All such realizability gain factors which must be introduced for the component transfer functions to make the associated branch impedances realizable must be accounted for by a compensating gain factor for the over-all ladder network.

c) Summary

If a transfer function F is to be synthesized by a constant resistance ladder stage, the necessary and sufficient conditions on the transfer function may be stated as follows:

F must be a positive real function (hence minimum phase with no unstable poles) with a non-vanishing real part, and with no poles, finite or infinite, on the imaginary axis and a gain factor must be applied to it such that Equation (III-7a) is satisfied. Actually, K is chosen so that $K\,\mathrm{Re}\left[\frac{1}{F(j\,\omega)}\right]_{\text{minimum}} = 1$ to simplify the network synthesis operation.

Table II-1 and Appendix A enumerate the allowable forms for F to be a positive real, non-minimum (PRNM) function as required. Only rational functions in the Laplace variable s, with at most a 2nd order numerator or denominator, are considered.* This is because these are the types of factors introduced in the over-all transfer function by the approximation technique used for meeting the gain-phase requirements. The following subsection defines the procedures to be followed for determining the required realizability gain factor F, i.e., a K such that $K \,\mathrm{Re}\left[\frac{1}{F j\omega}\right]_{\text{minimum}} = 1.$

6. Determination of the Realizability Gain Factor for the Stage Transfer Functions

For first order component transfer functions, i.e., $F(s) = \frac{1}{s+a}$ or $\frac{s+a}{s+b}$, the choice of a positive value for K such that $K \left| \frac{1}{F(j\omega)} \right|_{\text{minimum}} = 1$ is equivalent to the value for K satisfying $K \,\mathrm{Re}\left[\frac{1}{F(j\omega)}\right]_{\text{minimum}} = 1.$ If $F_1(s) = \frac{1}{s+a}$, then the required value for K_1 is $\frac{1}{a}$. If $F_2(s) = \frac{s+a}{s+b}$, then if $a > b$, $K_2 = \frac{a}{b}$, while if $a < b$, $K_2 = 1$. The branch impedance functions for such simple stage transfer functions, given by $Z_b(s) = \frac{K}{F(s)} - 1$ and $Z_a(s) = 1 + \frac{1}{Z_b(s)}$, can be determined by inspection.

Formulas are next derived for the realizability gain factor K for the following component transfer functions:

---

* It is again observed that the order of the numerator of F can at most equal that of the denominator for F to be realizable by a passive network.

1) a simple lead over a quadratic,

$$\frac{V_2}{V_1} = F_3(s) = \frac{s + \omega_L}{s^2 + 2\zeta\,\omega_n\, s + \omega_n^2}$$

(III-8)

2) a biquadratic

$$\frac{V_2}{V_1} = F_4(s) = \frac{s^2 + 2\,\zeta_1\,\omega_{n1}\, s + \omega_{n1}^2}{s^2 + 2\,\zeta_2\,\omega_{n2}\, s + \omega_{n2}^2}$$

It is assumed that $F_3(s)$ and $F_4(s)$ are PRNM functions as defined in Appendix A, so that the required value for K can be found.

a) Lead-Quadratic Case, $\frac{V_2}{V_1} = F_3(s) = \frac{s + \omega_L}{s^2 + 2\,\zeta\,\omega_n\, s + \omega_n^2}$

For the sake of uniformity in equation development for this case and the biquadratic case, the following substitution of coefficients will be made:

$$\text{let} \quad F_3(s) = \frac{\gamma_1\, s + \gamma_2}{s^2 + n_1 s + n_2} \qquad \text{(III-9)}$$

where $\gamma_1 = 1;\ \gamma_2 = \omega_L;\ n_1 = 2\,\zeta\,\omega_n;\ n_2 = \omega_n^2.$

The reciprocal of the frequency characteristic of the transfer function is

$$\frac{1}{F_3(j\omega)} = \frac{n_2 - \omega^2 + n_1\, j\,\omega}{\gamma_2 + \gamma_1\, j\,\omega}\ . \qquad \text{(III-10)}$$

Meeting the constraint that the minimum value of the real part of Equation (III-10) times the realizability gain factor K_3 shall be equal to 1 requires that

$$K_3 \left[\frac{\gamma_2\, n_2 + (\gamma_1\, n_1 - \gamma_2)\,\omega^2}{\gamma_2^2 + \gamma_1\,\omega^2}\right]_{\text{minimum}} = K_3 \Big[\, f\,(\omega)\, \Big]_{\text{minimum}} = 1. \qquad \text{(III-11)}$$

Choosing K_3 to be the larger of the values K_3' and K_3'' such that

$K_3'\left[\frac{-\gamma_2 n_2}{\gamma_2^2}\right] = 1$ and $K_3''\left[\frac{\gamma_1 n_1 - \gamma_2}{\gamma_1^2}\right] = 1$ will ensure that the expression

$K_3\left[f(\omega)\right]$ will be ≥ 1 for all ω with a minimum value $= 1$ at some ω_1. Thus K_3 must be chosen to be the largest of the values which make

$$K_3'\left[\frac{\gamma_2 n_2}{\gamma_2^2}\right] = K_3'\left[\frac{\omega_n^2}{\omega_L}\right] = 1\text{ , and}$$

$$K_3''\left[\frac{\gamma_1 n_1 - \gamma_2}{\gamma_1^2}\right] = K_3''(2\zeta\omega_n - \omega_L) = 1\text{ .} \tag{III-12}$$

Hence $K_3 = \text{maximum}\left(\frac{\omega_L}{\omega_n^2}\text{ , }\frac{1}{2\zeta\omega_n - \omega_L}\right)$ (III-13)

b) <u>Biquadratic Case</u>, $\frac{V_2}{V_1} = F_4 g(s) = \frac{s^2 + 2\zeta_1\omega_{n1}{}^2 + \omega_{n1}{}^2}{s^2 + 2\zeta_2\omega_{n2}s + \omega_{n2}{}^2}$

Let $F_4(s) = \frac{s^2 + \gamma_1 s + \gamma_2}{s^2 + n_1 s + n_2}$ (III-14)

where $\gamma_1 = 2\zeta_1\omega_{n1}$; $\gamma_2 = \omega_{n1}{}^2$; $n_1 = 2\zeta_2\omega_{n2}$; $n_2 = \omega_{n2}{}^2$.

The reciprocal of the frequency characteristic of this transfer function is:

$$\frac{1}{F_4(j\omega)} = \frac{n_2 - \omega^2 + n_1 j\omega}{\gamma_2 - \omega^2 + \gamma_1 j\omega} \tag{III-15}$$

and the real part of $\frac{1}{F_4(j\omega)}$ is

$$f(\omega) \equiv \text{Re} \frac{1}{F_4(j\omega)} = \frac{\omega^4 - (\gamma_2 + n_2 - \gamma_1 n_1)\omega^2 + \gamma_2 n_2}{\omega^4 - (2\gamma_2 - \gamma_1^2)\omega^2 + \gamma_2^2} \qquad \text{(III-16)}$$

To find a K_4 such that $K_4 \left[f(\omega) \right]_{\text{minimum}} = 1$, the procedure employed is to compute the minimum value for $f(\omega)$ and then take K_4 to be the reciprocal of this value. (Again it was established in the reference that by previously requiring $\frac{V_2}{V_1}$ to be positive real, non-minimum, i.e., $\text{Re}\left[\frac{V_2}{V_1}\right] > 0$, then $f(\omega)$ is > 0 for all ω and therefore such a K does exist.)

To find the minimum value of $f(\omega)$, the first derivative of Equation (III-16) is set equal to 0, and solutions are obtained for the frequencies ω_i, at which the local minimum or maximum values of the function occur. Following this procedure gives

$$\omega_1 = 0$$

and

$$\omega_{2,3} = \sqrt{\frac{-\gamma_2(\gamma_2 - n_2) \pm \sqrt{\gamma_1^2 \gamma_2 \left[(\gamma_1 n_2 - \gamma_2 n_1)(\gamma_1 - n_1) + (\gamma_2 - n_2)^2\right]}}{\gamma_1(\gamma_1 - n_1) - (\gamma_2 - n_2)}} \qquad \text{(III-17)}$$

Choosing only real, positive values for ω_i and substituting in Equation (III-16) determines $f(\omega)$ at its local minimum and maximum points.

Selecting the absolute minimum of these values, $F_o(\omega)$, the gain factor K_4 that will make $K_4 f_o(\omega) = 1$ is of course

$$K_4 = \left[\frac{1}{f_o(\omega)}\right] \qquad \text{(III-18)}$$

c) Illustrative Examples

An example of each of the foregoing cases is chosen from the natural factors of the over-all NASA transfer function which is to be synthesized.

For an "acceptable" (i.e., PRNM) lead-quadratic factor $\frac{s + 20}{s^2 + 23.4s + 2025}$, a K must be chosen such that the branch impedances of the ladder stage that has the modified transfer function $\frac{1}{K} \frac{s + 20}{s^2 + 23.4s + 2025}$ are physically realizable. Using Equation (III-13), K must be $\geq \frac{\omega_L}{\omega_n^2} = \frac{20}{2025} = 0.0099$ and $\geq \frac{1}{2\zeta\omega_n - \omega_L} = \frac{1}{23.4-20} = 0.294$. In this case the stronger constraint is $K \geq 0.294$, so that K is taken as 0.294. (Since the lead-quadratic term is modified by the factor $\frac{1}{K}$ in order to make it realizable, a compensation gain factor equal to K has to be provided external to the ladder network.)

An acceptable biquadratic factor is $\frac{s^2 + 20s + 400}{s^2 + 63s + 2025}$. A K is chosen using the following procedure. From Equations (III-17), $\omega_{1, 2, 3} = 0;\ 39.82;\ 10.65$. Substituting in Equation (III-16) gives the respective values for $f(\omega)$, 5.06; 0.724; 5.418. Since the minimum is 0.724, from Equation (III-18), $K = \frac{1}{.724} = 1.38$.

Again, the associated branch impedances for each of the preceding cases are obtained by $Z_b = \frac{K}{F(s)} - 1$ and $Z_a = 1 + \frac{1}{Z_b}$.

7. Determination of the Required Compensation Gain for the Over-all Ladder Network

A compensation gain factor, K_T, must be provided for the over-all ladder network so that its DC gain is unity. To simplify the following discussion concerning the determination of K_T, consider as an example the required component transfer function:

$$F_i'(s) = \frac{L'(a)}{L'(b)} = \frac{\frac{1}{a}\, s + 1}{\frac{1}{b}\, s + 1} \tag{III-19}$$

While this is the canonical form in which transfer functions are usually specified with a unity DC gain, actually the following transfer function is synthesized:

$$\frac{1}{K_i} F_i(s) = \frac{a}{bK_i} F_i'(s) = \frac{s + a}{K_i (s + b)} \equiv \frac{L(a)}{K_i\, L(b)} \tag{III-20}$$

The realizability gain factor, K_i, is determined as the smallest positive number which satisfies the inequality:

$$K_i \operatorname{Re}\left[\frac{1}{F_i(s)}\right] = K_i \operatorname{Re} \frac{s + b}{s + a} \geq 1 \tag{III-21}$$

The value for K_i is thus chosen so that for the ω_1 at which $R_e \left[\frac{1}{F(j\omega)}\right]$ is a minimum $K_i \operatorname{Re}\left[\frac{1}{F(j\omega_1)}\right] = 1$.

The branch impedances of the associated stage of the constant resistance ladder network will then be caused to be positive real, minimum resistance (PRMR) functions. This is required by the synthesis procedures to be used for these branch impedances.

For a lag-lead factor, i.e., $a > b$, it is noted that $K_i = \frac{a}{b}$ and $\omega_1 = 0$. For a lead-lag factor, that is, $a < b$, $K_i = 1$ and $\omega_1 = \infty$.

The relation between the desired component transfer function $F_i'(s)$ and the synthesized transfer function $F_i(s)/K_i$ is:

$$F_i'(s) = \frac{b}{a} K_i \left[F_i(s)/K_i\right] \tag{III-22}$$

Thus, the required compensation gain for this factor is $K_{T_i} = \frac{b}{a} K_i$, since

$F_i'(s)$ has the desired DC gain of unity. In general, assume that the desired over-all transfer function $F_D'(s)$ with a DC gain of unity and acceptable factoring is given by:

$$F_D'(s) = F_1(s)\, F_2'(s) \ldots F'_n(s) = \frac{N_1'(s)}{D_1'(s)} \cdot \frac{N_2'(s)}{D_2'(s)} \cdots \frac{N_n'(s)}{D_n'(s)} \qquad \text{(III-23)}$$

Then, the synthesized transfer function $F_D(s)$ will be given by:

$$F_D(s) = \left[\frac{1}{K_1}\,\frac{N_1(s)}{D_1(s)}\right]\left[\frac{1}{K_2}\,\frac{N_1(s)}{D_1(s)}\right] \cdots \left[\frac{1}{K_n}\,\frac{N_n(s)}{D_n(s)}\right], \qquad \text{(III-24)}$$

and the required over-all compensation gain factor K_T for the ladder network is:

$$K_T = \left[K_1\,\frac{D_1(0)}{N_1(0)}\right]\left[K_2\,\frac{D_2(0)}{N_2(0)}\right] \cdots \left[K_n\,\frac{D_n(0)}{N_n(0)}\right] \qquad \text{(III-25)}$$

$$= K_{T_1}\, K_{T_2} \cdots K_{T_n}\,.$$

B. BRANCH IMPEDANCE SYNTHESIS PROCEDURES

1. General

This section describes in detail the three procedures that are used to synthesize the 1-port branch networks of a constant resistance ladder network with the desired compensation function as its over-all transfer function. Table III-1 of section III delineated the realizable forms of the impedance functions for these branch networks and indicated the three synthesis procedures to be used: continued fraction, Bott-Duffin and modified Bott-Duffin. A fourth procedure is implicit in the synthesis operation; it is the recognition of simple first order impedance elements by inspection.

2. Continued Fraction Expansion

a) General

For those cases where it is applicable, i.e., where a zero or pole of the given impedance function, Z (s), is on the imaginary axis, the method of continued fractions yields a particularly simple realization. The paragraphs below outline the method and illustrate its use in the constant resistance ladder network synthesis problem.

Consider the 1-port network presented in Figure III-4(a) whose impedance is Z. Assume that Z can be expressed as the sum of a simple impedance function Z_1 (i.e., $Z_1 = K\,s$, $\frac{K}{s}$, $K(s^2 + a^2)$ or $\frac{K}{s^2} + a^2$ and a remainder $Z_2{}'$. (This is always true if Z has a zero or pole on the imaginary axis.) Then, as in Figure III-4(b):

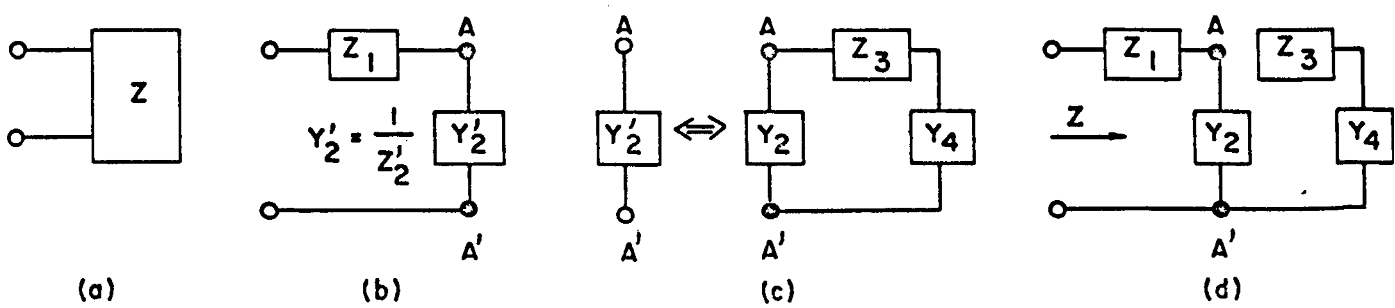


Figure III-4. Some Impedance Configurations

$$Z = Z_1 + Z_2{}' = Z_1 + \frac{1}{Y_2{}'} . \tag{III-26}$$

The admittance $Y_2{}'$ is next considered to be composed of the elements depicted in Figure III-4(c). Then, $Y_2{}'$ may be expressed as:

$$Y_2{}' = Y_2 + \frac{1}{Z_3 + Z_4} = Y_2 + \frac{1}{Z_3 \;\; \frac{1}{Y_4}} . \tag{III-27}$$

Substituting Equation (III-27) into Equation (III-26), the impedance of the network shown in Figure III-4 (d) is:

$$Z = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_3 + \cfrac{1}{Y_4}}} . \tag{III-28}$$

It is seen that the philosophy used to obtain Equation (III-28) may be generalized. Thus, consider the general network shown in Figure III-5.

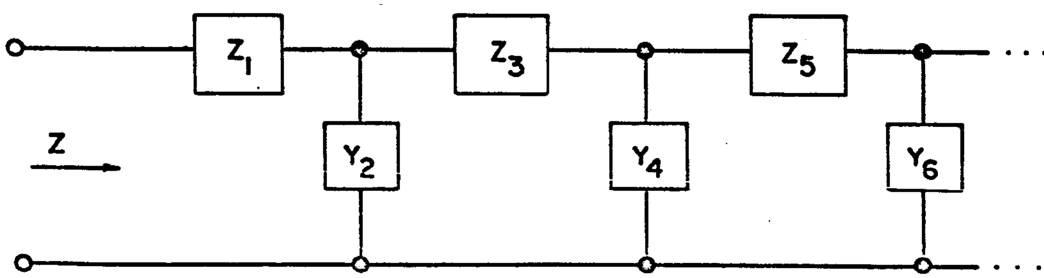


Figure III-5. General Network

The corresponding impedance function Z(s) may be written

$$Z = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_3 + \cfrac{1}{Y_4 + \cfrac{1}{Z_5 + \cfrac{1}{Y_6 + \cdots}}}}} \tag{III-29}$$

Inversely, if the given impedance function is expanded in the continued fraction form of Equation (III-29), the elements of the ladder network with this impedance function will be as defined in Figure III-5.

This method will now be illustrated for the two types of functions that were synthesized by this technique in the constant resistance ladder network synthesis problem.

b) Continued Fraction Expansion for the Form: $Z(s) = \dfrac{a\,s^2 + b\,s + c}{s + \omega_2}$

Assume that the voltage transfer function for one stage of the desired constant resistance ladder network comprises a simple lead term in the numerator, and a quadratic lag term in the denominator. Then, the equations relating the network transfer function of this stage to its branch network impedance functions, as presented in section III-A, lead to the form presented above for one of the impedances, namely Z_b. This impedance has a pole at $S = j\omega = \infty$. When this impedance is found, the related branch impedance (Z_a) is determined from the relation, $Z_a = 1 + Z_b{}'$. The method of impedance synthesis used is now explicitly outlined.

By simple long division:

$$Z_b = \frac{a\,s^2 + b\,s + c}{s + \omega_L} = \left\lfloor as + (b - a\,\omega_L) \right\rfloor + \frac{1}{\dfrac{s + \omega_L}{K}} \qquad \text{(III-30)}$$

where $K = c - (b - a\,\omega_L)\,\omega_L$.

It is observed that Equations (III-26) and (III-30) have the same form, and hence the following identification may be made:

$$Z_1 = a\,s + (b - a\,\omega_L) \qquad \text{(III-31)}$$

$$Y_2{}' = \frac{s + \omega_L}{K}. \qquad \text{(III-32)}$$

The form of Equation (III-31) suggests the series combination of an inductor, L_1, and a resistor, R_1, in series. Equating the impedance of such a configuration with Equation (III-31):

$$Z_1 = L_1\, s + R_1 = a\, s + (b - a\, \omega_L)$$

which leads to the identification:

$$L_1 = a\,;\;\; R_1 = b - a\, \omega_L. \tag{III-33}$$

Equation (III-32) may be rewritten:

$$\frac{1}{Y_2'} = Z_2' = \frac{K}{s + \omega_L}. \tag{III-32a}$$

The form of this equation suggests the use of a parallel combination of a capacitor, C_2, and a resistor, R_2. Equating the impedance of such a configuration with Equation (III-32a):

$$\frac{\frac{1}{C_2}}{s + \frac{1}{R_2 C_2}} = \frac{K}{s + \omega_L}$$

which leads to the identifications:

$$C_2 = \frac{1}{K}\,;\;\; R_2 = \frac{K}{\omega_L} \tag{III-34}$$

Since the impedance is of the form of Equation (III-26), the network will be of the form presented in Figure III-4 (b); it is again given in Figure III-6 with the specific impedances obtained for this case.

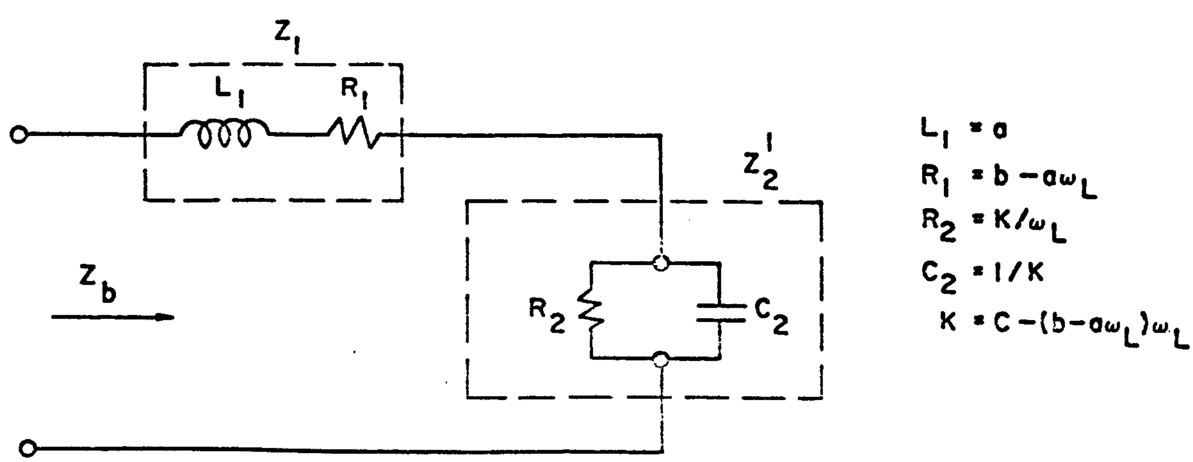


Figure III-6. Branch Network Configuration

For $Z_b = \dfrac{a\,s^2 + b\,s + c}{s + \omega_L}$

The related branch impedance function, Z_a, for the Z_b under study is now considered. It is given as:

$$Z_a = 1 + Z_b^{-1} = 1 + \frac{s + \omega_L}{a s^2 + b s + c} \qquad \text{(III-35)}$$

Substituting Equation (III-30) into Equation (III-35):

$$Z_a = 1 + \frac{1}{a s + (b - a\,\omega_L) + \dfrac{K}{s + \omega_L}} \qquad \text{(III-36)}$$

It is observed that Z_a has the form:

$$Z_a = Z_1 + \frac{1}{Y_2 + \dfrac{1}{Z_3}} \qquad \text{(III-37)}$$

where:

$$Z_1 = 1$$

$$Z_2 = \frac{1}{a s + (b - a\,\omega_L)} \;; \qquad Z_3 = \frac{s + \omega_L}{K}. \qquad \text{(III-38); (III-39)}$$

Equation (III-37) obviously dictates the use of a 1-ohm resistor for Z_1. Z_2 is synthesized by use of a parallel resistor-capacitor combination where the component values, say, R_2 and C_2, are found by a procedure identical to that used for Z_2' in the synthesis of Zb. The values so obtained are:

$$R_2 = \frac{1}{b + a\,\omega_L}, \quad C_2 = a. \qquad \text{(III-40)}$$

Z_3 is obtained in a manner analogous to that used for Z_1 in the synthesis of Z_a. The series inductor-resistor combination has the element values:

$$L_3 = \frac{1}{K}$$

$$R_3 = \frac{\omega_L}{K} \qquad \text{(III-41)}$$

The resulting impedance configuration is presented in Figure III-7.

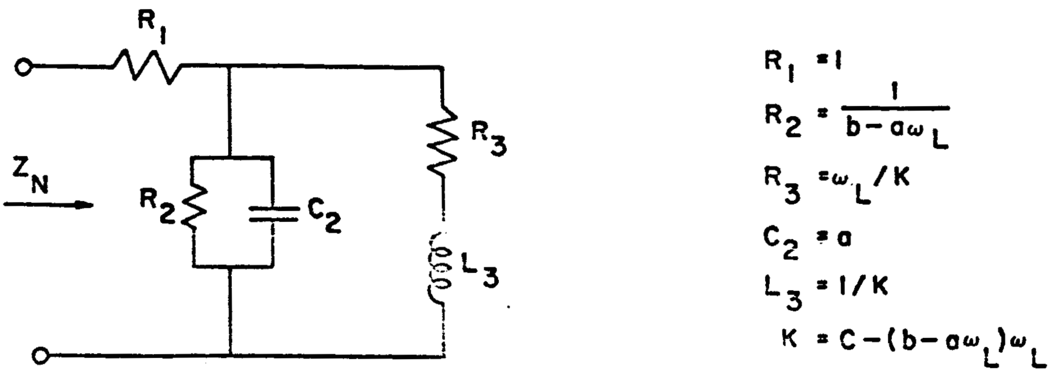


Figure III-7. Branch Network for $Z_a = 1 + \frac{s + \omega_L}{a\,s^2 + b\,s + c}$

c) Continued Fraction Expansion for the Form: $Z(s) = \frac{s^2 + b\,s + c}{s^2 + a\,s}$

For the case where a stage of the constant resistance ladder network is characterized by a voltage transfer function comprised of a lead quadratic term and two simple lag terms, the corresponding value of Z_b is, in many cases:

$$Z_b = \frac{s^2 + b\,s + c}{s^2 + a\,s} = \frac{1}{\dfrac{c + bs + s^2}{as + s^2}}$$

$$= \frac{1}{\left[\dfrac{c}{as} + \dfrac{ab - c}{a^2}\right] + \dfrac{1}{\dfrac{a}{\left[1 - \dfrac{ab - c}{a^2}\right]}\, s + \dfrac{1}{\left[1 - \left(\dfrac{ab - c}{a^2}\right)\right]}}} \qquad \text{(III-42)}$$

This may be identified with the continued function expansion:

$$Z_b = Z_1 + \frac{1}{Y_2 + \dfrac{1}{Z_3}}$$

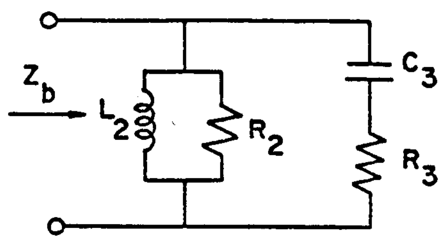
where:

$$Z_1 = 0 \qquad \text{(III-43)}$$

$$Z_2 = Y_2^{-1} = \left[\frac{c}{as} + \frac{ab - c}{a^2}\right]^{-1} \tag{III-44}$$

$$Z_3 = \frac{a}{\left[1 - \left(\frac{ab - c}{a^2}\right)\right]s} + \frac{1}{\left[1 - \left(\frac{ab - c}{a^2}\right)\right]} \tag{III-45}$$

The form of Equation (III-44) suggests the choice of a parallel inductor-resistor combination. If these choices are made and the coefficients of the resulting impedance functions are equated with the appropriate coefficients in Equations (III-44) and (III-45), the configuration presented in Figure III-8 is obtained.



$Z_1 = 0$

$L_2 = a/c$

$R_2 = \frac{a^2}{ab-c}$

$C_3 = \left[1 - \left(\frac{ab-c}{a^2}\right)\right]/a$

$R_3 = \left[1 - \left(\frac{ab-c}{a^2}\right)\right]^{-1}$

Figure III-8. Branch Network For $Z_b = \frac{s^2 + b\,s + c}{s^2 + a\,s}$

The related branch impedance Z_a is given as:

$$Z_a = 1 + Z_b^{-1}$$

$$= \left[\frac{c}{as} + \left(1 + \frac{ab - c}{a^2}\right)\right] + \frac{1}{\frac{a}{\left[1 - \left(\frac{ab - c}{a^2}\right)\right]s} + \frac{1}{\left[1 - \left(\frac{ab - c}{a^2}\right)\right]}} \tag{III-46}$$

Equation (III-46) is recognized as having the form:

$$Z_a = Z_1 + \frac{1}{Y_2}$$

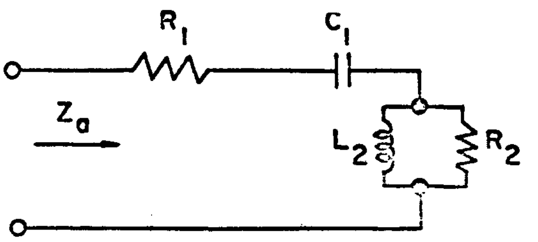
where:

$$Z_1 = \frac{c}{a\,s} + \left(1 + \frac{ab - c}{a^2}\right) \tag{III-47}$$

and

$$Z_2 = Y_2^{-1} = \frac{1}{\dfrac{a}{\left[1 - \left(\dfrac{ab - c}{a^2}\right)\right] s} + \dfrac{1}{\left[1 - \left(\dfrac{ab - c}{a^2}\right)\right]}} \tag{III-48}$$

The form of Equation (III-47) suggests a series resistor-capacitor combination while the form of Equation (III-48) suggests a parallel inductor-resistor combination. Computing the impedance functions associated with these element combinations and equating the coefficients of the resulting expressions with the appropriate coefficients in Equations (III-47) and (III-48), the impedance configuration of Figure III-9 is obtained.



$$R_1 = 1 + \frac{ab-c}{a^2} \qquad L_2 = \frac{1 - \left(\dfrac{ab-c}{a^2}\right)}{a}$$

$$C_1 = \frac{a}{c} \qquad R_2 = 1 - \left(\frac{ab-c}{a^2}\right)$$

Figure III-9 Branch Network For $Z_a = 1 + \dfrac{s^2 + as}{s^2 + bs + c}$

3. Bott-Duffin Synthesis of Positive Real Minimum Impedances

For positive real, minimum impedance functions, it is not possible to use the continued fraction synthesis technique. Since such a function has no poles or zeros on the imaginary axis, the simplification process associated with removal of such poles or zeros cannot be carried out.

Various techniques exist for synthesizing PR minimum impedance functions, including Bruné, Miyata and Bott-Duffin. The Bruné approach is not favored since it involves the use of ideal transformers. A study was conducted of the Miyata and Bott-Duffin procedures. It revealed that the latter yields a simpler network realization for the classes of component transfer functions associated with the various stages of the desired constant resistance ladder network. Hence, the Bott-Duffin synthesis technique has been chosen for the positive real, minimum, biquadratic impedance functions encountered in this application.

Suppose the following conjecture is made: It is possible to synthesize any positive real minimum impedance Z(s) by the impedance topology presented in Figure III-10. If the above is to be true, certain conditions must be fulfilled, namely:

a) If Z(s) is positive real, then each of the Z_i's composing the topology must be positive real.

b) If any useful result is to be obtained, the Z_i's must not be of higher order than the original Z(s).

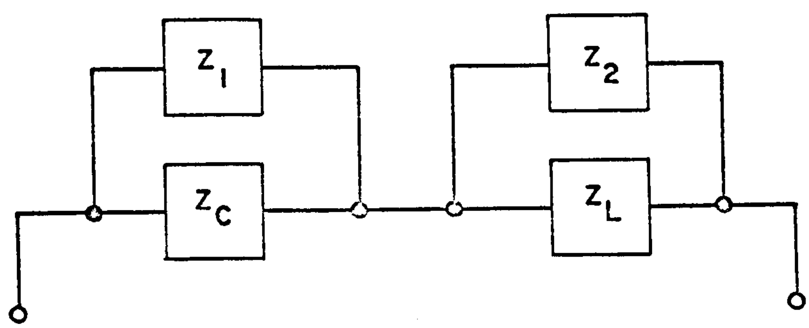


Figure III-10. Preliminary Impedance Topology

The Bott-Duffin method of impedance synthesis is based on the topology of Figure III-10 and the Richards function, R(s), which guarantees that the above requirements are satisfied.

The Richards function is defined as follows: Given a positive real function Z of the complex variable s, the function is:

$$R(s) = \frac{k\, Z(s) - s\, Z(k)}{k\, Z(k) - s\, Z(s)} \qquad (k = \text{positive number}) \tag{III-49}$$

where R(s) has the properties

a) R(s) is positive real

b) R(s) is not of higher order than Z(s).

Equation (III-49) may be solved for Z(s) as:

$$Z(s) = \frac{1}{\dfrac{1}{Z(k)\, R(s)} + \left[\dfrac{1}{k\, Z(k)}\right] s} + \frac{1}{\dfrac{1}{\left[\dfrac{Z(k)}{k}\right] s} + \dfrac{R(s)}{Z(k)}} \,. \tag{III-50}$$

From Figure III-10,

$$Z(s) = \frac{1}{Y_1 + Y_c} + \frac{1}{Y_2 + Y_L} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_c}} + \frac{1}{\frac{1}{Z_2} + \frac{1}{Z_L}} \tag{III-51}$$

Equating coefficients between Equations (III-50) and (III-51):

$$Z_1 = Z\,(k)\; R(s) \tag{III-52}$$

$$Z_2 = \frac{Z(k)}{R(s)} \tag{III-53}$$

$$Z_c = \left[\frac{1}{kZ(k)}\; s\right]^{-1} = \frac{1}{c_1\, s} \tag{III-54}$$

where:

$$c_1 = \frac{1}{k\, Z(k)} \tag{III-54a}$$

$$Z_L = \frac{Z(k)}{k}\; s = L_2\, s \tag{III-55}$$

where:

$$L_2 = \frac{Z(k)}{k} \tag{III-55a}$$

Since, if R(s) is PR, then both its reciprocal and R(s) multiplied by a constant are positive real, and the Z_i are also positive real.

As indicated by the preceding development, the impedance topology of Figure III-10 may be redrawn as shown in Figure III-11.

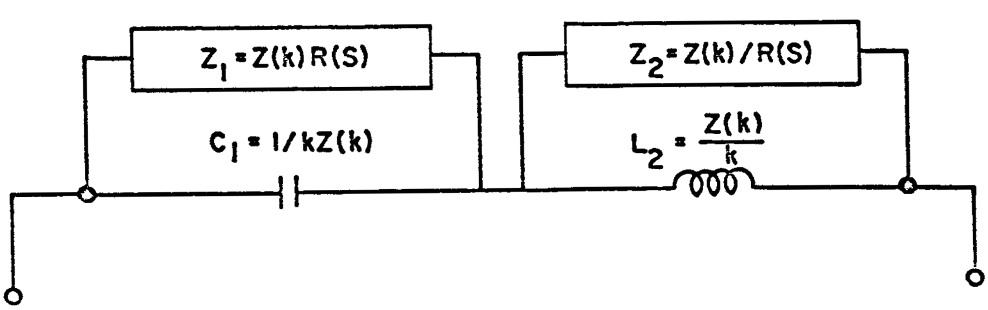


Figure III-11. First Impedance Topology

The next problem is to determine the required value of k. The approach used is to consider the frequency, ω_1, at which Z is purely reactive, i. e., Re $Z(s) \big|_{s=j\omega_1}$ vanishes. If at this frequency the reactance is positive, R(s) is chosen so that Z_1 is a short circuit and Z_2 is an open circuit. Hence, L_2 represents Z at ω_1 and an appropriate value of L_2 may be calculated from the known reactance and ω_1 values. If the reactance is negative, a similar procedure may be carried out with C_1 representing Z at ω_1, instead of L_2. The two cases are worked out in detail in the subsequent paragraphs.

<u>Case I</u> - $Z(j\omega_1) = j\,\underline{\overline{X}}_1$ and $\underline{\overline{X}}_1 > 0$

From Equations (III-51), (III-54), and (III-55):

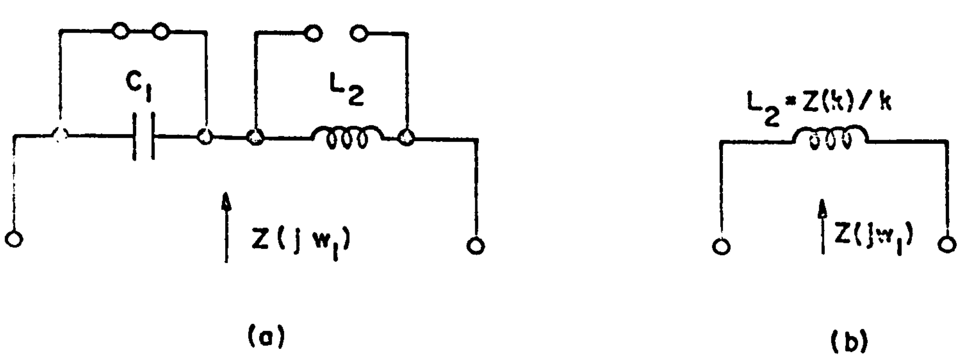
$$Z(j\omega_1) = \frac{1}{\dfrac{1}{Z_1(j\omega_1)} + j\omega_1 C_1} + \frac{1}{\dfrac{1}{Z_2(j\omega_1)} + \dfrac{1}{j\omega_1 L_2}} \qquad \text{(III-56)}$$

From Equations (III-52) and (III-53), if R(s) is required to have a zero at $\omega = \omega_1$, then:

$$Z_1(j\omega_1) = Z(k)\,R(j\omega_1) = 0, \qquad \text{(III-57)}$$

$$Z_2(j\omega_1) = Z(k)\,R(j\omega_1) = \infty. \qquad \text{(III-58)}$$

In this case, the impedance topology at $\omega = \omega_1$ may be represented as shown in Figure III-12.



(Network (a) Equivalent to Network (b))

Figure III-12. Impedance Topology at $\omega = \omega_1$ $(Z(j\omega_1) = j\underline{\overline{X}}_1,\ \underline{\overline{X}}_1 > 0)$

Thus $Z(j\omega_1) = L_2 s \big|_{s=j\omega_1}$ as desired (III-59)

$$\text{or,} \quad L_2 = \frac{\bar{X}_1}{\omega_1}. \tag{III-60}$$

Since both ω_1 and $\bar{X}_1$ are considered to be known, the required L_2 may be calculated.

An acceptable value for k (k_o, real and positive) can be calculated from Equations (III-55a) and (III-60) as:

$$\frac{Z(K_o)}{k_o} = L_2 = \frac{\bar{X}_1}{\omega_1}. \tag{III-61}$$

Equations (III-52), (III-53), and (III-54a) may now be evaluated, using this k_o, to yield the value of C_1 and the functional forms of Z_1 and Z_2.

Since $Z_1^{-1}(s)$ has a pole at $j\omega_1$, then, for $\omega_1 \neq 0$, it must also have a pole at $-j\omega_1$. Let $Z_1(s)$ be written as follows:

$$Z_1^{-1}(s) = \frac{F_1(s)}{(s^2 + \omega_1^2)\, F_2(s)} \tag{III-62}$$

where, again, $Z_1(s)$ is a PRM function and $F_1(s)$ and $F_2(s)$ have no common factors. Let $Z_1^{-1}(s)$ be expanded as:

$$Z_1^{-1}(s) = \frac{\alpha_1}{s+j\omega_1} + \frac{\alpha_2}{s-j\omega_1} + \frac{1}{Z_3(s)} \tag{III-63}$$

It is easily shown that the residues, α_i are:

$$\alpha_1 = \frac{F_1(-j\omega_1)}{-2j\omega_1 F_2(-j\omega_1)}, \qquad \alpha_2 = \frac{F_1(j\omega_1)}{2j\omega_1 F_2(j\omega_1)} \tag{III-64}$$

Equations (III-64) are of the form:

$$\alpha_1 = a + jb, \quad \alpha_2 = a - jb. \tag{III-65}$$

Since the PR condition requires that the α_i be real, the only choice is:

$$\alpha_1 = \alpha_2 = \alpha = a. \tag{III-66}$$

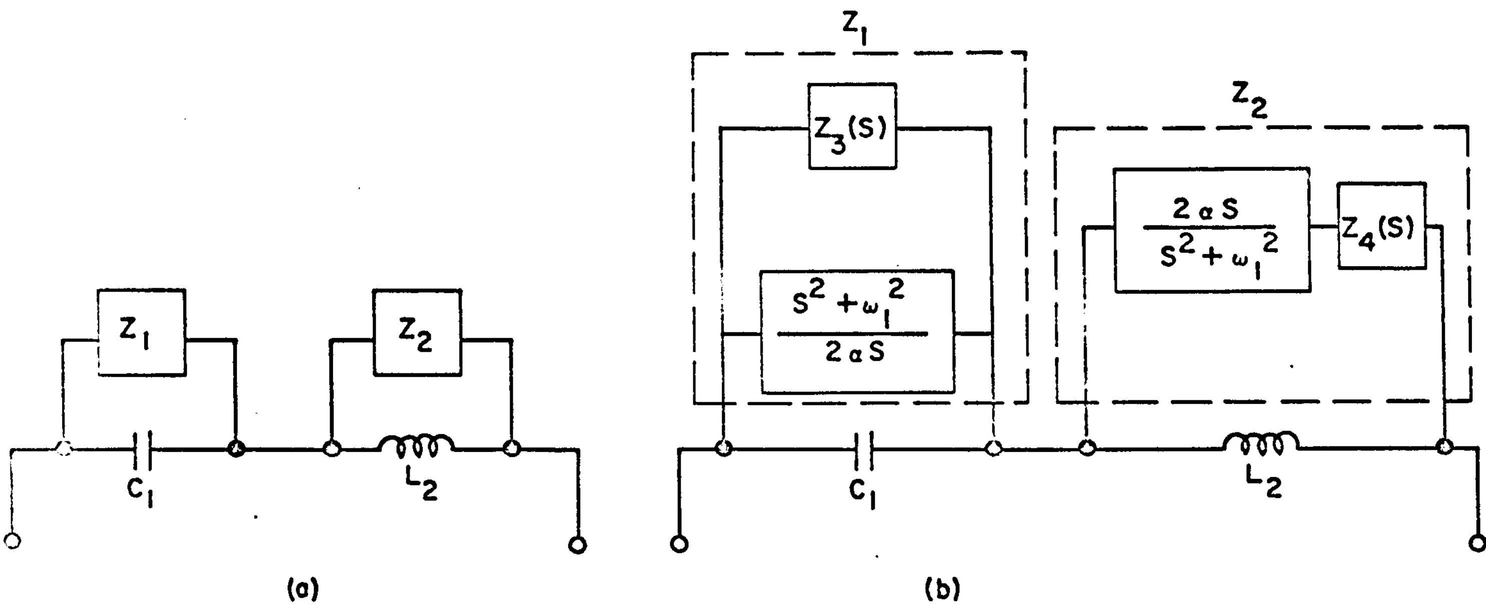
Substituting Equation (III-66) into (III-63):

$$Z_1^{-1}(s) = \frac{2\alpha s}{s^2 + \omega_1^2} + \frac{1}{Z_3(s)} . \tag{III-67}$$

Similarly,

$$Z_2(s) = \frac{2\alpha s}{s^2 + \omega_1^2} + Z_4(s) . \tag{III-68}$$

These results are illustrated by the circuit diagram of Figure III-13.



NETWORK (a) EQUIVALENT TO NETWORK (b)

Figure III-13. Second Impedance Configuration ($Z(j\,\omega_1) = j\,\bar{X}_1$, $\bar{X}_1 > 0$)

Assuming a series L_3, C_3 combination in parallel with Z_3:

$$Z_{L_3C_3} = \frac{1}{C_3\, s} + L_3\, s = \frac{L_3\, C_3\, s^2 + 1}{C_3\, s} = \frac{s^2 + \frac{1}{L_3\, C_3}}{\frac{1}{L_3}\, s} .$$

Hence, the following identifications may immediately be made:

$$\frac{1}{L_3} = 2\,\alpha \text{ or } L_3 = \frac{1}{2\alpha}\,, \tag{III-69}$$

$$\frac{1}{L_3 C_3} = \omega_1^{\,2} \text{ or } C_3 = \frac{1}{L_3\,\omega_1^{\,2}} = \frac{2\alpha}{\omega_1^{\,2}}\,. \tag{III-70}$$

Assuming for a parallel L_4, C_4 combination in series with Z_4:

$$Z_{L_4 C_4} = \frac{\dfrac{L_4}{C_4}}{\dfrac{1}{C_4\,s} + L_4\,s} = \frac{L_4\,s}{L_4 C_4\,s^2 + 1} = \frac{\dfrac{s}{C_4}}{s^2 + \dfrac{1}{L_4\,C_4}}\,.$$

Hence:

$$\frac{1}{C_4} = 2\,\alpha \quad \text{or} \quad C_4 = \frac{1}{2\,\alpha} \tag{III-71}$$

$$\frac{1}{L_4\,C_4} = \omega_1^{\,2} \text{ or } L_4 \; \frac{1}{C_4\,\omega_1^{\,2}} = \frac{2\alpha}{\omega_1^{\,2}}\,. \tag{III-72}$$

The circuit diagram corresponding to Equations (III-69) - (III-72) is presented in Figure III-14.

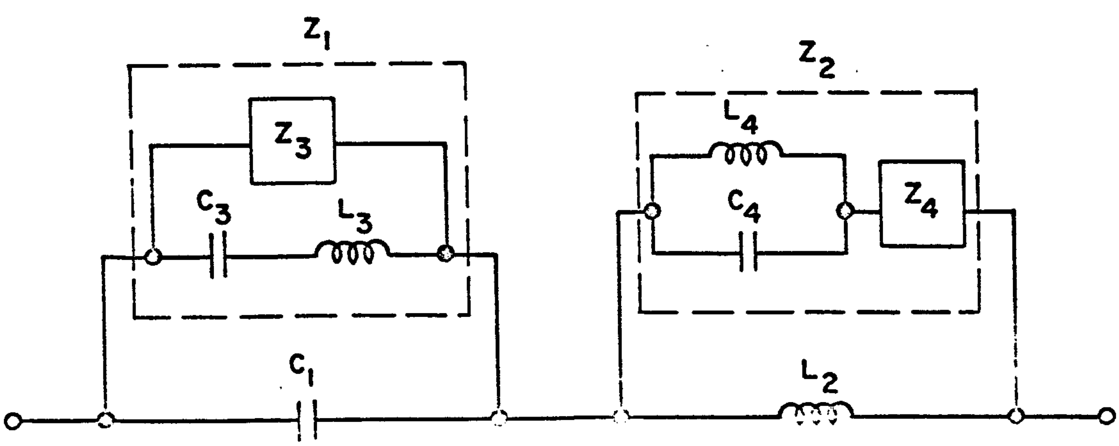


Figure III-14. Final Impedance Configuration ($Z\,(j\,\omega_1) = j\,\underline{X}_1$, $\underline{X}_1 > 0$)

In order for this approach to represent a valid synthesis procedure, the impedances Z_3 and Z_4 must be simpler than the original Z. By the method used to generate Z_3 and Z_4, i.e., via Equations (III-67) and (III-68), Z_3 and Z_4 each possess two less poles and zeros than Z_1 and Z_2, which are of the same order of complexity as Z as may be seen from Equations (III-52) and (III-53). Hence, Z_3 and Z_4 each have two less poles and zeros than Z.

The above statements are illustrated by the following equation:

$$F(s) = \frac{(s+a)(s+b)}{(s^2+C^2)(s+d)} = \frac{a_1 s}{s^2+C^2} + \frac{a_3}{s+d},$$

where:

$$a_3 = \frac{(a-d)(b-d)}{d^2+C^2}.$$

It is seen that the last term on the right has two less poles and two less zeros than the original function F (s).

To synthesize high order impedance functions, this method may be iterated using Z_3 and Z_4 as starting points for a second Bott-Duffin cycle. This is not required with the constant resistance network approach since Z(s) is at most a second order numerator divided by a second order denominator. In this case, Z_3 and Z_4 will prove to be constant, i.e., resistances.

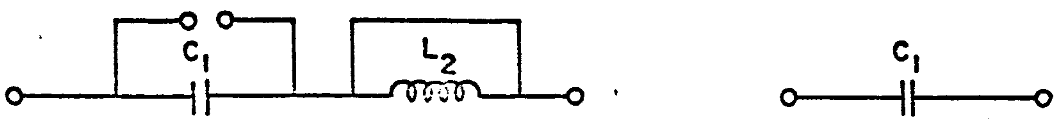
Case II - $Z(j\omega_1) = -j\,\overline{X}_1,\ \overline{X}_1 > 0$

Suppose R (s) is required to possess a pole at $s = j\omega_1$; then, from Equations (III-52) and (III-53):

$$Z_1(j\omega_1) = Z(k)\,R(s)\,\Big|_{s=j\omega_1} = \infty \qquad \text{(III-73)}$$

$$Z_2(j\omega_1) = Z(k)\,R(s)\,\Big|_{s=j\omega_1} = 0 \qquad \text{(III-74)}$$

For this case the circuit representation is shown in Figure III-15.



(a) (b)

NETWORK (a) EQUIVALENT TO NETWORK (b)

Figure III-15. Circuit Representation at $\omega = \omega_1$

$(Z\,(j\,\omega_1) = -\,j\,\overline{X}_1,\ \overline{X}_1 > 0)$

From Figure III-15 it is seen that:

$$Z\,(j\,\omega_1) = -\,j\,\overline{X}_1 = -j\left(\frac{1}{C\,\omega_1}\right) \text{ or } \frac{1}{C_1} = \omega_1\,\overline{X}_1, \tag{III-75}$$

where $\overline{X}_1$ and ω_1 are both known so that C_1 may be calculated.

If R (s) has a pole at $s = j\,\omega_1$, then the denominator of Equation (III-49) must vanish at this frequency, i.e.:

$$0 = k\,Z\,(k) - j\,\omega_1\,Z\,(j\,\omega_1) = k\,Z\,(k) - \omega_1\,\overline{X}_1 \tag{III-76}$$

An acceptable value for k (k_o, real and positive) is obtained from Equation III-76.

Using this value of k_o, L_2 may be obtained from Equation (III-55a) and the functional forms for Z_1 and Z_2 from Equations (III-52) and (III-53).

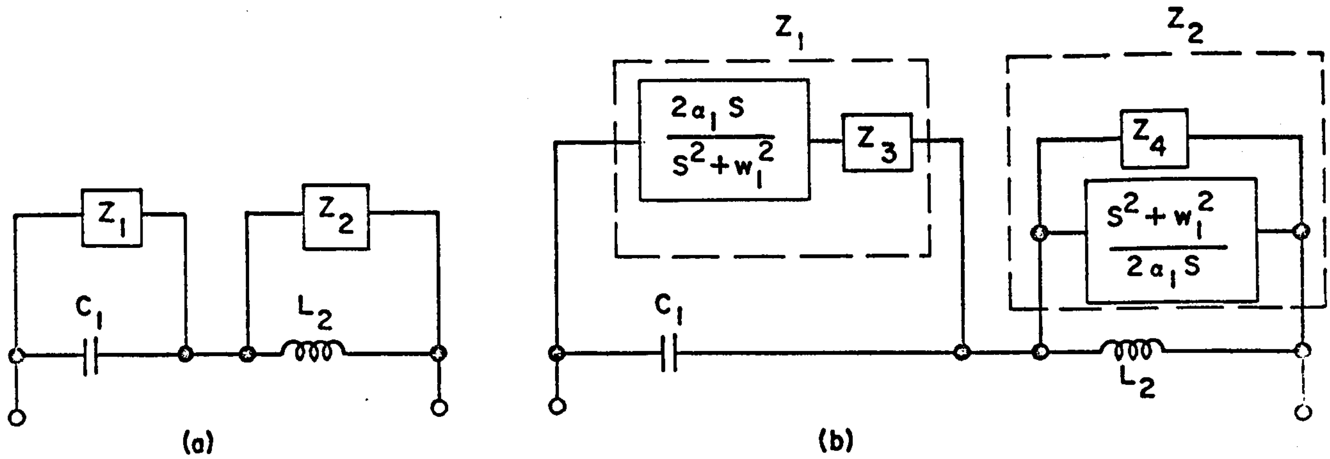
Since $Z_1\,(j\,\omega)$ has a pole at ω_1, as indicated by Equation (III-73), Z_1 may be expressed as:

$$Z_1 = \frac{2\,\alpha_1\,s}{s^2 + \omega_1^{\,2}} + Z_3. \tag{III-77}$$

Since $Z_2\,(j\,\omega)$ has a zero at ω_1, as indicated by Equation (III-74), Z_2 may be expressed as:

$$Z_2^{-1} = \frac{2\alpha_1 s}{s^2 + \omega_1^2} + \frac{1}{Z_4} . \qquad \text{(III-78)}$$

The progress made so far is illustrated in Figure III-16.



NETWORK (a) EQUIVALENT TO NETWORK (b)

Figure III-16. Second Impedance Configuration

For a parallel L_3 C_3 configuration in series with Z_3:

$$Z_{L_3 C_3} = \frac{\frac{1}{C_3} s}{s^2 + \frac{1}{L_3 C_3}} .$$

Hence, equating elements:

$$\frac{1}{C_3} = 2\alpha_1 \text{ or } C_3 = \frac{1}{2\alpha_1}, \qquad \text{(III-79)}$$

$$\frac{1}{L_3 C_3} = \omega_1^2 \text{ or } L_3 = \frac{1}{C_3 \omega_1^2} = \frac{2\alpha_1}{\omega_1^2} . \qquad \text{(III-80)}$$

For a series L_4 C_4 combination in parallel with Z_4:

$$Z_{L_4 C_4} = \frac{s^2 + \frac{1}{L_4 C_4}}{\frac{1}{L_4} s} .$$

For a series $L_4 C_4$ combination in parallel with Z_4:

$$Z_{L_4 C_4} = \frac{s^2 + \frac{1}{L_4 C_4}}{\frac{1}{L_4} s} .$$

The following identifications may similarly be made:

$$\frac{1}{L_4} = 2 \alpha_1 \qquad \text{or } L_4 = \frac{1}{2 \alpha_1} \tag{III-81}$$

$$\frac{1}{L_4 C_4} = \frac{1}{\omega_1^2} \qquad \text{or } C_4 = \frac{1}{L_4 \omega_1^2} = \frac{2 \alpha_1}{\omega_1^2} \tag{III-82}$$

The circuit diagram corresponding to Equations (III-79) - (III-82) is presented in Figure III-17.

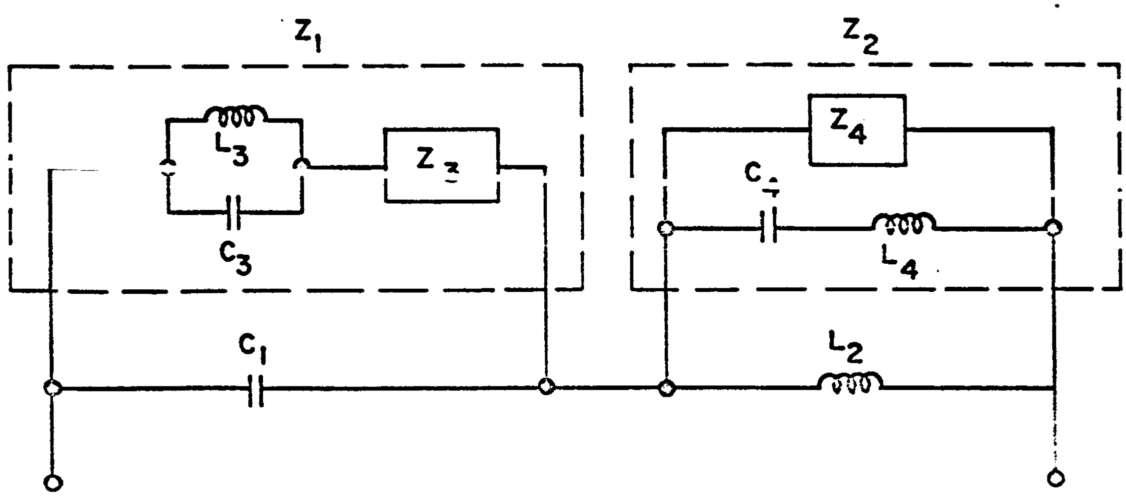


Figure III-17. Final Impedance Configuration ($Z(j\omega_1) = -j \underline{X}_1$, $\underline{X}_1 > 0$)

Finally, the complexity of Z_3 and Z_4 relative to Z for this case is the same as that for the previous case, i.e., Z_3 and Z_4 each have two less poles and zeros than Z. Thus, for the Z's under consideration, Z_3 and Z_4 will again prove to be constant, and the synthesis is completed with the application of one Bott-Duffin cycle.

4. Modified Bott-Duffin Synthesis Procedure

Use of the modified Bott-Duffin synthesis procedure remedies the sensitive, balanced bridge configuration of the Bott-Duffin network and reduces by one the number of elements required for a Bott-Duffin realization of a biquadratic impedance function. As in the Bott-Duffin approach, two separate cases are treated in the synthesis of a PRM impedance function Z(s): one where $Z(j\omega_1) = j\overline{X}_1$ and the other where $Z(j\omega_1) = -j\overline{X}_1$, $\overline{X}_1 > 0$.

<u>Case I</u> $Z(j\omega_1) = j\overline{X}_1$, $\overline{X}_1 > 0$

The Bott-Duffin approach outlined previously gives rise to an impedance which is, in essence, a bridge configuration as is evidenced by Figure III-18. In order for the bridge to be balanced, it is required that $V_{1A} = V_{1A'}$ and $V_{A2} = V_{A'2}$. This condition may be written:

$$\frac{V_{1A}}{V_{A2}} = \frac{V_{1A'}}{V_{A'2}} \qquad \text{(III-83)}$$

where:

$$V_{1A} = i_1 Z_c,\ V_{1A'} = i_2 Z_1,\ V_{A2} = i_1 Z_2,\ V_{A'2} = i_2 Z_L\ . \qquad \text{(III-84)}$$

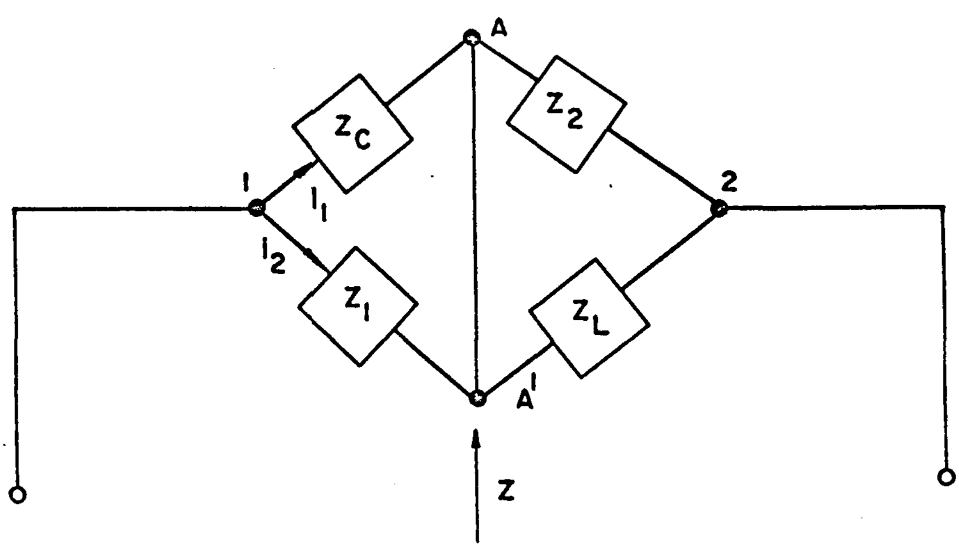


Figure III-18. Equivalent Impedance of Figure III-10

Substituting Equations (III-84) into (III-83) the balance condition is:

$$Z_1 Z_2 = Z_C Z_L. \tag{III-85}$$

Substituting Equations (III-52), (III-53), (III-54), and (III-55) into (III-85):

$$\left[Z(k)\,R(s)\right]\left[\frac{Z(k)}{R(s)}\right] = \left[\frac{Z(k)\,s}{k}\right]\left[\frac{k\,Z(k)}{s}\right],$$

or

$$Z^2(k) = Z^2(k) .$$

Hence, the bridge is balanced and $V_{AA'} = 0$ (independently of whether AA′ are connected or not). Since this is the case, any impedance may be connected across AA′ without changing Z.

Figure III-19 presents the final impedance configuration of Figure III-14 redrawn in such a way as to illustrate the procedure to be outlined.

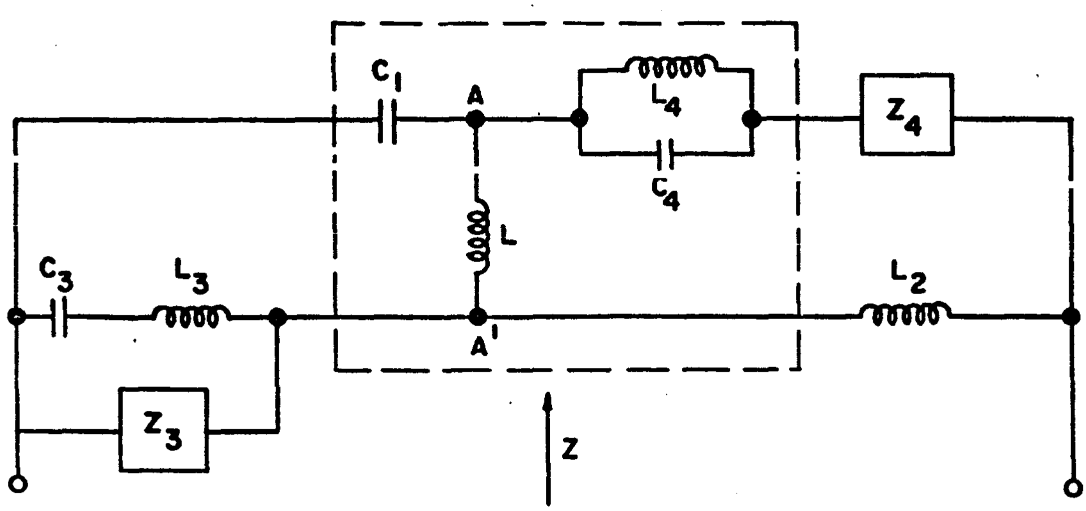


Figure III-19. Modified Impedance Topology of Figure III-14

It is seen that an impedance L has been connected across the terminals and the impedance of Figure III-14 has been manipulated to put into evidence the T network enclosed in the dotted line. It is desired to transform this network into an equivalent delta configuration.

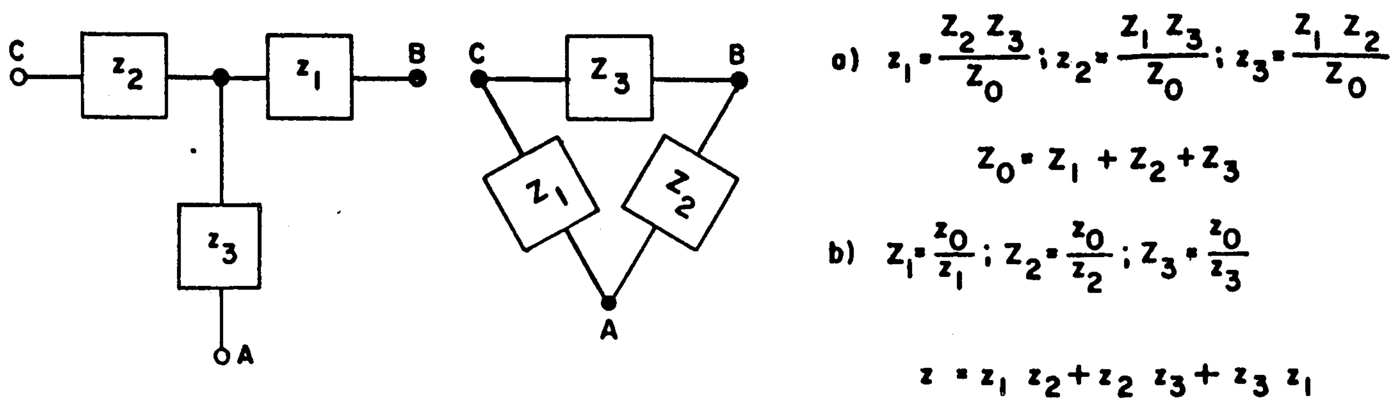


Figure III-20. Delta-Tee Transformation

Figure III-20 summarizes the familiar Δ - T transformation. For the network of interest, the following identifications are made:

$$z_1 = \frac{1}{C_1 s} \tag{III-86}$$

$$z_2 = \frac{L_4 s \, \frac{1}{C_4 s}}{L_4 s + \frac{1}{C_4 s}} = \frac{L_4 s}{L_4 C_4 s^2 + 1} \tag{III-87}$$

$$z_3 = L s$$

$$z_o = \left[\frac{1}{C_1 s}\right]\left[\frac{L_4 s}{L_4 C_4 s^2 + 1}\right] + \left[L s\right]\left[\frac{L_4 s}{L_4 C_4 s^2 + 1}\right] + \left[\frac{1}{C_1 s}\right]\left[L s\right]$$

$$= \frac{\frac{L_4 + L}{C_1}\left[i L_4 \frac{(C_1 + C_2)}{L_1 + L_4} s^2 + 1\right]}{L_4 C_4 s^2 + 1} = \frac{L_4 + L}{C_1} \quad \frac{\tau_1 s^2 + 1}{\tau s^2 + 1} \tag{III-89}$$

Since L is arbitrary, a favorable choice is one that makes $\tau_1 = \tau$. Equating these quantities in Equation (III-89):

$$L L_4 \frac{(C_1 + C_4)}{L_1 + L_4} = L_4 C_4$$

which leads to the result:

$$L = L_4\, C_4 / C_1 \, . \tag{III-90}$$

Substituting the value for L obtained in Equation (III-90) into (III-89):

$$z_o = \frac{L + L_4}{C_1} = \frac{L_4}{C_1^{\,2}} \; (C_1 + C_4) \, . \tag{III-91}$$

From Equations (III-86), (III-88), (III-90), (III-91) and the expressions for Z_1 and Z_3 in Figure III-20:

$$Z_1 = \frac{z_o}{z_1} = C_1\, s\, \frac{L_4}{C_1^{\,2}} \; (C_1 + C_4) = L_4\, C_4 \left(\frac{C_1 + C_4}{C_1\, C_4} \right) s \tag{III-92}$$

and

$$Z_3 = \frac{z_o}{z_3} = \frac{L_4\, (C_1 + C_4)}{L\, s\, C_1^{\,2}} = \frac{L_4\, (C_1 + C_4)}{\left(\frac{L_4\, C_4}{C_1} \right) C_1^{\,2}\, s} = \frac{1}{\frac{C_1\, C_4\, s}{C_1 + C_4}} \, . \tag{III-93}$$

The following definitions are now made:

$$C_o = \frac{C_1\, C_4}{C_1 + C_4} \tag{III-94}$$

$$\omega_1^{\,2} = \frac{1}{L_4\, C_4} \, . \tag{III-95}$$

Substituting Equations (III-94) and (III-95) into (III-92) and (III-93):

$$Z_1 = \frac{s}{C_o\, \omega_1^{\,2}} \equiv L_o\, s \tag{III-96}$$

and

$$Z_3 = \frac{1}{C_o\, s} \, . \tag{III-97}$$

The remaining impedance, Z_2, is obtained from Figure III-20 and Equations (III-87) and (III-91) as:

$$Z_2 = \frac{z_o}{z_2} = \left[\frac{L_4\, C_4\, s^2 + 1}{L_4\, s} \right] \left[\frac{L_4}{C_1^{\,2}} (C_1 + C_4) \right] = \frac{L_4\, C_4\, s^2 + 1}{\frac{C_1^{\,2}\, s}{C_1 + C_4}} \, . \tag{III-98}$$

Consider now a series impedance composed of an inductor L' and a capacitor C'. The impedance, Z', of this arm is $(L' C' s^2 + 1)(C' s)^{-1}$. Equation Z' with Z_2 in Equation (III-98):

$$\frac{L_4 C_4 s^2 + 1}{\dfrac{C_1^2 s}{C_1 + C_4}} = \frac{L' C' s^2 + 1}{C' s} .$$

This expression leads to:

$$C' = \frac{C_1^2}{C_1 + C_4} = \frac{C_1}{C_2} \cdot \frac{C_1}{C_4} \cdot C_o \qquad \text{(III-99)}$$

and

$$L' = \frac{L_4 C_4}{C'} = \frac{L_4 C_4^2}{C_1 C_o} = \frac{C_4}{C_1 C_o \omega_1^2} . \qquad \text{(III-100)}$$

The circuit of Figure III-19 may now be drawn as shown in Figure III-21.

Consider the parallel configuration shown in the dashed lines of Figure III-21b. From Equations (III-99) and (III-100) it is seen that $L' C' = \frac{1}{\omega_1^2}$ and, from Equation (III-69) $L_3 C_3 = \frac{1}{\omega_1^2}$.

Thus,

$$L' C' = L_3 C_3 = \frac{1}{\omega_1^2} .$$

Hence

$$Z' = L' s + \frac{1}{C' s} = \frac{1}{C' s} \left[L' C' s^2 + 1 \right] = \frac{1}{C' s} \left[\frac{s^2}{\omega_1^2} + 1 \right]$$

$$Z_3 = L_3 s + \frac{1}{C_3 s} = \frac{1}{C_3 s} \left[L_3 C_3 s^2 + 1 \right] = \frac{1}{C_3 s} \left[\frac{s^2}{\omega_1^2} + 1 \right].$$

The parallel combination, Z_p, of these impedances is, therefore:

$$Z_p = \frac{Z' Z_3}{Z' + Z_3} = \frac{\frac{s^2}{\omega_1^2} + 1}{s (C' + C_3)} . \tag{III-101}$$

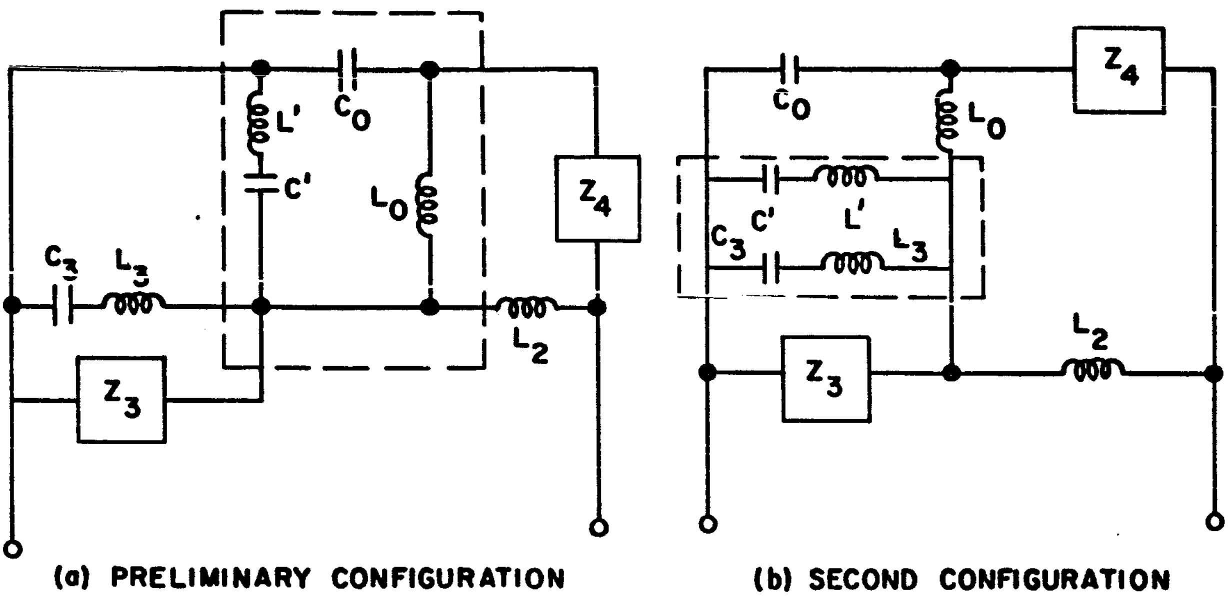


Figure III-21. Modified Bott-Duffin Impedance Configuration

In order to identify the above impedance with a realizable impedance, consider the series combination of an inductor L* and a capacitor C*. The impedance is:

$$Z^* = L^* s + \frac{1}{C^* s} = \frac{L^* C^* s^2 + 1}{C^* s} . \tag{III-102}$$

Equating coefficients between Equations (III-101) and (III-102):

$$C^* = C' + C_3$$

and

$$L^* = \frac{1}{\omega_1^2 C^*} = \frac{L_3 C_3}{C^*} = \frac{L_3 C_3}{C' + C_3} = \frac{L_3}{1 + \frac{C'}{C_3}} . \tag{III-103}$$

However, if $L' C' = L_3 C_3$, then $\frac{C'}{C_3} = \frac{L_3}{L'}$. Hence, Equation (III-103) becomes:

$$L^* = \frac{L_3}{1 + \frac{L_3}{L'}} = \frac{1}{\frac{1}{L_3} + \frac{1}{L'}} . \qquad \text{(III-104)}$$

The final Modified Bott-Duffin network assumes the form presented in Figure III-22b. Figure III-22a presents the original Bott-Duffin network for purposes of comparison.

Note that this procedure has two advantages:

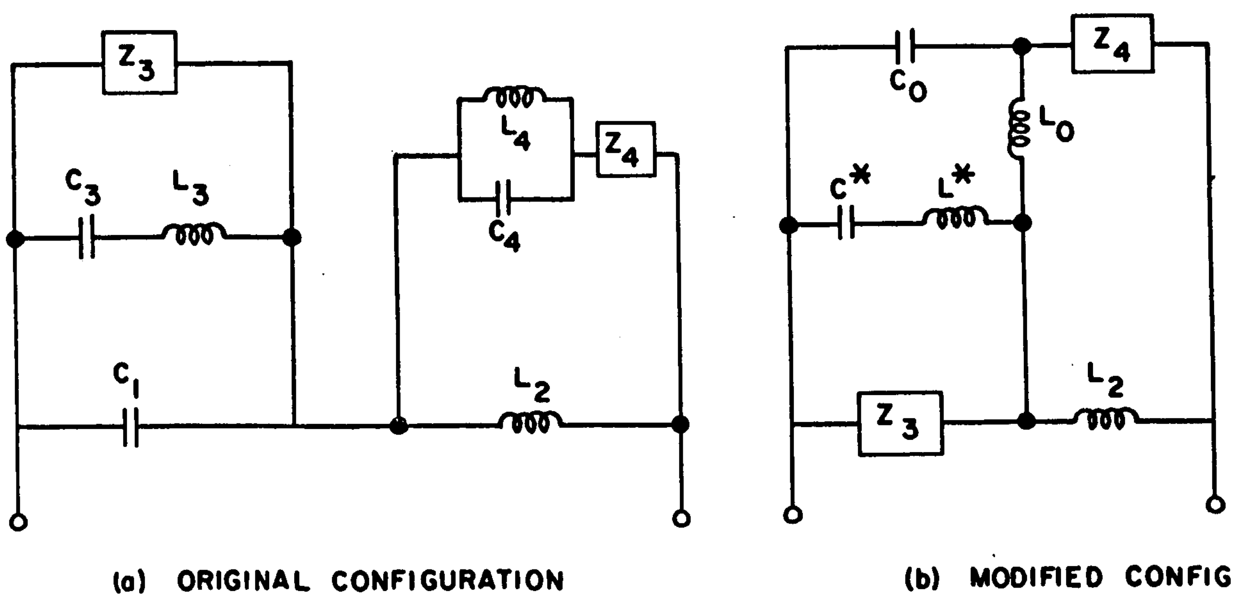
a) one element has been eliminated;

b) the impedance is no longer in the form of a balanced bridge and hence not so sensitive to component tolerances.

<u>Case II</u> $Z(j\omega_1) = -j\,\underline{\overline{X}}_1,\ \underline{\overline{X}}_1 > 0$

For the case where $Z(j\omega_1) = -j\,\underline{\overline{X}}_1$, a procedure similar to one utilized above may be followed. The results are presented in Figure III-23.

An example of the application of the Bott-Duffin and modified Bott-Duffin synthesis procedures to a biquadratic impedance function is presented in Appendix B.



$C_0 = \frac{C_1 C_4}{C_1 + C_4}$

$L_0 = \frac{1}{C_0 \omega_1^2}$

$C^* = \frac{C_1^2}{C_1 + C_4}$

$L^* = \frac{1}{\omega_1^2 C^*}$

Figure III-22. Final Modified Impedance Configuration for $Z(j\omega_1) = j\,\underline{\overline{X}}_1,\ \underline{\overline{X}}_1 > 0$

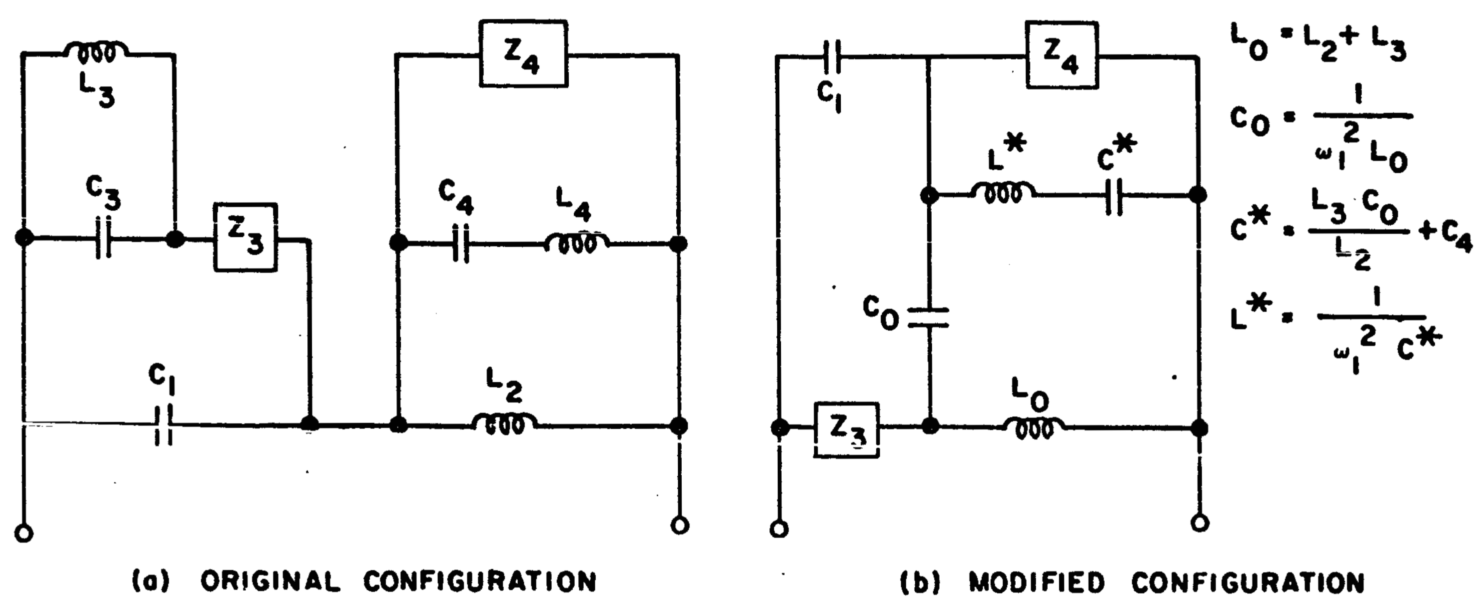


Figure III-23. Final Modified Impedance Configuration for $Z(j\omega_1) = -j\,\overline{X}_1,\ \overline{X}_1 > 0$

5. <u>Impedance Scaling for the Assumed Load Resistance</u>

Section III-a demonstrated that for a constant resistance ladder network, the branch impedances Z_a' and Z_b' (Figure III-23) are related to the component transfer function F(s) and the load resistance R_L by the equations:

$$Z_b' = R_L \left[\frac{1}{F(s)} - 1\right] \text{and } Z_a' = R_L \left[1 + \frac{R_L}{Z_b'}\right].$$

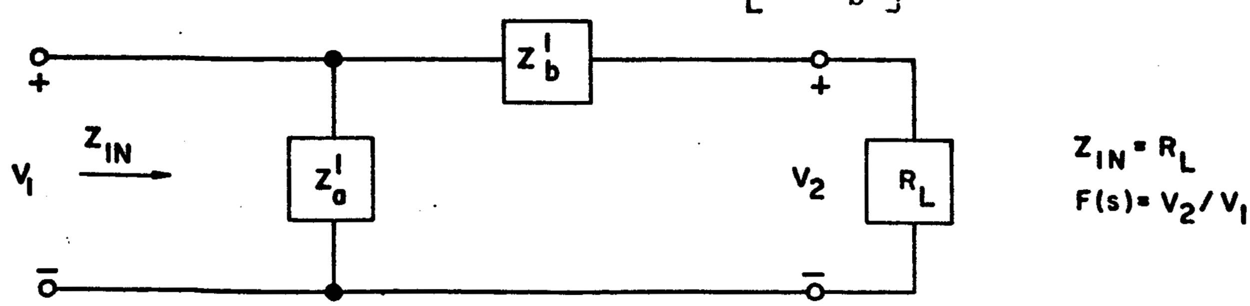


Figure III-24. Constant Resistance Ladder Network

For the initial step in synthesis of the branch impedances, Z_a' and Z_b', R_L is normalized to unity. Thus, the initial impedances, Z_a and Z_b, are given by:

$$Z_b = \frac{1}{F(s)} - 1 \text{ and } Z_a = 1 + \frac{1}{Z_b} . \qquad \text{(III-105)}$$

The desired impedances are then obtained from Z_a and Z_b by:

$$Z_b' = R_L \left[\frac{1}{F(s)} - 1 \right] = R_L Z_b \qquad \text{(III-106)}$$

$$Z_a' = R_L \left[1 + \frac{R_L}{Z_b'} \right] = R_L \left[1 + \frac{1}{Z_b} \right] = R_L Z_a \qquad \text{(III-107)}$$

The assumed load resistance for the NASA compensation network is R_L = 800 ohms. Hence, the impedance elements of Z_a and Z_b must be scaled up by a factor of 800. This is accomplished as follows:

Resistors:

$$R_{Z_a, Z_b} \rightarrow 800 R_{Z_a, Z_b} = R_{Z_a', Z_b'} . \qquad \text{(III-108)}$$

Inductors:

$$L_{Z_a, Z_b} \rightarrow 800 L_{Z_a, Z_b} = L_{Z_a', Z_b'} . \qquad \text{(III-109)}$$

Capacitors:

$$C_{Z_a, Z_b} \rightarrow \frac{C_{Z_a, Z_b}}{800} = C_{Z_a', Z_b'} . \qquad \text{(III-110)}$$

SECTION IV

TRANSFER FUNCTION FACTORING AND LADDER NETWORK SYNTHESIS

A. GENERAL

This section covers the following items:

1) factoring the over-all desired compensation function so as to yield positive real, non-minimum component transfer functions that can be realized as separate stages of a constant resistance ladder network;

2) synthesis of the branch impedances of the various stages of the ladder network using the previously described synthesis procedures;

3) modification of the factoring process so that the resulting ladder network will have more reasonably sized passive elements and less DC attenuation. (Any such DC attenuation in the network must be compensated by a gain factor in front of the network, and it is desired to minimize the required signal amplification.)

B. INITIAL TRANSFER FUNCTION FACTORING

The desired compensation transfer function obtained by approximating the gain-phase requirements given by NASA is:

$$F_D(s) = \frac{L(20)\ Q\left(\frac{0.5}{20}\right) Q\left(\frac{0.4}{126}\right) Q\left(\frac{0.4}{126}\right) Q\left(\frac{0.4}{126}\right)}{L(4)\ L\ (2000)\ L(2000)\ Q\left(\frac{0.26}{45}\right) Q\left(\frac{0.7}{45}\right) Q\left(\frac{0.96}{45}\right)} . \qquad \text{(IV-1)}$$

(This is actually the modified form of $F_D^1(s)$, with a non-unity DC gain to be subsequently compensated.) The numerator and denominator terms of $F_D(s)$ are linear or quadratic factors in s with real, positive coefficients. The notation L (ω_L) denotes a linear term, $S + \omega_L$, while Q (ξ/ω_N) denotes $S^2 + 2\ \xi\ \omega_N S + \omega_N^2$.

The numerator and demominator terms must be grouped so that each component transfer function $\frac{N_i}{D_i}$, of the over-all transfer function is a positive real, non-minimum function. Figure IV-1 presents a simplified schematic of the initial and subsequent iteration processes utilized in factoring the desired F_D (s), subject to the above conditions. The material presented in the following paragraphs of this section is basically an elucidation of this figure.

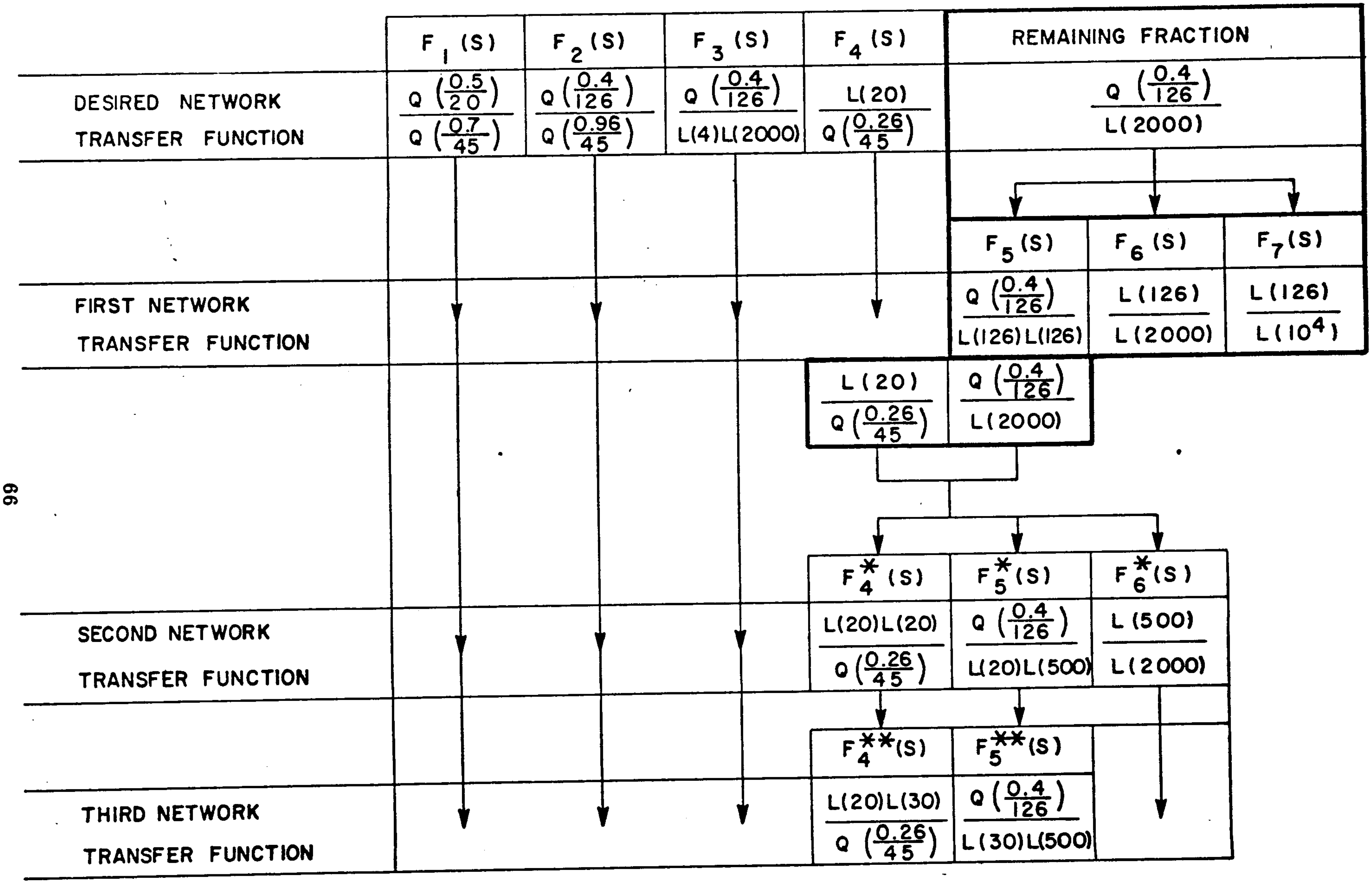


Figure IV-1. Factoring Flow Diagram for the Over-all Transfer Function

The denominator term $Q\left(\frac{0.26}{45}\right)$ can only be combined with the numerator term L(20) to yield a PRNM function as defined in Appendix A. Of the two remaining numerator forms present, that is, $Q\left(\frac{0.5}{20}\right)$ and $Q\left(\frac{0.4}{126}\right)$, the denominator term $Q\left(\frac{0.7}{45}\right)$ can only be combined with the numerator term $Q\left(\frac{0.5}{20}\right)$ to yield a PRNM function. The denominator term $Q\left(\frac{0.96}{45}\right)$ can be combined with one of the three $Q\left(\frac{0.4}{126}\right)$ factors in the numerator. The second $Q\left(\frac{0.4}{126}\right)$ factor of the numerator can be combined with the denominator terms, L(4) and L(2000).

Of the original factors, the fraction $\dfrac{Q\left(\frac{0.4}{126}\right)}{L(2000)}$ then remains. It is a non-realizable transfer function, since the order of the numerator exceeds the order of the denominator.

It now becomes necessary to introduce realizability factors of the form $\frac{L(a)}{L(a)}$ and $\frac{1}{L(b)}$, where a and b are real and positive and where b is large enough to cause any significant gain-phase variation to lie beyond the frequency range of interest. These introduced factors must be chosen so that when they are grouped with the remaining uncombined factors of the over-all transfer function, the PRNM requirement for each grouping will be satisfied.

The factors $\frac{L(126)}{L(126)}$, $\frac{L(126)}{L(126)}$, and $\frac{1}{L(10^4)}$ were initially chosen, because it was felt that the resulting network realizations would not contain any large passive elements and $L(10^4)$ was well beyond the frequency range of interest. These factors were grouped with $\dfrac{Q\left(\frac{0.4}{126}\right)}{L(2000)}$ into PRNM component transfer functions as follows:

$$\frac{Q\left(\frac{0.4}{126}\right)}{L(126)\,L(126)}, \quad \frac{L(126)}{L(2000)} \text{ and } \frac{L(126)}{L(10^4)} .$$

Thus, the over-all desired transfer function has been approximated by the product of component transfer functions as follows:

$$F_D(s) \cong F^{(1)}(s) = \prod_{i=1}^{7} F_i = \frac{Q\left(\frac{0.5}{20}\right)}{Q\left(\frac{0.7}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{Q\left(\frac{0.96}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{L(4)\,L(2000)} \frac{L(20)}{Q\left(\frac{0.26}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{L(126)\,L(126)}$$

$$\frac{L(126)}{L(2000)} \frac{L(126)}{L(10^4)} \qquad \text{(IV-2)}$$

This is the transfer function for the first ladder network to be synthesized.

C. INITIAL LADDER NETWORK SYNTHESIS

To meet the realizability requirements for the branch impedances of the individual stages of the ladder network, realizability gain factors, K_i, are introduced so that $K_i \, \mathrm{Re}\left[\frac{1}{F_i(j\omega)}\right]_{\mathrm{minimum}} = 1.0$ as described in Appendix A. (Again, a gain factor K_T is to be provided external to the over-all ladder network, to compensate for the Ki's thus introduced to provide a DC gain of unity.)

Each component transfer function of Equation (IV-2) has been synthesized as a separate stage of a constant resistance ladder network with the branch impedance of the i th stage, Z_{a_i} and Z_{b_i}, assuming unity load resistance, given by:

$$Z_{b_i} = \frac{K_i}{F_i} - 1 \text{ and } Z_{a_i} = 1 + \frac{1}{Z_{b_i}}.$$

In computing the element values for the branch impedance of each of these stages, the specific techniques associated in Table III-1 with the various forms of the component transfer functions (inspection, continued fractions, Bott-Duffin) were employed as previously described in detail in Section III. The impedance elements were scaled by a factor of 800 to account for a load resistance of 800 ohms.

The resultant over-all ladder network configuration 1, with a transfer function as given by Equation (IV-2), is shown in Figure IV-2. It contains 59 elements and the over-all gain factor, K_T, that will yield the final required compensation network given as 537,600. The component transfer functions for each of the individual stages of the ladder network are indicated, as well as the associated K_i and K_{T_i} values.

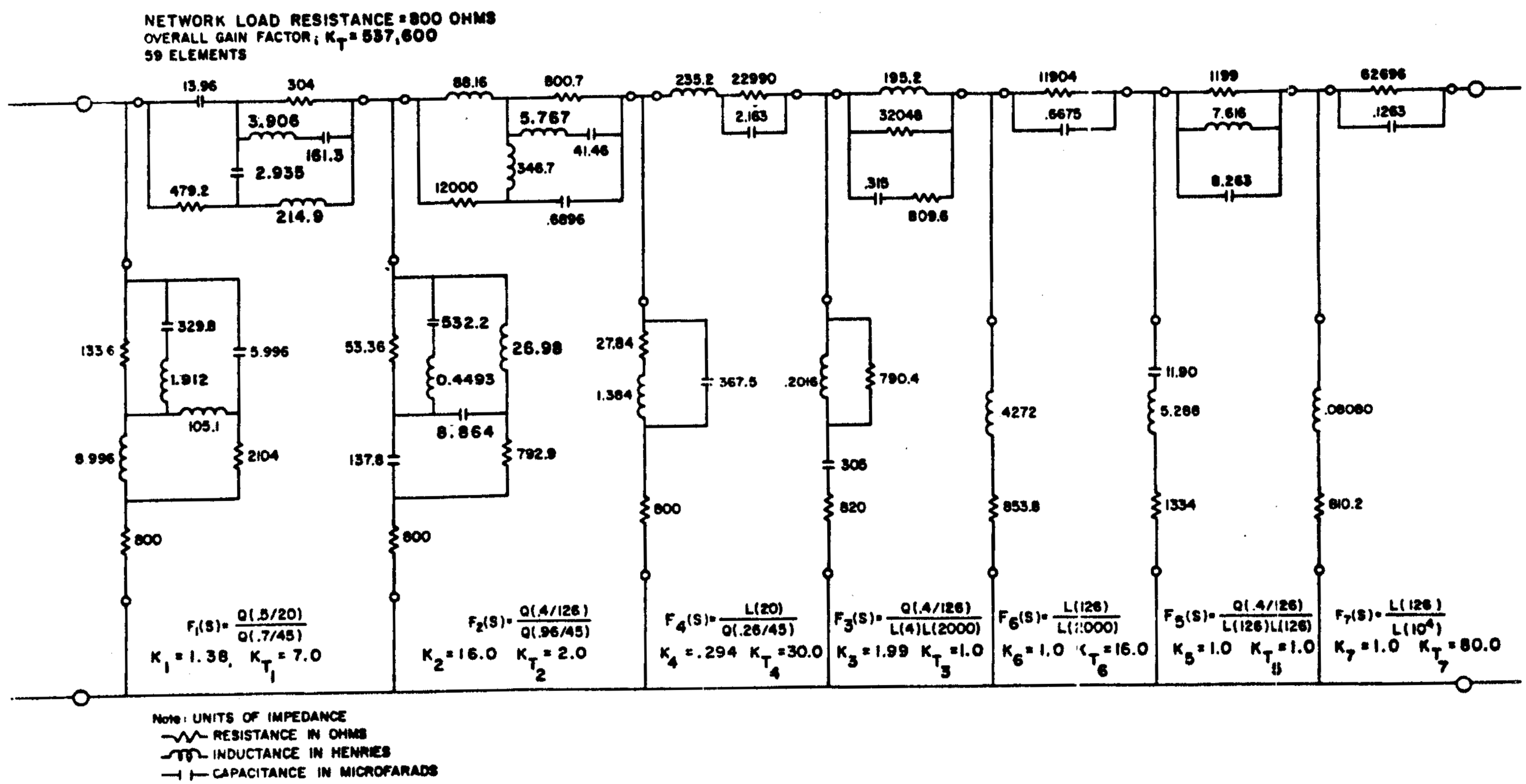


Figure IV-2. Ladder Network -1

D. IMPROVED LADDER NETWORKS

A modification of the first factor grouping was suggested by the relative magnitude of the K_{T_i}'s (K_{T_4}, K_{T_6} and K_{T_7}) in an effort to reduce the required over-all K_T. For the original remaining fraction $\frac{Q\left(\frac{0.4}{126}\right)}{L(2000)}$, factors of the form $\frac{L(a)}{L(a)}$ and $\frac{1}{L(b)}$ were considered for grouping as $\frac{Q\left(\frac{0.4}{126}\right)}{L(2000)\ L(a)} \cdot \frac{L(a)}{L(b)}$ with b again $= 10^4$.

For the transfer function form $\frac{Q\ (\zeta_s/\omega_N)}{L(\omega_1)L(\omega_2)}$, the minimum value of the real part of the reciprocal generally occurs at $\omega = 0$, and this minimum value $f_o(\omega) = \frac{\omega_1\,\omega_2}{\omega_N^2} = \frac{2000a}{(126)^2}$. The optimum value for the gain constants K_i and K_{T_i} are unity. This result is achieved by setting:

$$2000\ a = 15,876$$
$$\text{or}\ a = 7.94.$$

However, with $b = 10^4$, the over-all gain factor K_{T_i} for $\frac{L(7.94)}{L(10^4)} = 1260$. This suggested dropping the factor $\frac{1}{L(10^4)}$, and combining L (7.94) with the fraction $F_4(s) = \frac{L(20)}{Q\left(\frac{0.26}{45}\right)}$. Unfortunately, $\frac{L(20)\ L(7.94)}{Q\left(\frac{0.26}{45}\right)}$ was not PRNM, and so this value for a had to be discarded.

Examination of the factors at this point indicated the following possibilities:

$$\frac{L(20)\ L(a)}{Q\left(\frac{0.26}{45}\right)}\ \text{with either}\ \frac{Q\left(\frac{0.4}{126}\right)}{L(2000)\ L(a)}\ \text{or}\ \frac{Q\left(\frac{0.4}{126}\right)}{L(a)\ L(b)}\ \frac{L(b)}{L(2000)}\ .$$

Empirical numerical studies were conducted varying a and b. Some combinations were immediately discarded because the factors were not PRNM.

In evaluating the realizability gain factor K_i for some of these transfer functions, three transfer functions of the $\frac{Q\left(\frac{0.4}{126}\right)}{L(\omega_1)\ L(\omega_2)}$ form gave odd results; the real part of two of them had a minimum value at infinite frequency and for the third had the same apparent minimum value at two different frequencies. Time did not permit further investigation of the reasons for these peculiar results. One acceptable combination was discarded because the realization of the network required a passive element of impractical size.

Minimizing the individual K_{T_i} values was one of the basic criteria throughout. The transfer function for the second network configuration was considered a vast improvement over the first. This transfer function is

$$F_D(s) \cong F^{(2)}(s) = \prod_{i=1}^{6} F_i(s) = \frac{Q\left(\frac{0.5}{20}\right)}{Q\left(\frac{0.7}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{Q\left(\frac{0.96}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{L(4)\,L(2000)} \frac{L(20)\,L(20)}{Q\left(\frac{0.26}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{L(20)\,L(500)} \frac{L(500)}{L(2000)}$$

and the corresponding network is presented in Figure IV-3. Seven ladder stages of the first network were reduced to six, giving an $\frac{LL}{Q}$ form for $F_4^*(s)$ which required a Bott-Duffin synthesis with 15 elements. Although this increased the over-all number of network elements to 65 from the previous number of 59, the tremendous reduction in over-all gain factor K_T to 1736 was considered far more important.

Methods of refinement, which became apparent as these investigations continued, led to a further change in the factor $\frac{L(a)}{L(a)}$ from a = 20 to a = 30. This third configuration, shown in Figure IV-4, has a transfer function given by:

$$F_D(s) \cong F^{(3)}(s) = \frac{Q\left(\frac{0.5}{20}\right)}{Q\left(\frac{0.7}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{Q\left(\frac{0.96}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{L(4)\,L(2000)} \frac{L(20)\,L(30)}{Q\left(\frac{0.26}{45}\right)} \frac{Q\left(\frac{0.4}{126}\right)}{L(30)\,L(500)} \frac{L(500)}{L(2000)}$$

This network has the same number of elements as network 2 (65), but the over-all compensation gain factor K_T has been reduced to 700.

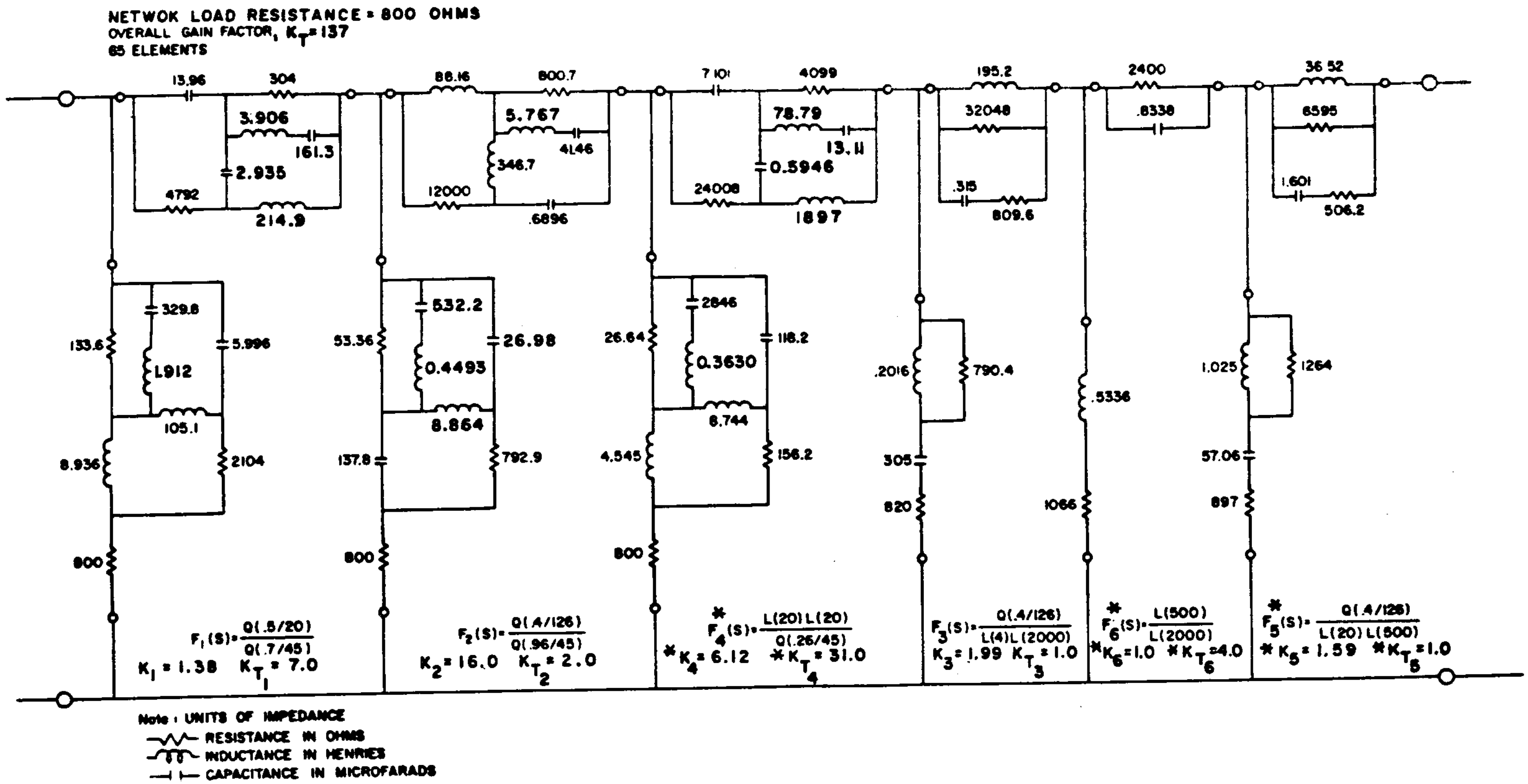


Figure IV-3. Ladder Network -2

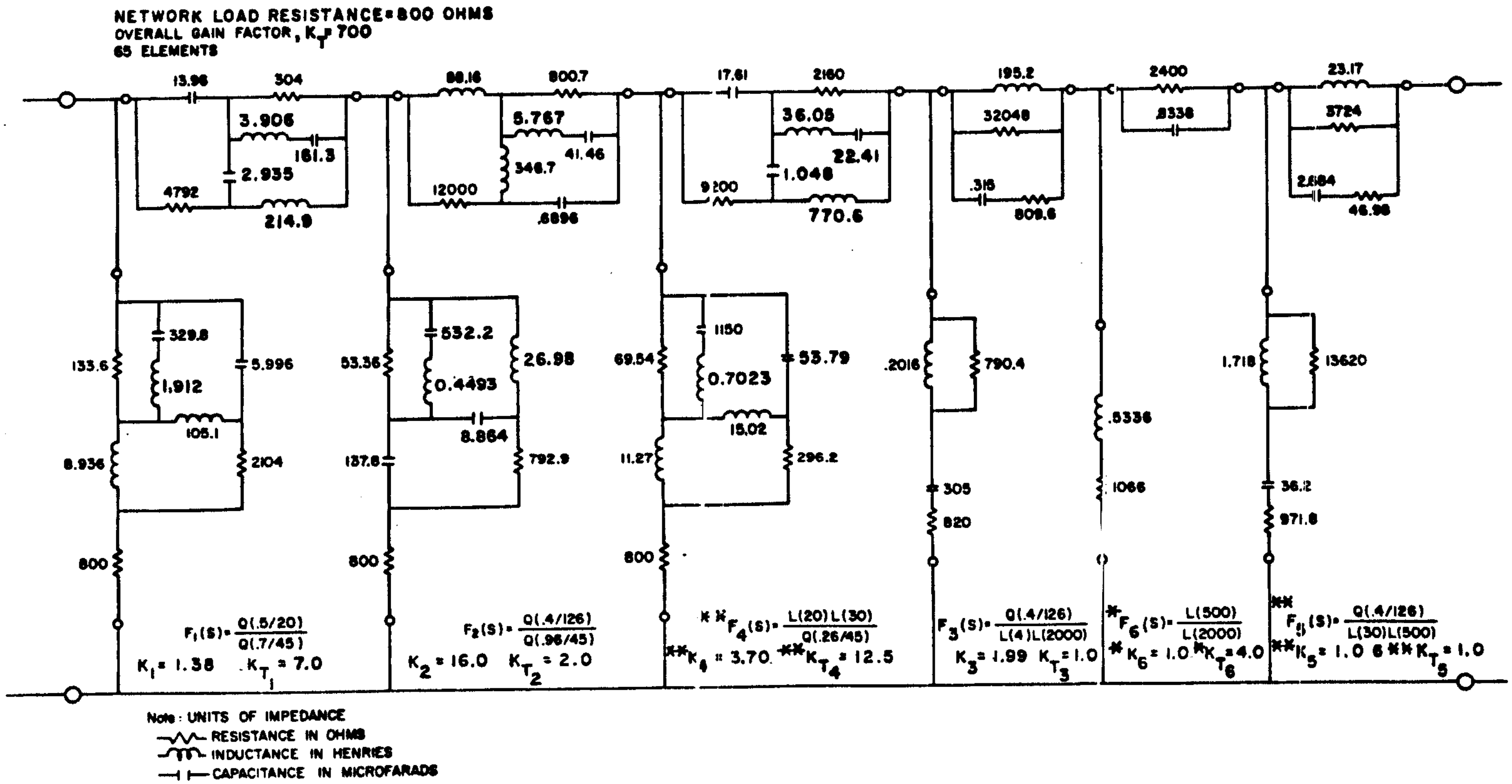


Figure IV-4. Ladder Network -3

SECTION V

SUMMARY AND CONCLUSIONS

Three constant-resistance ladder networks, that have voltage transfer functions meeting the gain-phase compensation requirements set by NASA with an assumed load resistance of 800 ohms, have been synthesized. By iteration of the transfer function factoring and network synthesis operation, a reasonably practical network configuration has been realized. This network, presented in Figure IV-4, has 65 passive elements in its six cascaded stages. The required over-all compensation gain for it is 700. The maximum inductor size is 770.6 henries and the maximum capacitor size is 532 microfarads.

A schematic of the complete operation in going from the NASA gain-phase compensation requirements to the final, improved network configuration is given in Figure IV-4 . The requirements for factoring the desired network transfer function so that these component transfer functions can be realized as stages of a constant resistance ladder network are derived in section III-A and are summarized in section IV For all realizable forms of these factors, section III-B describes the synthesis procedures to be used for the associated branch impedances (Table III-1), and section III-B describes the application of these techniques in realizing the desired transfer function. The evolution of the ladder compensation network to a more optimum configuration is described in section IV and illustrated in Figure IV-1.

From the studies which have been completed, it would seem that further refinements are possible, starting from the first step of the over-all synthesis operation. The curve-fitting procedure should be re-evaluated with the difficulties of realization of various types of factors taken into consideration.

Emphasis should be placed on minimizing the number of quadratic denominator terms, because generally they must be synthesized by the Bott-Duffin technique yielding a network of 15 elements, unless they are combinable with a linear numerator term. The denominator quadratic form was inefficient in the

NASA network where the power of the numerator and denominator polynonials were the same. However, it could possibly be effectively utilized when such is not the case, and it could be more easily synthesized by the continued fraction approach.

Quadratic form numerator terms should be combined with two linear form denominator terms where possible, especially when additional realizability factors are introduced by K_i and K_{T_i} values are more easily optimized. The synthesis of this form also may be accomplished via the continued fraction method.

Linear term fractions are simple to synthesize and should be combined where possible, avoiding large K_{T_i} values. The feasibility of this type of early planning remains to be investigated.

A satisfactorily effective procedure has been evolved for factoring a given transfer function and synthesizing it by a constant resistance ladder network. However, additional work in this area could hopefully lead to simpler, more straightforward procedures for realization of a network which is generally optimum in terms of its required compensation gain and the number and size of its passive elements.

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APPENDIX A

POSITIVE REAL NON-MINIMUM FUNCTIONS

A function, F(s), is called positive real (PR) if F(s) is real when s is real and if the real part of F(s), Re F(s), is non-negative for Re(s) ≥ 0. This definition is equivalent to:

1) $F(j\,\omega)$ has no poles or zeros in the right half plane,

2) any poles or zeros of $F(j\,\omega)$ on the imaginary axis are simple with a real positive residue,

3) $\operatorname{Re} F(j\,\omega) \geq 0$ for $0 \leq \omega \leq \infty$.

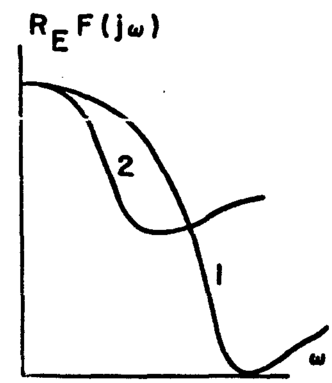


Figure A-1. Positive Real Function

Two simple PR functions are shown in Figure A-1. The (1) curve depicts a PR function with a vanishing real part, termed a positive real, minimum resistance function (PRMR). The (2) curve depicts a PR function with a non-vanishing real part, termed a positive real, non-minimun function (PRNM).

Figure A-2 presents the PRNM forms which are needed to apply the constant resistance technique. For many transfer functions, a suitable grouping of terms yields a product of PRNM terms. If this is not possible, the simple technique next discussed may be used.

Suppose the given transfer function is:

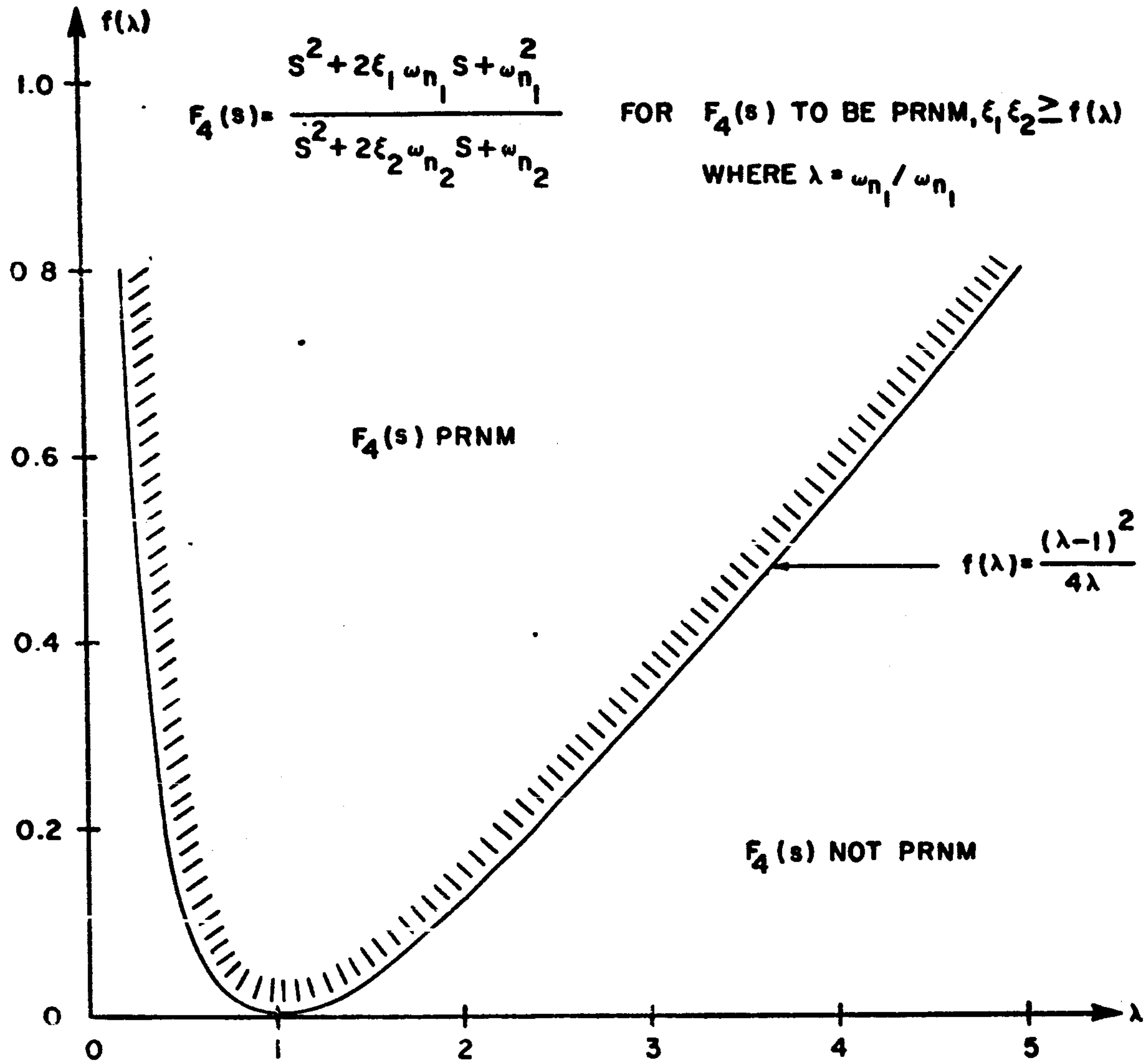
$$F(s) = \frac{K}{s^2 + 2\xi\,\omega_\eta\,s + \omega_\eta^2} \qquad \text{(A-1)}$$

This is not a PRNM function; however, a factor F(s) can be introduced as shown:

$$F(s) = F(s)\ \frac{G(s)}{G(s)} = \frac{s + a}{s^2 + 2\,\xi\,\omega_\eta s + \omega_\eta^2} \bullet \frac{Ka}{s + a}$$

where a is chosen so that (5) of Figure A-2 is satisfied.

In this way any stable transfer function may be realized as a product of PRNM functions. Note that while the individual stage transfer functions must be PRNM, the composite transfer function does not have to be PRNM.



OTHER PRNM FORMS

1. $F_1(s) = \dfrac{K}{S+a}$
2. $F_1^{-1}(s)$
3. $F_2(s) = \dfrac{K(S+a)}{(S+b)}$
4. $F_2^{-1}(s)$
5. $F_3(s) = \dfrac{S+\omega_L}{S^2 + 2\xi \omega_n S + \omega_n^2}$ $\quad 2\xi\omega \geq \omega_L$

Figure A-2. Positive Real, Non-Minimum (PRNM) Forms

APPENDIX B

BOTT-DUFFIN SYNTHESIS OF A BIQUADRATIC IMPEDANCE FUNCTION

As an example of the Bott-Duffin synthesis procedure, consider the following impedance function:

$$Z_b(s) = \frac{0.38\, s^2 + 66.94\, s + 2394.5}{s^2 + 20\, s + 400} \ . \tag{B-1}$$

The real part of $Z_b(s)$ is obtained from Equation (B-1) as:

$$\mathrm{Re}\ Z_b(s) = \frac{(2394.5 - 0.38\,\omega^2)(400 - \omega^2) - 1338.8\,\omega^2}{(400 - \omega^2)^2 + 400\,\omega^2}, \tag{B-2}$$

from which it is found that

$$\mathrm{Re}\ Z_b\,(j\omega) = 0 \text{ at } \omega = 39.82 \equiv \omega_1 \ . \tag{B-3}$$

Substituting the value obtained in (B-3) into (B-1), it is found that:

$$\mathrm{Im}\ Z_b\,(j\omega_1) = -j\,\underline{\overline{X}}_1 = -j\,2.248. \tag{B-4}$$

Substituting Equations (B-3) and (B-4) into Equation (III-75)

$$C_1 = \frac{1}{\omega_1\,\underline{\overline{X}}_1} = 0.01117 \text{ farad}. \tag{B-5}$$

From Equations (B-3) and (B-4), $\omega_1\,\underline{\overline{X}}_1 = 89.52$; substituting this value into Equation (III-76), there obtains:

$$k^3 - 59.4\,k^2 + 1589.7\,k - 94232 = 0. \tag{B-6}$$

This then is solved for the positive value of k which is found to be

$$k_o = 59.3. \tag{B-7}$$

Substituting this value into Equation (B-1):

$$Z\,(k_o) = 1.5096. \tag{B-8}$$

Substituting (B-7) and (B-8) into Equation (III-55a):

$$L_2 = \frac{Z(k_o)}{k_o} = 0.02546 \text{ henry}. \tag{B-9}$$

From Equations (III-49) and (B-7) and (B-8):

$$R(s) = 3.97 \left[\frac{s^2 + 64.37\, s + 1586}{s^2 + 1586} \right]. \qquad \text{B-10}$$

From Equations (B-10), (III-52), and (III-53):

$$Z_1 = 5.99 \left[\frac{s^2 + 64.37\, s + 1586}{s^2 + 1586} \right]$$

$$Z_2 = 0.38 \left[\frac{s^2 + 1586}{s^2 + 64.375\, s + 1586} \right].$$

Using the expansions given in Equations (III-77) and (III-78):

$$Z_1 = \frac{385.58}{s^2 + 1586} + 5.99$$

$$Z_2^{-1} = \frac{169.29}{s^2 + 1586} + 2.63.$$

Hence:

$$Z_3 = 5.99$$

$$Z_4 = 0.38.$$

From Equations (III-79), (III-80), (III-81), and (III-82):

$C_3 = 0.00259$ farad

$L_3 = 0.2431$ henry

$C_4 = 0.1067$ farad

$L_4 = 0.00591$ henry

From the equations of Figure III-23, the modified elements are:

$C_o = 0.002348$ farad

$L_o = 0.2686$ henry

$L^* = 0.004883$ henry

$C^* = 0.1291$ farad

As discussed in Section III, these passive element values correspond to a normalized load resistance of 1 ohm. The elements scaled up to account for the actual 800-ohm load resistance are as follows:

C_1	= 13.96	microfarads
Z_3	= 4792	ohms
Z_4	= 304	ohms
C_o	= 2.935	microfarads
L_o	= 214.9	henries
L^*	= 3.906	henries
C^*	= 161.3	microfarads