

TRANSIENT DYNAMICS &  
THERMAL STRESS FOR NUCLEAR  
ROCKET HEAT - EXCHANGER

by

T. C. Liu

and

C. N. Shen

Research Grant Number Nsg 14-59 Supplement (2)  
Sponsored by  
Atomic Energy Commission  
and  
National Aeronautics and Space Administration

at

Department of Mechanical Engineering  
Rensselaer Polytechnic Institute  
Troy, New York

Mechanical Engineering Dept. Report No. AC5  
SEPT, 7, 1966

## CONTENTS

### SUMMARY

- (1) Introduction
- (2) Type of Reactors
- (3) Heat Transfer Problems for Nuclear Reactors
  - (a) Two-Dimensional Approach for Steady-state Solutions
  - (b) Equivalent One-Dimensional Approach for Steady-state Solutions
  - (c) Two-Dimensional Approach for Transient Solutions
  - (d) Equivalent One-Dimensional Approach for Transient Solutions
  - (e) **Perturbation** Method for the Transient Solutions
- (4) Thermal Stresses
  - (a) General Thermal Stresses
  - (b) Three-Dimensional Quasi-Static Thermal Stresses
- (5) Sudden Increase of Flow Rate at Constant Full Power
  - (a) Solution of Temperature Distribution
  - (b) State Variable Constraint due to Temperature Distribution in x-Direction
  - (c) Numerical Example
  - (d) Some Observations
- (6) Shut-down of Power at Constant Full Flow
  - (a) The Heat Transfer Problem
  - (b) Thermal Stresses due to Temperature Distribution in x-Direction
  - (c) Numerical Example

(7) References

Appendix A Historical Review

Appendix B Reactor Calculations for a Nuclear Rocket

Appendix C Derivation of Energy Balance Equation

Appendix D Transient Solutions of Heat Transfer Problem  
by One-Dimensional Approach

Appendix E Transient Solutions by Perturbation

Appendix F General Equations for Thermal Stresses

Appendix G Thermal Elasticity

Appendix H Compatibility Condition

#### SUMMARY

Computation of the thermal stresses in a nuclear rocket is of importance from the standpoint of ensuring the necessary strength during reactor operation. This information is necessary to determine the control problem of a nuclear rocket.

An increase of hydrogen flow (coolant flow) is desirable in operating a reactor so as to eventually lower the steady-state core temperature. This increase of hydrogen flow not only introduces a higher temperature gradient in fuel element but also takes a higher heat transfer coefficient. Thus, excessive hydrogen flow at the rated full power condition yields the most critical thermal stresses.

The equivalent one-dimensional approach for the transient heat transfer problem in a nuclear reactor is discussed in details. From this approach, the temperature distribution of fuel element and coolant channel are obtained in simple form and with sufficient accuracy. The transient thermal stresses are proportional to the difference of temperature between the fuel element and coolant channel. The analytical solutions of the thermal transient and stresses are given for a sudden change of the hydrogen flow. From several numerical examples, it is

shown that the transient thermal stresses can be much higher than the steady state full power thermal stresses. The maximum transient to steady-state full power thermal stresses should be limited to a proper ratio to insure that the design of the reactor is within the margin of safety.

(1) Introduction

The steady-state thermal stresses are proportional to the power density for a given geometry of the core. Due to a sudden increase of hydrogen flow, the transient thermal stresses can be much higher than the steady-state full power thermal stresses during a period of time. If the rated power of a reactor increases, the power density also increases while the size of a reactor only increases slightly. Thus the ratio of transient to steady-state full power thermal stresses is very much limited if the design of the steady-state thermal stresses is near the margin of safety. The modern control theory gives a method of limiting these thermal stresses by controlling the hydrogen flow correctly.

The study of optimum control was concentrated on lumped parameter processes described by ordinary differential equations. Thus the common practice of introducing a state variable constraint on the optimum control of nuclear rockets is to limit the temperature rate or the derivative of temperature in the time domain for the purpose of limiting thermal stresses.

It is found from the present study that the spatial temperature gradients are more important. This is based on the study of thermal stresses of a nuclear reactor by introducing a distributed parameter model. For example, if there is a sudden increase of hydrogen flow at the steady-state conditions, it is found that the criterion for

limiting the thermal stresses is to constrain the product of the heat transfer coefficient and the difference of the temperature of the fuel plate and hydrogen coolant.

The present interest is to examine the transient thermal stresses in slab or cylindrical geometry due to the increase of hydrogen flow under full power condition. With this information it is hoped that the transient thermal stresses in the fuel element could be controlled as steady as possible while subject to a time varying increase of hydrogen flow under full power condition.

A historical review is given in Appendix A on the subjects of heat transfer, thermal stresses, optimum control in distributed parameter systems and the optimum control with state variable constraints.

## (2) Types of Reactors

The types of reactors which can be considered for a nuclear rocket are those with either a solid core, liquid core or gaseous core. Since a liquid core reactor requires a more complicated control system and since the technology for gaseous core reactors (1) is far behind at the present time, the solid core reactor is most suitable for a space vehicle with a large payload in the near future.

Three general types of solid core reactor are thermal, homogeneous; thermal, heterogeneous; and fast reactors. In a homogeneous thermal reactor, graphite or BeO may be used as the moderator and the fissionable material is mixed with the moderator. The heat of fission is liberated in the moderator-fuel region and is removed by hydrogen propellant. The highest operating temperature is limited by the material of the moderator. In comparison with BeO graphite possesses

- (i) better mechanical properties at high temperature,
- (ii) higher melting point, and
- (iii) poorer moderating power.

However, the specific impulse of hydrogen is proportional to the square root of the exhaust temperature. Very high temperature is required to obtain high specific impulse. To withstand such an extreme temperature graphite is really the only contender (1) (2) (3).

Most authors propose the core of a homogeneous reactor for space flight should have a graphite matrix (2) (4) (5) impregnated with U235. The atom ratio (5) (7) of carbon is



about 500 which ensures the criticality of the reactor and the retention of the properties of graphite in the fuel region. For this atom ratio, the estimated size of reactor including reflector is about 4 ft. in diameter and 4 ft. in height with 20% void as given in Appendix B. A 6 inch BeO reflector surrounds the surface of the active core. Two feasible constructions are suggested as follows:

(i) A right circular cylinder composed of a homogeneous graphite-uranium matrix in which circular holes are drilled parallel to the axis for coolant flow<sup>(2)(6)</sup>.

(ii) Coolant channels are parallel to the axis of the core. A number of plate type elements (composed of C-U matrix) are bound together to form a unit. A number of these units forms a cylindrical core<sup>(4)(6)</sup>. The second type construction is considered here because of the simpler geometry and mathematical analysis.

The design temperature of graphite is about 5,000°R. The temperature of hydrogen at exit should, therefore, be around 4,500 °R. The inlet temperature of hydrogen is taken as 180 °R which is slightly higher than the critical temperature<sup>(8)</sup> (60°R). For plate type construction, the thickness of the fuel element can be calculated by assuming that,

(i) The number of coolant channels is 212 for compromise of structure and heat transfer problems, and

(ii) The void is 20% for compromise of volume of reactor due to criticality and heat transfer problems. The thickness of a fuel element and the width of coolant channel are obtained from Appendix B as 0.12 in. and 0.03 in., respectively.

This value is in the range given by Levoy and Newgard<sup>(4)</sup> as reasonable. It is found that the thickness of the coolant channel is smaller than that of a fuel element.

(3) Heat Transfer Problems for Nuclear Reactors

The heat conduction pattern is approximately similar from channel to channel if the power generation is uniform in the radial direction and if the heat escape from the outer surface is negligible. The core can, therefore, be represented by a unit plate. Thus the heat conduction equation for the fuel element is given as<sup>(9)</sup>

$$\nabla k \nabla T_U(\bar{r}, \tau) + P(\bar{r}, \tau) = C \rho \frac{\partial}{\partial \tau} T_U(\bar{r}, \tau), \quad (3.1)$$

where  $T_U$  = temperature distribution in a fuel element,

$P$  = heat source per unit volume per unit time,

$k$  = thermal conductivity of the fuel element,

$\rho$  = density of the fuel element,

$C$  = specific heat of the fuel element,

and  $\bar{r}$  = position vector.

The differential equation which governs the heat flow in the coolant is the following energy balance (see appendix C).

The hydrogen coolant flows in the  $z$  direction axially.

$$A C_v \frac{\partial}{\partial \tau} [\rho_G T_G(z, \tau)] + C_g W(\tau) \frac{\partial}{\partial z} T_G(z, \tau) = Dh(\tau) [T_U(\bar{R}, \tau) - T_G(\bar{R}, \tau)], \quad (3.2)$$

where  $T_G$  = bulk temperature of the coolant, assumed uniform in  $x$  direction,

$A$  = cross-sectional area of the coolant channel,

$h(\tau)$  = convection heat transfer coefficient,

$C_v$  = constant volume specific heat of coolant,

$C_g$  = constant pressure specific heat of coolant,

$D$  = perimeter of coolant channel, twice the width of fuel plates,

$W(\tau)$  = mass flow rate of coolant per channel,

$\rho_g$  = density of coolant,

$\vec{R}$  = position vector of the points on the interface between the fuel element and the coolant.

The following properties for the reactor are assumed,

- (i) Material properties are constant,
- (ii) Heat flow in the axial direction of the fuel elements is small, i.e.,  $\frac{\partial^2}{\partial z^2} T_U = 0$ , or  $\nabla k \nabla T_U \rightarrow k \frac{\partial^2}{\partial x^2} T_U$ ,
- (iii) Small amount of stored energy in hydrogen as coolant, i.e.,  $C_v$  is negligible.

Under the above assumptions the heat conduction and energy balance equations for the plate-type fuel element (as shown in Figure 1) can be written as

$$k \frac{\partial^2}{\partial x^2} T_U(x, z, \tau) + P(x, z, \tau) = \rho C \frac{\partial}{\partial \tau} T_U(x, z, \tau), \quad (3.3)$$

$$C_g W(\tau) \frac{\partial}{\partial z} T_G(z, \tau) = Dh(\tau) [T_U(b, z, \tau) - T_G(z, \tau)], \quad (3.4)$$

with boundary conditions

$$\left. \frac{\partial}{\partial x} T_U(x, z, \tau) \right|_{x=0} = 0, \quad (3.5)$$

$$\left. \frac{\partial}{\partial x} T_U(x, z, \tau) \right|_{x=b} = -\frac{h(\tau)}{k} [T_U(b, z, \tau) - T_G(z, \tau)], \quad (3.6)$$

$$T_G(z, \tau) \Big|_{z=0} = T_{Go} \quad (3.7)$$

and the initial condition

$$T_U(x, z, \tau) \Big|_{\tau=0} = T_U(x, z, 0) \quad (3.8a)$$

or

$$T_G(z, \tau) \Big|_{\tau=0} = T_G(z, 0) \quad (3.8b)$$

Equations (3.3) through (3.8) give the heat transfer characteristics of the reactor. In order to obtain a meaningful solution an inductive approach is used to solve the problem which follows.

(a) Two Dimensional Approach for Steady-State Solutions

The temperature distribution is sought in a steady state condition under constant power input  $P_o$ . Let  $T_{US}$  and  $T_{GS}$  be the steady state temperature of the fuel element and the coolant, respectively. The differential equation can be obtained by setting  $W(\tau) = W_o$ ,  $h(\tau) = h_o$  and neglecting the transient term in the original equations. Thus, we have

$$k \frac{\partial^2}{\partial x^2} T_{US}(x, z) + P_o = 0, \quad (3.9a)$$

$$C_g W_o \frac{\partial}{\partial z} T_{GS}(z) = Dh_o [T_{US}(b, z) - T_{GS}(z)], \quad (3.9b)$$

with boundary conditions

$$\frac{\partial}{\partial x} T_{US}(x, z) \Big|_{x=0} = 0, \quad (3.10a)$$

$$\left. \frac{\partial}{\partial x} T_{US}(x,z) \right|_{x=b} = \frac{h_o}{k} [T_{US}(b,z) - T_{GS}(z)], \quad (3.10b)$$

and

$$T_{GS}(z) \Big|_{z=0} = T_{Go} . \quad (3.10c)$$

The solution for this straight forward problem is

$$T_{US}(x,z) = -\frac{P_o}{2k} x^2 + P_o b \left( \frac{b}{2k} + \frac{1}{h_o} \right) + T_{Go} + \frac{DbP_o}{C_g W_o} z, \quad (3.11a)$$

and

$$T_{GS}(z) = T_{Go} + \frac{DbP_o}{C_g W_o} z . \quad (3.11b)$$

#### (b) Equivalent One-Dimensional Approach for Steady-State Solutions

The boundary value problem mentioned above can be reduced to a one-dimensional problem by integrating Equation (3.9a) from zero to  $b$  and using boundary conditions (3.10a) and (3.10b). The temperature distribution in the  $x$  direction does not appear further. Thus the temperature distribution on the interface becomes our only interest. The differential equations become,

$$h_o [T_{US}(b,z) - T_{GS}(z)] = P_o b , \quad (3.12a)$$

$$C_g W_o \frac{\partial}{\partial z} T_{GS}(z) = Dh_o [T_{US}(b,z) - T_{GS}(z)] , \quad (3.12b)$$

with the boundary condition

$$T_{GS}(z) \Big|_{z=0} = T_{Go} . \quad (3.13)$$

The solutions of this boundary value problem are

$$T_{US}(b, z) = \frac{P_o b}{h_o} + T_{Go} + \frac{DbP_o}{C_g W_o} z, \quad (3.14a)$$

and

$$T_{GS}(z) = T_{Go} + \frac{DbP_o}{C_g W_o} z. \quad (3.14b)$$

(c) Two-Dimensional Approach for Transient Solutions

We seek the solutions of a reactor problem which changes from the steady state power level  $P_o$  to a new level  $P$ , while the flow rate of coolant changes from  $W_o$  to  $W$ . This boundary value problem can be expressed as

$$k \frac{\partial^2}{\partial x^2} T_U(x, z, \tau) + P = \rho C \frac{\partial}{\partial \tau} T_U(x, z, \tau), \quad (3.15a)$$

$$C_g W \frac{\partial}{\partial z} T_G(z, \tau) = Dh [T_U(b, z, \tau) - T_G(z, \tau)], \quad (3.15b)$$

with boundary conditions

$$\left. \frac{\partial}{\partial x} T_U(x, z, \tau) \right|_{x=0} = 0, \quad (3.16a)$$

$$\left. \frac{\partial}{\partial x} T_U(x, z, \tau) \right|_{x=b} = -\frac{h}{k} [T_U(b, z, \tau) - T_G(z, \tau)], \quad (3.16b)$$

$$\left. T_G(z, \tau) \right|_{z=0} = T_{Go}, \quad (3.16c)$$

and the initial condition

$$T_U(x, y, \tau) \Big|_{\tau=0} = -\frac{q}{2k} + p_o b \left( \frac{b}{2k} + \frac{1}{h_o} \right) + T_{Go} + \frac{DbP_o}{C_g W_o} z, \quad (3.17a)$$

or

$$T_G(z, \tau) \Big|_{\tau=0} = T_{Go} + \frac{DbP_o}{C_g W_o} z. \quad (3.17b)$$

This boundary value problem may be solved by using the Laplace transform in  $z$ . The solution may be obtained in double series form. Because of the complicated algebraic form we do not intend to go further with this solution.

The following one-dimensional approach for transient solutions gives a much simpler form which is sufficiently accurate.

#### (d) Equivalent One-Dimensional Approach for Transient Solutions

After integrating Equation (3.15a) in the  $x$ -direction from zero to  $b$ , applying the boundary conditions (3.16a) and (3.16b), and letting

$$n(z) = \frac{\int_0^b T_U(x, z, \tau) dx}{bT_U(b, z, \tau)} \quad (3.18)$$

the following equations are obtained.

$$h(\tau) [T_U(b, z, \tau) - T_G(z, \tau)] = Pb - \rho C b n(z) \frac{\partial}{\partial \tau} T_U(b, z, \tau), \quad (3.19a)$$

$$C_g W(\tau) \frac{\partial}{\partial z} T_G(z, \tau) = Dh(\tau) [T_U(b, z, \tau) - T_G(z, \tau)], \quad (3.19b)$$



with the boundary condition

$$T_G(z, \tau) \Big|_{z=0} = T_{Go}$$

and the initial condition

$$T_U(b, z, \tau) \Big|_{\tau=0} = T_{Go} + \frac{bP_o}{h_o} + \frac{DbP_o}{CgW_o} z, \quad (3.21a)$$

or

$$T_G(z, \tau) \Big|_{\tau=0} = T_{Go} + \frac{DbP_o}{CgW_o} z. \quad (3.21b)$$

By applying the Laplace transform in  $z$  and letting  $h(\tau) = h$ ,  $W(\tau) = W$  and  $n(z) = n$ , one obtains (see Appendix D)

$$T_G(Z, t) = T_{Go} + \frac{Pb}{h} \beta Z + \frac{P_o b}{h} e^{-\beta Z - ht} \left[ \left( \frac{P}{P_o} - \frac{h\beta_o}{h_o \beta} \right) \sum_{n=1}^{\infty} n \left( \frac{\beta Z}{ht} \right)^{\frac{n}{2}} I_n(2\sqrt{\beta Z ht}) \right. \\ \left. + \left( 1 - \frac{\beta_o}{\beta} \right) \frac{h}{h_o} \sum_{n=1}^{\infty} \left( \frac{\beta Z}{ht} \right)^{\frac{n}{2}} I_n(2\sqrt{\beta Z ht}) \right], \quad (3.22a)$$

and

$$h[T_U(b, Z, t) - T_G(Z, t)] = Pb - P_o b e^{-\beta Z - ht} \left[ \left( \frac{P}{P_o} - \frac{h}{h_o} \right) I_o(2\sqrt{\beta Z ht}) \right. \\ \left. + \left( \frac{P}{P_o} - \frac{h\beta_o}{h_o \beta} \right) \sum_{n=1}^{\infty} \left( \frac{\beta Z}{ht} \right)^{\frac{n}{2}} I_n(2\sqrt{\beta Z ht}) \right], \quad (3.22b)$$

where

$$Z = \frac{Dz}{Cg}, \quad t = \frac{\tau}{\rho C h n}, \quad \beta = \frac{h}{W}, \quad (3.22c)$$

and  $I_n$  = modified Bessel function.

## (e) Perturbation Method for the Transient Solutions

We can assume the temperature distribution to be the sum of the steady state temperature and a perturbed temperature, i.e.,

$$T_U = T_{US} + T_u$$

and 
$$T_G = T_{GS} + T_g$$

Where  $T_u$  and  $T_g$  are the perturbed temperatures of the fuel and coolant, respectively.

(i) If the flow rate of coolant is kept constant and the power is dropped suddenly, we have,

$$W(\tau) = W_0$$

and 
$$h(\tau) = h_0$$

Let us assume  $n(z) = n$  to be constant. Subtracting the Equations (3.12) from Equations (3.19), and letting  $\frac{\tau}{\rho C b n} = t$ ,  $\frac{Dz}{Cg} = Z$ , one obtains

$$\frac{\partial}{\partial t} T_u(b, Z, t) = -h_0 [T_u(b, Z, t) - T_g(Z, t)] - b(P_0 - P), \quad (3.23a)$$

$$W_0 \frac{\partial}{\partial Z} T_g(Z, t) = h_0 [T_u(b, Z, t) - T_g(Z, t)], \quad (3.23b)$$

with the boundary condition

$$T_g(Z, t) \Big|_{Z=0} = 0, \quad (3.24)$$

and the initial condition

$$T_u(b, Z, t) \Big|_{t=0} = 0, \quad (3.25a)$$

or

$$T_g(Z, t) \Big|_{t=0} = 0. \quad (3.25b)$$

This boundary value problem is readily solved by applying the Laplace transform in  $t$ . The solution shown in Appendix E is

$$T_U(b, Z, s) = \frac{1}{s} \left[ T_{Go} + \frac{P_o b}{h_o} (1 + \beta_o Z) - \frac{b(P_o - P)}{s(s+h_o)} \left[ \frac{h_o}{s} (1 - e^{-\frac{\beta_o}{s+h_o} Z}) - 1 \right] \right], \quad (3.26a)$$

or

$$T_U(b, Z, t) = T_{Go} + \frac{P_o b}{h_o} (1 + \beta_o Z) - (P_o - P) \frac{b}{h_o} [h_o t$$

$$- e^{-\beta_o Z - h_o t} \sum_{n=1}^{\infty} (n-1) \left( \frac{h_o t}{\beta_o Z} \right)^{\frac{n}{2}} I_n(2\sqrt{\beta_o Z h_o t})], \quad (3.26b)$$

where  $\beta_o = \frac{h_o}{W_o}$ .

It should be noted that since the power term here is linear, no approximation has been made for the above solution. The solutions still hold for large perturbations.

(ii) If the power of the reactor is kept constant and the flow rate of coolant increases suddenly, we have

$$h = h_o + \Delta h,$$

and  $W = W_o + \Delta W$ . (3.27)

Furthermore, we assume that the change of flow-rate is small enough so that the cross product terms of perturbations can be neglected. Substituting Equation (3.27) into Equation (3.19) subtracting Equation (3.12), and using the relation in Equation (3.14), one obtains

$$\frac{\partial}{\partial t} T_u(b, Z, t) = -h_o [T_u(b, Z, t) - T_g(Z, t)] - \left(\frac{\Delta h}{h_o}\right) b P_o,$$

$$W_o \frac{\partial}{\partial Z} T_g(Z, t) = h_o [T_u(b, Z, t) - T_g(Z, t)] + P_o b \left(\frac{\Delta h + \Delta W}{h_o W_o}\right), \quad (3.28a)$$

$$\text{where } Z = \frac{Dz}{Cg}, \quad t = \frac{\tau}{\rho C b n}. \quad (3.28b)$$

The heat transfer coefficient and the flow-rate of coolant are approximately related by (10)(11)

$$\frac{h}{h_o} = \left(\frac{W}{W_o}\right)^{0.8} \quad (3.28)$$

$$\text{Thus } \frac{\Delta h}{h_o} \approx 0.8 \frac{\Delta W}{W_o}$$

$$\text{and } \frac{\Delta h}{h_o} - \frac{\Delta W}{W_o} \approx -0.2 \frac{\Delta W}{W_o}$$

If the term  $\left(\frac{\Delta h}{h_o} - \frac{\Delta W}{W_o}\right)$  is negligible Equations (3.28) are equivalent to Equations (3.23) provided that

$$(P_o - P)b = \frac{\Delta h}{h_o} P_o b,$$

$$\text{or } 1 - \frac{P}{P_o} = \frac{\Delta h}{h_o}.$$

Therefore, a small increase of  $W$  or  $h$  gives a similar effect as a small decrease in power level.

(4) Thermal Stresses

## (a) General Thermal Stresses

The general form of thermoelastic equations has the strain term in the energy equation and the inertia terms in the equations of motion (see Appendix F). Fortunately, in most engineering applications, such as thermal-stress for the fuel element of a reactor, it is possible to disregard these coupling terms. Thus, the general thermal-stress problem becomes the quasi-static thermal-stress problem by simply dropping these coupling terms. The uncoupled system may be considered as two distinct problems, a problem of heat transfer and a problem of quasi-static thermoelasticity.

## (b) Three Dimensional Quasi-static Thermal Stresses

(i) We are interested in solving the quasi-static three dimensional thermal-stress problem. If the temperature distribution is assumed to be known the following fifteen functions are to be determined for the thermoelastic problem:

6 stress components:  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ,  $\sigma_{xy}$ ,  $\sigma_{yz}$ ,  $\sigma_{zx}$ ,

6 strain components:  $\epsilon_{xx}$ ,  $\epsilon_{yy}$ ,  $\epsilon_{zz}$ ,  $\epsilon_{xy}$ ,  $\epsilon_{yz}$ ,  $\epsilon_{zx}$ ,

3 displacement components:  $u$ ,  $v$ ,  $w$ .

There are, throughout the body, the following fifteen equations to be satisfied

3 equilibrium equations: Eq. (G.1)

6 stress-strain relations: Eq. (G.2)

6 strain-displacement relations: Eq. (G.3)

with proper boundary conditions.

These differential equations are given in Appendix G.

(ii) It is sometimes convenient to express the relations in terms of stresses. In examining the fifteen equations mentioned, the displacement and strain variables can be eliminated as shown in Appendix H. The resultant relations are called the compatibility conditions<sup>(12)</sup> of the thermoelastic problem.

$$(1 + \nu)\nabla^2\sigma_{xx} + \frac{\partial^2\theta}{\partial x^2} + \alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 + \frac{\partial^2}{\partial x^2}\right)(T-T_0) = 0, \quad (4.1)$$

$$(1 + \nu)\nabla^2\sigma_{yy} + \frac{\partial^2\theta}{\partial y^2} + \alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 + \frac{\partial^2}{\partial y^2}\right)(T-T_0) = 0, \quad (4.2)$$

$$(1 + \nu)\nabla^2\sigma_{zz} + \frac{\partial^2\theta}{\partial z^2} + \alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 + \frac{\partial^2}{\partial z^2}\right)(T-T_0) = 0, \quad (4.3)$$

$$(1 + \nu)\nabla^2\sigma_{xz} + \frac{\partial^2\theta}{\partial x\partial z} + \alpha E\frac{\partial^2}{\partial x\partial z}(T-T_0) = 0, \quad (4.4)$$

$$(1 + \nu)\nabla^2\sigma_{yx} + \frac{\partial^2\theta}{\partial y\partial x} + \alpha E\frac{\partial^2}{\partial y\partial x}(T-T_0) = 0, \quad (4.5)$$

$$\text{and } (1 + \nu)\nabla^2\sigma_{zy} + \frac{\partial^2\theta}{\partial z\partial y} + \alpha E\frac{\partial^2}{\partial z\partial y}(T-T_0) = 0, \quad (4.6)$$

$$\text{where } \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right),$$

$$\theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

and  $\alpha$  = coefficient of linear thermal expansion.

The solution of the thermal stress problem must satisfy the equilibrium equations Eq. (G1) with the proper boundary conditions. The stress components in an isotropic body must also satisfy the six equations given above.

(iii) Consider a plate-type fuel element of constant thickness  $2b$  as shown in Figure (1). The fuel element is free of surface traction and the temperature varies in the  $x$ -direction only, i.e.,  $T = T(x)$ . It is reasonable to suppose that under these conditions the stress components will be of the following form.

$$\sigma_{zz} = \sigma_{yy} = f(x) \quad (4.7)$$

and

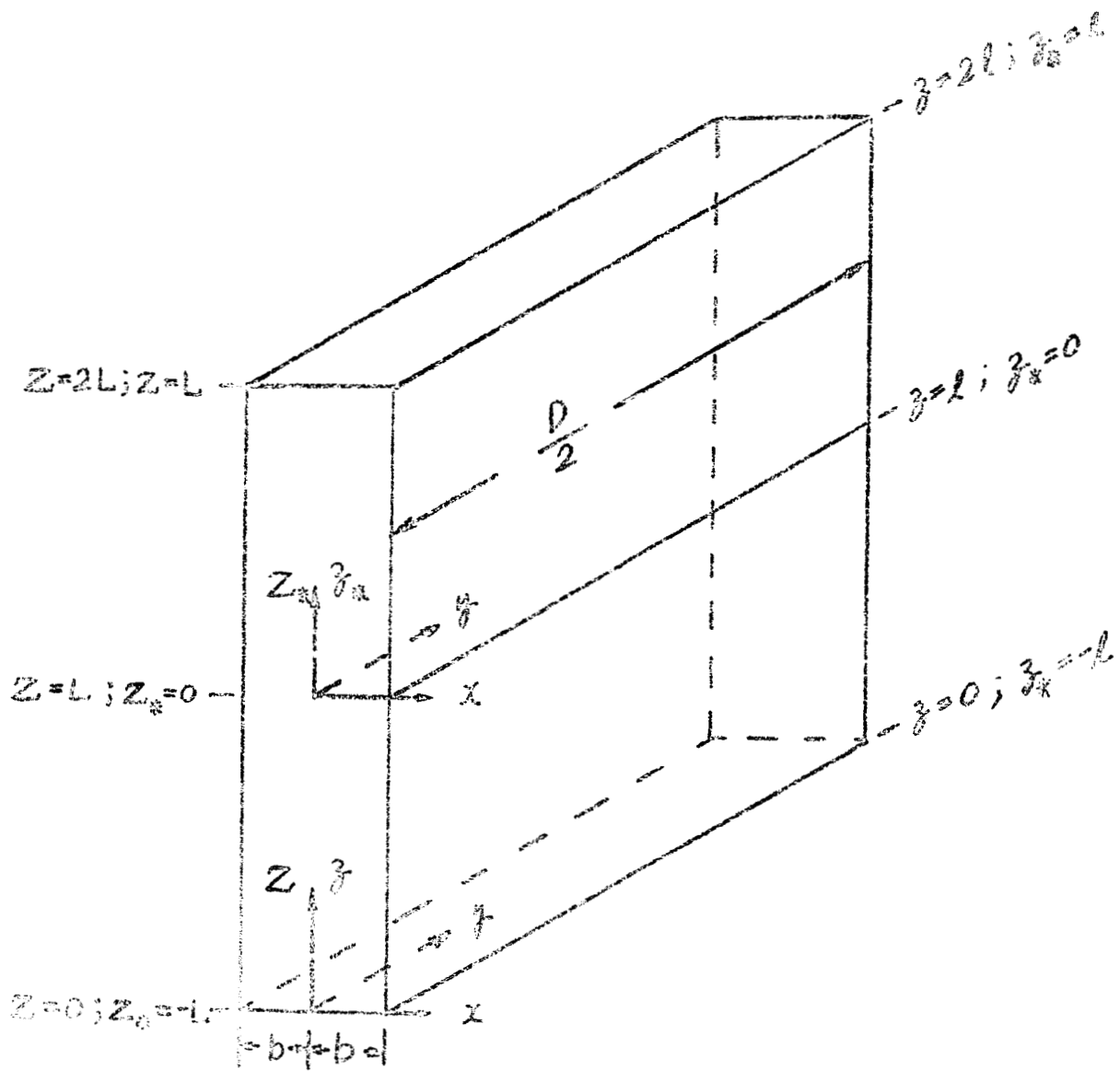
$$\sigma_{xx} = \sigma_{zx} = \sigma_{yx} = \sigma_{zy} = 0 \quad (4.8)$$

Equation (4.8) holds because no surface traction appears on the planes  $x = \pm b$ . Since the plate is thin, the region is extended to interior planes. Substituting into the equilibrium equations, Eq. (G1), shows that these stresses satisfy the equilibrium equations. Substituting into the compatibility conditions, Eq. (4.1), (4.2), or (4.3), we obtain

$$\frac{\partial^2}{\partial x^2} \left\{ f(x) + \frac{\alpha E}{1-\nu} [T(x) - T_0] \right\} = 0, \quad (4.9)$$

for any one of the three equations.





$$z = \frac{Dz}{C_g}, \quad L = \frac{DL}{C_g}, \quad z = z_* + L$$

Fig 1 Fuel Slab

The solutions of Eqs. (4.9) is (12)

$$\sigma_{yy} = \sigma_{zz} = \frac{\alpha E}{1-\nu} T(x) + c_1 + c_2 x, \quad (4.10)$$

where  $c_1$  and  $c_2$  are constants to be determined by the boundary conditions. The resultant force or the moment per unit of length produced by  $\sigma_{yy}$  or  $\sigma_{zz}$  vanishes over any plane  $y = \text{constant}$  or  $z = \text{constant}$ , respectively, which requires that

$$\int_{-b}^b \sigma_{zz} dx = \int_{-b}^b x \sigma_{zz} dx = \int_{-b}^b \sigma_{yy} dx = \int_{-b}^b x \sigma_{yy} dx = 0. \quad (4.11)$$

Thus, using Equation (4.10) we have

$$\frac{\alpha E}{1-\nu} \int_{-b}^b T(x) dx + c_1 \int_{-b}^b dx + c_2 \int_{-b}^b x dx = 0, \quad (4.12)$$

and

$$\frac{\alpha E}{1-\nu} \int_{-b}^b x T(x) dx + c_1 \int_{-b}^b x dx + c_2 \int_{-b}^b x^2 dx = 0. \quad (4.13)$$

Solving for  $c_1$  and  $c_2$ , one obtains

$$c_1 = \frac{1}{2b} \int_{-b}^b \frac{\alpha E}{1-\nu} T(x) dx, \quad (4.14)$$

and 
$$c_2 = \frac{3}{2b^3} \int_{-b}^b \frac{\alpha E}{1-\nu} x T(x) dx. \quad (4.15)$$

Then the stresses are<sup>(12)</sup>

$$\sigma_{zz} = \sigma_{yy} = \frac{\alpha E}{1-\nu} \left[ -T(x) + \frac{1}{2b} \int_{-b}^b T(x) dx + \frac{3x}{2b^3} \int_{-b}^b xT(x) dx \right]. \quad (4.16)$$

It is found that if the temperature distribution is symmetric, the constant  $c_2$  will be zero. Thus, the stress reduces to

$$\sigma_{zz} = \sigma_{yy} = \frac{\alpha E}{1-\nu} \left[ -T(x) + \frac{1}{b} \int_0^b T(x) dx \right], \quad (4.17)$$

where  $T(x) = T(-x)$ .

(5) Sudden Increase of Flow Rate at Constant Full Power

## (a) Solutions of Temperature Distribution

A reactor operating at full power  $P_0$  under steady state condition has a constant coolant flow rate  $W_0$ . This flow rate suddenly increases to  $W_1$ , the numerical value of which is determined from the equation (3.28) by setting  $h = 1.1h_0$ . The differential equations of this particular problem are given in Equations (3.19), (3.20) and (3.21) by letting  $h(\tau) = h_1$ ,  $W(\tau) = W_1$ ,  $P = P_0$  and  $\eta(z) = \eta = 1$ . The solutions given in Equation (3.22) are

$$T_G(Z,t) = T_{G0} + \frac{P_0 b}{h_1} \left\{ \beta_1 Z e^{-\beta_1 Z - h_1 t} \left[ \left(1 - \frac{h_1 \beta_0}{h_0 \beta_1}\right) \sum_{n=1}^{\infty} n \left(\frac{\beta_1 Z}{h_1 t}\right)^{\frac{n}{2}} I_n(2\sqrt{\beta_1 Z h_1 t}) \right. \right. \\ \left. \left. - \left(1 - \frac{\beta_0}{\beta_1}\right) \frac{h_1}{h_0} \sum_{n=1}^{\infty} \left(\frac{\beta_1 Z}{h_1 t}\right)^{\frac{n}{2}} I_n(2\sqrt{\beta_1 Z h_1 t}) \right] \right\}, \quad (5.1a)$$

$$h_1 [T_U(b, Z, t) - T_C(Z, t)] = P_0 b \left\{ 1 - e^{-\beta_1 Z - h_1 t} \left[ \left(1 - \frac{h_1}{h_0}\right) I_0(2\sqrt{\beta_1 Z h_1 t}) \right. \right. \\ \left. \left. + \left(1 - \frac{h_1 \beta_0}{h_0 \beta_1}\right) \sum_{n=1}^{\infty} \left(\frac{\beta_1 Z}{h_1 t}\right)^{\frac{n}{2}} I_n(2\sqrt{\beta_1 Z h_1 t}) \right] \right\}, \quad (5.1b)$$

where  $Z = \frac{Dz}{C_g}$ ,  $t = \frac{\tau}{\rho c b}$ ,  $\beta_1 = \frac{h_1}{W_1}$ .

and  $I_n =$  modified Bessel function.

(b) State Variable Constraint due to the Temperature Distribution in x-Direction

Due to symmetry the temperature of the fuel element has a parabolic distribution in the x direction. The thermal stresses given in Eqs. (4.17) are

$$\sigma_{zz} = \sigma_{yy} = \frac{\alpha E}{1-\nu} \left[ -T(x) + \frac{1}{2b} \int_{-b}^b T(x) dx \right],$$

where  $T(x) = A_0 - A_2 x^2$ ,

$A_0, A_2 = \text{constants.}$

Thus, 
$$\frac{\partial T}{\partial x} = -2A_2 x. \quad (5.2)$$

For this parabolic temperature distribution, we find that the maximum thermal stress occurs at  $x = b$ . This stress is tensile if  $A_2$  is positive

$$\sigma_m = \sigma_{zz} \Big|_{\substack{x = b \\ z = z_m}} = \frac{2\alpha E b^2}{3(1-\nu)} A_2 \quad (5.3)$$

where  $z_m$  is the location of the maximum thermal stresses. Since the thermal stresses for the above parabolic distribution of temperature can be related to the gradient of temperature at the surface of the plate we can express Eq. (5.3) in the following form

$$\sigma_m = -\frac{\alpha E b}{3(1-\nu)} \frac{\partial T}{\partial x} \Big|_{\substack{x = b \\ z = z_m}} \quad (5.4)$$

The following relation is given in Eq. (3.16b) for the temperature gradient

$$\left. \frac{\partial T}{\partial x} \right|_{\substack{x = b \\ z = z_m}} = -\frac{h_1}{k} [T_U(b, z_m, t) - T_G(z_m, t)] . \quad (5.5)$$

The term  $h_1[T_U(b, z_m, t) - T_G(z_m, t)]$  may be obtained from Equation (5.1b) for the solutions. Thus, Eq. (5.4) can be written as

$$\sigma_m = \frac{\alpha F b}{3k(1-\nu)} \left\{ h_1 [T_U(b, z_m, t) - T_G(z_m, t)] \right\} , \quad (5.6)$$

where  $h_1[T_U(b, z_m, t) - T_G(z_m, t)]$  is a constraint in determining thermal stresses due to temperature distribution across the depth of the slab. The constraint may also be formulated as

$$\phi = \frac{3k(1-\nu)}{\alpha F b} \sigma_{zz} \leq \phi \text{ max.} , \quad (5.7)$$

where  $\phi = h_1 [T_U(b, z_m, t) - T_G(z_m, t)] . \quad (5.8)$

Thus the above state variable constraint is the product of the heat transfer coefficient and the difference of the temperatures of the core and the propellant.

### (c) Numerical Example

A numerical example is given here for the case in which the heat transfer coefficient increases from  $h_0$  to  $h_1 = 1.1 h_0$ . The physical parameters for this numerical example are (see Appendix B):

$$\begin{aligned}
 W_0 &= \frac{60}{212} \left[ \frac{\text{lb}_m}{\text{sec}} \text{ per channel} \right], & h_0 &= 0.82 \left[ \frac{\text{BTU}}{\text{sec ft}^2 \text{ } ^\circ\text{R}} \right], \\
 h_1 &= 1.1 h_0, & W_1 &= W_0 \left[ \frac{h_1}{h_0} \right]^{\frac{1}{0.8}} = (1.1)^{\frac{1}{0.8}} W_0 = 1.126 W_0, \\
 B_1 &= \frac{h_1}{W_1} = \frac{0.902}{0.319} = 2.829 \left[ \frac{\text{BTU}}{\text{ft}^2 \text{ } ^\circ\text{R} \text{ lb}_m} \right], & C &\approx 0.47 \left[ \frac{\text{BTU}}{\text{lb}_m \text{ } ^\circ\text{R}} \right], \\
 C_g &\approx 4.1 \left[ \frac{\text{BTU}}{\text{lb}_m \text{ } ^\circ\text{R}} \right], & \rho &\approx 101 \left[ \frac{\text{lb}_m}{\text{ft}^3} \right], \\
 k &= 56 \sim 76 \left[ \frac{\text{BTU}}{\text{hr ft } ^\circ\text{R}} \right] \approx 0.02 \left[ \frac{\text{BTU}}{\text{sec ft } ^\circ\text{R}} \right], \\
 n &\approx 1, & b &= 0.005 \text{ [ft]}, \\
 \frac{D}{2} &\approx 3 \text{ [ft]}, & v &\approx 0.25, \\
 2L &= \text{axial length} = 3 \text{ [ft]}, & 2L &= \frac{D(3)}{C_g} = 4.4 \left[ \frac{\text{ft lb}_m \text{ } ^\circ\text{R}}{\text{BTU}} \right], \\
 \beta L &= 12.45, & T_{US} &\Big|_{\substack{x=0 \\ z=2L}} = 4878.4 \text{ [}^\circ\text{R]}, \\
 T_G &\Big|_{z=2L} = 4537.1 \text{ [}^\circ\text{R]}.
 \end{aligned}
 \tag{28}$$

The thermal stress at steady-state conditions is obtained from Equations (5.6) and (5.1) as

$$\begin{aligned}
 \sigma_{ss} &= (\sigma_m)_{t \rightarrow \infty} = \frac{Eb^2 P_0}{3k(1-\nu)} \\
 &= 0.555(10)^{-3} (\alpha E P_0).
 \end{aligned}
 \tag{5.9}$$

Thus the ratio of thermal stresses is (at  $Z_m = 2L$ )

$$\frac{\sigma_m}{\sigma_{ss}} = 1 - e^{-2\beta_1 L - h_1 t} \left[ \left(1 - \frac{h_1}{h_0} \frac{\beta_0}{\beta_1}\right) \sum_{n=1}^{\infty} \left(\frac{2\beta_1 L}{h_1 t}\right)^{\frac{n}{2}} I_n(2\sqrt{2\beta_1 L h_1 t}) - \left(\frac{h_1}{h_0} - 1\right) I_0(2\sqrt{2\beta_1 L h_1 t}) \right] \quad (5.10)$$

The Bessel Functions are evaluated by a short table. (13)

The thermal stresses at exit and the temperatures computed by Equation (5.1) are plotted vs the time  $\tau$  shown in Fig. 2. It is found that the increase of thermal stresses is about the same percentage as that of flow rate, which is approximately 12% higher if  $h_1 = 1.1 h_0$ .

#### (d) Some Observations

The steady-state design of a reactor may be rated at  $\sigma\%$  over its allowable maximum thermal stress at full power, i.e.,  $(100 + \sigma)\%$  is allowed for the transient thermal stresses.

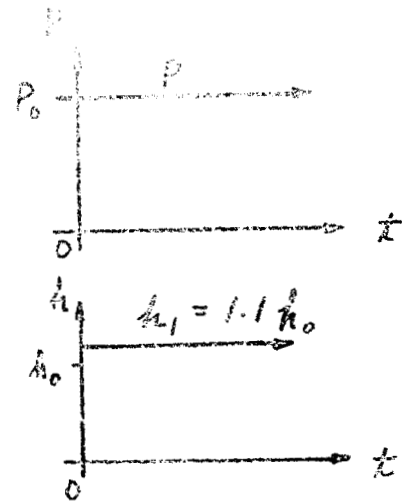
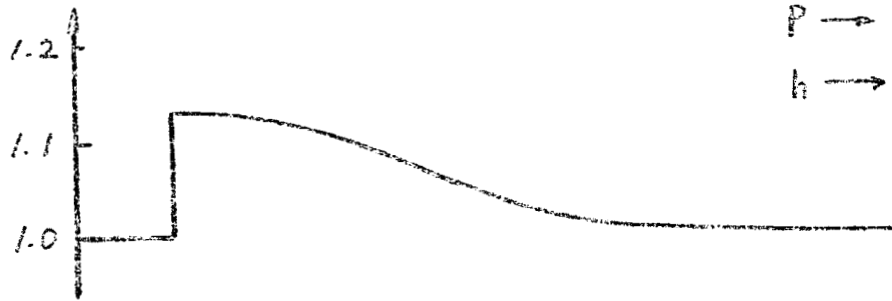
Thus

$$\phi_{\max} = h_0 [T_U(b, Z_m, t) - T_G(Z_m, t)]_{ss} \frac{100 + \sigma}{100} \quad (5.11)$$

Under the full power steady state conditions the temperatures  $T_U(b, Z_m, t)$  and  $T_G(Z_m, t)$  take on some constant values. These values remain approximately the same while the flow rate and the heat transfer coefficient suddenly increase from  $W_0$  and  $h_0$  to  $W_1$  and  $h_1$  respectively.



$$\frac{\sigma_m}{\sigma_{55}} = \frac{\sigma_m}{0.555(10)^{-3} \alpha E P_0}$$



$$P \rightarrow P = P_0$$

$$h \rightarrow h_1 = 1.1 h_0$$

Temperature

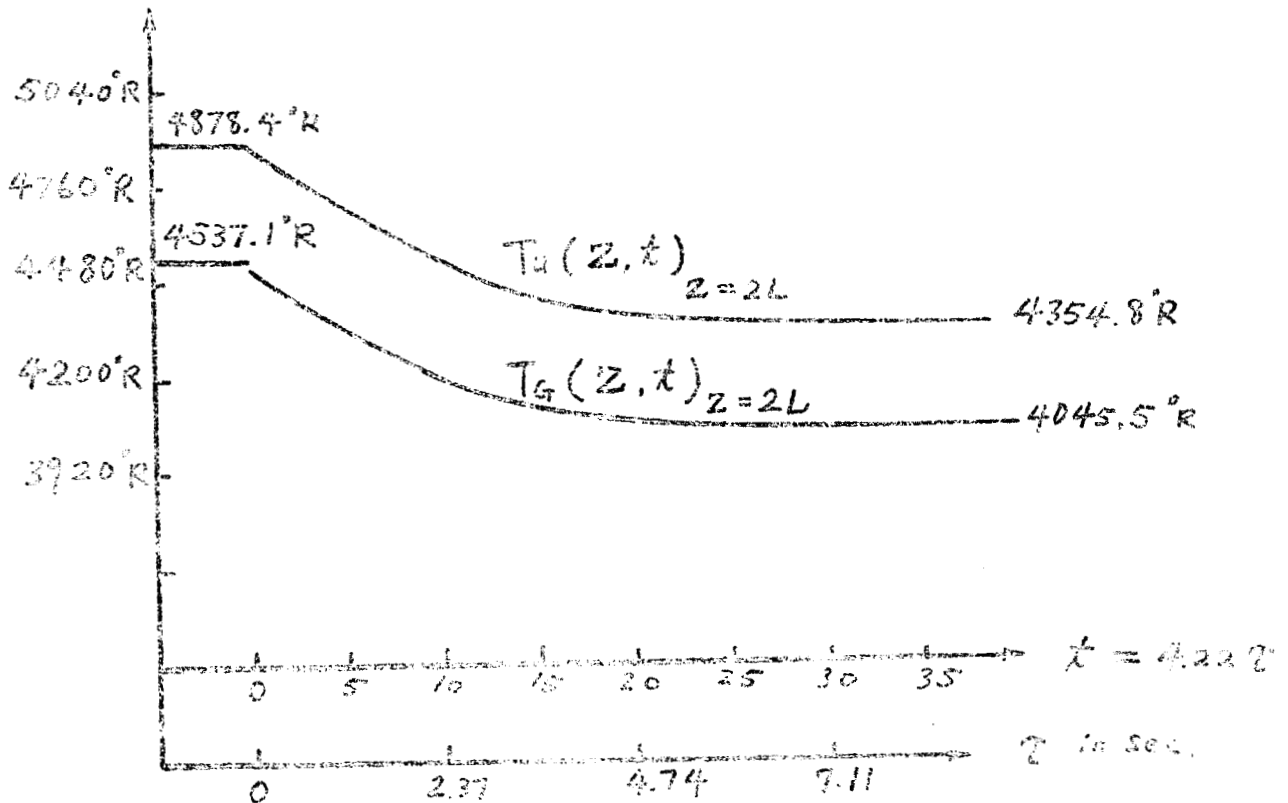


Fig 2 Thermal Stress Due to Sudden Change of Hydrogen Flow

Thus  $T_U(b, z_m, t) - T_G(z_m, t) \approx [T_U(b, z_m, t) - T_G(z_m, t)]_{ss}$  :

$$(5.12)$$

Combining Equations (5.8), (5.11) and (5.12) one obtains

$$\frac{\phi}{\phi_{max}} = \frac{h_1}{h_0} \frac{100}{100 + \sigma} \quad (5.13)$$

Thus the instantaneous maximum stress occurs at the instant of sudden increase of flow. If the condition (5.7) holds

we have  $h_1 < \frac{100 + \sigma}{100} h_0$  , (5.14)

or  $w_1^{0.8} < \frac{100 + \sigma}{100} w_0^{0.8}$  . (5.15)

by virtue of equation (3.28). From Equation (5.15) one may conclude that a sudden increase of flow may cause overstress. The maximum flow  $w_1$  is limited by the quantity given in Equation (5.15). Re-examining equation (3.19) indicates that immediate reduction of the function  $\phi$  is not obtainable by adjusting the other control variable  $P$  the power input to the reactor.

The temperature rate in time domain by Equation (3.19a) can be zero if the instantaneous power density  $P$  rises to the same numerical number given by the term  $\frac{h}{b} (T_U - T_G)$ , which contributes to the maximum thermal stresses due to the increase of hydrogen flow. Thus the maximum of this thermal stress can be at a time when the temperature rate in the time domain is zero.

(6) Shut-down of Power at Constant Full Flow

## (a) The Heat Transfer Problem

A reactor under steady state conditions has a constant flow rate  $W_0$ . The power drops from a steady state value  $P_0$  to a new level  $P$ . The process dynamics are given in Equations (3.23), (3.24) and (3.25). The solutions of the problem are shown in Eq. (3.26).

## (b) Thermal Stress due to Temperature Distribution in x-Direction

Equation (5.6) derived in Section 5 is valid here. The value of  $h_0 [T_U(b, z, t) - T_C(z, t)]$  can be obtained from Equation (3.22b) by setting  $\beta = \beta_0$ ,  $h = h_0$ .

## (c) Numerical Example

An example is given for a problem on reactor shutdown. The power of the reactor is initially at a high level  $P_0$  and suddenly dropped to a constant low level  $P$  whereas the flow rate of coolant is kept constant. The physical parameters for this example are

$$\begin{aligned}
 P &= 10^{-5} P_0, & 2l &= 3 \text{ [ft]} \\
 & & &= \text{axial length of the reactor,} \\
 h_0 &\approx 0.82 \left[ \frac{\text{BTU}}{\text{sec ft}^2 \text{ } ^\circ\text{F}} \right], & W_0 &\approx \frac{60}{212} \left[ \frac{\text{lb}_m}{\text{sec}} \text{ per Channel} \right], \\
 C_g &\approx 4.1 \left[ \frac{\text{BTU}}{\text{lb}_m \text{ } ^\circ\text{R}} \right], & C &\approx 0.47 \left[ \frac{\text{BTU}}{\text{lb}_m \text{ } ^\circ\text{R}} \right], \\
 D &\approx 6 \text{ [ft]}, & \rho &\approx 101 \left[ \frac{\text{lb}_m}{\text{ft}^3} \right], \\
 b &= 0.005 \text{ [ft]}, & n &= 1, \\
 k &\approx 0.02 \left[ \frac{\text{BTU}}{\text{sec ft } ^\circ\text{R}} \right], & \tau_p &= 100 \text{ [sec]}.
 \end{aligned}$$

(i) Numerical results for thermal stresses due to temperature variations in x-direction

With the non-dimensional time  $\tau/\tau_f$  and  $t/t_f$  as abscissa, the temperatures  $T_U$  and  $T_G$  at the exit  $z = 2l$  computed by Equation (3.26), are plotted in Figure 3. With the same abscissa the thermal stresses at  $z = 2l$  due to temperature variation in x-direction are shown in Figure 4. The Bessel Functions are evaluated by tables. (13)(14)

It is observed from Figure 4 that the thermal stress decreases monotonically from the highest value, i.e., the steady state stress, to a very small value in a time interval of about 11 sec. This stress will, therefore, cause no trouble to the reactor.

(ii) Numerical result for temperature variations in z-direction

Several curves of temperature  $T_U$  vs  $z$  with different  $\tau$  are plotted in Figure 5.

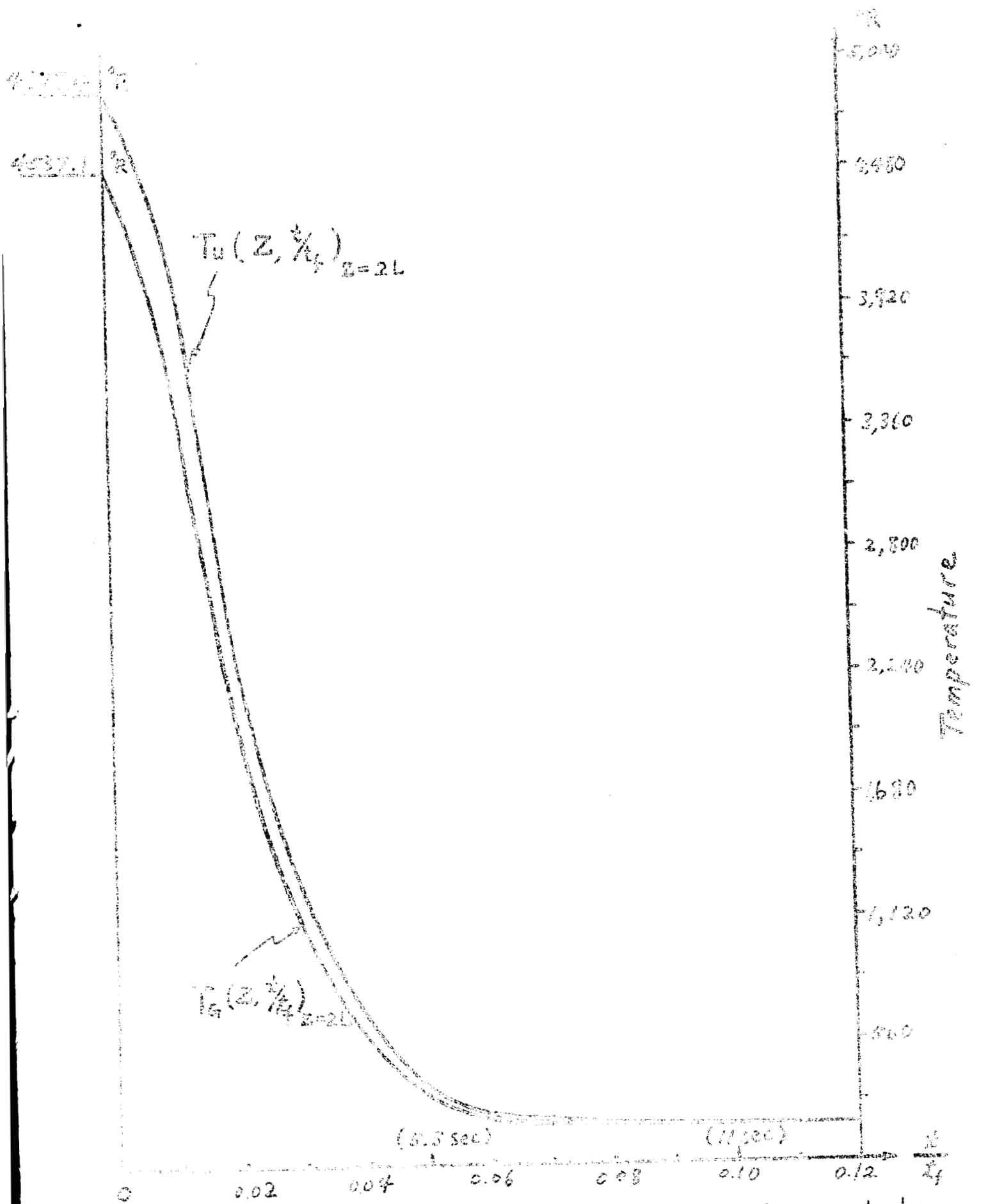


Fig 5 Temperature Distribution During Power Shut Down

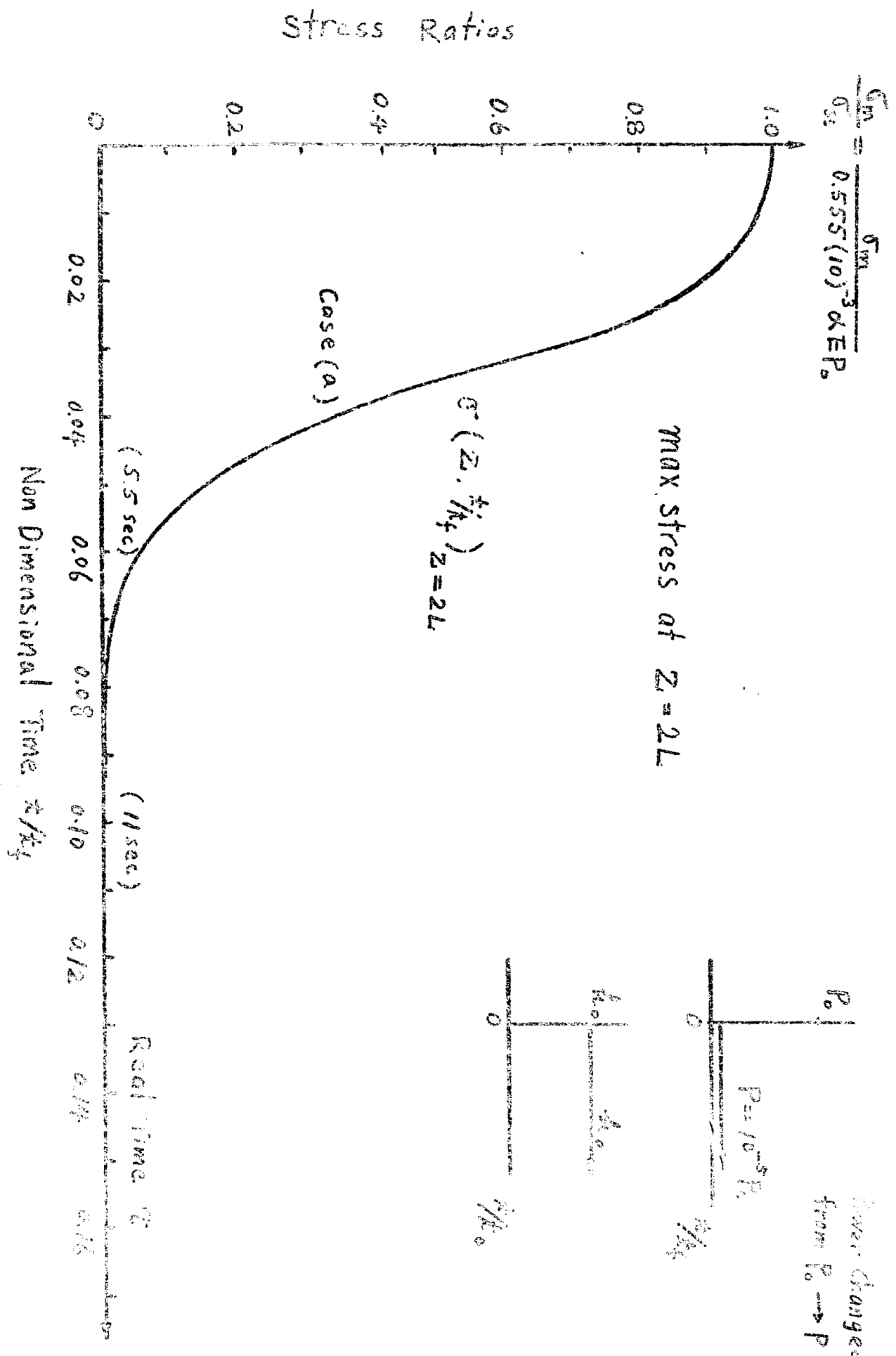
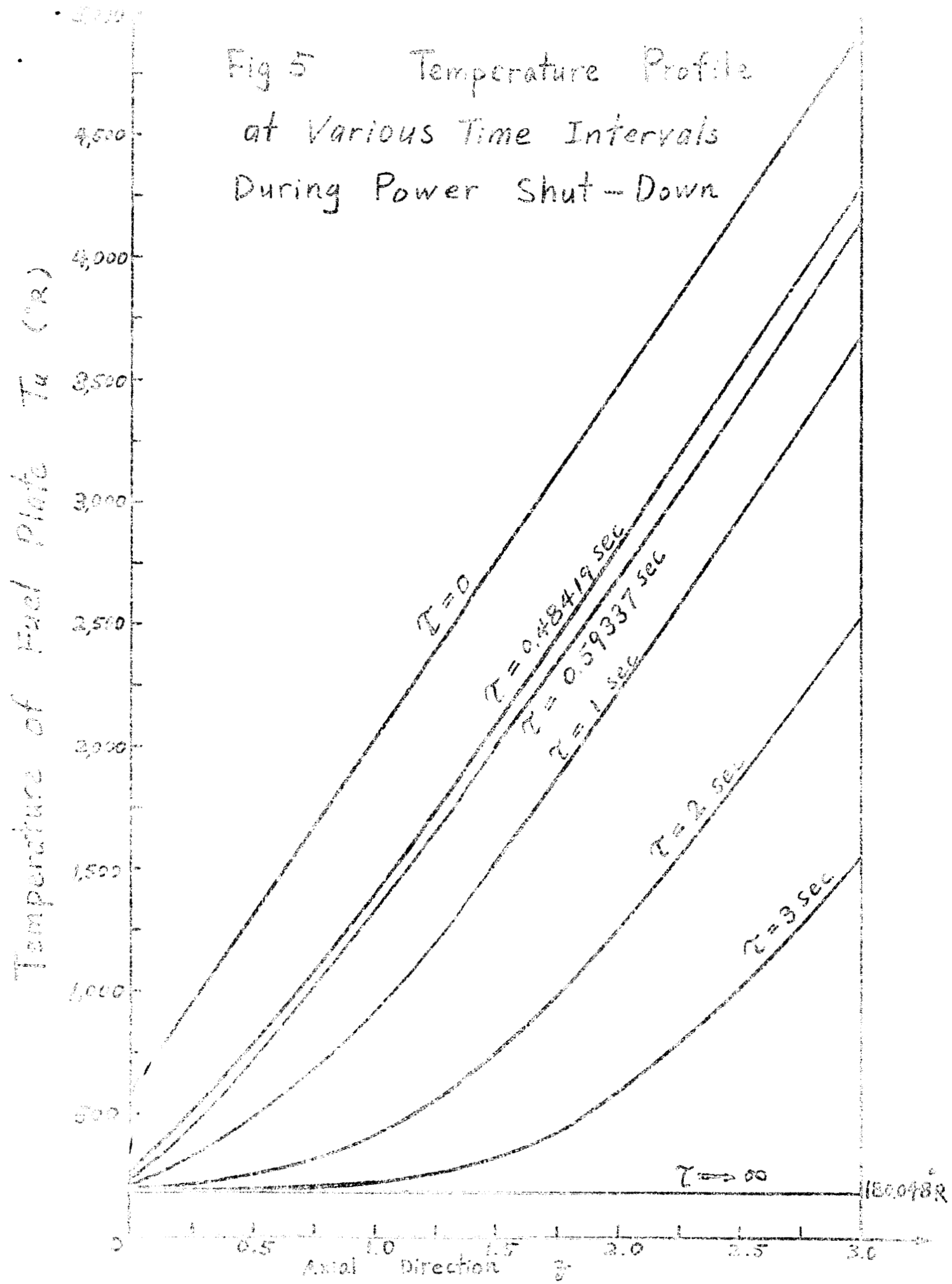


Fig 4 Thermal Stress Due to Temperature Distribution in X-Direction During Power Shut-Down

Fig 5 Temperature Profile  
 at Various Time Intervals  
 During Power Shut-Down



References

30

- (1) Ragsdale, R. G., Rom, F. E., "Advanced Concepts for Nuclear Rocket Propulsion" NASA SP-20 Nuclear Rocket Propulsion, P. 12, pp. 5-6.
- (2) Yoshitani, Y., "The Thermal Performance of Homogeneous Graphite Reactors" S.M. thesis MIT 1960.
- (3) MacMillan, D. P., "High-Temperature Materials for Rocket Reactors" Nucleonics Vol. 19, No. 4(1961) pp. 87-88.
- (4) Levoy, M. M., Newgard, J. J., "Rocket-Reactor Design" Nucleonics, July 1958, p. 66, p. 68.
- (5) Smith, H. P. Jr., "Dynamics and Control of Nuclear Rocket Engine" Ph.D. thesis of MIT 1960, p. 14.
- (6) Newgard, J. J., Levoy, M. M., "Consideration in the Design of a Nuclear Rocket" Nucl. Sci. and Eng. 7, 377-387 (1960) p. 381.
- (7) Reynolds, H. L., Proc. 2nd Inter. Conf. Peaceful use Atomic Energy, Geneva 12, p/2408, 632 (1958).
- (8) Hodgman, C.D., Handbook of Chemistry and Physics, Chemical Rubber Publishing Co., 37th edition, 1956.
- (9) Carslaw, H. S., Jaeger, J. C., "Conduction of Heat in Solids" Oxford University Press, 2nd edition 1959.
- (10) Kreith, F., "Principles of Heat Transfer" International Textbook Co., Scranton, 1960.
- (11) Eckert, E. R. G., "Introduction to the Transfer of Heat and Mass" McGraw-Hill Book Co. Inc., 1950.
- (12) Boley, B. A., Weiner, J. H., "Theory of Thermal Stresses, John Wiley & Sons, Inc., 1960, p. 258, p. 278, p. 261, p. 280.
- (13) Fox, L., "A Short Table for Bessel Functions of Integer Orders and Large Arguments" University Press, Cambridge 1954.
- (14) "Functions of Positive Integer Order, Mathematical Tables", Vol. X, Cambridge University Press, England, 1952.



## Appendix A Historical Review

### (a) Heat Transfer

The determination of the temperature distribution in a solid is a boundary value problem of heat conduction. The heat conduction principle was first proposed by Fourier in 1822. Awbery (A1) was the first to derive the analytical solution of a conduction problem of a solid with constant heat source. An analytical solution was obtained by Paterson (A2) for the conduction problem having a source term coupled with temperature. In 1946, Carslaw & Jaeger (A3) published a book in which most of the conduction problems were systematically discussed. Recently several papers by Siegel (A4) (A5) studied the heat transfer for fluid flow inside parallel plates and circular tubes under time and position dependent wall temperatures. An example for a nuclear reactor was also given. Doggett et al (A6) worked on a nuclear reactor problem with sinusoidal space and exponential time varying power generation. Most of the solutions given by the above authors were in a complicated series form.

### (b) Thermal Stresses

The formulation of elasticity including the effect of temperature variations was studied by Duhamel as early as 1835. However, investigations on the effects of temperature in the study of the inelastic behavior of material have begun only recently. Therefore, the subject of thermal stresses ranges from classical topics to modern theories which are still in a tentative form. S. Timoshenko (A7) derived the formulae for the thermal stresses in a uniform plate with a

ferential equations. However, in many engineering applications, one has to deal with distributed parameter systems. Butkovskii et al (A17) specified several optimal control problems and showed one example which could be solved by Pontryagin's maximum principle. Later Butkovskii (A18) gave the optimal conditions for a system described by nonlinear integral equations. The optimal conditions for the controlled processes described by a hyperbolic differential equation were given in Egorov's (A19) paper. Lure (20) gave the optimal theorems for the Mayer-Bolza problem in distributed parameter systems by the method of calculus of variations. Sirazetdinov (A21) developed the maximum principle for a system governed by a quasi-linear first order partial differential equation.

(d) State variable constraints

As early as 1937 Valentine (A22) outlined problems with inequality constraints on the control variables. These problems were discussed by Breakwell (A23) in detail. Inequality constraints on state variables of an optimal problem were presented by Gamkrelidze (A24) and Berkovitz (A25). Recently, the necessary conditions for extremal solutions on optimal programming problems with inequality constraints were given by Bryson et al. (A26)

Solutions of these problems by the method of steepest ascent were proposed. (A27) (A28) Numerical solutions are obtained through the use of integral penalty functions. A more direct method was recently presented by Denham and Bryson. (A29)

References for Appendix A

- (A1) Awbery, J.G., "The Flow of Heat in a Body Generating Heat" Phil. Mag., Series 7, Vol. 4, 1927.
- (A2) Paterson, S., "The Conduction of Heat in a Medium Generating Heat", Phil. Mag., Series 7, Vol. 32, 1941.
- (A3) Carslaw, H.S., Jaeger, J.C., "Conduction of Heat in Solids", Oxford Univ. Press, 1st edition, 1946.
- (A4) Siegel, R., "Heat Transfer for Laminar Flow in Ducts with Arbitrary Time Variations in Wall Temperature", ASME Trans., Series E, Vol. 82, P 241, June, 1960.
- (A5) Siegel, R., Perlmutter, M., "Laminar Heat Transfer in a Channel with Unsteady Flow and Wall Heating Varying with Position and Time", ASME Trans., Series C, Vol. 85, No. 4, P 358, Nov., 1963.
- (A6) Doggett, W.C., Arnold, E.L., "Axial Temperature Distribution for a Nuclear Reactor with Sinusoidal Space and Exponential Time-varying Power Generation", ASME Trans., Series C, Vol. 83, No. 4, P 423, Nov., 1961.
- (A7) Timoshenko, S., "Theory of Elasticity", McGraw-Hill Book Co., Inc. New York, 1934.
- (A8) Melan, E., Parkus, H., "Warmespannungen", Wien Springer-verlag, 1953.
- (A9) Parkus, H., "Instationare Warmespannungen", Wien Springer-verlag, 1959.
- (A10) Mendelston, A., Manson, S. S., "Approximate Solution to Thermal-Stock Problems in Plates, Hollow Spheres, and Cylinders with Heat Transfer at Two Surfaces", ASME Trans., Vol. 78, No. 3, April, 1956.
- (A11) Mendelson, A., Hirschberg, M., "Analysis of Elastic Thermal Stresses in Thin Plate with Spanwise and Chordwise Variations of Temperature and Thickness", NACA TN-3778, Nov., 1956.
- (A12) Krämerov, A.Ya., Fridman, Ya.B., Iwanov, S.A., "Thermal Stresses in Reactor Structures", Atomnaya Energ., Vol. 8, 1960. (Russian)
- (A13) Boley, B.A., Weiner, J.H., "Theory of Thermal Stresses", John Wiley & Sons, Inc., New York, 1960.
- (A14) Nowacki, "Thermo-elasticity", Addison-Wesley Publishing Co., Inc., 1962.

Appendix B Reactor Calculations for a Nuclear Rocket.

(a) Calculation of Buckling

Assuming that the atom ratio (B1) (B2) is about 500 for a reactor with spherical core, we have

$$\frac{N_C}{N_U} = 500 . \quad (B1)$$

The atom ratio can be calculated from the following relations:

$$N_C = \frac{1}{12} \rho_C A_O \quad (B2)$$

$$N_U = \frac{1}{235} \rho_U A_O \quad (B3)$$

$$\rho_U = \frac{M_U}{V_V} \quad (B4)$$

$$\bar{\rho}_C = (1-\epsilon) \rho_C + \epsilon \rho_H \quad (B5)$$

where

- $N_C$  = number of carbon atoms per unit volume of core,
- $N_U$  = number of U235 atoms per unit volume of core,
- $A_O$  = Avogadro number  $6.02 \times 10^{23}$ ,
- $V$  = volume of core without void,
- $V_V$  = volume of core with void,
- $M_U$  = mass of U235 in volume  $V$ ,
- $\epsilon$  = void fraction (B3), 20% (B6)
- $\rho_C$  = density of carbon,  $1.6 \left[ \frac{gm}{cm^3} \right]$  (B4) (B7)
- $\rho$  = density of U235,
- $\rho_U$  = density of hydrogen,
- $\bar{\rho}_C$  = average density of carbon with void in the reactor,

$$\approx (1-0.2)(1.6) \approx 1.28 \left[ \frac{\text{gm}}{\text{cm}^3} \right]. \quad (\text{B8})$$

Combining Equations (E1)(B2) and (B3) we have for a core with no void,

$$\frac{235}{12} \frac{\rho_C}{M_U} V = 500. \quad (\text{B9})$$

The critical mass for a spherical reactor with graphite core and Beryllium reflector is given by Hansen, G.E. et al (B5) in LA2141 p. 57, as about 30,000 grams. The Be-reflector is estimated to be 11 cm thick.

$$\text{Thus } M_U = 30,000 \text{ [gm]}. \quad (\text{B10})$$

Substituting Eq. (B6) and (B10) into (B9) we have

$$V = 500 \frac{12}{235} \cdot \frac{30,000}{1.6} = 478,000 \text{ [cm}^3\text{]} .$$

Thus the volume of core with void is

$$V_v = V \frac{\rho_C}{\rho_C} = 478,000 \frac{1.6}{1.28} \approx 598,000 \text{ [cm}^3\text{]} .$$

Let  $R_s$  be the radius of the spherical core with the void and without counting reflector. The reactor does have reflectors.

$$\frac{4\pi R_s^3}{3} = V_v ,$$

$$R_s \approx 52.3 \text{ [cm]} .$$

The corresponding bare core can be found by adding the extrapolation length  $\lambda$ , which can be obtained from LA2141 p.53.

For the atom ratio of 500, one has

$$\delta = \lambda - 2T = \lambda - 22 ,$$

where  $T = \text{Be-reflector saving} = 11$  [cm].

Solving for  $\lambda$ , we have

$$\lambda = 30 \text{ [cm]}.$$

The total radius of the equivalent bare core is

$$R_s + \lambda = 52 + 30 = 82 \text{ [cm]}. \quad (\text{B11})$$

The above figure can be compared to a calculation from another source. The radius of bare sphere<sup>(B6)</sup> for a U235-Graphite reactor without void is approximately 60 cm for a critical mass of about 30 to 40 kg of U235. The equivalent radius with 20% void is about 66 cm.

Using the formula for buckling in spherical geometry<sup>(B7)</sup> we have

$$B^2 = \left( \frac{\pi}{R_s + \lambda} \right)^2 = 15 \times 10^{-4} \text{ [cm}^{-2}\text{]}. \quad (\text{B12})$$

The dimensions of a cylindrical core<sup>(B7)</sup> for the same buckling are related as

$$B^2 = \left( \frac{\pi}{H + \lambda_H} \right)^2 + \left( \frac{2.405}{R + \lambda_r} \right)^2, \quad (\text{B13})$$

where  $H =$  height of the cylinder ,

$R =$  radius of the cylinder ,

$\lambda_r =$  thickness of reflector in radial direction,

and  $\lambda_H =$  thickness of reflector in axial direction.

Assuming that the height of the reactor is equal to the diameter,

Equation (B13) becomes

$$B^2 = \left( \frac{\pi}{2R + 2\lambda_r} \right)^2 + \left( \frac{2.405}{R + \lambda_r} \right)^2 = \frac{\left( \frac{\pi}{2} \right)^2 + (2.405)^2}{(R + \lambda_r)^2}.$$

Equating the buckling for both geometries we obtain

$$\frac{R + \lambda_r}{R_s + \lambda_r} = \sqrt{\frac{(\frac{\pi}{2})^2 + (2.405)^2}{R + \lambda_r}} \approx 0.913 .$$

Substituting the quantity given in Eq. (B11) into the above equation one obtains

$$R + \lambda_r = 82 \times 0.913 \approx 75 \text{ [cm]} ,$$

and

$$R = 75 - 30 = 45 \text{ [cm]} \approx 1.5 \text{ [ft]} .$$

Thus the core of a reactor in cylindrical shape with 20% void is 3 ft in diameter and 3 ft in height without including reflector thickness.

(b) The Relationship between Void Fraction and Buckling.

Since the density of hydrogen is small in comparison with that of carbon, Eq. (B5) is approximated as

$$\begin{aligned} \bar{\rho}_C &\approx (1-\zeta)\rho_C \\ \text{or } \frac{\bar{\rho}_C}{\rho_C} &= (1-\zeta) . \end{aligned} \quad (\text{B14})$$

Therefore, for different void fractions, the average density carbon, can be obtained:

$$\frac{\bar{\rho}_{CO}}{\rho_C} = \frac{1-\zeta_0}{1-\zeta} . \quad (\text{B15})$$

Because both reactors are in critical condition, we have

$$\frac{\tau_o B_o^2}{\tau B^2} = \left(\frac{\rho C}{\rho_{Co}}\right)^2 \left(\frac{B_o}{B}\right)^2 \approx 1$$

where  $\tau$  = Fermi age  $= (\Sigma)^{-2}$  (B7), (B16)

$\Sigma$  = Macroscopic cross section  $= \rho$  (B7).

Then the relation between the buckling and void fraction is obtained

$$\frac{B}{B_o} = \frac{\rho C}{\rho_{Co}} = \frac{1-\zeta}{1-\zeta_o} \quad (B17)$$

With Equation (B12), one has

$$\frac{(R+\lambda)_o}{R+\lambda} = \frac{(R_s+\lambda)_o}{R_s+\lambda} = \frac{B}{B_o} = \frac{1-\zeta}{1-\zeta_o} \quad (B18)$$

For instance, changing the void fraction  $\zeta$  from 0.2 to 0.3 we have

$$B = B_o \frac{0.7}{0.8}$$

or  $R+\lambda = (R+\lambda)_o \frac{0.8}{0.7} = 75 \times \frac{0.8}{0.7} = 85.5$  [cm].

Thus,  $R = 85.6 - 30 = 55.6$  cm = 1.8 [ft] for core only.

(c) Calculation of Fuel Element

The thickness of fuel plates can be calculated by the following assumptions.

- (i) The number of channels is 212,
- (ii) The void is 20% ,
- (iii) The cross-sectional area of the reactor, 3 ft in diameter, is equivalent to a square, 2.66 ft X 2.66 ft.



(iv) The width of fuel plates is 2.66 ft.

If the thickness of fuel plate is  $2b$  and the width of coolant channel is  $d$ , the cross-sectional area of the reactor is

$$(2.66 \times 12)(2b+d)212[\text{in}^2] = (2.66 \times 12)^2[\text{in}^2], \quad (\text{B19a})$$

and the void of the reactor is

$$\frac{dx 2.66 \times 12}{(2b+d) \times 2.66 \times 12} = 0.2. \quad (\text{B19b})$$

The solution for  $d$  and  $b$  are thus

$$d = 0.03[\text{in}],$$

and

$$b = 0.06[\text{in}].$$

The 0.12 in fuel plate is within the range given by M. M. Leroy and J. J. Newgard in "Rocket-Reactor Design." (B8)

#### (d) Computation of Hydrogen Flow at Full Power

The hydrogen temperature at exit is assumed to be  $4,500^\circ\text{R}$ . If the inlet temperature of hydrogen is taken as  $180^\circ\text{R}$  and the total power of the reactor  $10^6\text{KW}$ , we can calculate the hydrogen flow rate per channel from the relation

$$P_t = \bar{C}_g (212)W(4500-180), \quad (\text{B20})$$

where  $\bar{C}_g$  = average specific heat of hydrogen per channel,

$$P_t = \text{full power} = 10^6 \text{ KW} \approx 9.5(10)^5 \left[ \frac{\text{BTU}}{\text{Sec}} \right].$$

Thus,  $W \approx$  flow rate of hydrogen per channel

$$\approx \frac{60}{212} = 0.283 \left[ \frac{\text{lbm}}{\text{Sec}} \right].$$

#### (e) Computation of Maximum Temperature at Full Power

The highest temperature of fuel plates under steady-state conditions is at the exit with the reactor at full power.

Thus the following relation can be obtained from Equations (3.11a) and (3.11b)

$$T_{US} \Big|_{\substack{x=0 \\ z=2\ell}} - T_{GS} \Big|_{z=2\ell} = \frac{P_o}{h_o} b_o \quad (B22)$$

where  $P_o = \frac{P_t}{V_v} = \frac{9.5 \times 10^5}{\frac{\pi}{4}(3)^2 \times 3 \times 0.8} \approx 5.6 \times 10^4 \left[ \frac{\text{BTU}}{\text{ft}^3 \text{Sec}} \right]$ ,

$V_v = \text{volume of carbon-uranium} = \frac{\pi}{4}(3)^2 \times 3 \times 0.8 \text{ [ft}^3\text{]},$

$b = 0.06 \text{ [in]} = 0.005 \text{ [ft]},$

and  $h_o = \text{heat transfer coefficient}.$

The heat transfer coefficient  $h_o$  is evaluated by the following equations. (B8)

$$Re = \frac{v \rho_g D_H}{\mu} = \frac{W D_H}{\mu A},$$

$$N_u = 0.023 (Re)^{0.8} (P_r)^{\frac{1}{3}},$$

and

$$h_o = \frac{N_u k}{D_H},$$

where

$Re = \text{Reynolds number},$

$v = \text{velocity of hydrogen},$

$\rho_g = \text{density of hydrogen},$

$W = v \rho_g A = \text{mass flow rate of hydrogen}$

$$\approx 123 \left[ \frac{\text{gm}}{\text{Sec}} \right] = 0.283 \left[ \frac{\text{lbm}}{\text{Sec}} \right],$$

$A = \text{cross-sectional area of coolant channel}$

$$= d \frac{D}{2} \approx 0.96 \text{ [in}^2\text{]} = 6.2 \text{ [cm}^2\text{]}.$$

$\mu = \text{average viscosity (B10)} \approx 214 \times 10^{-6} \left[ \frac{\text{gm}}{\text{Cm Sec}} \right],$

$$D_h = 4 \frac{\text{Wetted area}}{\text{Wetted Perimeter}} = (4) \frac{d \frac{D}{2}}{2(d + \frac{D}{2})} \approx 2d$$

$$= 0.06[\text{in}] = 0.152[\text{cm}] ,$$

Nu = Nusselt no. ;

Pr = Prandtl no.  $\approx 0.66$  for hydrogen (B9) ;

k = average heat conduction coefficient of hydrogen (B10)

$$\approx 14 \times 10^{-4} \left[ \frac{\text{Cal}}{\text{sec cm}^{\circ}\text{K}} \right] \approx 9.76 \times 10^{-5} \left[ \frac{\text{BTU}}{\text{sec ft}^{\circ}\text{F}} \right] .$$

From Equations (B23) and (B25), we have

$$Re \approx 1.4 \times 10^4 \quad \text{at full power ,}$$

and  $Nu \approx 42 .$

By using Equation (B25) and the values of Nusselt no. and k we obtain

$$h_o \approx 0.82 \left[ \frac{\text{BTU}}{\text{Sec ft}^2 \text{ }^{\circ}\text{R}} \right] \quad \text{at full power.}$$

Thus the maximum temperature of the fuel plate can be evaluated from Equation (B22)

$$T_{US} \Big|_{\substack{x=0 \\ z=2\ell}} = \frac{P_{ob}}{h_o} + T_{GS} \Big|_{z=2\ell} = \frac{5.6 \times 10^4 \times 0.005}{0.82} + 4500$$

$$= 342 + 4500 = 4842 [^{\circ}\text{R}] .$$

The above calculation shows that the highest temperature of the fuel element is less than the designed melting point of carbon. The actual melting point of carbon is about  $6850^{\circ}\text{R}$  (B11)

(B4)

Reference for Appendix B

- (B1) Renolds, H. L. "Critical Measurements and Calculations for Enriched-uranium Graphite-moderated Systems" Proc. 2nd Inter. Conf. Peaceful use Atomic Energy, Geneva 12 p/2408, 632 (1958).
- (B2) Smith, H. P., Jr. "Dynamics and Control of Nuclear Rocket Engine" Ph.D. Thesis of MIT 1960.
- (B3) Newgard, J. J., Levoy, M. M. "Consideration in the Design of Nuclear Rocket" Nucl. Sci. and Eng. 7,377-387 1960.
- (B4) Etherington, H., "Nuclear Eng. Handbook" McGraw Hill, 1958 p 10-55 p 10-19, 10-57.
- (B5) Hanson, G. E., et al "Beryllium-Reflected, Graphite-Moderated Critical Assemblies" AEC Report LA-2141 1957.
- (B6) Ragsdale, R. G., Rom, F. E., "Advanced Concepts for Nuclear Rocket Propulsion", Nuclear Rocket Propulsion NASA Sp-20, Dec. 1962.
- (B7) Glasstone, S. "Principles of the Nuclear Reactor Engineering" D. Van Nostrand Co., Inc., 1955, p 173, p 162, p 92.
- (B8) Levoy, M. M., Newgard, J. J. "Rocket-Reactor Design" Nucleonics July 1958.
- (B9) Kreith, F. "Principles of Heat Transfer" Inter. Book Co., 1960.
- (B10) Farmer, O. A., "Digital Programs for Para Hydrogen Properties" AEC Report LAMS-2762.
- (B11) Hodgman, C. D. "Handbook of Chemistry and Physics" Chemical Rubber Publishing Co., 37th Edition 1955-1956, p 2112.

- (b) net energy flow into the control volume per unit time due to the fluid flow,

$$- \frac{\partial}{\partial z} [\rho_g Av(u + \frac{1}{2} v^2)] dz \quad \left[ \frac{\text{BTU}}{\text{Sec}} \right].$$

- (c) work done to the control volume per unit time due to change of pressure and density,

$$- d(pAv) = - \frac{\partial}{\partial z} (pAv) dz \quad \left[ \frac{\text{BTU}}{\text{Sec}} \right].$$

- (d) heat input into control volume per unit time from the boundaries,

$$D dz h [T_U - T_G] \quad \left[ \frac{\text{BTU}}{\text{Sec}} \right].$$

Equating the rate of change of energy to the sum of the other energy rates, i.e., (a) = (b) + (c) + (d), we have

$$A \frac{\partial}{\partial t} [\rho_g (u + \frac{1}{2} v^2)] + \frac{\partial}{\partial z} [Av \rho_g (u + \frac{1}{2} v^2 + \frac{p}{\rho_g})] = Dh [T_U - T_G]. \quad (C1)$$

Let  $Av \rho_g = W(t), \quad \frac{1}{2} v^2 \ll u,$

$$u = C_v (T_G - T_{Go}), \text{ and } u + \frac{p}{\rho_g} = C_g (T_G - T_{Go})$$

where  $T_{Go}$  = reference temperature

we obtain

$$AC_v \frac{\partial}{\partial t} (\rho_g T_G) + C_g W(t) \frac{\partial}{\partial z} T_G = Dh [T_U - T_G]. \quad (C2)$$

Reference for Appendix C.

- (C1) J. F. Lee, F. W. Sears, "Thermodynamics" Addison - Wesley Publishing Co., Inc. 1959.

Appendix D Transient Solutions of Heat Transfer  
 Problem by One-Dimensional Approach

By the introduction of the following variables

$$t = \frac{\tau}{\rho C b \eta} \quad \text{and} \quad z = \frac{Dz}{C_g},$$

Equations (3.19) can be transformed into

$$h(t)[T_U(b, z, t) - T_G(z, t)] = Pb - \frac{\partial}{\partial t} T_U(b, z, t), \quad (D1a)$$

$$\frac{\partial}{\partial z} T_G(z, t) = \frac{h(t)}{W(t)} [T_U(b, z, t) - T_G(z, t)], \quad (D1b)$$

with the boundary condition

$$T_G(z, t) \Big|_{z=0} = T_{Go}, \quad (D2)$$

and the initial condition

$$T_U(b, z, t) \Big|_{t=0} = T_{Go} + \frac{P_0 b}{h_0} + \frac{P_0 b z}{W_0}, \quad (D3)$$

where  $h_0 = h(t=0)$  and  $W_0 = W(t=0)$ .

Applying the Laplace transform in  $z$ , and defining

$$T_U(b, z, t) = L[T_U(b, z, t)],$$

and

$$T_G(s, t) = L[T_G(z, t)],$$

we obtain,

$$\frac{\partial}{\partial t} T_U(b, s, t) = -h(t)[T_U(b, s, t) - T_G(s, t)] + \frac{Pb}{s}, \quad (D4a)$$

$$sT_G(s, t) - T_G(0, t) = \beta(t)[T_U(b, z, t) - T_G(s, t)], \quad (D4b)$$

with the initial condition

$$T_U(b, s, t) \Big|_{t=0} = \frac{T_{Go}}{s} + \left(\frac{s+\beta}{s^2}\right) \frac{P_0 b}{h_0}. \quad (D5)$$

Equation (D4b) gives the following relation

$$T_G(s,t) = \frac{\beta(t)}{s+\beta(t)} T_U(b,s,t) + \frac{1}{s+\beta(t)} T_G(o,t). \quad (D6)$$

Substituting the above equation into Equation (D4a) gives

$$\frac{\partial}{\partial t} T_U(b,s,t) + \frac{sh(t)}{s+\beta(t)} T_U(b,s,t) = \frac{h(t)}{s+\beta(t)} T_G(o,t) + \frac{Pb}{s}. \quad (D7)$$

The solution of Equation (D7) is

$$T_U(b,s,t) = e^{-\int \frac{sh(t)}{s+\beta(t)} dt} \left\{ \int_0^t \left[ \frac{h(t)}{s+\beta(t)} T_{Go} + \frac{Pb}{s} \right] e^{\int \frac{sh(t)}{s+\beta(t)} dt} dt + \frac{T_{Go}}{s} + \left( \frac{s+\beta_0}{s^2} \right) \frac{P_{ob}}{h_0} \right\}. \quad (D8)$$

If  $h(t) = \text{constant}$  and  $\beta(t) = \text{constant}$ , then the above equation can be readily integrated

$$T_U(b,s,t) = \frac{1}{s} T_{Go} + \frac{s+\beta}{s^2} \frac{Pb}{h} - b \left[ \frac{s+\beta}{s^2} \frac{P}{h} + \frac{s+\beta_0}{s^2} \frac{P_{ob}}{h_0} \right] e^{-\frac{hs}{s+\beta} t}. \quad (D9)$$

Substituting into Equation (D6) gives

$$T_G(s,t) = \frac{1}{s} T_{Go} + \frac{\beta}{s^2} \frac{Pb}{h} - \frac{\beta Pb}{hs^2} e^{-\frac{hs}{s+\beta} t} + \frac{(\beta-\beta_0) P_{ob}}{\beta h_0} \frac{1}{s} e^{-\frac{hs}{s+\beta} t} \\ + \frac{\beta_0 P_{ob}}{h_0} \frac{1}{s^2} e^{-\frac{hs}{s+\beta} t} + \frac{(\beta-\beta_0) P_{ob}}{\beta h_0} \frac{1}{s+\beta} e^{-\frac{hs}{s+\beta} t}. \quad (D10)$$

The inverse Laplace transform of Equations (D9) and (D10) involves the determination of the following functions:

$$(a) \quad L^{-1} \left[ \frac{1}{s+\beta} e^{-\frac{hs}{s+\beta} t} \right] = e^{-ht} L^{-1} \left[ \frac{1}{s+\beta} e^{\frac{h\beta t}{s+\beta}} \right], \quad (D11)$$

$$(b) \quad L^{-1} \left[ \frac{1}{s} e^{-\frac{hs}{s+\beta} t} \right] = e^{-ht} L^{-1} \left[ \frac{1}{s} e^{\frac{h\beta t}{s+\beta}} \right], \quad (D12)$$

$$(c) \quad L^{-1}\left[\frac{1}{s^2} e^{-\frac{hs}{s+\beta} t}\right] = e^{-ht} L^{-1}\left[\frac{1}{s^2} e^{\frac{h\beta t}{s+\beta}}\right] \quad (D13)$$

The above functions can be written in the following form

$$R(s) = \frac{1}{(s+\gamma)^m} e^{\frac{a}{s+\beta}} \quad (D14)$$

where a proper choice of  $m$ ,  $a$ , and  $\gamma$  represents either Equation (D11), (D12) or (D13). Thus if the inverse of Equation (D14) is known, we are able to obtain the inverse of Equations (D11), (D12) and (D13). The inverse of Equation (D14) is derived as follows:

$$L^{-1}\left[\frac{1}{s+\gamma} e^{\frac{a}{s+\beta}}\right] = L^{-1}\left[\frac{s+\beta}{s+\gamma} \frac{1}{s+\beta} e^{\frac{a}{s+\beta}}\right] = L^{-1}[G(s)F(s)] = \int_0^Z g(\zeta) f(Z-\zeta) d\zeta \quad (D15)$$

$$\text{where } G(s) = \frac{s+\beta}{s+\gamma} \quad (D16)$$

$$g(\zeta) = L^{-1}G(s) = L^{-1}\left[1 - \frac{\gamma-\beta}{s+\gamma}\right] = \delta(\zeta) - (\gamma-\beta)e^{-\gamma\zeta} \quad (D17)$$

$$F(s) = \frac{1}{s+\beta} e^{\frac{a}{s+\beta}} \quad (D18)$$

$$\text{and } f(\zeta) = L^{-1}F(s) = L^{-1}\left[\frac{1}{s+\beta} e^{\frac{a}{s+\beta}}\right] = e^{-\beta\zeta} L^{-1}\left[\frac{1}{s} e^{\frac{a}{s}}\right] \quad (D19)$$

By applying series expansion to Equation (D19) we have (D1)(D2)  
(D3)

$$\begin{aligned} L^{-1}\left[\frac{1}{s} e^{\frac{a}{s}}\right] &= L^{-1}\left[\frac{1}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{a}{s}\right)^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} L^{-1}\left[\frac{1}{s^{n+1}}\right] \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{\zeta^n}{n!} = \sum_{n=0}^{\infty} \frac{(\sqrt{a\zeta})^{2n}}{n!n!} = I_0(2\sqrt{a\zeta}) \quad (D20) \end{aligned}$$

thus

$$f(\zeta) = e^{-\beta\zeta} I_0(2\sqrt{a\zeta}) \quad (D21)$$



Substituting Equations (D17) and (D21) into Equations (D15)

gives

$$\begin{aligned} L^{-1}\left[\frac{1}{s+\gamma}e^{\frac{a}{s+\beta}}\right] &= \int_0^Z [\delta(\zeta) - (\gamma-\beta)e^{-\gamma\zeta}]e^{-\beta(Z-\zeta)}I_0(2\sqrt{a(Z-\zeta)})d\zeta \\ &= e^{-\beta Z}I_0(2\sqrt{aZ}) - (\gamma-\beta)e^{-\beta Z} \int_0^Z e^{-(\gamma-\beta)\zeta}I_0(2\sqrt{a(Z-\zeta)})d\zeta. \end{aligned} \quad (D22)$$

Let  $2\sqrt{a(Z-\zeta)} = \xi$ , one then obtains

$$\zeta = Z - \frac{\xi^2}{4a}, \quad \text{and} \quad d\zeta = -\frac{d\xi}{2a}. \quad (D23)$$

Applying this change of variables to Equations (D22), we have

$$\begin{aligned} L^{-1}\left[\frac{1}{s+\gamma}e^{\frac{a}{s+\beta}}\right] &= e^{-\beta Z}I_0(2\sqrt{aZ}) \\ &+ (\gamma-\beta)e^{-\beta Z} \int_0^Z \frac{e^{-(\gamma-\beta)(Z-\frac{\xi^2}{4a})}}{2\sqrt{aZ}} \frac{\xi}{2a} I_0(\xi) d\xi \\ &= e^{-\beta Z}I_0(2\sqrt{aZ}) + \frac{(\beta-\gamma)}{2a}e^{-\gamma Z} \int_0^{2\sqrt{aZ}} e^{-\frac{\beta-\gamma}{4a}\xi^2} \xi I_0(\xi) d\xi. \end{aligned} \quad (D24)$$

To evaluate the integral in Equation (D24), relations between the modified Bessel functions are needed. The integral relation is derived from the recurrence formulae as follows (D4)(D5)

$$2 \frac{d}{d\xi} I_{m+1}(\xi) = I_m(\xi) + I_{m+2}(\xi), \quad (D25a)$$

$$I_m(\xi) - I_{m+2}(\xi) = \frac{2(m+1)}{\xi} I_{m+1}(\xi). \quad (D25b)$$

Solving for  $I_{m+2}(\xi)$  from Equation (D25b) and substituting into Equation (D25a) gives

$$\frac{d}{d\xi} I_{m+1}(\xi) + \frac{m+1}{\xi} I_{m+1}(\xi) = I_m(\xi). \quad (D26)$$

Multiplying  $\xi^{m+1}$  in the above equation, we obtain

$$\xi^{m+1} \frac{d}{d\xi} I_{m+1}(\xi) + (m+1)\xi^m I_{m+1}(\xi) = \xi^{m+1} I_m(\xi),$$

i.e., 
$$\frac{d}{d\xi} [\xi^{m+1} I_{m+1}(\xi)] = \xi^{m+1} I_m(\xi). \quad (D27)$$

Therefore, the integral relation of Modified Bessel functions is

$$\int \xi^{m+1} I_m(\xi) d\xi = \xi^{m+1} I_{m+1}(\xi). \quad (D28)$$

Applying the above integral relation to Equation (D24) and integrating by parts gives

$$\begin{aligned} L^{-1} \left[ \frac{1}{s+\gamma} e^{\frac{a}{s+\beta}} \right] &= e^{-\beta Z} I_0(2\sqrt{aZ}) \\ &+ \frac{\beta-\gamma}{2a} e^{-\gamma Z} \left\{ \xi I_1(\xi) e^{-\frac{\beta-\gamma}{4a}\xi^2} \Big|_0^{2\sqrt{aZ}} + \frac{\beta-\gamma}{2a} \int_0^{2\sqrt{aZ}} \xi^2 I_1(\xi) e^{-\frac{\beta-\gamma}{4a}\xi^2} d\xi \right\} \\ &= e^{-\beta Z} I_0(2\sqrt{aZ}) + \frac{\beta-\gamma}{2a} e^{-\gamma Z} [2\sqrt{aZ} I_1(2\sqrt{aZ}) e^{-(\beta-\gamma)Z}] \\ &\quad + \left(\frac{\beta-\gamma}{2a}\right)^2 e^{-\gamma Z} \int_0^{2\sqrt{aZ}} \xi^2 I_1(\xi) e^{-\frac{\beta-\gamma}{4a}\xi^2} d\xi \\ &= e^{-\beta Z} I_0(2\sqrt{aZ}) + e^{-\beta Z} (\beta-\gamma) \left(\sqrt{\frac{Z}{a}}\right) I_1(2\sqrt{aZ}) \\ &\quad + \left(\frac{\beta-\gamma}{2a}\right)^2 e^{-\gamma Z} \int_0^{2\sqrt{aZ}} \xi^2 I_1(\xi) e^{-\frac{\beta-\gamma}{4a}\xi^2} d\xi \\ &= e^{-\beta Z} \left\{ I_0(2\sqrt{aZ}) + (\beta-\gamma) \left(\sqrt{\frac{Z}{a}}\right) I_1(2\sqrt{aZ}) + [(\beta-\gamma) \sqrt{\frac{Z}{a}}]^2 I_2(2\sqrt{aZ}) + \dots \right\} \\ &= e^{-\beta Z} \sum_{n=0}^{\infty} \left[ (\beta-\gamma) \sqrt{\frac{Z}{a}} \right]^n I_n(2\sqrt{aZ}). \quad (D29) \end{aligned}$$

Differentiating the above equation with respect to  $\gamma$ , we have

$$L^{-1}\left[\frac{1}{(s+\gamma)^2}e^{\frac{a}{s+\beta}}\right] = e^{-\beta t} \sum_{n=1}^{\infty} n(\beta-\gamma)^{n-1} \left(\sqrt{\frac{z}{a}}\right)^n I_n(2\sqrt{az}). \quad (D30)$$

If this differentiation is applied  $(m-1)$  times, one obtains

$$L^{-1}\left[\frac{1}{(s+\gamma)^m}e^{\frac{a}{s+\beta}}\right] = e^{-\beta t} \sum_{n=m-1}^{\infty} \frac{n!}{(n-m+1)!} (\beta-\gamma)^{n-m+1} \left(\sqrt{\frac{z}{a}}\right)^n I_n(2\sqrt{az}) \quad (D31)$$

Applying these derived inverse formulae to Equation (D10) gives Equation (3.22a) in the text.

References for Appendix D

- (D1) Journal für Math. LXIX. (1868) p. 227.
- (D2) Churchill, R. V., "Operational Mathematics" McGraw-Hill  
1958.
- (D3) Brown & Nilsson, Introduction to Linear Systems Analysis  
Wiley 1962 p 390.
- (D4) Watson, G. N., A Treatise on the Theory of Bessel  
Functions Cambridge Univ. Press, 2nd Edition 1958 p 79.
- (D5) Hildebrand, F. B., Advanced Calculus for Engineers  
Prentice Hall 7th Edition 1957 p 164.

## Appendix E Transient Solutions by Perturbation

The perturbed temperatures of fuel and coolant are given in Equations (3.23) and repeated here for convenience.

$$\frac{\partial T_u}{\partial t}(b, Z, t) = -h_o[T_u(b, Z, t) - T_g(Z, t)] - b(P_o - P) \quad (E1)$$

$$\frac{\partial T_g}{\partial Z}(Z, t) = \beta_o[T_u(b, Z, t) - T_g(Z, t)], \quad (E2)$$

with boundary condition

$$T_g(Z, t) \Big|_{Z=0} = 0 \quad (E3)$$

and the initial condition

$$T_u(b, Z, t) \Big|_{t=0} = 0 \quad (E4a)$$

$$\text{or } T_g(Z, t) \Big|_{Z=0} = 0 \quad (E4b)$$

The above zero initial condition simplifies the following Laplace transform in  $t$

$$pT_u(b, Z, p) = -h_o[T_u(b, Z, p) - T_g(Z, p)] - \frac{b}{p}(P_o - P), \quad (E5a)$$

$$\frac{\partial T_g}{\partial Z}(Z, p) = \beta_o[T_u(b, Z, p) - T_g(Z, p)], \quad (E5b)$$

with the boundary condition

$$T_g(Z, p) \Big|_{Z=0} = 0 \quad (E6)$$

where

$$T_u(b, Z, p) = L[T_u(b, Z, t)]$$

and

$$T_g(Z, p) = L[T_g(Z, t)]$$

The solutions of Equation (E5) in terms of the Laplace Transform Operator  $p$  are

$$T_g(Z,p) = - \frac{b(P_o - P)}{p^2} (1 - e^{-\frac{\beta_o p}{p+h_o} Z}) \quad (E7a)$$

and

$$T_u(b,Z,p) = - \frac{bh_o(P_o - P)}{p^2(p+h_o)} (1 - e^{-\frac{\beta_o p}{p+h_o} Z}) - \frac{b(P_o - P)}{p(p+h_o)} \quad (E7b)$$

Using the relation

$$T_U(b,Z,p) = \frac{1}{p} T_{US} + T_u(b,Z,p)$$

we obtain

$$T_U(b,Z,p) = \frac{1}{p} [T_{co} + \frac{p_o b}{h_o} (1 + \beta_o Z)] - \frac{b(P_o - P)}{p(p+h_o)} [\frac{h_o}{p} (1 - e^{-\frac{\beta_o p}{p+h_o} Z}) + 1] \quad (E8)$$

Applying the inverse Laplace transform for operator  $p$  derived in Appendix D gives

$$T_U(b,Z,t) = T_{co} + \frac{p_o b}{h_o} (1 + \beta_o Z) - \frac{b}{h_o} (P_o - P) [h_o t - e^{-\beta_o Z} h_o t \sum_{n=1}^{\infty} (n-1) (\frac{h_o t}{\beta_o Z})^{\frac{n}{2}} I_n(2\sqrt{\beta_o Z h_o t})] \quad (E9)$$

## Appendix F General Equations for Thermal Stresses

The basic differential equations for the linearly coupled thermoelastic theory are<sup>(F1)</sup>

$$k\nabla^2 T + P = \rho C \frac{\partial T}{\partial t} + (3\lambda + 2\mu)\alpha T_0 \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \quad (F1)$$

$$\left. \begin{aligned} \frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{xy} + \frac{\partial}{\partial z} \sigma_{xz} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{yz} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial}{\partial x} \sigma_{xz} + \frac{\partial}{\partial y} \sigma_{yz} + \frac{\partial}{\partial z} \sigma_{zz} &= \rho \frac{\partial^2 w}{\partial t^2}, \end{aligned} \right\} \quad (F2)$$

where  $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ ,  $\mu = \frac{E}{2(1+\nu)}$ ,

$E$  = Young's modulus,

$\nu$  = Poisson's ratio,

$T$  = temperature distribution,

$T_0$  = reference temperature for which the material is stress-free,

$\sigma$  = stresses,

$u, v, w$  = displacements in  $x, y, z$  direction, respectively,

$\alpha$  = coefficient of linear thermal expansion,

$\rho$  = density,

$C$  = specific heat,

$P$  = heat generated per unit volume per unit time.

The heat conduction equation, Equation (F1), is coupled with

the equilibrium equation, Equation (F2). This coupled boundary value problem is of considerable mathematical difficulty. Fortunately, in engineering applications, it is possible to introduce certain assumptions to simplify this problem. The principal simplifications are the omission of the mechanical coupling term in the heat conduction equation and the inertia terms in the equilibrium equations. Usually, the theory based on none of these simplifications is called coupled theory. The uncoupled theory is based on an approximation which omits the mechanical coupling term in the heat conduction equation whereas the uncoupled quasi-static theory omits both the mechanical coupling and inertia terms.

Reference for Appendix F.

(F1) Boley, B. A., Weiner, J. H., "Theory of Thermal Stresses"  
John Wiley & Sons, Inc., 1960.



## Appendix G

## Thermal Elasticity

For the quasi-static three-dimensional thermal stress problem in rectangular coordinates the three equations of equilibrium with no body forces are<sup>(G1)(G2)</sup>

$$\begin{aligned}\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{xy} + \frac{\partial}{\partial z} \sigma_{xz} &= 0, \\ \frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{yz} &= 0, \quad (G1)\end{aligned}$$

and

$$\frac{\partial}{\partial x} \sigma_{xz} + \frac{\partial}{\partial y} \sigma_{yz} + \frac{\partial}{\partial z} \sigma_{zz} = 0.$$

These equations, containing six components of stress  $\sigma_{xx}$ ,  $\sigma_{xy}$ , ...,  $\sigma_{zz}$ , are not sufficient for the determination of these components. In order to obtain the solution we must consider the elastic deformations and the thermal expansions of the body. By using Hooke's law, the six stress-strain relations are<sup>(G1)</sup>

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] + \alpha(T - T_0), \\ \epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] + \alpha(T - T_0), \\ \epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] + \alpha(T - T_0), \\ \epsilon_{xy} &= \frac{\sigma_{xy}}{2G}, \quad \epsilon_{yz} = \frac{\sigma_{yz}}{2G}, \quad \epsilon_{xz} = \frac{\sigma_{xz}}{2G}, \quad (G2)\end{aligned}$$

where

- E = Young's modulus,
- G = shear modulus,
- $\epsilon$  = strain,
- T = temperature distribution,
- $T_0$  = initial temperature distribution,
- and  $\alpha$  = coefficient of linear thermal expansion.

It should be noted that the six components of strain at each point are completely determined by the three displacement components  $u$ ,  $v$ ,  $w$ , subject to the following six strain displacement relations

$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}, \\ \epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.\end{aligned}\tag{G3}$$

#### References for Appendix G

- (G1) Boley, E.A., Weiner, J. H., "Theory of Thermal Stresses," John Wiley & Sons, Inc., 1960.
- (G2) Timoshenko, S., "Theory of Elasticity," McGraw-Hill Book Co., Inc., 1934.

Appendix H Compatibility Conditions

The strain-displacements relations reduce to the following six differential equations in terms of strains by differentiating Equation (G3) in Appendix G.

$$\left. \begin{aligned} \frac{\partial^2}{\partial y^2} \epsilon_{xx} + \frac{\partial^2}{\partial x^2} \epsilon_{yy} &= \frac{\partial^2}{\partial x \partial y} \epsilon_{xy} , \\ \frac{\partial^2}{\partial z^2} \epsilon_{yy} + \frac{\partial^2}{\partial y^2} \epsilon_{zz} &= \frac{\partial^2}{\partial y \partial z} \epsilon_{yz} , \\ \frac{\partial^2}{\partial x^2} \epsilon_{zz} + \frac{\partial^2}{\partial z^2} \epsilon_{xx} &= \frac{\partial^2}{\partial x \partial z} \epsilon_{xz} , \end{aligned} \right\} \text{(H1a)}$$

$$\left. \begin{aligned} 2 \frac{\partial^2}{\partial y \partial z} \epsilon_{xx} &= \frac{\partial}{\partial x} \left( - \frac{\partial}{\partial x} \epsilon_{yz} + \frac{\partial}{\partial y} \epsilon_{xz} + \frac{\partial}{\partial z} \epsilon_{xy} \right) , \\ 2 \frac{\partial^2}{\partial x \partial z} \epsilon_{yy} &= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \epsilon_{yz} - \frac{\partial}{\partial y} \epsilon_{xz} + \frac{\partial}{\partial z} \epsilon_{xy} \right) , \\ 2 \frac{\partial^2}{\partial x \partial y} \epsilon_{zz} &= \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} \epsilon_{yz} + \frac{\partial}{\partial y} \epsilon_{xz} - \frac{\partial}{\partial z} \epsilon_{xy} \right) . \end{aligned} \right\} \text{(H1b)}$$

By using Hooke's law, the strains in terms of stresses are

$$\left. \begin{aligned} \epsilon_{yy} &= \frac{1}{E} [(1+\nu)\sigma_y - \nu\theta] + \alpha E (T - T_0) , \\ \epsilon_{zz} &= \frac{1}{E} [(1+\nu)\sigma_z - \nu\theta] + \alpha E (T - T_0) , \\ \epsilon_{yz} &= \frac{2(1+\nu)}{E} \sigma_{yz} , \end{aligned} \right\} \text{(H2)}$$

where  $\theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ .

Substituting Equation (H2) into the 2nd relation of Equation (H1a) gives

$$(1+\nu)\left(\frac{\partial^2}{\partial z^2}\sigma_{yy} + \frac{\partial^2}{\partial y^2}\sigma_{zz}\right) - \nu\left(\frac{\partial^2\theta}{\partial z^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \alpha E\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right)(T-T_0) = 2(1+\nu)\frac{\partial^2}{\partial y\partial z}\sigma_{yz} \quad (H3)$$

Differentiating the second of the equilibrium equations [i.e. Equation (G1)] with respect to  $z$  and the third with respect to  $y$  and adding them together, we have

$$2\frac{\partial^2}{\partial y\partial z}\sigma_{yz} = -\frac{\partial^2}{\partial z^2}\sigma_{zz} - \frac{\partial^2}{\partial y^2}\sigma_{yy} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{xy}\right) \quad (H4)$$

Applying the first of the equilibrium equations, one obtains

$$2\frac{\partial^2}{\partial y\partial z}\sigma_{yz} = -\frac{\partial^2}{\partial z^2}\sigma_{zz} - \frac{\partial^2}{\partial y^2}\sigma_{yy} + \frac{\partial^2}{\partial x^2}\sigma_{xx} \quad (H5)$$

Substituting Equation (H5) into Equation (H3) we have

$$(1+\nu)\left(\nabla^2\theta - \nabla^2\sigma_{xx} - \frac{\partial^2\theta}{\partial x^2}\right) - \nu\left(\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2}\right) = -\alpha E\left(\nabla^2 - \frac{\partial^2}{\partial x^2}\right)(T-T_0). \quad (H6)$$

Two analogous equations can be obtained from the other conditions of compatibility Equation (H1a). Adding all three equations gives

$$(1-\nu)\nabla^2\theta = -2\alpha E\nabla^2(T-T_0). \quad (H7)$$

Substituting the above into equation (H6) one obtains

$$(1+\nu)\nabla^2\sigma_{xx} + \frac{\partial^2\theta}{\partial x^2} = -\alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 + \frac{\partial^2}{\partial x^2}\right)(T-T_0). \quad (H8)$$

By changing the subscripts from  $x$  to  $y$  or  $z$  two more equations of the same type can be obtained by using the relation given in Equation (H1a). By a similar procedure three other relations are derived from Equation (H1b). One of these relations is

$$(1+\nu)\nabla^2\sigma_{yz} + \frac{\partial^2\theta}{\partial y\partial z} = -\alpha E\frac{\partial^2}{\partial y\partial z}(T-T_0). \quad (H9)$$