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NONLINEAR FLUID OSCILLATIONS IN A PARTIALLY FILLED CYLINDRICAL SECTOR CONTAINER

by J. A. Baird, Jr.

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NONLINEAR FLUID OSCILLATIONS IN A PARTIALLY FILLED CYLINDRICAL SECTOR CONTAINER

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ABSTRACT

This investigation is a study of finite-amplitude free oscillations of an inviscid incompressible fluid in a cylindrical sector container. The analysis is made for a standing wave whose motion to the first approximation is that of the first nonaxisymmetric mode. The effects of surface tension are not considered. The method of Krylov and Bogoliubov is used to satisfy the nonlinear boundary condition. Numerical results are presented for 90° sector tanks. Author

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LIST OF SYMBOLS

a	Radius of cylindrical sector
A^i, a^i	Acceleration
$\overline{B}_2, \overline{B}_3$	See Equations 51 and 52
$C_j{}^i$	Rotational transformation between Z^j and X^i systems
F^k, f^k	Body forces
G	Gravitational Acceleration
G_c	Frequency correction factor
g_{ij}	Metric tensor
g^{ij}	Reciprocal tensor
g	determinate of g_{ij}
h	Fluid depth
I(F)	$\int_0^a r\, F(r)\, dr$
J_n	Bessel function of 1st kind and order n
$\overline{\ell}$	Position vector of origin of η^i
n, λ	Constants in Solution of Laplace's Equation
p	Pressure
p_o	Pressure at free surface
R^k	Position vector of origin of X^i system
r, θ, z	Cylindrical coordinates

U^i, u^i	Velocity of origin of X^i system
V^i, v^i	Velocity
X^i	Moving Cartesian coordinate system
Z^i	Newtonian coordinate system
α	Integer which determines sector angle
γ_{km}	mth zero of $J'_{k\alpha}$
δ^{ij}	Knonecker delta
ϵ	Expansion parameter
ϵ_{ijk}	Permutation tensor
ζ	Wave height
η^i	Orthogonal curvilinear coordinate system
ρ	Mass density
ϕ	Velocity potential
ψ	Velocity potential
ψ_r	Indicates $\partial\psi/\partial r$
ω^p	Angular velocity of X^k system
$\omega_{k\alpha, m}$	Natural frequencies of fluid oscillation
$'$	Differentiation with respect to argument
$\cdot$	d/dt

CHAPTER I

INTRODUCTION

Purpose and Scope of Investigation

This investigation is a study of finite-amplitude free oscillations of an inviscid incompressible fluid in a cylindrical sector container. The analysis is made for a standing wave whose motion to the first approximation is that of the first nonaxisymmetric mode. The effects of surface tension are not considered. No limitations are placed on the liquid depth in the formulation of the problem, however it is found that at certain discrete depths the solution becomes invalid.

The formulation of the problem results in a governing linear partial differential equation along with three linear boundary conditions and one nonlinear boundary condition. The main difficulty in finding a solution is the satisfaction of this nonlinear boundary condition which must be applied to a moving boundary whose shape is itself unknown. The method of solution is to approximate this boundary condition by a Taylor series expansion which retains terms to the third order of the wave height. The solution which satisfies this approximate nonlinear boundary condition is found by the method of Krylov and Bogoliubov. It is emphasized that this solution is only for periodic waves since in general the solutions to nonlinear problems are nonperiodic.

The solution to this problem is of interest primarily for two reasons. First, it is fundamental in the study of fluid mechanics and nonlinear vibrations as an initial step toward the more complex problem of forced oscillations. Second, the particular container shape chosen is of interest for application to large space vehicles. The increasing size of space vehicles and their large diameters has lowered the natural frequencies of the liquid propellant and thus shifted them closer to the control frequency of the vehicle. Since propellant oscillations can create forces which can affect the stability and control of the vehicle, it is important to both thoroughly understand the phenomenon and to investigate means of raising the propellant natural frequencies and thus remove them from close proximity to the control frequency. One proposed means of increasing propellant natural frequencies is to use compartmented cylindrical tanks. The cylindrical sector tank investigated here can be used as a "building block" to make up such a compartmented cylindrical container.

Previous Work

In the past many investigations were devoted to gravity waves. The majority of these works, however, treated finite-amplitude waves in deep water and in shallow water as separate problems rather than considering a solution for general depth. Also until recently most of the study was devoted to progressive waves. To the author's knowledge the first theoretical study of finite-amplitude standing waves was conducted in 1952 by W. G. Penny and A. I. Price[1*] who analyzed such waves in a rectangular tank of infinite depth.

Finite-amplitude axisymmetric gravity waves in a circular container were studied by Mack[2] in 1958, who considered period free oscillations. In 1959 Tadjbakhsh and Keller[3] analyzed the same problem as Penny and Proce except that a solution was found for finite depth. Their solution was extended by Concus[4] who also included the effects of surface tension. A perturbation solution for nonlinear free oscillations of an inviscid incompressible fluid in a circular container was found by DiMaggio and Rehm[5] in 1965. Their solution was carried out for a standing wave whose motion to the first approximation was that of the first antisymmetric mode.

Although all of the previously mentioned work falls into the realm of nonlinear oscillations one linear analysis should also be recognized. In 1963 Bauer[8] presented a solution based on a linearized free surface condition for the cylindrical sector tank. The natural frequencies predicted by Bauer proved to be too large when compared

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*Numbers refer to references listed in Bibliography.

to experimental values. Bauer's explanation that in tanks of this type the frequency was strongly influenced by nonlinear effects was one of the stimuli which prompted this investigation.

The author wishes to express his appreciation to Dr. C. H. Chang.

CHAPTER II

DERIVATION OF GOVERNING EQUATIONS

In order to make the formulation of the problem as useful as possible, the governing equations and boundary conditions for the motion of an incompressible, inviscid fluid in a moving container are developed in a general manner. The use of tensor notation and orthogonal curvilinear coordinates makes the formulation valid for containers of arbitrary shape and six degrees of freedom. The effects of surface tension are not considered.

The coordinate systems used are shown in Figure 1. The z^i system is a rectangular Cartesian inertial system. The x^i system is a rectangular Cartesian coordinate system which moves relative to the z^i system. The x^i system is fixed in the container with its origin located at some arbitrary point of the container. The rotational transformation between the x^i and z^i systems is given by

$$z^i = C_j^{\ i} x^j$$

where summation convention is used and the $C_j^{\ i}$ are defined by the relation

$$C_j^{\ i} = C_j^{\ i}(t) = \cos\,(z^i,\ x^j) \quad .$$

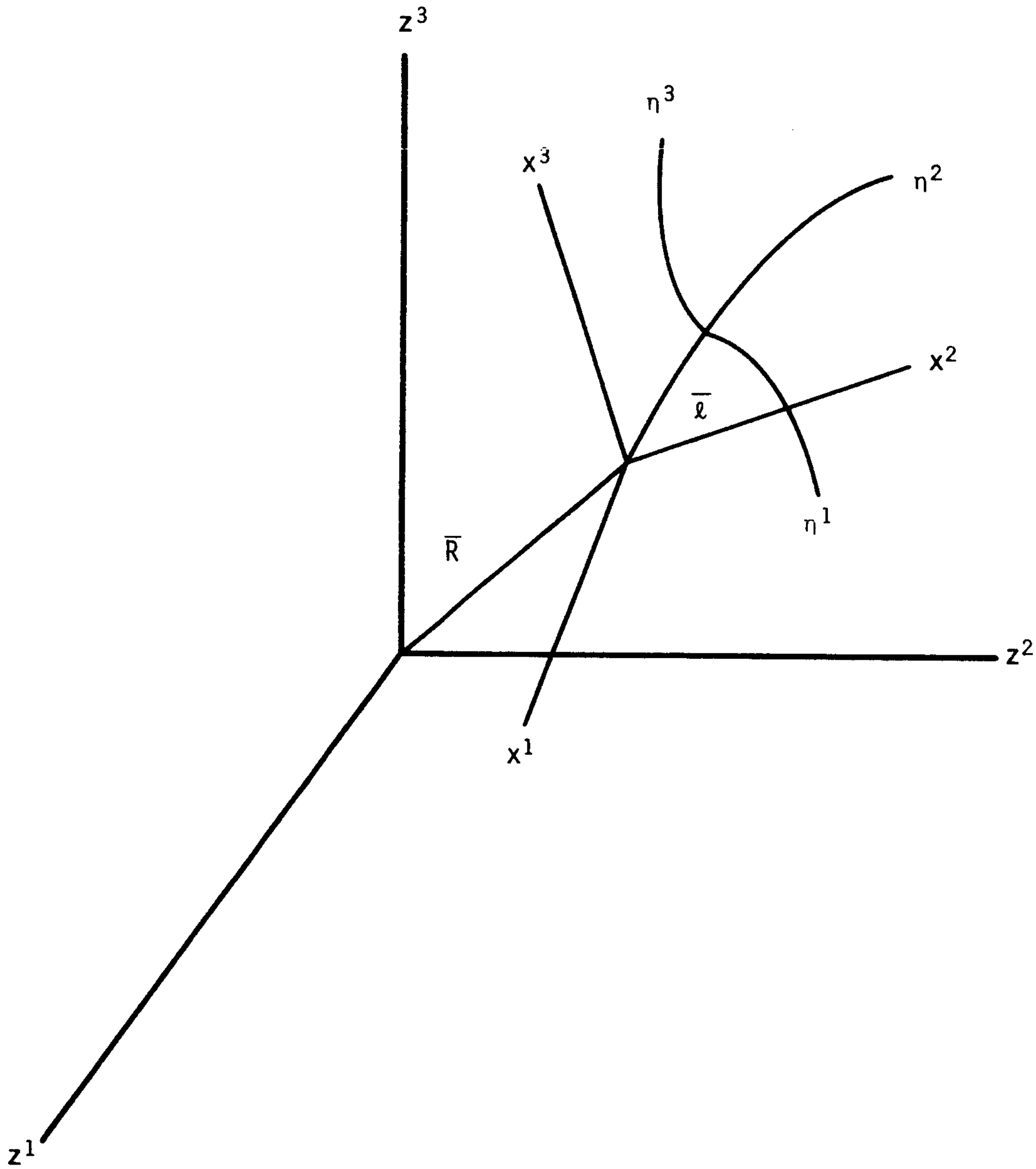

Figure 1. Coordinate System

The η^i system is an orthogonal curvilinear coordinate system which is used to describe the container shape. The location of the origin of the η^i system is given by a constant vector $\bar{\ell}$ measured in the x^i system. The use of the x^i system allows the translational and angular velocities of the container to be expressed as quantities which have physical significance. The vector $\bar{\ell}$ is included so that the origin of the curvilinear system does not have to be the center of rotation.

The position of a particle in the x^k system may be described in the z^k system by

$$z^k = R^k + C_j{}^k x^j \quad .$$

From this point on, capital letters will be used to denote quantities described in the z^k (inertial) system.

The velocity V^k is

$$V^k = \frac{d z^k}{dt} = \frac{d R^k}{dt} + \frac{d}{dt} (C_j{}^k x^j) \quad .$$

Denoting d/dt by a dot ($\cdot$) and performing the indicated differentiation yields

$$V^k = \dot{R}^k + \dot{C}_j{}^k x^j + C_j{}^k \dot{x}^j \quad .$$

The $\dot{C}_j{}^k$ can be eliminated in favor of a quantity which has physical meaning by defining

$$\omega_{j\ell} = C_j{}^k \dot{C}_\ell{}^k \quad .$$

The $\omega_{j\ell}$ can be shown to be a second order antisymmetric tensor which has the dual vector ω^p given by

$$\omega^p = -\frac{1}{2}\,\epsilon^{j\ell p}\,\omega_{j\ell} \quad .$$

Physically ω^p is the angular velocity vector of the x^k frame with respect to the z^k system expressed in the x^k system. Using this definition, the velocity can be written

$$V^k = \dot{R}^k - \epsilon_{\ell jm}\,\omega^m\,C_\ell{}^k\,x^j + C_j{}^k\,\dot{x}^j \quad .$$

Noting that the transformation from the z^k system to the x^k system for any vector quantity, B is

$$b^i = C_i{}^k\,B^k$$

and defining

$$\dot{r}^k = U^k$$

the velocity may be expressed in the x^i system as

$$v^i = u^i - \epsilon_{ijm}\,\omega^m\,x^j + \dot{x}^i \quad . \qquad (1)$$

The absolute acceleration is found from

$$A^i = \frac{d}{dt}\,V^i = \dot{C}_k{}^i\,V^k + C_k{}^i\,\dot{V}^k$$

which when expressed in the x^k system, becomes

$$a^j = \dot{v}^j + \epsilon_{jmk}\,\omega^m\,v^k \quad . \qquad (2)$$

The governing equations for an incompressible inviscid fluid in the z^i (inertial) system are Euler's equation of motion

$$A^k = F^k - \frac{1}{\rho} \frac{\partial p}{\partial z^k} \tag{3}$$

and the continuity equation

$$\frac{\partial V^k}{\partial z^k} = 0 \tag{4}$$

where ρ is the mass density of the fluid and p is the pressure. If the fluid motion is initially irrotational, it can be shown by Kelvin's theorem that it remains irrotational. For irrotational flow, there exists a potential function ϕ such that

$$\frac{\partial \phi}{\partial z^k} = V^k \quad . \tag{5}$$

The continuity equation can be written in terms of the velocity potential ϕ as

$$\frac{\partial}{\partial z^\ell} \left(\frac{\partial \phi}{\partial z^k} \right) \delta^{\ell k} = 0 \quad . \tag{6}$$

By noting that

$$\frac{\partial}{\partial z^k} = \frac{\partial}{\partial x^i} \frac{\partial x^i}{\partial z^k} = C_i{}^k \frac{\partial}{\partial x^i} \quad ,$$

Equations 3, 5 and 6 can be expressed in the x^i system as

$$a^j = f^j - \frac{1}{\rho} \frac{\partial p}{\partial x^j} \tag{7}$$

$$\frac{\partial}{\partial x^i}\left(\frac{\partial \phi}{\partial x^j}\right)\delta^{ij} = 0 \tag{8}$$

$$v^j = \frac{\partial \phi}{\partial x^j} \quad . \tag{9}$$

Defining the body force as

$$f^j = -\frac{\partial \Omega}{\partial x^j}$$

and using Equation 2, Euler's equation of motion in the x^k system becomes

$$-\frac{\partial \Omega}{\partial x^i} - \frac{1}{\rho}\frac{\partial p}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial v^j}{\partial x^i}\dot{x}^i + \epsilon_{j\ell m}\,\omega^\ell\, v^m$$

which can be integrated to yield

$$-\Omega - \frac{(p-p_o)}{\rho} = \frac{\partial \phi}{\partial t} + \frac{1}{2}(v^j - u^j)^2 + \epsilon_{j\ell m}\,\omega^m\, x^\ell\, v^j + C(t) \quad . \tag{10}$$

It has been shown[6] that C(t) can be taken as zero without any essential loss of generality.

Since most containers are more readily described in some coordinate system other than rectangular Cartesian, the governing equations and boundary conditions are now formulated in orthogonal curvilinear coordinates.

The surfaces bounding the region occupied by the fluid can be described by equations of the form

$$f(\eta_i, t) = 0 \tag{11}$$

where the η_i are orthogonal curvilinear coordinates. These surfaces can be either container walls or fluid free surfaces.

The governing Equations 8 and 10 with $\phi = \phi(\eta^i)$ become in the η^i system

$$(g)^{-\frac{1}{2}} \frac{\partial}{\partial \eta^k} \left[(g)^{\frac{1}{2}} g^{k\ell} \frac{\partial \phi}{\partial \eta} \right] = 0 \qquad \text{in the region} \tag{12}$$

$$\frac{\partial \phi}{\partial t} = - \frac{p - p_o}{\rho} - \Omega - \frac{1}{2} \frac{\partial \phi}{\partial \eta^i} \frac{\partial \eta^i}{\partial x^j} \frac{\partial \phi}{\partial \eta^\ell} \frac{\partial \eta^\ell}{\partial x^k} \delta^{jk}$$

$$+ \frac{\partial \phi}{\partial \eta^i} \frac{\partial \eta^i}{\partial x^j} u^j - \frac{1}{2} u^j u^i \delta_{ij} - \epsilon_{j\ell m} \omega^m x^\ell \frac{\partial \phi}{\partial \eta^i} \frac{\partial \eta^i}{\partial x^j} \tag{13}$$

where $g \equiv$ determinate $g_{k\ell}$ and $g_{k\ell}$ is the metric tensor. Equations 12 and 13 are the governing equations for the interior of the fluid region. The conditions which must be satisfied on the surfaces given by Equation 11 can be established from the fact that the surfaces must be material. A necessary and sufficient condition that the surfaces be material is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \dot{x}^i = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \eta^j} \frac{\partial \eta^j}{\partial x^i} \dot{x}^i = 0 \quad . \tag{14}$$

It can be shown[6] that this relation requires that the fluid velocity normal to the surface be equal to the normal component of velocity of the surface itself.

A unit vector $\overline{\nu}$ normal to the surface is described in the η^i system by

$$\overline{\nu} = \frac{\text{grad } f}{|\text{grad } f|} = \frac{(\partial f/\partial \eta^i)\, g^i}{|\text{grad } f|}$$

which has physical components given by

$$\nu_{(j)} = \frac{(g_{\underline{jj}})^{-\frac{1}{2}} (\partial f/\partial \eta^j)}{|\text{grad } f|} \quad . \tag{15}$$

No summation is indicated on quantities such as $g_{\underline{jj}}$. The physical components of the fluid velocity in the η^i system are

$$v_{(j)} = (g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial \phi}{\partial \eta^j} \tag{16}$$

and the physical components of the velocity of the surface are

$$v^{(j)} = (g_{\underline{jj}})^{\frac{1}{2}} \frac{\partial \eta^j}{\partial x^i} V^i \quad . \tag{17}$$

The condition for the equality of the normal velocity components is found by performing the scalar product of Equations 15 and 16 and equating this to the scalar product of Equations 15 and 17 which yields

$$|\text{grad } f|^{-1} (g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial f}{\partial \eta^i} \frac{\partial \phi}{\partial \eta^j} \delta^{ij} = V^i \frac{\partial f}{\partial \eta^j} \frac{\partial \eta^j}{\partial x^i} |\text{grad } f|^{-1} .$$

Through the use of Equations 1 and 14, this equation can be written as

$$|\text{grad} f|^{-1} (g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial f}{\partial \eta^i} \frac{\partial \phi}{\partial \eta^j} \delta^{ij} = |\text{grad} f|^{-1} \left[\frac{\partial f}{\partial \eta^\ell} \frac{\partial \eta^\ell}{\partial x^i} (u^i - \epsilon_{ijm} \omega^m x^j) - \frac{\partial f}{\partial t}\right] . \tag{18}$$

This equation represents the necessary boundary conditions on the surfaces. The term $\partial f/\partial t$ is the relative velocity of the surface in the x^i system.

The two types of surfaces that are of interest in most problems are surfaces where

$$\eta^i = K^i(t) \tag{19}$$

which represent the container walls and the free fluid surface given by

$$x^3 = \zeta(\eta^1, \eta^2, \eta^3, t) + \ell^3 \quad . \tag{20}$$

On the types of surfaces described by Equation 19, the boundary conditions given by Equation 18 become on $\eta^p = K^p$

$$(g_{\underline{pp}})^{-\frac{1}{2}} \frac{\partial \eta^p}{\partial \eta^p} = \left[\frac{\partial \eta^p}{\partial x^i} (u^i - \epsilon_{ijm}\, \omega^m\, x^j) - \frac{\partial \eta^p}{\partial t} \right] (g_{\underline{pp}})^{\frac{1}{2}} \quad . \tag{21}$$

After multiplying both sides by $|\text{grad } f|$, Equation 18 becomes on $x^3 = \zeta + \ell^3$

$$(g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial x^3}{\partial \eta^i} \frac{\partial \phi}{\partial \eta^j} \delta^{ij} - \frac{\partial x^3}{\partial \eta^\ell} \frac{\partial \eta^\ell}{\partial x^i} (u^i - \epsilon_{ijm}\, \omega^m\, x^j)$$

$$= (g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial \zeta}{\partial \eta^i} \frac{\partial \phi}{\partial \eta^j} \delta^{ij} - \frac{\partial \zeta}{\partial \eta^\ell} \frac{\partial \eta^\ell}{\partial x^i} (u^i - \epsilon_{ijm}\, \omega^m\, x^j) + \frac{\partial \zeta}{\partial t} \quad . \tag{22}$$

The complete formulation of the nonlinear boundary value problem in terms of the unknown velocity potential ϕ and free surface displacement ζ is given by Equations 12, 21, 22 and Equation 13 evaluated on $x^3 = \zeta + \ell^3$.

Since the transformation to curvilinear coordinates is normally given in the form

$$x^k = x^k(\eta^i) \quad ,$$

the partial derivatives $\partial x^k / \partial \eta^i$ are more easily found than $\partial \eta^i / \partial x^k$. By using the relation

$$\frac{\partial \eta^p}{\partial x^k} = \frac{\partial x^k}{\partial \eta^\ell} g^{\ell p} = \frac{\partial x^k}{\partial \eta^p} g^{\underline{pp}} \quad ,$$

the derivatives $\partial \eta^p / \partial x^k$ may be eliminated. The formulation of the problem is now given by

$$(g)^{-\frac{1}{2}} \frac{\partial}{\partial \eta^k} \left[(g)^{\frac{1}{2}} g^{k\ell} \frac{\partial \phi}{\partial \eta^\ell} \right] = 0 \quad \text{in the region} \tag{23}$$

on $\eta^p = K^p$

$$(g_{\underline{pp}})^{-\frac{1}{2}} \frac{\partial \phi}{\partial \eta^p} = (g^{\underline{pp}})^{\frac{1}{2}} \frac{\partial x^i}{\partial \eta^p} (u^i - \epsilon_{ijm}\, \omega^m x^j) - \frac{\partial \eta^p}{\partial t} (g_{\underline{pp}})^{\frac{1}{2}} \tag{24}$$

on $x^3 = \zeta + \ell^3$

$$(g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial x^3}{\partial \eta^i} \frac{\partial \phi}{\partial \eta^j} \delta^{ij} - \frac{\partial x^3}{\partial \eta^p} \frac{\partial x^3}{\partial \eta^\ell} g^{\underline{\ell\ell}} (u^i - \epsilon_{ijm}\, \omega^m x^j)\, \delta^{p\ell}$$

$$= (g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial \zeta}{\partial \eta^i} \frac{\partial \phi}{\partial \eta^j} \delta^{ij} - \frac{\partial \zeta}{\partial \eta^p} \frac{\partial x^i}{\partial \eta^\ell} g^{\underline{\ell\ell}} (u^i - \epsilon_{ijm}\, \omega^m x^j)\, \delta^{p\ell} + \frac{\partial \zeta}{\partial t} \tag{25}$$

on $x^3 = \zeta + \ell^3$

$$\frac{\partial\phi}{\partial t} = -\Omega - \frac{1}{2}\frac{\partial\phi}{\partial\eta^i}\frac{\partial x^j}{\partial\eta^m}\frac{\partial\phi}{\partial\eta^\ell}\frac{\partial x^k}{\partial\eta^n} g^{\underline{ii}} g^{\underline{\ell\ell}} \delta^{jk} \delta^{im} \delta^{\ell n} - \frac{1}{2} u^j u^i \delta_{ij}$$

$$- \frac{\partial\phi}{\partial\eta^i}\frac{\partial x^j}{\partial\eta^\ell} g^{\underline{ii}} u^j \delta^{i\ell} - \epsilon_{j\ell m}\omega^m x^\ell \frac{\partial\phi}{\partial\eta^i}\frac{\partial x^j}{\partial\eta^\ell} g^{\underline{ii}} g^{i\ell} \quad . \tag{26}$$

The boundary conditions may be simplified by defining

$$\phi = \Psi + \mu^k x^k \quad .$$

The formulation of the problem now becomes

$$(g)^{-\frac{1}{2}} \frac{\partial}{\partial\eta^k}\left[(g)^{\frac{1}{2}} g^{k\ell} \frac{\partial\Psi}{\partial\eta^\ell}\right] = 0 \quad \text{in the region} \tag{27}$$

on $\eta^P = K^P$

$$(g_{\underline{pp}})^{-\frac{1}{2}} \frac{\partial\Psi}{\partial\eta^P} = -(g^{\underline{PP}})^{\frac{1}{2}} \frac{\partial x^i}{\partial\eta^P} \epsilon_{ijm}\omega^m x^j - \frac{\partial\eta^P}{\partial t}(g_{\underline{pp}})^{\frac{1}{2}} \tag{28}$$

on $x^3 = \zeta + \ell^3$

$$(g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial x^3}{\partial\eta^i}\frac{\partial\Psi}{\partial\eta^j}\delta^{ij} + \frac{\partial x^3}{\partial\eta^P}\frac{\partial x^i}{\partial\eta^\ell} g^{p\ell} \epsilon_{ijm}\omega^m x^j \delta^{p\ell}$$

$$= (g_{\underline{ii}}\, g_{\underline{jj}})^{-\frac{1}{2}} \frac{\partial\zeta}{\partial\eta^i}\frac{\partial\Psi}{\partial\eta^j}\delta^{ij} + \frac{\partial\zeta}{\partial\eta^P}\frac{\partial x^i}{\partial\eta^\ell} g^{\underline{\ell\ell}} \epsilon_{ijm}\omega^m x^j \delta^{p\ell} + \frac{\partial\zeta}{\partial t} \tag{29}$$

on $x^3 = \zeta + \ell^3$

$$\Psi_t = -\dot{u}^k x^k - \Omega - \frac{1}{2}\frac{\partial\Psi}{\partial\eta^i}\frac{\partial x^j}{\partial\eta^m}\frac{\partial\Psi}{\partial\eta^\ell}\frac{\partial x^k}{\partial\eta^n} g^{\underline{ii}} g^{\underline{\ell\ell}} \delta^{jk}\delta^{im}\delta^{\ell n}$$

$$- \epsilon_{j\ell m}\omega^m x^\ell \frac{\partial\Psi}{\partial\eta^i}\frac{\partial x^j}{\partial\eta^n} g^{nj} - \epsilon_{j\ell m}\omega^m x^\ell u^j \quad . \tag{30}$$

CHAPTER III

GOVERNING EQUATIONS FOR A CYLINDRICAL SECTOR

Consider now a container whose shape is a sector of a cylinder with arbitrary angle π/α as shown in Figure 2. The x^i coordinate system is fixed in the container as shown in Figure 2. The η^i coordinate system is chosen to be cylindrical coordinates whose origin lies on the x^3 axis at the undisturbed free surface. The transformation between the x^i and η^i systems is given by

$$x^1 = \eta^1 \cos \eta^2 \tag{31}$$

$$x^2 = \eta^1 \sin \eta^2 \tag{32}$$

$$x^3 = \eta^3 + \frac{h}{2} \quad . \tag{33}$$

For the case of free fluid oscillations with the body forces considered to be gravity forces,

$$\Omega = G\,\eta^3$$

$$\omega^i = u^i = 0$$

where G is the gravitational acceleration.

If ζ is given by

$$\zeta = \zeta\,(x^1,\ x^2,\ t),\quad K^1 = a$$

$$K^2 = 0,\ \frac{\pi}{\alpha}\ \text{ and }\ K^3 = -\,h \quad ,$$

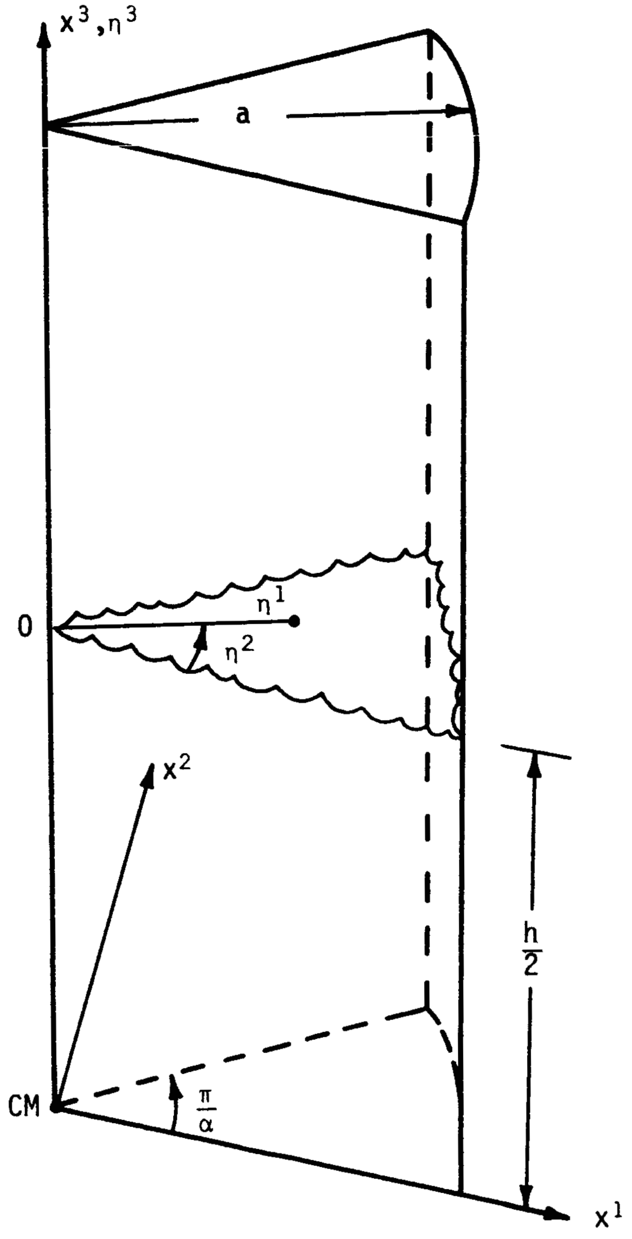

Figure 2. Cylindrical Sector Container

then Equations 27, 28, 29 and 30 become

$$\frac{\partial^2\Psi}{(\partial\eta^1)^2} + (\eta^1)^{-1}\frac{\partial\Psi}{\partial\eta^1} + (\eta^1)^{-2}\frac{\partial^2\Psi}{(\partial\eta^2)^2} + \frac{\partial^2\Psi}{(\partial\eta^3)^2} = 0 \text{ in R} \tag{34}$$

$$\frac{\partial\Psi}{\partial\eta^1} = 0 \quad \text{on} \quad \eta^1 = a \tag{35}$$

$$(\eta^1)^{-1}\frac{\partial\Psi}{\partial\eta^2} = 0 \quad \text{on} \quad \eta^2 = 0 \quad \text{and } \eta^2 = \frac{\pi}{\alpha} \tag{36}$$

$$\frac{\partial\Psi}{\partial\eta^3} = 0 \quad \text{on} \quad \eta^3 = -h \tag{37}$$

$$\frac{\partial\Psi}{\partial\eta^3} = \frac{\partial\zeta}{\partial\eta^1}\frac{\partial\Psi}{\partial\eta^1} + (\eta^1)^{-2}\frac{\partial\zeta}{\partial\eta^2}\frac{\partial\Psi}{\partial\eta^2} + \frac{\partial\zeta}{\partial t} \quad \text{on} \quad \eta^3 = \zeta \tag{38}$$

$$G\zeta = -\frac{\partial\Psi}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial\Psi}{\partial\eta^1}\right)^2 + (\eta^1)^{-2}\left(\frac{\partial\Psi}{\partial\eta^2}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta^3}\right)^2\right] \quad \text{on} \quad \eta^3 = \zeta\ . \tag{39}$$

The boundary conditions at the free surface consist of Equations 38 and 39. Following the method of Hutton,[11] these equations may be combined to form one approximate nonlinear boundary condition. Equation 39 can be written

$$-G\zeta = \Gamma\ [r, \theta, \zeta(r, \theta, t), t] \quad \text{on } z = \zeta \tag{40}$$

where

$$\Gamma = \Psi_t + \frac{1}{2}\ [(\Psi_r)^2 + r^{-2}(\Psi_\theta)^2 + (\Psi_z)^2]\quad . \tag{41}$$

By differentiating both sides of Equation 40, the following equations result:

$$
\begin{aligned}
-G\zeta_t &= \Gamma_t + \Gamma_\zeta \zeta_t \\
-G\zeta_r &= \Gamma_r + \Gamma_\zeta \zeta_r \\
-G\zeta_\theta &= \Gamma_\theta + \Gamma_\zeta \zeta_\theta
\end{aligned}
\tag{42}
$$

where

$$\Gamma_t = \Psi_{tt} + \Psi_r \Psi_{rt} + r^{-2} \Psi_\theta \Psi_{\theta t} + \Psi_z \Psi_{zt} \tag{43}$$

$$\Gamma_\theta = \Psi_{\theta t} + \Psi_r \Psi_{r\theta} + r^{-2} \Psi_\theta \Psi_{\theta\theta} + \Psi_z \Psi_{z\theta} \tag{44}$$

$$\Gamma_r = \Psi_{rt} + \Psi_r \Psi_{rr} + r^{-2} \Psi_\theta \Psi_{r\theta} - r^{-3} \Psi_\theta^2 + \Psi_z \Psi_{rz} \tag{45}$$

$$\Gamma_z = \Psi_{zt} + \Psi_r \Psi_{rz} + r^{-2} \Psi_\theta \Psi_{\theta z} + \Psi_z \Psi_{zz} \quad . \tag{46}$$

If Equation 38 is multiplied by $-(G+\Gamma_\zeta)$, the result is

$$
\begin{aligned}
&(G + \Gamma_\zeta)\Psi_z - (G + \Gamma_\zeta)\zeta_r \Psi_r - r^{-2} \Psi_\theta (G + \Gamma_\zeta)\zeta_\theta \\
&\quad - (G + \Gamma_\zeta)\zeta_t = 0 \qquad \text{on } z = \zeta
\end{aligned}
\tag{47}
$$

or

$$G\Psi_z + \Gamma_z \Psi_z + \Gamma_r \Psi_z + r^{-2} \Gamma_\theta \Psi_\theta + \Gamma_t = 0 \qquad \text{on } z = \zeta \quad . \tag{48}$$

With the use of Equations 43, 44, 45, and 46, Equation 48 can be written

$$
\begin{aligned}
&G\Psi_z + 2\Psi_z \Psi_{zt} + 2\Psi_z \Psi_r \Psi_{zr} + 2r^{-2} \Psi_z \Psi_\theta \Psi_{\theta z} + (\Psi_z)^2 \Psi_{zz} \\
&\quad + 2\Psi_r \Psi_{rt} + (\Psi_r)^2 \Psi_{rr} + 2r^{-2} \Psi_r \Psi_\theta \Psi_{r\theta} - r^{-3} (\Psi_\theta)^2 \Psi_r \\
&\quad + 2r^{-2} \Psi_\theta \Psi_{\theta t} + r^{-4} (\Psi_\theta)^2 \Psi_{\theta\theta} + \Psi_{tt} = 0 \qquad \text{on } z = \zeta \quad .
\end{aligned}
\tag{49}
$$

The free surface boundary conditions now consist of Equation 49 which depends on ζ implicitly and Equation 39 which depends on ζ both explicitly and implicitly. The wave height ζ can be eliminated between these two equations if they are each expanded in a Taylor Series about $\eta = 0$ and then combined. For the sake of brevity the algebraic manipulations are not included, but may be found in Reference 11. The resulting equation with terms to $0(\zeta^3)$ included is

$$\Psi_{tt} + G\,\Psi_z + \overline{B}_2 + \overline{B}_3 + 0(\zeta^4) = 0 \quad \text{on } z = 0 \tag{50}$$

where

$$\begin{aligned}\overline{B}_2 = {} & 2\,\Psi_r\,\Psi_{rt} + 2\,r^{-2}\,\Psi_\theta\,\Psi_{\theta t} + 2\,\Psi_z\,\Psi_{zt} \\ & - G^{-1}\,\Psi_{ztt}\,\Psi_t - \Psi_{zz}\,\Psi_t\end{aligned} \tag{51}$$

$$\begin{aligned}\overline{B}_3 = {} & (\Psi_r)^2\,\Psi_{rr} + r^{-4}\,(\Psi_\theta)^2\,\Psi_{\theta\theta} + (\Psi_z)^2\,\Psi_{zz} \\ & - r^{-3}\,\Psi_r\,(\Psi_\theta)^2 + 2\,\Psi_r\,\Psi_z\,\Psi_{rz} + 2\,r^{-2}\,\Psi_r\,\Psi_\theta\,\Psi_{r\theta} \\ & + 2\,r^{-2}\,\Psi_z\,\Psi_\theta\,\Psi_{\theta z} - \frac{1}{2}\,G^{-1}\,\Psi_{ttz}\,[(\Psi_r)^2 + r^{-2}\,(\Psi_\theta)^2 + (\Psi_z)^2] \\ & - \frac{1}{2}\,\Psi_{zz}\,[(\Psi_r)^2 + r^{-2}\,\Psi_\theta^2 + \Psi_z^2] \\ & - 2\,G^{-1}\,\Psi_t\;\;\Psi_{rz}\,\Psi_{rt} + \Psi_r\,\Psi_{rzt} + r^{-2}\,\Psi_{\theta z}\,\Psi_{\theta t} + r^{-2}\,\Psi_\theta\,\Psi_{\theta zt} \\ & + \Psi_{zz}\,\Psi_{zt} + \Psi_z\,\Psi_{zzt}] \\ & + G^{-2}\,\Psi_{zt}\,\Psi_t\,[\Psi_{ztt} + G\,\Psi_{zz}] \\ & + \frac{1}{2}\,G^{-2}\,\Psi_t^2\,[\Psi_{ttzz} + G\,\Psi_{zzz}] \quad .\end{aligned} \tag{52}$$

Redefining the variables as

$$\eta^1 = r, \quad \eta^2 = \theta, \quad \eta^3 = z \quad ,$$

the boundary value problem for free oscillations in a cylindrical sector container can now be restated as

$$\Psi_{rr} + (r)^{-1}\,\Psi_r + (r)^{-2}\,\Psi_{\theta\theta} + \Psi_{zz} = 0 \quad \text{in} \quad R \tag{53}$$

$$\Psi_r = 0 \quad \text{on} \quad r = a \tag{54}$$

$$(r)^{-1}\,\Psi_\theta = 0 \quad \text{on} \quad \theta = 0, \frac{\pi}{\alpha} \tag{55}$$

$$\Psi_z = 0 \quad \text{on} \quad z = -h \tag{56}$$

$$\Psi_{tt} + G\,\Psi_z + \overline{B}_2 + \overline{B}_3 + 0(\zeta^4) = 0 \quad \text{on} \quad z = 0 \quad . \tag{57}$$

CHAPTER IV

SOLUTION

General Method

The problem as formulated now consists of finding a solution to Laplace's equation which satisfies three linear and one nonlinear boundary conditions. The basic approach to finding a solution will be to use classical methods to find a solution of Laplace's equation which satisfies the three linear boundary conditions and then to satisfy the nonlinear boundary condition asymptotically using the method of Krylov and Bogoliubov.

The solution to Laplace's equation in cylindrical coordinates is well known and has the following form:

$$\Psi = [C_1 \sinh(\lambda z) + C_2 \cosh(\lambda z)] [C_3 \sin(n\theta) + C_4 \cos(n\theta)] [C_5 J_n(\lambda r) + C_6 \Psi_n(\lambda r)] \tag{58}$$

where λ and n are constants to be determined. The above solution which will be used in this analysis requires that n be an integer. A solution is available for n not an integer but it will not be considered here. If Ψ is to be bounded at $r = 0$, then C_6 must be taken as zero. Application of Equation 55 yields at $\theta = 0$

$$[C_1 \sinh(\lambda z) + C_2 \cosh(\lambda z)]\, C_3\, n\, J_n(\lambda r) = 0 \;,$$

and at $\theta = \pi/\alpha$

$$[C_1 \sinh(\lambda z) + C_2 \cosh(\lambda z)] \left[-C_4\, n \sin\left(\frac{n\pi}{\alpha}\right)\right] J_n(\lambda r) = 0$$

which requires that

$$C_3 = 0 \quad \text{and} \quad \sin\left(\frac{n\pi}{\alpha}\right) = 0$$

or

$$n = k\alpha, \qquad k = 0, 1, 2 \ldots \quad .$$

It should be noted here that in order for n to be an integer, α must be an integer.

In order to satisfy the first boundary condition, Equation 54, it is necessary that

$$J'_{k\alpha}(\lambda a) = 0$$

Thus,

$$\lambda a = \gamma_{km} = \text{mth zero of } J'_{k\alpha} \quad .$$

The third linear boundary condition yields the relation

$$C_1 = C_2 \frac{\sinh(\lambda_{k\alpha,m} h)}{\cosh(\lambda_{k\alpha,m} h)} \quad .$$

The solution of Laplace's equation which satisfies the three linear boundary conditions is thus given by

$$\Psi_{k\alpha,m} = A_{k\alpha,m} \cos (k\alpha\theta)\, J_{k\alpha} (\lambda_{k\alpha,m} r) \frac{\cosh [\lambda_{k\alpha,m}(z+h)]}{\cosh (\lambda_{k\alpha,m} h)} \quad . \quad (59)$$

Solution for Linearized Free Surface Condition

If now only small wave heights and slopes are considered, the linearized fourth boundary condition, Equation 57, reads

$$\Psi_{tt} + G\, \Psi_z = 0 \quad \text{on} \quad z = 0 \quad . \quad (60)$$

This condition applied to Equation 59 yields

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \Big\{\big[(A_{k\alpha,m})_{tt} + G\, \lambda_{k\alpha,m} \tanh (\lambda_{k\alpha,m} h)\, A_{k\alpha,m}\big] \times \cos (k\alpha\theta)\, J_{k\alpha} (\lambda_{k\alpha,m} r)\Big\} = 0 \quad .$$

If $A_{k\alpha,m} = K \cos (\omega_{k\alpha,m} t)$, then the linear solution gives

$$\Psi = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} K \cos (\omega_{k\alpha,m} t) \cos (k\alpha\theta) \times J_{k\alpha} (\lambda_{k\alpha,m} r) \frac{\cosh [\lambda_{k\alpha,m}(z+h)]}{\cosh (\lambda_{k\alpha,m} h)} \quad (61)$$

$$\omega^2_{k\alpha,m} = G\, \lambda_{k\alpha,m} \tan h\, (\lambda_{k\alpha,m} h) \quad . \quad (62)$$

The fundamental mode of the linear solution will now be chosen as the one which has the frequency $\omega_{\alpha,o}$. Then the linear solution is given by

$$\Psi = K \cos(\omega_{\alpha,o} t) \cos(\alpha\theta) J_\alpha (\lambda_{\alpha,o} r) \frac{\cosh[\lambda_{\alpha,o}(z+h)]}{\cosh(\lambda_{\alpha,o} h)} \quad (63)$$

where

$$\omega^2_{\alpha,o} = G \lambda_{\alpha,o} \tanh(\lambda_{\alpha,o} h) \quad . \quad (64)$$

The expression for the wave height (linear theory) is

$$\zeta = - G^{-1} \Psi_t \quad \text{on} \quad z = 0$$

$$\zeta = K G^{-1} \omega_{\alpha,o} \sin(\omega_{\alpha,o} t) \cos(\alpha\theta) J_\alpha (\lambda_{\alpha,o} r) \quad . \quad (65)$$

Nonlinear Solution

To find a solution which satisfies the approximate nonlinear boundary condition (Equation 57), the method of Krylov and Bogoliubov is followed. This method seeks a solution of the form

$$\Psi = \sum_{i=1}^{\infty} \epsilon^i \Psi^{(i)} \quad (66)$$

where ϵ is a small positive parameter which will be defined later and

$$\Psi^{(i)} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A^i_{k\alpha,m}(\Omega, K) \cos(k\alpha\theta) J(\lambda_{k\alpha,m} r) \frac{\cosh[\lambda_{k\alpha,m}(z+h)]}{\cosh(\lambda_{k\alpha,m} h)} \quad . \quad (67)$$

The quantities Ω and K are defined by

$$\dot{\Omega} = \omega_{\alpha,o} + \epsilon\, B^1(K) + \epsilon^2\, B^2(K) + \ldots \tag{68}$$

$$\dot{K} = \epsilon\, D^1(K) + \epsilon^2\, D^2(K) + \ldots \tag{69}$$

An addition restriction that is necessary to insure uniqueness is that the expression for $A^{i}_{k\alpha,m}$ contain no first harmonics for $i \geqq 1$. To this end the following conditions are specified:

$$\int_0^{2\pi} \Psi^{(i)}(K,\Omega) \cos\Omega \, d\Omega = 0 \qquad i \geqq 1 \tag{70}$$

$$\int_0^{2\pi} \Psi^{(i)}(K,\Omega) \sin\Omega \, d\Omega = 0 \qquad i \geqq 1 \quad . \tag{71}$$

From a physical point of view the imposition of these conditions is equivalent to selecting K as the full amplitude of the first fundamental harmonic of the oscillation.

The following notation is now employed for convenience.

$$\left.\begin{aligned} \omega &\equiv \omega_{\alpha,o} \\ J_{k\alpha,m} &\equiv J_{k\alpha}(\lambda_{k\alpha,m}\, r) \\ \lambda &\equiv \lambda_{\alpha,o} \\ I(F) &= \int_0^a r\, F(r)\, dr \quad . \end{aligned}\right\} \tag{72}$$

Noting that $\Psi = \Psi[K(t), \Omega(t)]$, the time derivatives are

$$\Psi_t = \Psi_\Omega \dot{\Omega} + \Psi_K \dot{K} \tag{73}$$

$$\Psi_{tt} = \Psi_\Omega \ddot{\Omega} + \Psi_{\Omega\Omega} (\dot{\Omega})^2 + 2\,\dot{\Omega}\,\dot{K}\,\Psi_{K\Omega} + (\dot{K})^2\,\Psi_{KK} + \ddot{K}\,\Psi_K \quad . \tag{74}$$

From Equations 68 and 69, it is found that

$$\begin{aligned}\ddot{\Omega} = \dot{\Omega}_K \dot{K} &= [\epsilon\, B_K^1 + \epsilon^2 B_K^2 + \ldots][\epsilon\, D^1 + \epsilon^2 D^2 + \ldots] \\ &= \epsilon^2 [B_K^1 D^1] + \epsilon^3 [B_K^2 D^1 + B_K^1 D^2] + 0(\epsilon^4)\end{aligned} \tag{75}$$

$$\begin{aligned}\ddot{K} = \dot{K}_K \dot{K} &= [\epsilon\, D_K^1 + \epsilon^2 D_K^2 + \ldots][\epsilon\, D^1 + \epsilon^2 D^2 + \ldots] \\ &= \epsilon^2 [D^1 D_K^1] + \epsilon^3 [D^1 D_K^2 + D^2 D_K^1] + 0(\epsilon^4)\end{aligned} \tag{76}$$

$$(\dot{\Omega})^2 = \omega^2 + \epsilon\,[2\omega B^1] + \epsilon^2 [2\omega B^2 + (B^1)^2] + 0(\epsilon^3) \tag{77}$$

$$(\dot{K})^2 = \epsilon^2\, [D^1]^2 + \epsilon^3 [2\, D^1 D^2 \;+ 0(\epsilon^4) \tag{78}$$

$$\dot{\Omega}\,\dot{K} = \epsilon\,[\omega D^1] + \epsilon^2 [B^1 D^1 + \omega D^2] + 0(\epsilon^3) \quad . \tag{79}$$

Now substituting Equations 75, 76, 77, 78, 79, into Equations 73 and 74 results in

$$\begin{aligned}\Psi_t &= [\epsilon\,\Psi_\Omega^1 + \epsilon^2 \Psi_\Omega^2 + \epsilon^3 \Psi_\Omega^3 + \ldots][\omega + \epsilon\, B^1 + \epsilon^2 B^2 + \ldots] \\ &\quad + [\epsilon\,\Psi_K^1 + \epsilon^2 \Psi_K^2 + \epsilon^3 \Psi_K^3 + \ldots][\epsilon\, D^1 + \epsilon^2 D^2 + \ldots] \\ &= \epsilon\,[\omega\,\Psi_\Omega^1] + \epsilon^2 [B^1 \Psi_\Omega^1 + \omega\,\Psi_\Omega^2 + D^1 \Psi_K^1] \\ &\quad + \epsilon^3 [B^2 \Psi_\Omega^1 + B^1 \Psi_\Omega^2 + \omega\,\Psi_\Omega^3 + D^2 \Psi_K^1 + D^1 \Psi_K^2] + 0(\epsilon^4)\end{aligned} \tag{80}$$

$$\Psi_{tt} = [\epsilon \overset{1}{\Psi}_{\Omega} + \epsilon^2 \overset{2}{\Psi}_{\Omega} + \ldots] [\epsilon^2 (\overset{1}{B}_K D^1) + \epsilon^3 (\overset{2}{B}_K D^1 + \overset{1}{B}_K D^2) + \ldots]$$

$$+ [\epsilon \overset{1}{\Psi}_{\Omega\Omega} + \epsilon^2 \overset{2}{\Psi}_{\Omega\Omega} + \epsilon^3 \overset{3}{\Psi}_{\Omega\Omega} + \ldots] \{\omega^2 + \epsilon [2\omega B^1] + \epsilon^2 [2\omega B^2 + (B^1)^2]\}$$

$$+ 2 [\epsilon (\omega D^1) + \epsilon^2 (B^1 D^1 + \omega D^2)] [\epsilon \overset{1}{\Psi}_{K\Omega} + \epsilon^2 \overset{2}{\Psi}_{K\Omega} + \epsilon^3 \overset{3}{\Psi}_{K\Omega}]$$

$$+ [\epsilon^2 (D^1)^2 + \epsilon^3 (2 D^1 D^2)] [\epsilon \overset{1}{\Psi}_{KK} + \epsilon^2 \overset{2}{\Psi}_{KK} + \epsilon^3 \overset{3}{\Psi}_{KK}]$$

$$+ [\epsilon^2 (D^1 \overset{1}{D}_K) + \epsilon^3 (D^1 \overset{2}{D}_K + D^2 \overset{1}{D}_K)] [\epsilon \overset{1}{\Psi}_K + \epsilon^2 \overset{2}{\Psi}_K + \epsilon^3 \overset{3}{\Psi}_K]$$

$$= \epsilon [\omega^2 \overset{1}{\Psi}_{\Omega\Omega}] + \epsilon^2 [2\omega B^1 \overset{1}{\Psi}_{\Omega\Omega} + \omega^2 \overset{2}{\Psi}_{\Omega\Omega} + 2\omega D^1 \overset{1}{\Psi}_{K\Omega}]$$

$$+ \epsilon^3 [\overset{1}{B}_K D^1 \overset{1}{\Psi}_{\Omega} + 2\omega B^2 \overset{1}{\Psi}_{\Omega\Omega} + (B^1)^2 \overset{1}{\Psi}_{\Omega\Omega} + 2\omega B^1 \overset{2}{\Psi}_{\Omega\Omega} + \omega^2 \overset{3}{\Psi}_{\Omega\Omega}$$

$$+ 2\omega D^1 \overset{2}{\Psi}_{K\Omega} + 2 B^1 D^1 \overset{1}{\Psi}_{K\Omega} + 2\omega D^2 \overset{1}{\Psi}_{K\Omega} + (D^1)^2 \overset{1}{\Psi}_{KK}$$

$$+ D^1 \overset{1}{D}_K \overset{1}{\Psi}_K] \quad . \tag{81}$$

Using Equations 80 and 81 in the first part of the nonlinear boundary condition (Equation 57), the following is obtained:

$$\Psi_{tt} + G \Psi_z = \epsilon [\omega^2 \overset{1}{\Psi}_{\Omega\Omega} + G \overset{1}{\Psi}_z]$$

$$+ \epsilon^2 [2\omega B^1 \overset{1}{\Psi}_{\Omega\Omega} + \omega^2 \overset{2}{\Psi}_{\Omega\Omega} + 2\omega D^1 \overset{1}{\Psi}_{K\Omega} + G \overset{2}{\Psi}_z]$$

$$+ \epsilon^3 [\overset{1}{B}_K D^1 \overset{1}{\Psi}_{\Omega} + 2 \ B^2 \overset{1}{\Psi}_{\Omega\Omega} + (B^1)^2 \overset{1}{\Psi}_{\Omega\Omega} + 2 \ B^1 \overset{2}{\Psi}_{\Omega\Omega}$$

$$+ \omega^2 \overset{3}{\Psi}_{\Omega\Omega} + 2\omega D^1 \overset{2}{\Psi}_{K\Omega} + 2 \ B^1 D^1 \overset{1}{\Psi}_{K\Omega} + 2\omega D^2 \overset{1}{\Psi}_{K\Omega}$$

$$+ (D^1)^2 \overset{1}{\Psi}_{KK} + D^1 \overset{1}{D}_K \overset{1}{\Psi}_K + G \overset{3}{\Psi}_z] + 0(\epsilon^4) \quad . \tag{82}$$

The remainder of the nonlinear boundary condition is given by $\overline{B}_2$ and $\overline{B}_3$ which through the use of Equations 80 and 81 become

$$\overline{B}_2 = \epsilon^2 \; [2\omega \, \Psi^1_{\Omega r} \, \Psi^1_r + 2 \, r^{-2} \omega \; \Psi^1_{\Omega\theta} \Psi^1_\theta$$

$$+ 2 \, \omega \, \Psi^1_{\Omega z} \, \Psi^1_z - G^{-1} \omega^3 \, \Psi^1_\Omega \, \Psi^1_{\Omega\Omega z} - \omega \, \Psi^1_\Omega \, \Psi^1_{zz}$$

$$+ \epsilon^3 \; [2 \, B^1 \, \Psi^1_{\Omega r} \, \Psi^1_r + 2 \omega \, \Psi^2_{\Omega r} \, \Psi^1_r + 2 \, D^1 \, \Psi^1_{Kr} \, \Psi^1_r$$

$$+ 2 \, \omega \, \Psi^1_{\Omega r} \, \Psi^2_r + 2 \, r^{-2} \; B^1 \, \Psi^1_{\Omega\theta} \, \Psi^1_\theta + 2 \, r^{-2} \, \omega \; \Psi^2_{\Omega\theta} \, \Psi^1_\theta$$

$$+ 2 \, r^{-2} \, D^1 \; \Psi^1_{K\theta} \, \Psi^1_\theta + 2 \, r^{-2} \; \omega \, \Psi^2_\theta \, \Psi^1_{\Omega\theta} + 2 \, B^1 \, \Psi^1_{\Omega z} \; \Psi^1_z$$

$$+ 2 \, \omega \, \Psi^2_{\Omega z} \; \Psi^1_z + 2 \, D^1 \, \Psi^1_{Kz} \; \Psi^1_z + 2 \, \omega \, \Psi^1_{\Omega z} \; \Psi^2_z$$

$$- 2 \, G^{-1} \omega^2 \, B^1 \; \Psi^1_{\Omega\Omega z} \, \Psi^1_\Omega - G^{-1} \omega^3 \; \Psi^2_{\Omega\Omega z} \, \Psi^1_\Omega$$

$$- 2 \, G^{-1} \omega^2 \, D^1 \; \Psi^1_{K\Omega z} \, \Psi_\Omega - G^{-1} \, \omega^2 \, B^1 \, \Psi^1_\Omega \, \Psi^1_{\Omega\Omega z}$$

$$- G^{-1} \, \omega^3 \, \Psi^2_\Omega \; \Psi^1_{\Omega\Omega z} - G^{-1} \omega^2 \; D^1 \, \Psi^1_K \; \Psi^1_{\Omega\Omega z}$$

$$- \omega \, \Psi^1_\Omega \, \Psi^2_{zz} - B^1 \, \Psi^1_\Omega \, \Psi^1_{zz} - \omega \, \Psi^2_\Omega \, \Psi^1_{zz} - D^1 \, \Psi^1_K \; \Psi^1_{zz}] + 0(\epsilon^4) \qquad (83)$$

$$\overline{B}_3 = \epsilon^3 \; [(\Psi^1_r)^2 \; \Psi^1_{rr} + r^{-4} \, (\Psi^1_\theta)^2 \; \Psi^1_{\theta\theta} + (\Psi^1_z)^2 \, \Psi^1_{zz}$$

$$- r^{-3} \, \Psi^1_r \; (\Psi^1_\theta)^2 + 2 \; \Psi^1_r \; \Psi^1_z \; \Psi^1_{rz} + 2 \; r^{-2} \; \Psi^1_r \, \Psi^1_\theta \, \Psi^1_{r\theta}$$

$$+ 2 \; r^{-2} \; \Psi^1_z \; \Psi^1_\theta \, \Psi^1_{\theta z} - \frac{1}{2} \; G^{-1} \; \omega^2 \, \Psi^1_{\Omega\Omega z} \; (\Psi^1_r)^2$$

$$- \frac{1}{2} \; G^{-1} \; r^{-2} \; \omega^2 \; \Psi^1_{\Omega\Omega z} \; (\Psi^1_\theta)^2 - \frac{1}{2} \; G^{-1} \; \omega^2 \; \Psi^1_{\Omega\Omega z} \; (\Psi^1_z)^2$$

$$- \frac{1}{2} \; \Psi^1_{zz} \; (\Psi^1_r)^2 - \frac{1}{2} \; r^{-2} \; \Psi^1_{zz} \; (\Psi^1_\theta)^2 - \frac{1}{2} \; \Psi^1_{zz} \; (\Psi^1_z)^2$$

$$- 2 \; G^{-1} \; \omega^2 \, \Psi^1_\Omega \; \Psi^1_{rz} \; \Psi^1_{\Omega r} - 2 \; G^{-1} \; \omega^2 \, \Psi_\Omega \; \Psi^1_r \; \; \Psi^1_{\Omega rz}$$

$$- 2 \; G^{-1} \; r^{-2} \; \omega^2 \; \Psi^1_\Omega \; \Psi^1_{\theta z} \; \Psi^1_{\Omega\theta} - 2 \; G^{-1} \; r^{-2} \; \omega^2 \, \Psi^1_\Omega \; \Psi^1_\theta \; \Psi^1_{\theta z \Omega}$$

$$- 2 \; G^{-1} \omega^2 \; \Psi^1_\Omega \; \Psi^1_{zz} \; \Psi^1_{z\Omega} - 2 \; G^{-1} \; \omega^2 \, \Psi^1_\Omega \; \Psi^1_z \, \Psi^1_{\Omega zz}$$

$$+ G^{-2} \; \omega^4 \, \Psi^1_{\Omega z} \; \Psi^1_\Omega \; \Psi^1_{\Omega\Omega z} + G^{-1} \omega^2 \; \Psi^1_{\Omega z} \; \Psi^1_\Omega \; \Psi^1_{zz}$$

$$+ \frac{1}{2} \; G^{-2} \, \omega^4 \; (\Psi^1_\Omega)^2 \; \Psi^1_{\Omega\Omega zz} + \frac{1}{2} \; G^{-1} \; \omega^2 \, (\Psi^1_\Omega)^2 \; \Psi^1_{zzz}] + 0 \, (\epsilon^4) \qquad (84)$$

Upon substituting expressions 82, 83, and 84 into the nonlinear boundary condition and equating coefficient of ϵ^n, the following results are obtained. All three equations are evaluated on z = 0. For the first approximation,

$$\omega^2 \Psi^1_{\Omega\Omega} + G \Psi^1_z = 0 \quad . \tag{85}$$

For the second approximation,

$$\begin{aligned} \omega^2 \Psi^2_{\Omega\Omega} + G \Psi^2_z = & - 2 B^1 \omega \Psi^1_{\Omega\Omega} - 2 \omega D^1 \Psi^1_{K\Omega} - 2 \omega \Psi^1_{\Omega r} \Psi^1_r \\ & - 2 r^{-2} \omega \Psi^1_{\Omega\theta} \Psi^1_\theta - 2 \omega \Psi^1_{\Omega z} \Psi^1_z + G^{-1} \omega^3 \Psi^1_\Omega \Psi^1_{\Omega\Omega z} \\ & + \omega \Psi^1_\Omega \Psi^1_{zz} \quad . \end{aligned} \tag{86}$$

For the third approximation,

$$\begin{aligned} \omega^2 \Psi^3_{\Omega\Omega} + G \Psi^3_\Omega = & - B^1_K D^1 \Psi^1_\Omega - 2 \omega B^2 \Psi^1_{\Omega\Omega} \\ & - (B^1)^2 \Psi^1_{\Omega\Omega} - 2 \omega B^1 \Psi^2_{\Omega\Omega} - 2 \omega D^1 \Psi^2_{K\Omega} - 2 B^1 D^1 \Psi^1_{K\Omega} \\ & - 2 \omega D^2 \Psi^1_{K\Omega} - (D^1)^2 \Psi_{KK} - D^1 D^1_K \Psi^1_K - 2 B^1 \Psi^1_{\Omega r} \Psi^1_r \\ & - 2 \omega \Psi^2_{\Omega r} \Psi^1_r - 2 D^1 \Psi^1_{Kr} \Psi^1_r - 2 \omega \Psi^1_{\Omega r} \Psi^2_r \\ & - 2 r^{-2} B^1 \Psi^1_{\Omega\theta} \Psi^1_\theta - 2 r^{-2} \omega \Psi^2_{\Omega\theta} \Psi^1_\theta - 2 r^{-2} D^1 \Psi^1_{K\theta} \Psi^1_\theta \\ & - 2 r^{-2} \omega \Psi^2_\theta \Psi^1_{\Omega\theta} - 2 B^1 \Psi^1_{\Omega z} \Psi^1_z - 2 \omega \Psi^2_{\Omega\Omega} \Psi^1_z - 2 D^1 \Psi_{Kz} \Psi^1_z \\ & - 2 \omega \Psi^2_{\Omega z} \Psi^2_z + 2 G^{-1} \omega^2 B^1 \Psi^1_{\Omega\Omega z} \Psi^1_\Omega + G^{-1} \omega^3 \Psi^2_{\Omega\Omega z} \Psi^1_\Omega \\ & + 2 G^{-1} \omega^2 D^1 \Psi^1_{K\Omega z} \Psi^1_\Omega + G^{-1} \omega^2 B^1 \Psi^1_\Omega \Psi^1_{\Omega\Omega z} + G^{-1} \omega^3 \Psi^2_\Omega \Psi^1_{\Omega\Omega z} \\ & + G^{-1} \omega^2 D^1 \Psi^1_K \Psi^1_{\Omega\Omega z} + \omega \Psi^1_\Omega \Psi^2_{zz} + B^1 \Psi^1_\Omega \Psi^1_{zz} + \omega \Psi^2_\Omega \Psi^1_{zz} \end{aligned}$$

$$+ D^1 \Psi_K^1 \Psi_{zz}^1 - (\Psi_r^1)^2 \Psi_{rr}^1 - r^{-4} (\Psi_\theta^1)^2 \Psi_{\theta\theta}^1 - (\Psi_z^1)^2 \Psi_{zz}^1$$

$$+ r^{-3} \Psi_r^1 (\Psi_\theta^1)^2 - 2 \Psi_r^1 \Psi_z^1 \Psi_{rz}^1 - 2 r^{-2} \Psi_r^1 \Psi_\theta^1 \Psi_{r\theta}^1 - 2 r^{-2} \Psi_z^1 \Psi_\theta^1 \Psi_{\theta z}^1$$

$$+ \frac{1}{2} G^{-1} \omega^2 \Psi_{\Omega\Omega z}^1 (\Psi_r^1)^2 + \frac{1}{2} G^{-1} r^{-2} \omega^2 \Psi_{\Omega\Omega z}^1 (\Psi_\theta^1)^2 + \frac{1}{2} G^{-1} \omega^2 \Psi_{\Omega\Omega z}^1 (\Psi_z^1)^2$$

$$+ \frac{1}{2} \Psi_{zz}^1 (\Psi_r^1)^2 + \frac{1}{2} r^{-2} \Psi_{zz}^1 (\Psi_\theta^1)^2 + \frac{1}{2} \Psi_{zz}^1 (\Psi_z^1)^2$$

$$+ 2 G^{-1} \omega^2 \Psi_\Omega^1 \Psi_{rz}^1 \Psi_{\Omega r}^1 + 2 G^{-1} \omega^2 \Psi_\Omega^1 \Psi_r^1 \Psi_{\Omega rz}^1 + 2 G^{-1} r^{-2} \omega^2 \Psi_\Omega^1 \Psi_{\theta z}^1 \Psi_{\Omega\theta}^1$$

$$+ 2 G^{-1} r^{-2} \omega^2 \Psi_\Omega^1 \Psi_\theta^1 \Psi_{\theta z\Omega}^1 + 2 G^{-1} \omega^2 \Psi_\Omega^1 \Psi_{zz}^1 \Psi_{z\Omega}^1 + 2 G^{-1} \omega^2 \Psi_\Omega^1 \Psi_z^1 \Psi_{\Omega zz}^1$$

$$- G^{-2} \omega^4 \Psi_{\Omega z}^1 \Psi_\Omega^1 \Psi_{\Omega\Omega z}^1 - G^{-1} \omega^2 \Psi_{\Omega z}^1 \Psi_\Omega^1 \Psi_{zz}^1 - \frac{1}{2} G^{-2} \omega^4 (\Psi_\Omega^1)^2 \Psi_{\Omega\Omega zz}^1$$

$$- \frac{1}{2} G^{-1} \omega^2 (\Psi_\Omega^1)^2 \Psi_{zzz}^1 \quad . \tag{87}$$

In a similar manner expressions for the wave height ζ may be developed.

$$\zeta = - G^{-1} \Psi_t - \frac{1}{2} G^{-1} [(\Psi_r)^2 + r^{-2} (\Psi_\theta)^2 + (\Psi_z)^2] \quad \text{on} \quad z = \zeta$$

$$= - G^{-1} \{\epsilon [\omega \Psi_\Omega^1] + \epsilon^2 [B^1 \Psi_\Omega^1 + \omega \Psi_\Omega^2 + D^1 \Psi_K^1]$$

$$+ \epsilon^3 [B^2 \Psi_\Omega^1 + B^1 \Psi_\Omega^2 + \omega \Psi_\Omega^3 + D^2 \Psi_K^1 + D^1 \Psi_K^2]\}$$

$$- \frac{1}{2} G^{-1} \{[\epsilon \Psi_r^1 + \epsilon^2 \Psi_r^2] [\epsilon \Psi_r^1 + \epsilon^2 \Psi_r^2]$$

$$+ r^{-2} [\epsilon \Psi_\theta^1 + \epsilon^2 \Psi_\theta^2] [\epsilon \Psi_\theta^1 + \epsilon^2 \Psi_\theta^2]$$

$$+ [\epsilon \Psi_z^1 + \epsilon^2 \Psi_z^2] [\epsilon \Psi_z^1 + \epsilon^2 \Psi_z^2] \quad . \tag{88}$$

Expanding ζ as

$$\zeta = \sum_{i=1}^{\infty} \epsilon^i \zeta^i$$

and equating coefficients of ϵ gives the following equations for the wave heights. For the first approximation,

$$\zeta^1 = - G^{-1} \omega \Psi^1_\Omega \quad \text{on} \quad z = 0 \quad . \tag{89}$$

For the second approximation,

$$\zeta^2 = - G^{-1} B^1 \Psi^1_\Omega - G^{-1} \omega \Psi^2_\Omega - G^{-1} D^1 \Psi^1_K$$
$$- \frac{1}{2} G^{-1} [(\Psi^1_r)^2 + r^{-2} (\Psi^1_\theta)^2 + (\Psi^1_z)^2] \quad \text{on} \quad z = 0 \quad . \tag{90}$$

For the third approximation,

$$\zeta^3 = - G^{-1} [B^2 \Psi^1_\Omega + B^1 \Psi^2_\Omega + \omega \Psi^3_\Omega + D^2 \Psi^1_K + D^1 \Psi^2_K]$$
$$- \frac{1}{2} G^{-1} [2 \Psi^1_r \Psi^2_r + 2 r^{-2} \Psi^1_\theta \Psi^2_\theta + 2 \Psi^1_z \Psi^2_z] \quad \text{on} \quad z = 0 \quad . \tag{91}$$

It is easily seen that the first approximation represents the linear theory and is satisfied by choosing

$$\Psi^1 = K \cos \Omega \cos (\alpha\theta) J_{\alpha, o} \frac{\cosh [\lambda (z+h)]}{\cosh (\lambda h)} \tag{92}$$

$$\zeta^1 = K G^{-1} \omega \sin \Omega \cos (\alpha\theta) J_{\alpha, o} \tag{93}$$

To satisfy Equation 86 which represents the second order approximation, one uses Equation 67 and Equation 92 and seeks expressions for the A^2_{km} such that Equation 86 is satisfied. Substitution of Equation 67 and Equation 92 into Equation 86 gives

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \{[\omega^2 (A^2_{k\alpha, m})_{\Omega\Omega} + G \lambda_{k\alpha, m} \tanh(\lambda_{k\alpha, m} h) A^2_{k\alpha, m}] \cos(k\alpha\theta) J_{k\alpha, m}\}$$

$$= 2\,\omega\, B^1 K \cos\Omega \cos(\alpha\theta)\, J_{\alpha, o} + 2\,\omega\, D^1 \sin\Omega \cos(\alpha\theta)\, J_{\alpha, o}$$

$$+ 2 K^2 \omega \lambda^2 \cos\Omega \sin\Omega \cos^2(\alpha\theta) + 2\, r^{-2} K^2 \omega \alpha^2 \cos\Omega \sin\Omega \sin^2(\alpha\theta) J^2_{\alpha, o}$$

$$+ 2 K^2 \omega \lambda^2 \cos\Omega \sin\Omega \cos^2(\alpha\theta) \tanh^2(\lambda h) J^2_{\alpha, o}$$

$$+ G^{-1} K^2 \omega^3 \lambda \cos\Omega \sin\Omega \cos^2(\alpha\theta) \tanh(\lambda h) J^2_{\alpha, o}$$

$$- K^2 \omega \lambda^2 \cos\Omega \sin\Omega \cos^2(\alpha\theta) J^2_{\alpha, o} \quad . \qquad (94)$$

The identities

$$\cos^2(\alpha\theta) = \frac{1}{2}\,[1 + \cos(2\alpha\theta)] \qquad (95)$$

$$\sin^2(\alpha\theta) = \frac{1}{2}\,[1 - \cos(2\alpha\theta)] \qquad (96)$$

and use of Equation 62 enables Equation 94 to be written as

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \{[\omega^2 (A^2_{k\alpha, m})_{\Omega\Omega} + \omega^2_{k\alpha, m} A^2_{k\alpha, m}] \cos(k\alpha\theta)\, J_{k\alpha, m}\}$$

$$= 2\,\omega\, B^1 K \cos\Omega \cos(\alpha\theta)\, J_{\alpha, o} + 2\,\omega\, D^1 \sin\Omega \cos(\alpha\theta)\, J_{\alpha, o}$$

$$+ \frac{1}{2}\,\omega K^2 \lambda^2 \sin 2\Omega\, (J'_{\alpha, o})^2 + \frac{1}{2}\,\alpha^2 r^{-2}\,\omega K^2 \sin 2\Omega\, J^2_{\alpha, o}$$

$$+ \frac{1}{2}\, G^{-2} \omega^5 K^2 \sin 2\Omega\, J^2_{\alpha, o} + \frac{1}{4}\, G^{-2} \omega^5 K^2 \sin 2\Omega\, J^2_{\alpha, o}$$

$$- \frac{1}{4}\,\omega \lambda^2 K^2 \sin 2\Omega\, J^2_{\alpha, o} + \frac{1}{2}\,\omega K^2 \lambda^2 \sin 2\Omega \cos(2\alpha\theta)\, (J'_{\alpha, o})^2$$

$$- \frac{1}{2}\,\alpha^2 r^{-2} \omega K^2 \sin 2\Omega \cos(2\alpha\theta) J^2_{\alpha,o} + \frac{1}{2}\, G^{-2} \omega^5 K^2 \sin 2\Omega \cos(2\alpha\theta)\, J^2_{\alpha,o}$$

$$+ \frac{1}{4}\, G^{-2} \omega^5 K^2 \sin 2\Omega \cos(2\alpha\theta)\, J^2_{\alpha,o} - \frac{1}{4}\,\omega \lambda^2 K^2 \sin 2\Omega \cos(2\alpha\theta) J^2_{\alpha,o} . \qquad (97)$$

Equating coefficient of cos $(k\alpha\theta)$ yields four equations which after being multiplied by $r\,J_{o,j}$, $r\,J_{\alpha,j}$, $r\,J_{2\alpha,j}$, $r\,J_{k\alpha,j}$, respectively, and being integrated over the interval 0 to a will appear as

$$[\omega^2 (A^2_{o,j})_{\Omega\Omega} + \omega^2_{o,j} A^2_{o,j}] \frac{a^2}{2} J^2_o (\lambda_{o,j} a)$$
$$= \frac{1}{2} K^2 \omega \sin 2\Omega \{[\lambda^2 I \ (J'_{\alpha,o})^2 J_{o,j}] + \alpha^2 I [r^{-2} J^2_{\alpha,o} J_{o,j}]\}$$
$$+ \frac{1}{2} K^2 \sin 2\Omega \left[\frac{3}{2} G^{-2} \omega^5 - \frac{1}{2} \omega \lambda^2\right] I [J^2_{\alpha,o} J_{o,j}] \qquad (98)$$

$$[\omega^2 (A^2_{\alpha,j})_{\Omega\Omega} + \omega^2_{\alpha,j} A^2_{\alpha,j}] \left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,j} a^2}\right] \frac{a^2}{2} J^2_\alpha (\lambda_{\alpha,j} a)$$
$$= \omega [B^1 K \cos \Omega + D^1 \sin\Omega] \left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,o} a^2}\right] a^2 J^2_\alpha (\lambda_{\alpha,o} a) \qquad (99)$$

$$[\omega^2 (A^2_{2\alpha,j})_{\Omega\Omega} + \omega^2_{2\alpha,j} A^2_{2\alpha,j}] \left[1 - \frac{4\alpha^2}{\lambda^2_{\alpha,o} a^2}\right] \frac{a^2}{2} J^2_{2,\alpha} (\lambda_{2\alpha,j} a)$$
$$= \frac{1}{2} K^2 \omega \sin 2\Omega \ [\lambda^2 I (J'_{\alpha,o})^2 J_{2\alpha,j}] - \alpha^2 I [r^{-2} J^2_{\alpha,o} J_{2\alpha,j}]$$
$$+ \frac{1}{2} K^2 \sin 2\Omega \left[\frac{3}{2} G^{-2} \omega^5 - \frac{1}{2} \omega \lambda^2\right] I [J^2_{\alpha,o} J_{2\alpha,j}] \qquad (100)$$

$$[\omega^2 (A^2_{k\alpha,j})_{\Omega\Omega} + \omega^2_{k\alpha,j} A^2_{k\alpha,j}] = 0 \quad k \geqq 3 \quad . \qquad (101)$$

By using the first two Bessel function identities from Appendix A, these equations become

$$\omega^2 (A^2_{o,j})_{\Omega\Omega} + \omega^2_{o,j} A^2_{o,j} = K^2 C_{1j} \sin 2\Omega\, I [J^2_{\alpha,o} J_{o,j}] \qquad (102)$$

$$\omega^2 (A^2_{\alpha,j})_{\Omega\Omega} + \omega^2_{\alpha,j} A^2_{\alpha,j} = (B^1 K \cos\Omega + D^1 \sin\Omega) C_{2j} \tag{103}$$

$$\omega^2 (A^2_{2\alpha,j})_{\Omega\Omega} + \omega^2_{2\alpha,j} A^2_{2\alpha,j} = K^2 C_{3j} I [J^2_{\alpha,o} J_{2\alpha,j}] \sin 2\Omega \tag{104}$$

$$\omega^2 (A^2_{k\alpha,j})_{\Omega\Omega} + \omega^2_{k\alpha,j} A^2_{k\alpha,j} = 0 \quad k \geqq 3 \tag{105}$$

where

$$C_{1j} = \frac{1}{2} \omega a^{-2} [J^2_o (\lambda_{o,j} a)]^{-1} \left[\lambda^2 - \lambda^2_{o,j} + \frac{3\omega^4}{G^2}\right] \tag{106}$$

$$C_{2j} = 2\omega \left[1 - \frac{\alpha^2}{\lambda^2 a^2}\right] J^2_\alpha (\lambda a) \left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,j} a^2}\right]^{-1} [J^2_\alpha (\lambda_{\alpha,j} a)]^{-1} \tag{107}$$

$$C_{3j} = \frac{1}{2} \omega a^{-2} \left[1 - \frac{4\alpha^2}{\lambda^2_{2\alpha,j} a^2}\right]^{-1} [J_{2\alpha}(\lambda_{2\alpha,j} a)]^{-1} \left[\frac{3\omega^4}{G^2} + \lambda^2 - \lambda^2_{2\alpha,j}\right] . \tag{108}$$

To determine the $A^2_{k\alpha,j}$ from Equations 102, 103, 104, and 105, the $A^2_{k\alpha,j}$ is expanded in a Fourier series.

$$A^2_{k\alpha,j} = \frac{1}{2} a^o_{k\alpha,j} + \sum_{\ell=1}^{\infty} \left\{ a^\ell_{k\alpha,j} \cos \ell\Omega + b^\ell_{k\alpha,j} \sin \ell\Omega \right\} . \tag{109}$$

Upon substitution of Equation 109 into Equations 102, 103, 104, and 105, the following is obtained:

$$\frac{1}{2} a^o_{o,j} + \sum_{\ell=1}^{\infty} \left\{ [\omega^2_{o,j} - \ell^2 \omega^2] a^\ell_{o,j} \cos \ell\Omega + [\omega^2_{o,j} - \ell^2 \omega^2] b^\ell_{o,j} \sin \ell\Omega \right\}$$

$$= K^2 C_{1j} \sin 2\Omega I [J^2_{\alpha,o} J_{o,j}] \tag{110}$$

$$\frac{1}{2}\, a^{o}_{\alpha,j} + \sum_{\ell=1}^{\infty} \left\{ [\omega^{2}_{\alpha,j} - \ell^{2}\,\omega^{2}]\, a^{\ell}_{\alpha,j} \cos \ell\Omega + [\omega^{2}_{\alpha,j} - \ell^{2}\,\omega^{2}]\, b^{\ell}_{\alpha,j} \sin \ell\Omega \right\}$$

$$= (B^{1} K \cos \Omega + D^{1} \sin \Omega)\, C_{2j} \qquad (111)$$

$$\frac{1}{2}\, a^{o}_{2\alpha,j} + \sum_{\ell=1}^{\infty} \left\{ [\omega^{2}_{2\alpha,j} - \ell^{2}\omega^{2}]\, a^{\ell}_{2\alpha,j} \cos \ell\Omega + [\omega^{2}_{2\alpha,j} - \ell^{2}\omega^{2}]\, b^{\ell}_{2\alpha,j} \sin \ell\Omega \right\}$$

$$= K^{2}\, C_{3j}\, I\, [J_{\alpha,o}\, J^{2}_{2\alpha,j}] \sin 2\Omega \qquad (112)$$

$$\frac{1}{2}\, a^{o}_{k\alpha,j} + \sum_{\ell=1}^{\infty} \left\{ [\omega^{2}_{k\alpha,j} - \ell^{2}\,\omega^{2}]\, a^{\ell}_{k\alpha,j} \cos \ell\Omega \right.$$

$$\left. + [\omega^{2}_{k\alpha,j} - \ell^{2}\,\omega^{2}]\, b^{\ell}_{2\alpha,j} \sin \ell\Omega \right\} = 0 \qquad (113)$$

for $k \geqq 3$.

At this point the assumption will be made that the liquid depth is such that $\omega^{2}_{k\alpha,j} \neq 4\,\omega^{2}$. The significance of this assumption will be examined later. At this point the conditions imposed by Equations 70 and 71 require $a^{1}_{k\alpha,j} = b^{1}_{k\alpha,j} = 0$. Now equating coefficients of $\cos \ell\Omega$ and $\sin \ell\Omega$ gives the results

$$b^{2}_{o,j} = [\omega^{2}_{o,j} - 4\,\omega^{2}]^{-1} K^{2}\, C_{1j}\, I\, [J^{2}_{\alpha,o}\, J_{o,j}] \qquad (114)$$

$$b^{2}_{2\alpha,j} = [\omega^{2}_{2\alpha,j} - 4\,\omega^{2}]^{-1} K^{2}\, C_{3j}\, I\, [J^{2}_{\alpha,o}\, J_{2\alpha,j}] \qquad (115)$$

$$B^{1} = D^{1} = 0 \qquad (116)$$

$$a^{\ell}_{k\alpha, j} = 0 \qquad \ell = 0, 1, 2, \ldots \tag{117}$$

$$b^{\ell}_{k\alpha, j} = 0 \qquad \ell \neq 2 \quad . \tag{118}$$

The results of the second approximation are given by

$$\Psi^2 = \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \sin 2\Omega \, J_o \, (\lambda_{o,j} \, r) \, \frac{\cosh \left[\lambda_{o,j} \, (z+h)\right]}{\cosh \, (\lambda_{o,j} \, h)} \right\}$$

$$+ \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha, j} \sin 2\Omega \, J_{2\alpha, j} (\lambda_{2\alpha, j} \, r) \cos \, (2\alpha\theta) \, \frac{\cosh \left[\lambda_{2\alpha, j} \, (z+h)\right]}{\cosh \lambda_{2\alpha, j} \, h} \right\} \tag{119}$$

$$\zeta^2 = -\frac{1}{2} \, G^{-1} \cos^2 \Omega \, K^2 \left[\cos^2 (\alpha\theta) \, \lambda^2_{\alpha, o} \, (J'_{\alpha, o})^2 + r^{-2} \alpha^2 \sin^2 \alpha\theta \, J^2_{\alpha, o} \right.$$

$$\left. + \frac{\omega^4}{G^2} \cos^2 (\alpha\theta) \, J^2_{\alpha, o} \right] - 2 \, G^{-1} \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \cos 2\Omega \, J_{o,j} \right.$$

$$\left. + b^2_{2\alpha, j} \cos 2\Omega \, J_{2\alpha, j} \cos 2\alpha\theta \right\} \quad . \tag{120}$$

The third approximation is found by a similar procedure. Substituting Equations 67, 92, and 119, into Equation 87, yields

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left\{ [\omega^2 \, (A^3_{k\alpha, m})_{\Omega\Omega} + \omega^2_{k\alpha, m} \, A^3_{k\alpha, m}] \cos \, (k\alpha\theta) \, J_{k\alpha, m} \right.$$

$$= 2 \, K \, \omega \, B^2 \cos \Omega \cos \, (\alpha\theta) \, J_{\alpha, o} + 2 \, \omega \, D^2 \sin \Omega \cos \, (\alpha\theta) \, J_{\alpha, o}$$

$$- 4 \, K \, \omega \cos \Omega \cos 2\Omega \, \lambda \cos \, (\alpha\theta) \, J'_{\alpha, o} \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \, \lambda_{o,j} \, J'_{o,j} \right.$$

$$\left. + b^2_{2\alpha, j} \, \lambda_{2\alpha, j} \, J'_{2\alpha, j} \cos \, (2\alpha\theta) \right\}$$

$$+ 2\,\omega\, K \sin\Omega \cos(\alpha\theta)\, \lambda\, J'_{\alpha,o} \sin 2\Omega \sum_{j=0}^{\infty} \Big\{ b_{o,j}^2\, \lambda_{o,j}\, J'_{o,j}$$

$$+ b_{2\alpha,j}^2\, \lambda_{2\alpha,j}\, J'_{2\alpha,j} \cos(2\alpha\theta) \Big\}$$

$$- 4\, r^{-2}\, \omega\, K \cos\Omega\, \alpha \sin(\alpha\theta)\, J_{\alpha,o} \cos 2\Omega \sum_{j=0}^{\infty} \Big\{ b_{2\alpha,j}^2\, J_{2\alpha,j}(2\alpha) \sin(2\alpha\theta) \Big\}$$

$$+ 2\, r^{-2}\, \omega K \sin(\Omega)\, \alpha \sin(\alpha\theta)\, J_{\alpha,o} \sin 2\Omega \sum_{j=0}^{\infty} \Big\{ b_{2\alpha,j}^2\, J_{2\alpha,j}(2\alpha) \sin(2\alpha\theta) \Big\}$$

$$- 4\,\omega K \cos\Omega \cos(\alpha\theta)\, \lambda\, J_{\alpha,o} \tanh(\lambda h) \cos 2\Omega \sum_{j=0}^{\infty} \Big\{ b_{o,j}\, \lambda_{o,j}\, J_{o,j} \tanh(\lambda_{o,j} h)$$

$$+ b_{2\alpha,j}^2\, J_{2\alpha,j} \cos(2\alpha\theta)\, \lambda_{2\alpha,j} \tanh(\lambda_{2\alpha,j} h) \Big\}$$

$$+ 2\,\omega K \sin\Omega \cos(\alpha\theta)\, J_{\alpha,o}\, \lambda \tanh(\lambda h) \sin 2\Omega \sum_{j=0}^{\infty} b_{o,j}^2\, \lambda_{o,j} \tanh(\lambda_{o,j} h)\, J_{o,j}$$

$$+ b_{2\alpha,j}^2\, \lambda_{2\alpha,j}\, J_{2\alpha,j} \cos(2\alpha\theta) \tanh(\lambda_{2\alpha,j} h) \Big\}$$

$$+ \frac{4}{G}\, \omega^3 K \sin\Omega \cos(\alpha\theta)\, J_{\alpha,o} \sin 2\Omega \sum_{j=0}^{\infty} \Big\{ b_{o,j}^2\, \lambda_{o,j}\, J_{o,j} \tanh(\lambda_{o,j} h)$$

$$+ b_{2\alpha,j}^2\, \lambda_{2\alpha,j}\, J_{2\alpha,j} \cos(2\alpha\theta) \tanh(\lambda_{2\alpha,j} h) \Big\}$$

$$- \frac{2}{G}\, \omega^3 K \cos\Omega \cos(\alpha\theta)\, J_{\alpha,o}\, \lambda \tanh(\lambda h) \cos 2\Omega \sum_{j=0}^{\infty} \Big\{ b_{o,j}^2\, J_{o,j}$$

$$+ b_{2\alpha,j}^2\, J_{2\alpha,j} \cos(2\alpha\theta) \Big\} - \omega K \sin\Omega \cos(\alpha\theta)\, J_{\alpha,o} \sum_{j=0}^{\infty} \Big\{ b_{o,j}^2\, \lambda_{o,j}^2\, J_{o,j} \sin 2\Omega$$

$$+ b_{2,j}^2\, J_{2,j} \cos(2\alpha\theta)\, \lambda_{2\alpha,j}^2 \sin 2\Omega \Big\}$$

$$+ 2\omega K \cos\Omega \cos(\alpha\theta) J_{\alpha,o} \cos(2\Omega)\, \lambda^2 \sum_{j=0}^{\infty} \left\{ b_{o,j}^2 J_{o,j} + b_{2\alpha,j}^2 J_{2\alpha,j} \cos(2\alpha\theta) \right\}$$

$$- K^3 \cos^3\Omega \cos^3(\alpha\theta)\, \lambda^4 (J'_{\alpha,o})^2 (J''_{\alpha,o}) + \frac{K^3}{r^4} \cos^3\Omega \sin^2(\alpha\theta)\, \alpha^4 \cos(\alpha\theta) J_{\alpha,o}^3$$

$$- K^3 \lambda^4 \cos^3\Omega \cos^3(\alpha\theta) J_{\alpha,o}^3 \tanh^2(\lambda h)$$

$$+ \lambda r^{-3} K^3 \cos^3\Omega \cos(\alpha\theta) \sin^2(\alpha\theta)\, \alpha^2 J_{\alpha,o}^2 J'_{\alpha,o}$$

$$- 2 K^3 \cos^3\Omega \cos^3(\alpha\theta)\, \lambda^4 J_{\alpha,o} (J'_{\alpha,o})^2 \tanh^2(\lambda h)$$

$$- 2 r^{-2} K^3 \cos^3\Omega \sin^2(\alpha\theta)\, \alpha^2 \cos(\alpha\theta)\, \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$- 2 r^{-2} K^3 \cos^3\Omega \sin^2(\alpha\theta)\, \alpha^2 \cos(\alpha\theta) J_{\alpha,o}^3 \lambda^2 \tanh^2(\lambda h)$$

$$- \frac{1}{2} K^3 G^{-1} \omega^2 \cos^3\Omega \cos^3(\alpha\theta)\, \lambda^3 J_{\alpha,o} (J'_{\alpha,o})^2 \tanh(\lambda h)$$

$$- \frac{1}{2} G^{-1} r^{-2} \omega^2 K^3 \cos(\alpha\theta)\, \alpha^2 \sin^2(\alpha\theta) \cos^3\Omega\, \lambda \tanh(\lambda h) J_{\alpha,o}^3$$

$$- \frac{1}{2} G^{-1} \omega^2 K^3 \cos^3\Omega \cos^3(\alpha\theta) J_{\alpha,o}^3 \lambda^3 \tanh^3(\lambda h)$$

$$+ \frac{1}{2} K^3 \cos^3\Omega \cos^3(\alpha\theta) J_{\alpha,o} (J'_{\alpha,o})^2 \lambda^4$$

$$+ \frac{1}{2} r^{-2} K^3 \cos^3\Omega \cos(\alpha\theta)\, \alpha^2 \sin^2(\alpha\theta) J_{\alpha,o}^3 \lambda^2$$

$$+ \frac{1}{2} K^3 \cos^3\Omega \cos^3(\alpha\theta) J_{\alpha,o}^3 \lambda^4 \tanh^2(\lambda h)$$

$$+ 4 G^{-1} K^3 \omega^2 \cos\Omega \sin^2\Omega \cos^3(\alpha\theta) J_{\alpha,o} (J'_{\alpha,o})^2 \lambda^3 \tanh(\lambda h)$$

$$+ 4 G^{-1} \omega^2 K^3 \cos\Omega \sin^2\Omega \cos(\alpha\theta)\, \alpha^2 \sin^2(\alpha\theta) J_{\alpha,o}^3 \lambda \tanh(\lambda h)$$

$$+ 4 G^{-1} \omega^2 K^3 \cos\Omega \sin^2\Omega \cos^3(\alpha\theta) J_{\alpha,o}^3 \lambda^3 \tanh(\lambda h)$$

$$+ G^{-2} \omega^4 K^3 \cos\Omega \sin^2\Omega \cos^3(\alpha\theta) J_{\alpha,o}^3 \lambda^2 \tanh^2(\lambda h)$$

$$- G^{-1} \omega^2 K^3 \cos\Omega \sin^2\Omega \cos^3(\alpha\theta) J_{\alpha,o}^3 \lambda^3 \tanh(\lambda h)$$

$$+ \frac{1}{2} G^{-2} \omega^4 K^3 \cos\Omega \sin^2\Omega \cos^3(\alpha\theta) J^3_{\alpha,o} \lambda^2$$

$$- \frac{1}{2} G^{-1} \omega^2 K^3 \cos\Omega \sin^2\Omega \cos^3(\alpha\theta) J^3_{\alpha,o} \lambda^3 \tanh(\lambda h) \quad . \qquad (121)$$

By using identities A-7, A-8, A-9, A-10, A-11, A-12, and A-13 from Appendix A, Equation 121 may be written

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left\{ [\omega^2 (A^2_{k\alpha,m})_{\Omega\Omega} + \omega^2_{k\alpha,m} A^3_{k\alpha,m}] \cos(k\alpha\theta) J_{k\alpha,m} \right\}$$

$$= 2\,\omega B^2 K \cos\Omega \cos(\alpha\theta) J_{\alpha,o} + 2\,\omega D^2 \sin\Omega \cos(\alpha\theta) J_{\alpha,o}$$

$$- 2\,\omega K [\cos\Omega + \cos 3\Omega] \lambda \cos(\alpha\theta) J'_{\alpha,o} \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \lambda_{o,j} J'_{o,j} \right\}$$

$$+ \omega K [\cos\Omega - \cos 3\Omega] \lambda J'_{\alpha,o} \cos(\alpha\theta) \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \lambda_{o,j} J'_{o,j} \right\}$$

$$- 2\, G^{-2} \omega^3 K [\cos\Omega + \cos 3\Omega] J_{\alpha,o} \cos(\alpha\theta) \sum_{j=0}^{\infty} \left\{ b^2_{o,j} J_{o,j} \omega^2_{o,j} \right\}$$

$$+ 3\, G^{-2} K [\cos\Omega - \cos 3\Omega] J_{\alpha,o} \cos(\alpha\theta) \sum_{j=0}^{\infty} \left\{ b^2_{o,j} J_{o,j} \omega^2_{o,j} \right\}$$

$$- G^{-2} \omega^5 K [\cos\Omega + \cos 3\Omega] J_{\alpha,o} \cos(\alpha\theta) \sum_{j=0}^{\infty} \left\{ b^2_{o,j} J_{o,j} \right\}$$

$$- \frac{1}{2} \omega K [\cos\Omega - \cos 3\Omega] \cos(\alpha\theta) J_{\alpha,o} \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \lambda^2_{o,j} J_{o,j} \right\}$$

$$+ \omega K [\cos\Omega + \cos 3\Omega] \cos(\alpha\theta) J_{\alpha,o} \lambda^2 \sum_{j=0}^{\infty} \left\{ b^2_{o,j} J_{o,j} \right\}$$

$$- \omega K \lambda J'_{\alpha,o} \cos(\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} \lambda_{2\alpha,j} J'_{2\alpha,j} \right\}$$

$$- \omega K \lambda J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} \lambda_{2\alpha,j} J'_{2\alpha,j} \right\}$$

$$+ \frac{1}{2} \omega K \lambda J'_{\alpha,o} \cos(\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} \lambda_{2\alpha,j} J'_{2\alpha,j} \right\}$$

$$+ \frac{1}{2} \omega K \lambda J'_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} \lambda_{2\alpha,j} J'_{2\alpha,j} \right\}$$

$$- r^{-2} \omega K \alpha^2 J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$+ r^{-2} \omega K \alpha^2 J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$+ \frac{1}{2} r^{-2} \omega K \alpha^2 J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$- \frac{1}{2} r^{-2} \omega K \alpha^2 J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$- G^{-2} \omega^3 K J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \omega^2_{2\alpha,j} \right\}$$

$$- G^{-2} \omega^3 K J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \omega^2_{2\alpha,j} \right\}$$

$$+ \frac{3}{2} G^{-2} \omega^3 K J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \omega^2_{2\alpha,j} \right\}$$

$$+ \frac{3}{2} G^{-2} \omega^3 J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \omega^2_{2\alpha,j} \right\}$$

$$- \frac{1}{2} G^{-2} \omega^5 K J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$- \frac{1}{2} G^{-2} \omega^5 J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$- \frac{1}{4} \omega K J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \lambda^2_{2\alpha,j} \right\}$$

$$- \frac{1}{4} \omega K J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega - \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \lambda^2_{2\alpha,j} \right\}$$

$$+ \frac{1}{2} \omega K \lambda^2 J_{\alpha,o} \cos(\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$+ \frac{1}{2} \omega K \lambda^2 J_{\alpha,o} \cos(3\alpha\theta) [\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} J_{2\alpha,j} \right\}$$

$$+ \frac{3}{16} K^3 \cos(\alpha\theta) [3\cos\Omega + \cos 3\Omega] \Big\{ - \lambda^4 (J'_{\alpha,o})^2 J''_{\alpha,o}$$

$$- \lambda^2 G^{-2} \omega^4 J^3_{\alpha,o} - 2 \lambda^2 G^{-2} \omega^4 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$- \frac{1}{2} G^{-2} \omega^4 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2 - \frac{1}{2} G^{-4} \omega^8 J^3_{\alpha,o} + \frac{1}{2} \lambda^4 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$+ \frac{1}{2} G^{-2} \lambda^2 \omega^4 J^3_{\alpha,o} \Big\}$$

$$+ \frac{1}{16} K^3 \cos(3\alpha\theta) [3\cos\Omega + \cos 3\Omega] \Big\{ - \lambda^4 (J'_{\alpha,o})^2 J''_{\alpha,o}$$

$$- \lambda^2 G^{-2} \omega^4 J^3_{\alpha,o} - 2 \lambda^2 G^{-2} \omega^4 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$- \frac{1}{2} G^{-2} \omega^4 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2 - \frac{1}{2} G^{-4} \omega^8 J^3_{\alpha,o} + \frac{1}{2} \lambda^4 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$+ \frac{1}{2} G^{-2} \lambda^2 \omega^4 J^3_{\alpha,o}\Big\} + \frac{1}{16} K^3 \cos(\alpha\theta) [3 \cos \Omega + \cos 3\Omega] \Big\{ r^{-4} \alpha^4 J^3_{\alpha,o}$$

$$+ r^{-3} \alpha^2 \lambda J^3_{\alpha,o} J'_{\alpha,o} - 2 r^{-2} \alpha^2 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$- 2 G^{-2} r^{-2} \alpha^2 \omega^4 J^3_{\alpha,o} - \frac{1}{2} G^{-2} r^{-2} \omega^4 \alpha^2 J^3_{\alpha,o} + \frac{1}{2} r^{-2} \alpha^2 \lambda^2 J^3_{\alpha,o}\Big\}$$

$$- \frac{1}{16} K^3 \cos(3\alpha\theta) [3 \cos \Omega + \cos 3\Omega] \Big\{ r^{-4} \alpha^4 J^3_{\alpha,o}$$

$$+ r^{-3} \alpha^2 \lambda J^2_{\alpha,o} J'_{\alpha,o} - 2 r^{-2} \alpha^2 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2 - 2 G^{-2} r^{-2} \alpha^2 \omega^4 J^3_{\alpha,o}$$

$$- \frac{1}{2} G^{-2} r^{-2} \omega^4 \alpha^2 J^3_{\alpha,o} + \frac{1}{2} r^{-2} \alpha^2 \lambda^2 J^3_{\alpha,o}\Big\}$$

$$+ \frac{3}{16} K^3 \cos(\alpha\theta) [\cos \Omega - \cos 3\Omega] \Big\{ 2 G^{-2} \omega^4 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$+ 2 G^{-2} \omega^4 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2 + 4 G^{-2} \omega^4 \lambda^2 J^3_{\alpha,o}$$

$$+ G^{-4} \omega^8 J^3_{\alpha,o} - G^{-2} \omega^4 \lambda^2 J^2_{\alpha,o} + \frac{1}{2} G^{-2} \omega^4 \lambda^2 J^3_{\alpha,o}$$

$$- \frac{1}{2} G^2 \omega^4 \lambda^2 J^3_{\alpha,o}\Big\} + \frac{1}{16} K^3 \cos(3\alpha\theta) [\cos \Omega - \cos 3\Omega] \Big\{ 2 G^{-2} \omega^4 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2$$

$$+ 2 G^{-2} \omega^4 \lambda^2 J_{\alpha,o} (J'_{\alpha,o})^2 + 4 G^{-2} \omega^4 \lambda^2 J^3_{\alpha,o}$$

$$+ G^{-4} \omega^8 J^3_{\alpha,o} - G^{-2} \omega^4 \lambda^2 J^3_{\alpha,o}$$

$$+ \frac{1}{2} G^{-2} \omega^4 \lambda^2 J^3_{\alpha,o} - \frac{1}{2} G^{-2} \omega^4 \lambda^2 J^3_{\alpha,o}\Big\}$$

$$+ \frac{1}{4} G^{-2} r^{-2} \omega^4 K^3 \cos(\alpha\theta) [\cos \Omega - \cos 3\Omega] \alpha^2 J^3_{\alpha,o}$$

$$- \frac{1}{4} G^2 r^{-2} \omega^4 K^3 \cos(\alpha\theta) [\cos \Omega - \cos 3\Omega] \alpha^2 J^3_{\alpha,o} \quad . \qquad (122)$$

Now equating the coefficients of cos $(k\alpha\theta)$ two equations result. Multiplying these equations respectively by r $J_{\alpha n}$ and r $J_{3\alpha n}$ and integrating over the interval 0 to a will result in the following two equations:

$$\left\{\left[\omega^2 (A^3_{\alpha n})_{\Omega\Omega} + \omega_{\alpha n} A^3_{\alpha n}\right]\left[1 - \frac{\alpha^2}{\lambda^2_{\alpha n} a^2}\right] J^2_\alpha (\lambda_{\alpha n} a)\right\} \frac{a^2}{2}$$

$$= 2\,\omega\,[B^2 K \cos\Omega + D^2 \sin\Omega]\; I\,[J_{\alpha,o} J_{\alpha,n}]$$

$$- \omega K \lambda\,[\cos\Omega + \cos 3\Omega] \sum_{j=0}^{\infty} b^2_{o,j}\, \lambda_{o,j}\, I\,[J'_{o,j} J'_{\alpha,o} J_{\alpha,n}]$$

$$+ K\cos\Omega \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \left[\frac{\omega^3}{G^2}\,\omega^2_{o,j} - \frac{\omega^5}{G^2} - \frac{\omega}{2}\,\lambda^2_{o,j} + \omega\lambda^2 \right] I\,[J_{o,j} J_{\alpha,o} J_{\alpha,n}] \right\}$$

$$+ K\cos 3\Omega \sum_{j=0}^{\infty} \left\{ b^2_{o,j} \left[-\frac{5}{G^2}\,\omega^3\,\omega^2_{o,j} - \frac{\omega^5}{G^2} + \frac{1}{2}\,\omega\lambda^2_{o,j} + \omega\lambda^2 \right] I\,[J_{o,j} J_{\alpha,o} J_{\alpha,n}] \right\}$$

$$- \frac{1}{2} K\omega\lambda\,[\cos\Omega + 3\cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j}\, \lambda_{2\alpha,j}\, I\,[J'_{2\alpha,j} J'_{\alpha,o} J_{\alpha,n}] \right\}$$

$$- K\omega\alpha^2\,[\cos\Omega + 3\cos 3\Omega] \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j}\, I\,[r^{-2} J_{\alpha,o} J_{2\alpha,j} J_{\alpha,n}] \right\}$$

$$+ \frac{1}{2} K\cos\Omega \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} \left[\frac{\omega^3}{G^2}\,\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} - \frac{\omega}{2}\,\lambda^2_{2\alpha,j} + \omega\lambda^2 \right] I\,[J_{2\alpha,j} J_{\alpha,o} J_{\alpha,n}] \right\}$$

$$+ \frac{1}{2} K\cos 3\Omega \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} \left[-5\frac{\omega^3}{G^2}\,\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} + \frac{\omega}{2}\lambda^2_{2\alpha,j} + \omega\lambda^2 \right] I\,[J_{2\alpha,j} J_{\alpha,o} J_{\alpha,n}] \right\}$$

$$- \frac{9}{16} K^3 \cos\Omega\, \lambda^4\, I\,[(J'_{\alpha,o})^2 J''_{\alpha,o} J_{\alpha,n}]$$

$$+\left\{+\frac{1}{2}\left[3\lambda^4 - 7\frac{\lambda^2}{G^2}\omega^4\right] I\,[(J'_{\alpha,o})^2 J_{\alpha,o} J_{\alpha,n}]\right.$$

$$+\alpha^4 I\,[r^{-4} J^3_{\alpha,o} J_{\alpha,n}] + \alpha^2 \lambda I\,[r^{-3} J^2_{\alpha,o} J'_{\alpha,o} J_{\alpha,n}]$$

$$-2\alpha^2\lambda^2 I\,[r^{-2} J_{\alpha,o} (J'_{\alpha,o})^2 J_{\alpha,n}] + \frac{1}{6}\left[-7\alpha^2\frac{\omega^4}{G^2} + 3\lambda^2\alpha^2\right] I\,[r^{-2} J^3_{\alpha,o} J_{\alpha,n}]$$

$$\left.+\left[\frac{3}{2}\lambda^2\omega^4 - \frac{1}{2}\frac{\omega^8}{G^4}\right] I\,[J^3_{\alpha,o} J_{\alpha,n}]\right\}\frac{3}{16} K^3 \cos\Omega$$

$$+\frac{1}{16} K^3 \cos 3\Omega \left\{-3\lambda^4 I\,[(J'_{\alpha,o})^2 J''_{\alpha,o} J_{\alpha,n}]\right.$$

$$+\left[\frac{3}{2}\lambda^4 - \frac{39}{2}\frac{\lambda^2}{G^2}\right] I\,[J_{\alpha,o}(J'_{\alpha,o})^2 J_{\alpha,n}] + \alpha^4 I\,[r^{-4} J^3_{\alpha,o} J_{\alpha,n}]$$

$$+\alpha^2\lambda I\,[r^{-3} J^2_{\alpha,o} J'_{\alpha,o} J_{\alpha,n}] - 2\alpha^2\lambda^2 I\,[r^{-2} J_{\alpha,o}(J'_{\alpha,o})^2 J_{\alpha,n}]$$

$$+\frac{1}{2}\left[\lambda^2\alpha^2 - 13\alpha^2\frac{\omega^4}{G^2}\right] I\,[r^{-2} J^3_{\alpha,o} J_{\alpha,n}]$$

$$-\frac{1}{2}\left[21\omega^4\frac{\lambda^2}{G^2} + 9\frac{\omega^8}{G^4}\right] I\,[J^3_{\alpha,o} J_{\alpha,n}] \quad . \qquad (123)$$

$$\left\{\left[\omega^2 (A^3_{3\alpha,n})_{\Omega\Omega} + \omega^2_{3\alpha,n} A^3_{3\alpha,n}\right]\left[1 - \frac{9\alpha^2}{\lambda^2_{3\alpha,n} a^2}\right] J^2_{3\alpha}(\lambda_{3\alpha,n} a)\right\}\frac{a^2}{2}$$

$$= -\frac{1}{2}\lambda K \omega\,[\cos\Omega + 3\cos 3\Omega] \sum_{j=0}^{\infty}\left\{b^2_{2\alpha,j}\lambda_{2\alpha,j} I\,[J'_{2\alpha,j} J'_{\alpha,o} J_{3\alpha,n}]\right\}$$

$$+\frac{1}{2}\omega K \alpha^2\,[\cos\Omega + 3\cos 3\Omega] \sum_{j=0}^{\infty}\left\{b^2_{2\alpha,j} I\,[r^{-2} J_{2\alpha,j} J_{\alpha,o} J_{3\alpha,n}]\right\}$$

$$+\frac{1}{2} K \cos\Omega \sum_{j=0}^{\infty}\left\{b^2_{2\alpha,j}\left[\frac{\omega^3}{G^2}\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} - \frac{\omega}{2}\lambda^2_{2\alpha,j} + \omega\lambda^2\right] I\,[J_{2\alpha,j} J_{\alpha,o} J_{3\alpha,n}]\right\}$$

$$+\frac{1}{2}\,K\cos 3\Omega \sum_{j=0}^{\infty}\left\{b_{2\alpha,j}^{2}\left[-5\frac{\omega^{3}}{G^{2}}\,\omega_{2\alpha,j}^{2}-\frac{\omega^{5}}{G^{2}}\right.\right.$$

$$\left.\left.+\frac{\omega}{2}\,\lambda_{2\alpha,j}^{2}+\omega\,\lambda^{2}\right] I\,[J_{2\alpha,j}\,J_{\alpha,o}\,J_{3\alpha,n}]\right\}$$

$$+\frac{1}{16}\,K^{3}\cos\Omega\left\{-3\,\lambda^{4}\,I\,[(J'_{\alpha,o})^{2}\,J''_{\alpha,o}\,J_{3\alpha,n}]\right.$$

$$+\frac{1}{2}\left[3\,\lambda^{4}-7\,\lambda^{2}\,\frac{\omega^{4}}{G^{2}}\right] I\,[J_{\alpha,o}(J'_{\alpha,o})^{2}\,J_{3\alpha,n}]-3\,\alpha^{4}\,I\,[r^{-4}\,J_{\alpha,o}^{3}\,J_{3\alpha,n}]$$

$$-3\,\alpha^{2}\,\lambda\,I\,[r^{-3}\,J_{\alpha,o}^{2}\,J'_{\alpha,o}\,J_{3\alpha,n}]+6\,\alpha^{2}\,\lambda^{2}\,I\,[r^{-2}\,J_{\alpha,o}\,(J'_{\alpha,o})^{2}\,J_{3\alpha,n}]$$

$$+\frac{1}{2}\,[7\,\alpha^{2}\,\omega^{4}-3\,\alpha^{2}\,\lambda^{2}]\,I\,[r^{-2}\,J_{\alpha,o}^{3}\,J_{3\alpha,n}]$$

$$\left.+\frac{1}{2}\left[3\,\omega^{4}\,\frac{\lambda^{2}}{G^{2}}-\frac{\omega^{8}}{G^{4}}\right] I\,[J_{\alpha,o}^{3}\,J_{3\alpha,n}]\right\}$$

$$+\frac{1}{16}\,K^{3}\cos 3\Omega\left\{-\lambda^{4}\,I\,[(J'_{\alpha,o})^{2}\,J''_{\alpha,o}\,J_{3\alpha,n}]\right.$$

$$+\frac{1}{2}\left[\lambda^{4}-13\,\lambda^{2}\,\frac{\omega^{4}}{G^{2}}\right] I\,[J_{\alpha,o}\,(J'_{\alpha,o})^{2}\,J_{3\alpha,n}]$$

$$-\alpha^{4}\,I\,[r^{-4}\,J_{\alpha,o}^{3}\,J_{3\alpha,n}]-\alpha^{2}\,\lambda\,I\,[r^{-3}\,J_{\alpha,o}^{2}\,J'_{\alpha,o}\,J_{3\alpha,n}]$$

$$+2\,\alpha^{2}\,\lambda^{2}\,I\,[r^{-2}\,J_{\alpha,o}\,(J'_{\alpha,o})^{2}\,J_{3\alpha,n}]$$

$$+\frac{1}{2}\left[13\,\alpha^{2}\,\frac{\omega^{4}}{G^{2}}-\alpha^{2}\,\lambda^{2}\right] I\,[r^{-2}\,J_{\alpha,o}^{3}\,J_{3\alpha,n}]$$

$$\left.-\frac{1}{2}\left[7\,\lambda^{2}\,\frac{\omega^{4}}{G^{2}}+3\,\frac{\omega^{8}}{G^{4}}\right] I\,[J_{\alpha,o}^{3}\,J_{3\alpha,n}]\right\}\quad. \tag{124}$$

To simplify the algebra, the following constants are defined.

$$Q_{4jn} = \frac{2}{a^2}\left[G^{-2}\,\omega^3\,\omega^2_{o,j} - \frac{\omega^5}{G^2} - \omega\,\lambda^2_{o,j} + \frac{\omega}{2}\,\lambda^2 + \frac{1}{2}\,\omega\,\lambda^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,n}\,a^2}\right]^{-1}\left[J_\alpha\,(\lambda_{\alpha,n}\,a)\right]^{-2} \tag{125}$$

$$Q_{5jn} = \frac{2}{a^2}\left[-5\,\frac{\omega^3}{G^2}\,\omega^2_{o,j} - \frac{\omega^5}{G^2} - \omega\,\lambda^2_{o,j} - \frac{1}{2}\,\omega\,\lambda^2 - \frac{3}{2}\,\omega\,\lambda^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,n}\,a^2}\right]^{-1}\left[J_\alpha\,(\lambda_{\alpha,n}\,a)\right]^{-2} \tag{126}$$

$$Q_{6jn} = \frac{2}{a^2}\left[\frac{\omega^3}{G^2}\,\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} - \frac{3}{2}\,\omega\,\lambda^2_{2\alpha,j} - \omega\,\lambda^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,n}\,a^2}\right]^{-1}\left[J_\alpha\,(\lambda_{\alpha,n}\,a)\right]^{-2} \tag{127}$$

$$Q_{7jn} = \frac{2}{a^2}\left[-5\,\frac{\omega^3}{G^2}\,\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} + \frac{5}{2}\,\omega\,\lambda^2_{2\alpha,j} - 2\,\omega\,\lambda^2 + 3\,\omega\,\lambda^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,n}\,a^2}\right]^{-1}\left[J_\alpha\,(\lambda_{\alpha,n}\,a)\right]^{-2} \tag{128}$$

$$Q_{8n} = \left[\frac{3}{2}\,\lambda^2\,\frac{\omega^4}{G^2} - \frac{1}{2}\,\frac{\omega^8}{G^4} + \frac{1}{6}\,(3\,\lambda^2 - \lambda^2_{\alpha,n})\left(\frac{3}{2}\,\lambda^2 - \frac{7}{2}\,\frac{\omega^4}{G^2}\right)\right] \tag{129}$$

$$Q_{9n} = \left[-\frac{21}{2}\,\frac{\lambda^2}{G^2}\,\omega^4 - \frac{9}{2}\,\frac{\omega^8}{G^4} + \frac{1}{2}\,(3\,\lambda^2 - \lambda^2_{\alpha,n})\left(\frac{1}{2}\,\lambda^2 - \frac{13}{2}\,\frac{\omega^4}{G^2}\right)\right] \tag{130}$$

$$Q_{10jn} = \frac{2}{a^2}\left[\frac{\omega^3}{G^2}\,\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} - \frac{3}{2}\,\omega\,\lambda^2_{2\alpha,j} + \omega\,\lambda^2_{3\alpha,n}\right]\left[1 - \frac{9\,\alpha^2}{\lambda^2_{3\alpha,n}\,a^2}\right]^{-1}\left[J_{3\alpha}\,(\lambda_{3\alpha,n}\,a)\right]^{-2} \tag{131}$$

$$Q_{11jn} = \frac{2}{a^2}\left[-5\frac{\omega^3}{G^2}\omega^2_{2\alpha,j} - \frac{\omega^5}{G^2} - \frac{5}{2}\omega\lambda^2_{2\alpha,j} - 2\omega\lambda^2 - 3\omega\lambda^2_{3\alpha,n}\right]\left[1 - \frac{9\alpha^2}{\lambda^2_{3\alpha,n}a^2}\right]^{-1}[J_{3\alpha}(\lambda_{3\alpha,n}a)]^{-2} \tag{132}$$

$$Q_{12n} = \left[\frac{3}{2}\frac{\lambda^2}{G^2}\omega^4 - \frac{1}{2}\frac{\omega^8}{G^4} + \frac{1}{6}(3\lambda^2 - \lambda^2_{3\alpha,n})\left(\frac{3}{2}\lambda^2 - \frac{7}{2}\frac{\omega^4}{G^2}\right)\right] \tag{133}$$

$$Q_{13n} = \left[-\frac{7}{2}\frac{\lambda^2}{G^2}\omega^4 - \frac{3}{2}\frac{\omega^8}{G^4} + \frac{1}{6}(3\lambda^2 - \lambda^2_{3\alpha,n})\left(\frac{1}{2}\lambda^2 - \frac{13}{2}\frac{\omega^4}{G^2}\right)\right] \tag{134}$$

$$Q_{14n} = \frac{2}{a^2}\left[1 - \frac{\alpha^2}{\lambda^2_{\alpha,n}a^2}\right]^{-1}[J_\alpha(\lambda_{\alpha,n}a)]^{-2} \tag{135}$$

$$Q_{15n} = \frac{2}{a^2}\left[1 - \frac{9\alpha^2}{\lambda^2_{3\alpha,n}a^2}\right]^{-1}[J_{3\alpha}(\lambda_{3\alpha,n}a)]^{-2} \quad . \tag{136}$$

Once again it is assumed that the A^3_{kn} can be expanded in a Fourier series of the form

$$A^3_{k\alpha,n} = \frac{1}{2}c^o_{k\alpha,n} + \sum_{\ell=1}^{\infty}\left\{c^\ell_{k\alpha,n}\cos\ell\Omega + d^\ell_{k\alpha,n}\sin\ell\Omega\right\} \quad . \tag{137}$$

With the use of Equations A-9, A-10, A-11, A-12, and A-13, Equation 123 can be written as

$$\frac{1}{2}c^o_{\alpha,n} + \sum_{\ell=1}^{\infty}\left\{[\omega^2_{\alpha,n} - \ell^2\omega^2]\,c^\ell_{\alpha,n}\cos\ell\Omega + [\omega_{\alpha,n} - \ell^2\omega^2]\,d^\ell_{\alpha,n}\sin\ell\Omega\right\}$$

$$= 2\omega\,[B^2K\cos\Omega + D^2\sin\Omega]\;I\,[J_{\alpha,o}\,J_{\alpha,n}]$$

$$+ \sum_{j=0}^{\infty} \left\{ K \cos \Omega\, b_{o,j}^{2}\, Q_{4jn}\, I \left[J_{o,j}\, J_{\alpha,o}\, J_{\alpha,n} \right] \right\}$$

$$+ \sum_{j=0}^{\infty} \left\{ K \cos 3\Omega\, b_{o,j}^{2}\, Q_{5jn} \left[J_{o,j}\, J_{\alpha,o}\, J_{\alpha,n} \right] \right\}$$

$$+ \frac{1}{2} K \cos \Omega \sum_{j=0}^{\infty} \left\{ b_{2\alpha,j}^{2}\, Q_{6jn}\, I \left[J_{2\alpha,j}\, J_{\alpha,o}\, J_{\alpha,n} \right] \right\}$$

$$+ \frac{1}{2} K \cos 3\Omega \sum_{j=0}^{\infty} \left\{ b_{2\alpha,j}^{2}\, Q_{7jn}\, I \left[J_{2\alpha,j}\, J_{\alpha,o}\, J_{\alpha,n} \right] \right\}$$

$$+ \frac{3}{16} K^{3} \cos \Omega \Big\{ - 3 \lambda^{4} I \left[(J'_{\alpha,o})^{2} J_{\alpha,o}\, J_{\alpha,n} \right]$$

$$+ \alpha^{4} I \left[r^{-4} J_{\alpha,o}^{3}\, J_{\alpha,n} \right] + \alpha^{2} \lambda I \left[r^{-3} J_{\alpha,o}^{2}\, J'_{\alpha,o}\, J_{\alpha,n} \right]$$

$$- 2 \alpha^{2} \lambda^{2} I \left[r^{-2} J_{\alpha,o} (J_{\alpha,o})^{2} J_{\alpha,n} \right] + Q_{8n}\, I \left[J_{\alpha,o}^{3}\, J_{\alpha,n} \right] \Big\}$$

$$+ \frac{1}{16} K^{3} \cos 3\Omega \Big\{ - 3 \lambda^{4} I \left[(J'_{\alpha,o})^{2} J''_{\alpha,o}\, J_{\alpha,n} \right] + \alpha^{4} I \left[r^{-4} J_{\alpha,o}^{3}\, J_{\alpha,n} \right]$$

$$+ \alpha^{2} \lambda I \left[r^{-3} J_{\alpha,o}^{2}\, J'_{\alpha,o}\, J_{\alpha,n} \right] - 2 \alpha^{2} \lambda^{2} I \left[r^{-2} (J'_{\alpha,o})^{2} J_{\alpha,o}\, J_{\alpha,n} \right]$$

$$+ Q_{9n}\, I \left[J_{\alpha,n}^{3}\, J_{\alpha,n} \right] \Big\} \quad , \tag{138}$$

and Equation 124 may be written as

$$\frac{1}{2} c_{3\alpha,n}^{o} + \sum_{\ell=1}^{\infty} \left\{ \left[\omega_{3\alpha,n} - \ell^{2} \omega^{2} \right] c_{3\alpha,n} \cos \ell\Omega + \left[\omega_{3\alpha,n}^{2} - \ell^{2} \omega^{2} \right] d_{3\alpha,n}^{\ell} \sin \ell\Omega \right\}$$

$$= \frac{1}{2} K \cos \Omega \sum_{j=0}^{\infty} \left\{ b_{2\alpha,j}^{2}\, Q_{10jn}\, I \left[J_{2\alpha,j}\, J_{\alpha,o}\, J_{3\alpha,n} \right] \right.$$

$$+ \frac{1}{2} K \cos 3\Omega \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} Q_{11jn} I [J_{2\alpha,j} J_{\alpha,o} J_{3\alpha,n}] \right.$$

$$+ \frac{1}{16} K^3 \cos \Omega \left\{ - 3 \lambda^4 I [(J'_{\alpha,o})^2 J''_{\alpha,o} J_{3\alpha,n} - 3 \alpha^4 I [r^{-4} J^3_{\alpha,o} J_{3\alpha,n}] \right.$$

$$- 3 \alpha^2 \lambda I [r^{-3} J^2_{\alpha,o} J'_{\alpha,o} J_{3\alpha,n}] + 6 \alpha^2 \lambda^2 I [r^{-2} J_{\alpha,o} (J'_{\alpha,o})^2 J_{3\alpha,n}]$$

$$\left. + Q_{12n} I [J^3_{\alpha,o} J_{3\alpha,n}] \right\} + \frac{1}{16} K^3 \cos 3\Omega \left\{ - \lambda^4 I [(J'_{\alpha,o})^2 J''_{\alpha,o} J_{3\alpha,n}] \right.$$

$$- \alpha^4 I [r^{-4} J_{\alpha,o} J_{3\alpha,n}] - \alpha^2 \lambda I [r^{-3} J^2_{\alpha,o} J^1_{\alpha,o} J_{3\alpha,n}]$$

$$+ 2 \alpha^2 \lambda^2 I [r^{-2} J_{\alpha,o} (J'_{\alpha,o})^2 J_{3\alpha,n}] + Q_{13n} I [J^3_{\alpha,o} J_{3\alpha,n}] \quad . \qquad (139)$$

Now equating coefficient of cos $\ell\Omega$ and sin $\ell\Omega$ noting that

$c^1_{\alpha,n} = d^1_{\alpha,n} = c^1_{3\alpha,n} = d^1_{3\alpha,n} = 0$, the following expressions result:

$$2 \omega B^2 K = - \sum_{j=0}^{\infty} \left\{ K b^2_{o,j} Q_{4jn} I [J_{o,j} J_{\alpha,o} J_{\alpha,n}] \right\}$$

$$- \frac{1}{2} K \sum_{j=0}^{\infty} \left\{ b^2_{2\alpha,j} Q_{6jn} I [J_{2\alpha,j} J_{\alpha,o} J_{\alpha,n}] \right\}$$

$$- \frac{3}{16} K^3 Q_{14n} \left\{ - 3 \lambda^4 I [(J_{\alpha,o})^2 J''_{\alpha,o} J_{\alpha,n}] \right.$$

$$+ \alpha^4 I [r^{-4} J^3_{\alpha,o} J_{\alpha,n}] + \alpha^2 \lambda I [r^{-3} J^2_{\alpha,o} J'_{\alpha,o} J_{\alpha,n}]$$

$$\left. - 2 \alpha^2 \lambda^2 I [r^{-2} J_{\alpha,o} (J'_{\alpha,o})^2 J_{\alpha,n}] + Q_{8n} I [J^3_{\alpha,o} J_{\alpha,n}] \right\} \qquad (140)$$

$$c^3_{\alpha,n} = [\omega^2_{\alpha,n} - 9\,\omega^2]^{-1} \Bigg(K \sum_{j=0}^{\infty} \Big\{ b^2_{o,j}\, Q_{5jn}\, I\,[J_{o,j}\, J_{\alpha,o}\, J_{\alpha,n}] \Big\}$$

$$+ \frac{1}{2} K \sum_{j=0}^{\infty} \Big\{ b^2_{2\alpha,j}\, Q_{7jn}\, I\,[J_{2\alpha,j}\, J_{\alpha,o}\, J_{\alpha,n}] \Big\}$$

$$+ \frac{1}{16} K^3 Q_{14n} \Big\{ - 3\,\lambda^4\, I\,[(J'_{\alpha,o})^2\, J''_{\alpha,o}\, J_{\alpha,n}] + \alpha^4\, I\,[r^{-4}\, J^3_{\alpha,o}\, J_{\alpha,n}]$$

$$+ \alpha^2\,\lambda\, I\,[r^{-3}\, J^2_{\alpha,o}\, J'_{\alpha,o}\, J_{\alpha,n}] - 2\,\alpha^2\,\lambda^2\, I\,[r^{-2}\, (J'_{\alpha,o})^2\, J_{\alpha,o}\, J_{\alpha,n}]$$

$$+ Q_{9n}\, I\,[J^3_{\alpha,o}\, J_{\alpha,n}] \Big\} \Bigg) \qquad (141)$$

$$c^3_{3\alpha,n} = [\omega^2_{3\alpha,n} - 9\,\omega^2]^{-1} \Bigg(\frac{1}{2} K \sum_{j=0}^{\infty} \Big\{ b^2_{2\alpha,j}\, Q_{11jn}\, I\,[J_{2\alpha,j}\, J_{\alpha,o}\, J_{3\alpha,n}] \Big\}$$

$$+ \frac{1}{16} K^3 Q_{15n} \Big\{ - \lambda^4\, I\,[(J'_{\alpha,o})^2\, J''_{\alpha,o}\, J_{3\alpha,n}]$$

$$- \alpha^4\, I\,[r^{-4}\, J^3_{\alpha,o}\, J_{3\alpha,n}] - \alpha^2\,\lambda\, I\,[r^{-3}\, J^2_{\alpha,o}\, J'_{\alpha,o}\, J_{3\alpha,n}]$$

$$+ 2\,\alpha^2\,\lambda^2\, I\,[r^{-2}\, J_{\alpha,o}\, (J'_{\alpha,o})^2\, J_{3\alpha,n}] + Q_{13n}\, I\,[J^3_{\alpha,o}\, J_{3\alpha,n}] \Big\} \Bigg) \qquad (142)$$

$$D^2 = 0 \qquad (143)$$

$$c^{\ell}_{\alpha,n} = 0 \quad \ell \neq 3 \qquad (144)$$

$$d^{\ell}_{\alpha,n} = 0 \quad \ell = 0, 1, 2, \ldots \quad . \qquad (145)$$

The third order approximation for the velocity potential is now given by the expression

$$\Psi^3 = \sum_{n=0}^{\infty} \left\{ c^3_{\alpha,n} \cos 3\Omega \cos(\alpha\theta)\, J_{\alpha,n} \frac{\cosh[\lambda_{\alpha,n}(z+h)]}{\cosh(\lambda_{\alpha,n} h)} \right\}$$

$$+ \sum_{n=0}^{\infty} \left\{ c^3_{3\alpha,n} \cos 3\Omega \cos(3\alpha\theta)\, J_{3\alpha,n} \frac{\cosh[\lambda_{3\alpha,n}(z+h)]}{\cosh(\lambda_{3\alpha,n} h)} \right\} \quad (146)$$

where $c^3_{\alpha n}$ and $c^3_{3\alpha,n}$ are given by Equations 141 and 142.

The frequency correction term B^2 and the wave height ζ^3 are found from Equations 140 and 91 to be

$$\zeta^3 = K\, G^{-1} B^2 \sin\Omega \cos(\alpha\theta)\, J_{\alpha,o} + \omega \sum_{n=0}^{\infty} \left\{ c^3_{\alpha,n} \cos 3\Omega \cos(\alpha\theta)\, J_{\alpha,n} \right\}$$

$$+ \omega \sum_{n=0}^{\infty} \left\{ c^3_{3\alpha,n} \cos 3\Omega \cos(3\alpha\theta)\, J_{3\alpha,n} \right\}$$

$$- G^{-1} \lambda K \cos\Omega \cos(\alpha\theta)\, J_{\alpha,o} \sum_{j=0}^{\infty} \Big\{ b^2_{o,j} \lambda_{o,j} \sin 2\Omega\, J_{o,j}$$

$$+ b^2_{2\alpha,j} \lambda_{2\alpha,j} \sin 2\Omega\, J_{2\alpha,j} \cos(2\alpha\theta) \Big\}$$

$$- 2\, G^{-1} r^{-2} K \alpha \cos\Omega \sin(\alpha\theta)\, J_{\alpha,o} \sum_{j=0}^{\infty} \left\{ \alpha\, b^2_{2\alpha,j} \sin 2\Omega\, J_{2\alpha,j} \sin(2\alpha\theta) \right\}$$

$$- G^{-3} \omega^2 \cos\Omega \cos(\alpha\theta)\, J_{\alpha,o} \sum_{j=0}^{\infty} \Big\{ \omega^2_{o,j} b^2_{o,j} \sin 2\Omega\, J_{o,j}$$

$$+ \omega^2_{2\alpha,j} b^2_{2\alpha,j} \sin 2\Omega\, J_{2\alpha,j} \cos(2\alpha\theta) \Big\} \quad (147)$$

$$2\,\omega\,B^2 = -\sum_{j=0}^{\infty}\left\{b_{o,j}^2\,Q_{4jn}\,I\,[J_{o,j}\,J_{\alpha,o}\,J_{\alpha,n}]\right\}$$

$$-\frac{1}{2}\sum_{j=0}^{\infty}\left\{b_{2\alpha,j}^2\,Q_{6jn}\,I\,[J_{2\alpha,j}\,J_{\alpha,o}\,J_{\alpha,n}]\right.$$

$$+\frac{3}{16}\,K^2\,Q_{14n}\left\{3\,\lambda^4\,I\,[(J'_{\alpha,o})^2\,J''_{\alpha,o}\,J_{\alpha,n}]\right.$$

$$-\alpha^4\,I\,[r^{-4}\,J_{\alpha,o}^3\,J_{\alpha,n}] - \alpha^2\,\lambda\,I\,[r^{-3}\,J_{\alpha,o}^2\,J'_{\alpha,o}\,J_{\alpha,n}]$$

$$+\,2\,\alpha^2\,\lambda^2\,I\,[r^{-2}\,J_{\alpha,o}\,(J'_{\alpha,o})^2\,J_{\alpha,n}]$$

$$\left.-\,Q_{8n}\,I\,[J_{\alpha,o}^3\,J_{\alpha,n}]\right\} \quad . \tag{148}$$

Physical Significance of ϵ

In order to make the solution meaningful a physical significance must be found for the expansion parameter ϵ. To the first approximation the wave height ζ is given by

$$\zeta = \epsilon\,\zeta^1 = \epsilon\,K\,G^{-1}\,\sin\Omega\,\cos(\alpha\theta)\,J_\alpha(\lambda r) \quad . \tag{149}$$

Let $\zeta^1 = 1$ at $\Omega = \pi/2$, $\theta = 0$, and $r = a$, then $1 = K\,G^{-1}\,J_\alpha(\lambda a)$ or $K = G\,J_\alpha^{-1}(\lambda a)$, and thus

$$\zeta^1 = J_\alpha^{-1}(\lambda a)\,\cos(\alpha\theta)\,\sin\Omega\,J_\alpha(\lambda r) \quad .$$

Now from Equation 149

$$\zeta = \epsilon \, \zeta^1 = \epsilon \, J_\alpha^{-1} (\lambda a) \cos (\alpha\theta) \sin \Omega \, J_\alpha (\lambda r) \quad . \tag{150}$$

At $\Omega = \pi/2$, $\theta = 0$, and $r = a$, Equation 150 gives $\zeta = \epsilon$; thus, the parameter ϵ represents the wave height of the linear theory evaluated at $\Omega = \pi/2$, $\theta = 0$, and $r = a$. This is the same physical significance found by Mack and by DiMaggio and Rehm for circular cylindrical containers.

It is now convenient to nondimensionalize the solution by introducing the variables

$$\overline{r} = \frac{r}{a}, \quad \overline{z} = \frac{z}{a}, \quad \Omega = \Omega, \quad \gamma_{m,n} = \frac{\lambda_{m,n}}{a}$$

$$\overline{\zeta} = \frac{\zeta}{a}, \quad \overline{\Psi} = \frac{\Psi}{(\dot{\Omega}) a^2}, \quad \overline{G} = \frac{G}{(\dot{\Omega})^2 a}, \quad \overline{h} = \frac{h}{a}$$

$$\overline{K} = \frac{K}{(\dot{\Omega}) a^2}, \quad \overline{\epsilon} = \frac{\epsilon}{a} \quad .$$

The notation given by Equation 72 now becomes

$$\left.\begin{aligned} \omega &= \omega_{\alpha,o} \\ J_{k\alpha,m} &= J_{k\alpha} (\gamma_{k\alpha} \overline{r}) \\ \gamma &= \gamma_{\alpha,o} \\ \overline{I}(F) &= \int_0^1 \overline{r} \, F(\overline{r}) \, d\overline{r} \end{aligned}\right\} \tag{151}$$

Summary of Solution

In terms of the variables described above the various solutions may be summarized as follows. Note that the term $(\overline{G})^{-1}$ represents a dimensionless frequency parameter.

First Approximation

$$\overline{\Psi}^1 = \overline{K} \cos \Omega \cos (\alpha\theta) J_{\alpha,o} \frac{\cosh[\gamma(\overline{z}+\overline{h})]}{\cosh(\gamma \overline{h})} \tag{152}$$

$$\overline{\zeta}^1 = \overline{K} (\overline{G})^{-1} \sin \Omega \cos (\alpha\theta) J_{\alpha,o} \tag{153}$$

$$(\overline{G})^{-1} = \gamma \tanh(\gamma\overline{h}) \tag{154}$$

Second Approximation

$$\overline{\Psi}^2 = \sum_{j=0}^{\infty} \left\{ \overline{b}^2_{o,j} \sin 2\Omega \, J_{o,j} \frac{\cosh[\gamma(\overline{z}+\overline{h})]}{\cosh(\gamma\overline{h})} \right\}$$

$$+ \sum_{j=0}^{\infty} \left\{ \overline{b}^2_{2\alpha,j} \sin 2\Omega \, J_{2\alpha,j} \cos(2\alpha\theta) \frac{\cosh[\gamma_{2\alpha,j}(\overline{z}+\overline{h})]}{\cosh(\gamma_{2\alpha,j}\overline{h})} \right\} \tag{155}$$

$$\overline{\zeta}^2 = -\frac{1}{2} (\overline{K})^2 (\overline{G})^{-1} \cos^2 \Omega \, [\gamma^2 \cos^2(\alpha\theta) (J'_{\alpha,o})^2$$

$$+ (\overline{r})^{-2} \alpha^2 \sin^2(\alpha\theta) J^2_{\alpha,o} + (\overline{G})^{-2} \cos^2(\alpha\theta) J^2_{\alpha,o}]$$

$$- 2 (\overline{G})^{-1} \sum_{j=0}^{\infty} \left\{ \overline{b}^2_{o,j} \cos 2\Omega \, J_{o,j} + \overline{b}^2_{2\alpha,j} \cos 2\Omega \, J_{2\alpha,j} \cos 2\alpha\theta \right\} \tag{156}$$

$$(\overline{G})^{-1} = \gamma \tanh (\gamma \overline{h}) \quad , \tag{157}$$

where

$$\overline{b}^{\,2}_{o,j} = \frac{1}{2}(\overline{K})^2 \left[\frac{\omega^2_{o,j}}{\omega^2} - 4\right]^{-1} [J_o(\gamma_{o,j})]^{-2} [\gamma^2 - \gamma^2_{o,j} + 3(\overline{G})^{-2}] I [J^2_{\alpha,o} J_{o,j}] \tag{158}$$

$$\overline{b}^{\,2}_{2\alpha,j} = \frac{1}{2}(\overline{K})^2 \left[\frac{\omega^2_{2\alpha,j}}{\omega^2} - 4\right]^{-1} \left[1 - \frac{4\alpha^2}{\gamma^2_{2\alpha,j}}\right]^{-1} [J_{2\alpha}(\gamma_{2\alpha,j})]^{-2}$$

$$\times [3(\overline{G})^{-2} + \gamma^2 - \gamma^2_{2\alpha,j}] \overline{I} [J^2_{\alpha,o} J_{2\alpha,j}] \tag{159}$$

Third Approximation

$$\Psi^3 = \sum_{n=0}^{\infty} \left\{\overline{c}^{\,3}_{\alpha,n} \cos 3\Omega \cos(\alpha\theta) J_{\alpha,n} \frac{\cosh[\gamma_{\alpha,n}(\overline{z}+\overline{h})]}{\cosh(\gamma_{\alpha,n}\overline{h})}\right\}$$

$$+ \sum_{n=0}^{\infty} \left\{\overline{c}^{\,3}_{3\alpha,n} \cos 3\Omega \cos(3\alpha\theta) J_{3\alpha,n} \frac{\cosh[\gamma_{3\alpha,n}(\overline{z}+\overline{h})]}{\cosh(\gamma_{3\alpha,n}\overline{h})}\right\} \tag{160}$$

$$\overline{\zeta}^3 = \overline{K}\,(\overline{G})^{-1} B^2(\omega)^{-1} \sin\Omega \cos(\alpha\theta) J_{\alpha,o}$$

$$+ 3(\overline{G})^{-1} \sum_{m=0}^{\infty} \left\{\overline{c}^{\,3}_{\alpha,m} \cos 3\Omega \cos(\alpha\theta) J_{\alpha,m}\right\}$$

$$+ 3(\overline{G})^{-1} \sum_{m=0}^{\infty} \left\{\overline{c}^{\,3}_{3\alpha,m} \cos 3\Omega \cos(3\alpha\theta) J_{3\alpha,m}\right\}$$

$$- (\overline{G})^{-1} \gamma \overline{K} \cos\Omega \cos(\alpha\theta) J_{\alpha,o} \sum_{j=0}^{\infty} \left\{\overline{b}^{\,2}_{o,j} \gamma_{o,j} \sin 2\Omega J_{o,j}\right.$$

$$+ \overline{b}^{2}_{2\alpha,j}\, \gamma_{2\alpha,j} \sin 2\Omega\, J_{2\alpha,j} \cos(2\alpha\theta) \Big\}$$

$$- (\overline{G})^{-1} (\overline{r})^{-2}\, \overline{K}\, \alpha \cos\Omega \sin\alpha\theta\, J_{\alpha,o} \sum_{j=0}^{\infty} \left\{ \alpha\, \overline{b}^{2}_{2\alpha,j} \sin 2\Omega\, J_{2\alpha,j} \sin(2\alpha\theta) \right\}$$

$$- \overline{K}\, (\overline{G})^{-3} \cos\Omega \cos(\alpha\theta)\, J_{\alpha,o} \sum_{j=0}^{\infty} \Big\{ \omega^{2}_{o,j}\, (\omega)^{-2}\, \overline{b}^{2}_{o,j} \sin 2\Omega\, J_{o,j}$$

$$+ (\omega_{2\alpha,j})^{2}\, (\omega)^{-2}\, \overline{b}^{2}_{2\alpha,j} \sin 2\Omega\, J_{2\alpha,j} \cos(2\alpha\theta) \Big\} \qquad (161)$$

$$(\overline{G})^{-1} = \gamma \tanh(\gamma h)\, [\,1 + \epsilon^{2}\, \overline{G}_{c}] \qquad (162)$$

where

$$\overline{G}_{c} = - \sum_{j=0}^{\infty} \left\{ \overline{b}^{2}_{o,j}\, \overline{Q}_{4jn}\, \overline{I}\, [J_{o,j}\, J_{\alpha,o}\, J_{\alpha,n}] \right\}$$

$$- \frac{1}{2} \sum_{j=0}^{\infty} \left\{ \overline{b}^{2}_{2\alpha,j}\, \overline{Q}_{6jn}\, \overline{I}\, [J_{2\alpha,j}\, J_{\alpha,o}\, J_{\alpha,n}] \right\}$$

$$+ \frac{3}{16} (\overline{K})^{2}\, \overline{Q}_{14n} \Big\{ 3\, \gamma^{4}\, \overline{I}\, [(J'_{\alpha,o})^{2} J''_{\alpha,o}\, J_{\alpha,n}]$$

$$- \alpha^{4}\, \overline{I}\, [(\overline{r})^{-4}\, J^{3}_{\alpha,o}\, J_{\alpha,n}] - \alpha^{2} \gamma\, \overline{I}\, [(\overline{r})^{-3}\, J^{2}_{\alpha,o}\, J'_{\alpha,o}\, J_{\alpha,n}]$$

$$+ 2\, \alpha^{2} \gamma^{2}\, \overline{I}\, [(\overline{r})^{-2}\, J_{\alpha,o}\, (J'_{\alpha,o})^{2}\, J_{\alpha,n}] - \overline{Q}_{8n}\, \overline{I}\, [J^{3}_{\alpha,o}\, J_{\alpha,n}] \Big\} \qquad (163)$$

$$\bar{c}^{3}_{\alpha,n} = \left[\frac{\omega^2_{\alpha,n}}{\omega^2} - 9\right]^{-1} \Bigg(\bar{K} \sum_{j=0}^{\infty} \left\{\bar{b}^2_{o,j}\ \bar{Q}_{5jn}\ \bar{I}\ [J_{o,j}\ J_{\alpha,o}\ J_{\alpha,n}]\right\}$$

$$+ \frac{1}{2}\ \bar{K} \sum_{j=0}^{\infty} \left\{\bar{b}^2_{2\alpha,j}\ \bar{Q}_{7jn}\ \bar{I}\ [J_{2\alpha,j}\ J_{\alpha,o}\ J_{\alpha,n}]\right\}$$

$$+ \frac{1}{16}\ \bar{K}^3\ \bar{Q}_{14n} \Big\{-3\ \gamma^4\ \bar{I}\ [(J'_{\alpha,o})^2\ J''_{\alpha,o}\ J_{\alpha,n}]$$

$$+ \alpha^4\ \bar{I}\ [(\bar{r})^{-4}\ J^3_{\alpha,o}\ J_{\alpha,n}] + \alpha^2\ \gamma\ \bar{I}\ [(\bar{r})^{-3}\ J^2_{\alpha,o}\ J'_{\alpha,o}\ J_{\alpha,n}]$$

$$- 2\ \alpha^2\ \gamma^2\ I\ [(\bar{r})^{-2}\ (J'_{\alpha,o})^2\ J_{\alpha,o}\ J_{\alpha,n}] + \bar{Q}_{9n}\ \bar{I}\ [\ (J_{\alpha,o})^3\ J_{\alpha,n}]\Big\}\Bigg) \qquad (164)$$

$$\bar{c}^{3}_{3\alpha,n} = \left[\frac{\omega^2_{3\alpha,n}}{\omega^2} - 9\right]^{-1} \Bigg(\frac{1}{2}\ \bar{K} \sum_{j=0}^{\infty} \left\{\bar{b}^2_{2\alpha,j}\ \bar{Q}_{11jn}\ \bar{I}\ [J_{2\alpha,j}\ J_{\alpha,o}\ J_{3\alpha,n}]\right\}$$

$$+ \frac{1}{16}\ (\bar{K})^3\ \bar{Q}_{15n} \Big\{-\ \gamma^4\ \bar{I}\ [(J'_{\alpha,o})^2\ J''_{\alpha,o}\ J_{3\alpha,n}]$$

$$- \alpha^4\ \bar{I}\ [(\bar{r})^{-4}\ J^3_{\alpha,o}\ J_{3\alpha,n}] - \alpha^2\ \gamma\ \bar{I}\ [(\bar{r})^{-3}\ J^2_{\alpha,o}\ J'_{\alpha,o}\ J_{3\alpha,n}]$$

$$+ 2\ \alpha^2\ \gamma^2\ \bar{I}\ [(\bar{r})^{-2}\ J_{\alpha,o}\ (J'_{\alpha,o})^2\ J_{3\alpha,n}]$$

$$+ \bar{Q}_{13n}\ \bar{I}\ [J^3_{\alpha,o}\ J_{3\alpha,n}]\Big\}\Bigg) \qquad (165)$$

$$\bar{Q}_{4jn} = 2\left[(\bar{G})^{-2}\ \frac{\omega^2_{o,j}}{\omega^2} - (\bar{G})^{-2} - \gamma^2_{o,j} + \frac{1}{2}\ \gamma^2 + \frac{1}{2}\ \gamma^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\gamma^2_{\alpha,n}}\right]^{-1}[J_\alpha(\gamma_{\alpha,n})]^{-2} \qquad (166)$$

$$\bar{Q}_{5jn} = 2\left[-5\ (\bar{G})^{-2}\ \frac{\omega^2_{o,j}}{\omega^2} - (\bar{G})^{-2} - \gamma^2_{o,j} - \frac{1}{2}\ \gamma^2 - \frac{3}{2}\ \gamma^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\gamma^2_{\alpha,n}}\right]^{-1}[J_\alpha(\gamma_{\alpha,n})]^{-2} \qquad (167)$$

$$\bar{Q}_{6jn} = 2\left[(G)^{-2}\frac{\omega^2_{2\alpha,j}}{\omega^2} - (\bar{G})^{-2} - \frac{3}{2}\gamma^2_{2\alpha,j} - \gamma^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\gamma^2_{\alpha,n}}\right]^{-1}[J_\alpha(\gamma_{\alpha,n})]^{-2} \tag{168}$$

$$\bar{Q}_{7jn} = 2\left[-5(\bar{G})^{-2}\frac{\omega^2_{2\alpha,j}}{\omega^2} - (\bar{G})^2 + \frac{5}{2}\gamma^2_{2\alpha,j} - \gamma^2_{\alpha,n}\right]\left[1 - \frac{\alpha^2}{\gamma^2_{\alpha,n}}\right]^{-1}[J_\alpha(\gamma_{\alpha,n})]^{-2} \tag{169}$$

$$\bar{Q}_{8n} = \frac{3}{2}\gamma^2(\bar{G})^{-2} - \frac{1}{2}(\bar{G})^{-4} + \frac{1}{6}(3\gamma^2 - \gamma^2_{\alpha,n})\left[\frac{3}{2}\gamma^2 - \frac{7}{2}(G)^{-2}\right] \tag{170}$$

$$\bar{Q}_{9n} = -\frac{21}{2}\gamma^2(\bar{G})^{-2} - \frac{9}{2}(\bar{G})^{-4} + \frac{1}{2}(3\gamma^2 - \gamma^2_{\alpha,n})\left[\frac{1}{2}\gamma^2 - \frac{13}{2}(\bar{G})^{-2}\right] \tag{171}$$

$$\bar{Q}_{10jn} = 2\left[(\bar{G})^{-2}\frac{\omega^2_{2\alpha,j}}{\omega^2} - (\bar{G})^{-2} - \frac{3}{2}\gamma^2_{2\alpha,j} + \gamma^2_{3\alpha,n}\right]\left[1 - \frac{9\alpha^2}{\gamma^2_{3\alpha,n}}\right]^{-1}[J_{3\alpha}(\gamma_{3\alpha,n})]^{-2} \tag{172}$$

$$\bar{Q}_{11jn} = 2\left[-5(\bar{G})^{-2}\frac{\omega^2_{2\alpha,j}}{\omega^2} - (\bar{G})^{-2} - \frac{5}{2}\gamma^2_{2\alpha,j} - 2\gamma^2 - 3\gamma^2_{3\alpha,n}\right]\left[1 - \frac{9\alpha^2}{\gamma^2_{3\alpha,n}}\right]^{-1}[J_{3\alpha}(\gamma_{3\alpha,n})]^{-2} \tag{173}$$

$$\bar{Q}_{12n} = \frac{3}{2}\gamma^2(\bar{G})^{-2} - \frac{1}{2}(\bar{G})^{-4} + \frac{1}{6}(3\gamma^2 - \gamma^2_{3\alpha,n})\left[\frac{3}{2}\gamma^2 - \frac{7}{2}(\bar{G})^{-2}\right] \tag{174}$$

$$\bar{Q}_{13n} = \left[-\frac{7}{2}\gamma^2 (\bar{G})^{-2} - \frac{3}{2}(\bar{G})^{-4} + \frac{1}{6}(3\gamma^2 - \gamma^2_{3\alpha,n}) \right] \left[\frac{1}{2}\gamma^2 - \frac{13}{2}(\bar{G})^{-2} \right] \tag{175}$$

$$\bar{Q}_{14n} = 2\left[1 - \frac{\alpha^2}{\gamma^2_{\alpha,n}} \right]^{-1} [J_\alpha(\gamma_{\alpha,n})]^{-2} \tag{176}$$

$$\bar{Q}_{15n} = 2\left[1 - \frac{9\alpha^2}{\gamma^2_{3\alpha,n}} \right]^{-1} [J_{3\alpha}(\gamma_{3\alpha,n})]^{-2} \quad . \tag{177}$$

CHAPTER V

NUMERICAL RESULTS FOR 90° SECTOR TANK

Numerical results were obtained for $\alpha = 2$ which corresponds to the case of a 90° sector tank. The various definite integrals of products of Bessel functions were numerically evaluated using Simpson's rule. The values of these integrals correct to three significant figures are given in Appendix B.

Critical Depths

As previously mentioned the expressions for the second approximation contain the factors $\omega_{0j}^2/\omega^2 - 4$ and $\omega_{2\alpha,j}^2/\omega^2 - 4$ in the denominator. Similarly the expressions for the third approximation contain the factors $\omega_{2\alpha,j}^2/\omega^2 - 9$ and $\omega_{3\alpha,j}^2/\omega^2 - 9$ in the denominator. If for some particular values of j and $\bar{h}$ these factors become zero then the solution is invalid at that point. This type of behavior has also been noted for other container shapes.[2,5] It was found that certain values of j and $\bar{h}$ do cause these factors to become zero. These values have been referred to as critical depths. The critical depths for the 90° sector tank are given in Table 1.

TABLE 1

CRITICAL VALUES OF $\overline{h}$

j	$\frac{\omega_{0j}^2}{\omega^2} - 4 = 0$	$\frac{\omega_{4j}^2}{\omega^2} - 4 = 0$	$\frac{\omega_{2j}^2}{\omega^2} - 9 = 0$	$\frac{\omega_{6j}}{\omega^2} - 9 = 0$
1	---	0.3236	---	0.1421
2	0.1695	---	0.0808	0.2039
3	0.3910	---	0.1657	0.2691
4	---	---	0.2233	0.3282
5	---	---	0.2908	0.5126

It has been pointed out by Mack[2] that a physical meaning may be attached to these critical depths. It has been assumed that there is a first mode of order ϵ oscillating at frequency ω and that all other modes and harmonics are of order ϵ^2 or higher. However, when the depth equals one of the critical depths this assumption is not valid. At these depths the particular mode designated by j will also be of order ϵ. It is interesting to note that the experimental work of Fultz and Murty[7] show no unusual behavior at the critical depth in the case of a circular cylinder.

Frequency of Oscillation

The frequency equation of the first and second approximations was found to be

$$(\overline{G})^{-1} = \gamma \tanh (\gamma \overline{h})$$

which agrees with that found by Bauer[8] for the linear case. The frequency then is a function only of the depth through the second approximation. In the third approximation, however, the frequency is given by the expression

$$(\bar{G})^{-1} = \gamma \tanh(\gamma \bar{h})(1 + \epsilon^2 G_C)$$

where the frequency correction factor G_C is given by Equation 163 A plot of G_C against depth is given in Figure 3. It should be noticed that G_C changes sign, being positive for depths below 0.29 and negative for larger depths. This correction factor for the natural frequency is probably the most noteworthy result of this analysis. One of the prime reasons for compartmented containers is to raise the natural frequencies of the liquid. The fact that the natural frequency decreases with amplitude for dimensionless depths above 0.29 is extremely important since it tends to offset the reason for dividing the tanks into compartments.

The frequency correction factor remains almost constant for depths greater than $\bar{h} = 1.0$, but at the lower depths it becomes very large. No explanation of this behavior can be offered. Figure 4 compares the frequency correction factor for 90° cylindrical sector tanks with that found by Mack for the axisymmetric case of a cylindrical tank and with that found by DiMaggio and Rehm for the non-axisymmetric case of a cylindrical tank. It is interesting to note that for small depths DiMaggio and Rehm's solution also gives very large values while Mack's does not.

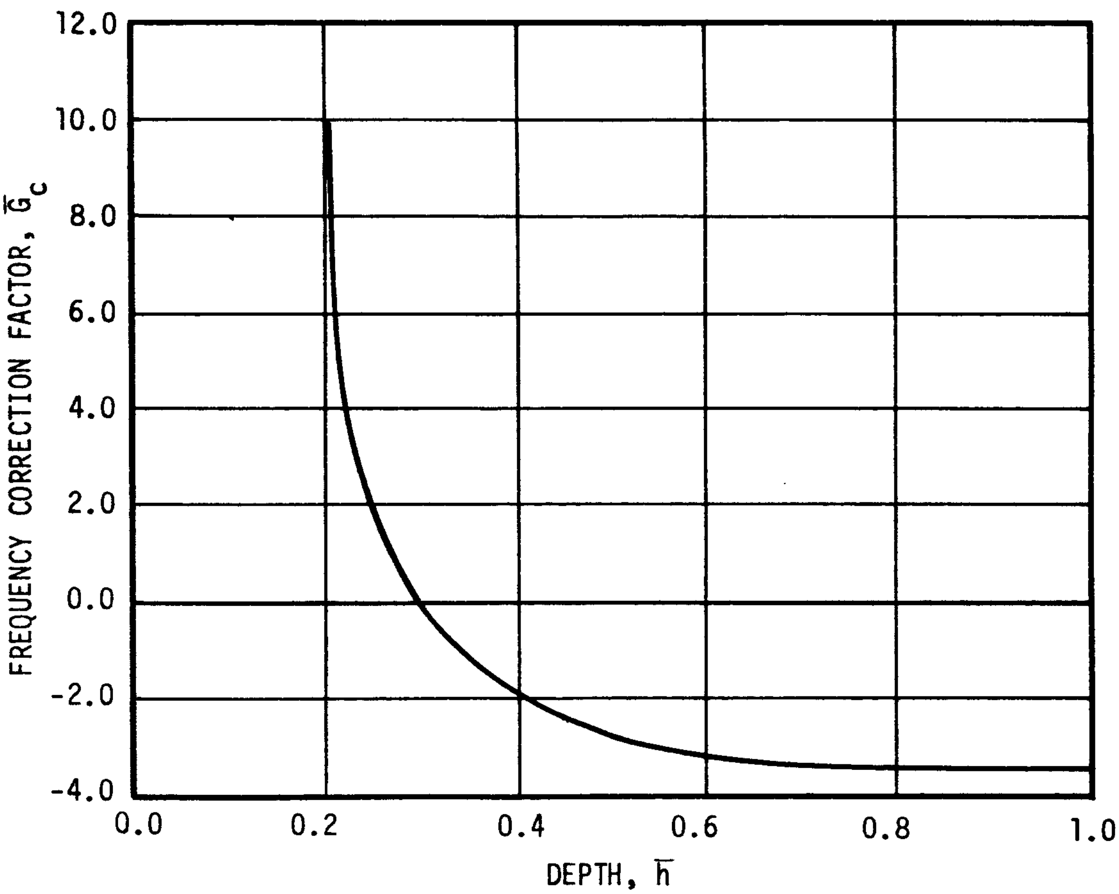

Figure 3. Frequency Correction Factor as a Function of Depth

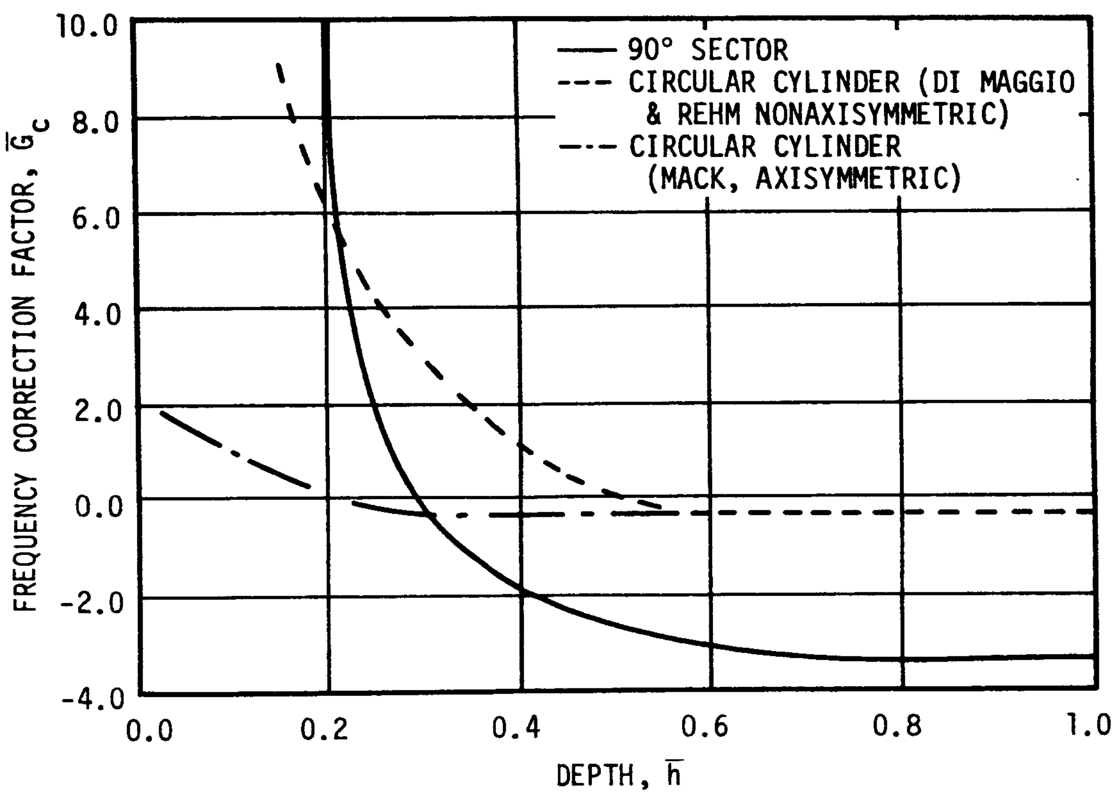

Figure 4. Comparison of Frequency Correction Factors For 90° Sector and Circular Cylinder

The experimental work of Abramson, Chu and Kana,[9] while not directly comparable to this analysis, has nevertheless shown that for large depths the frequency of 90° sector tanks does decrease with amplitude and tends to be more strongly nonlinear than circular cylinders. This result is also indicated by Figure 4.

A comparison of the dimensionless frequencies versus depth for the linear and nonlinear theories is shown in Figure 5. Figure 5 also shows the frequency of the first non-axisymmetric mode for a circular cylinder. The large values given by the nonlinear theory at small depths make the usefulness of the solution at these depths doubtful. From a practical point of view however, the greatest interest is in relatively large depths since this is where the greatest forces are generated. For depths above $\bar{h} = 0.4$, the nonlinear theory gives a frequency approximately 15 percent lower than that predicted by the linear theory. The experimental work presented in Reference 9 showed that for large amplitudes the frequency reduction varied from 10 to 19 percent depending on the amplitude. However, since wave amplitudes were not measured, an exact comparison is not possible.

Wave Profile

The linear wave height $\bar{\zeta}$ is given by

$$\bar{\zeta} = \epsilon \bar{\zeta}^1 = \epsilon \bar{K} (\bar{G})^{-1} \sin\Omega \cos(\alpha\theta) J_{\alpha,o} \quad .$$

By choosing $\bar{\zeta}^1 = 1$ at $\Omega = \pi/2$, $\theta = 0$ and $\bar{r} = 1$, $\bar{K}$ becomes

$$\bar{K} = \bar{G} [J_\alpha(\gamma_{\alpha,o})]^{-1}$$

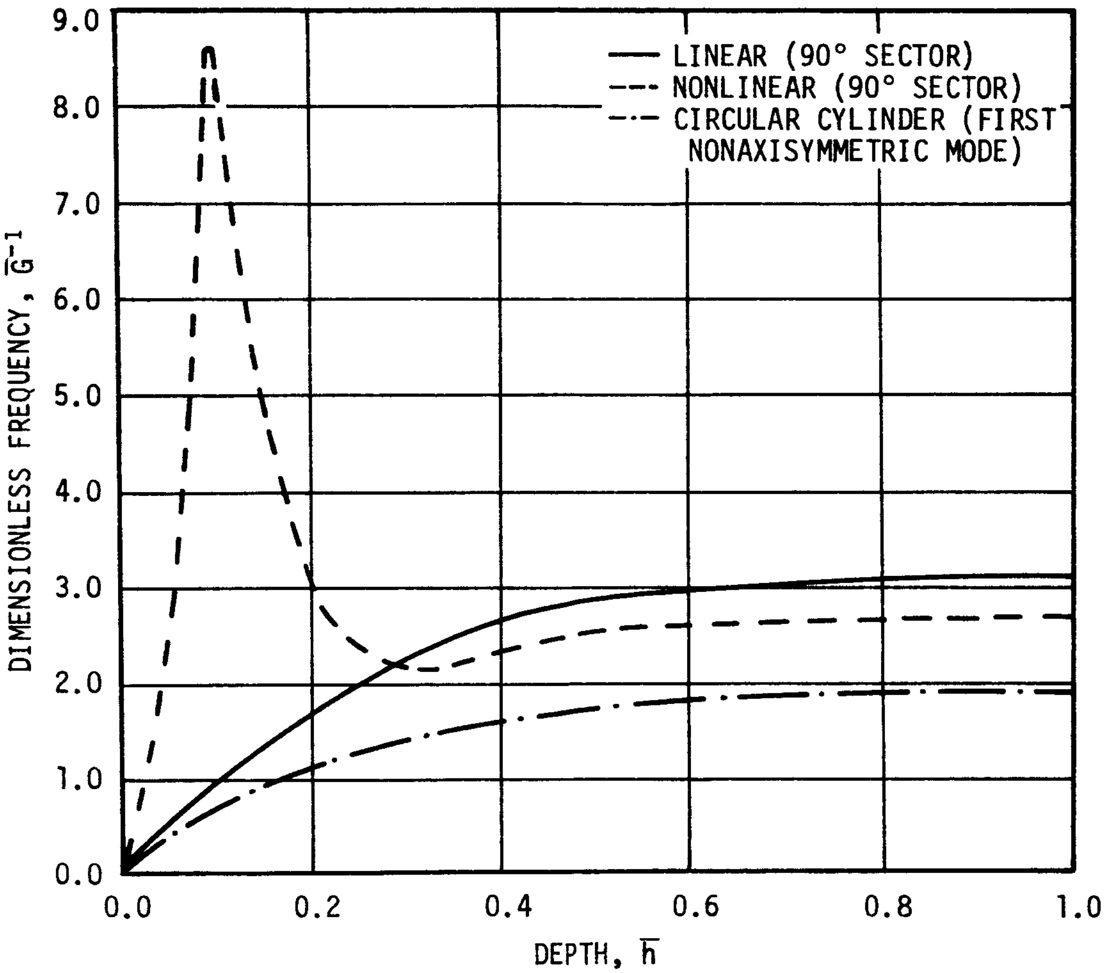

Figure 5. Comparison of Linear and Nonlinear Frequencies As a Function of Depth

and thus the parameter ϵ represents the wave height of the linear theory evaluated at $\Omega = \pi/2$, $\theta = 0$ and $F = 1$. Figures 6, 7, 8, 9, 10, and 11 give some wave profiles comparing the linear theory and the third approximation for several values of $\bar{h}$ and $\bar{\epsilon}$. As might be expected the linear and nonlinear theory tend to become equal as $\bar{\epsilon}$ decreases but significant differences may be noted for large values of $\bar{\epsilon}$.

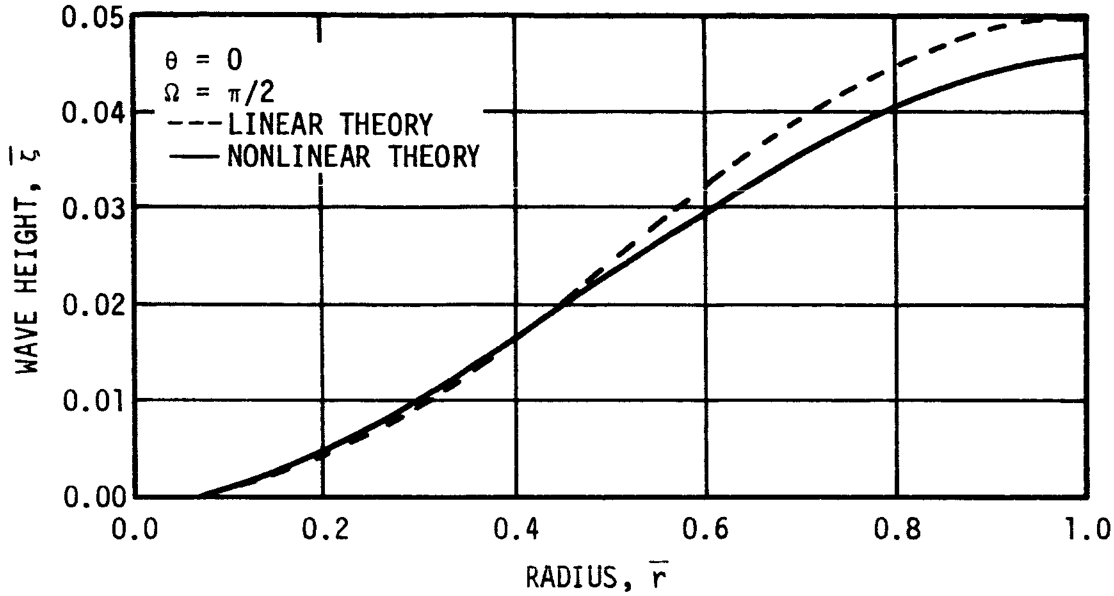

Figure 6. Wave Profile, $\bar{\varepsilon}$ = 0.05, $\bar{h}$ = 0.6

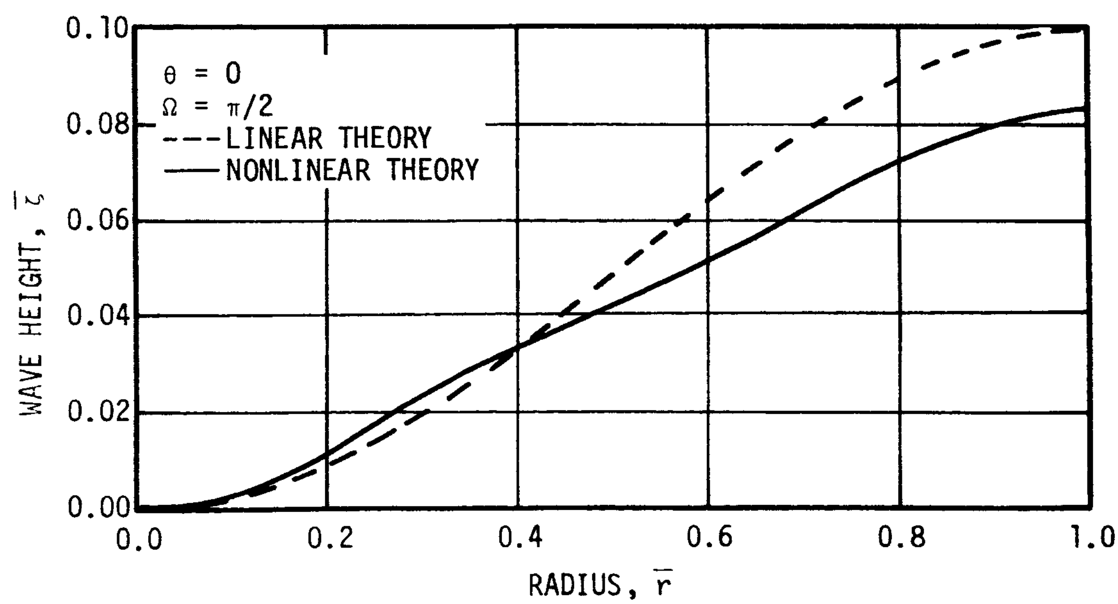

Figure 7. Wave Profile, $\bar{\varepsilon}$ = 0.1, $\bar{h}$ = 0.6

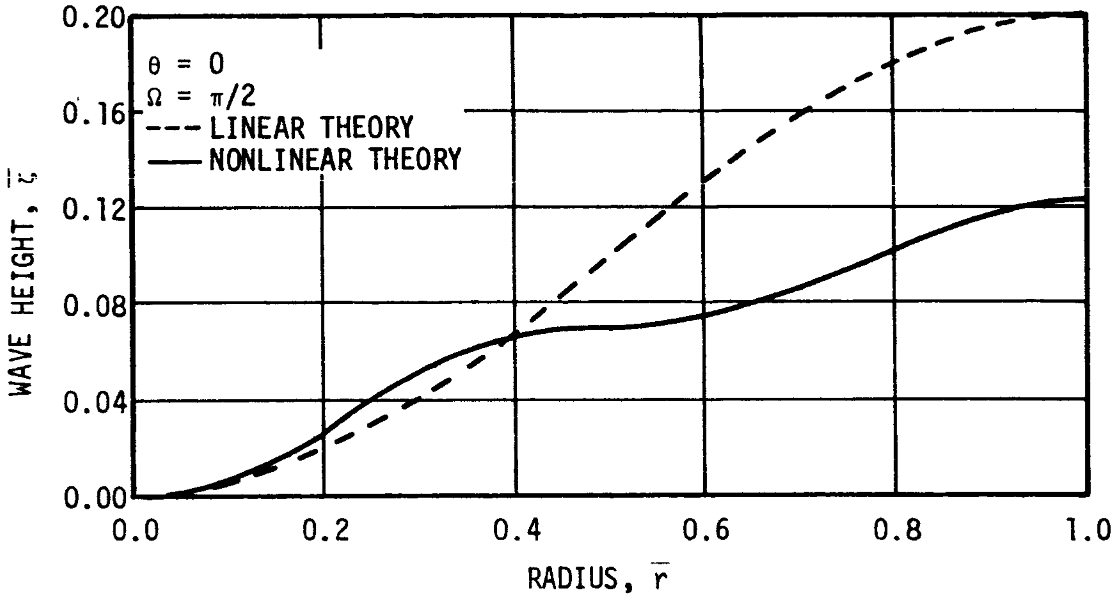

Figure 8. Wave Profile, $\bar{\varepsilon}$ = 0.2, $\bar{h}$ = 0.6

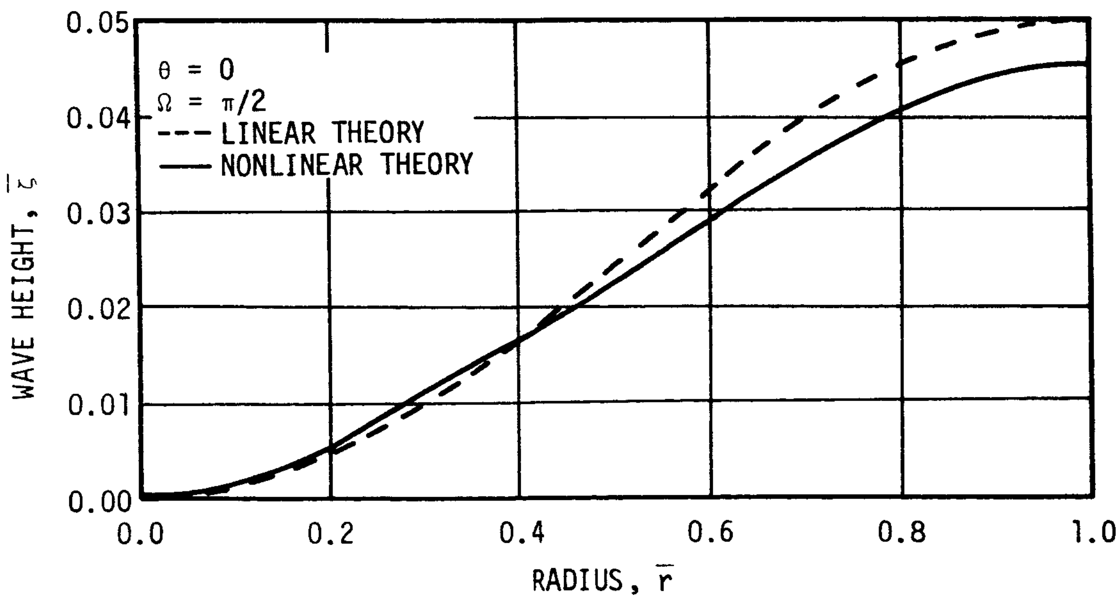

Figure 9. Wave Profile, $\bar{\varepsilon}$ = 0.05, $\bar{h}$ = 1.0

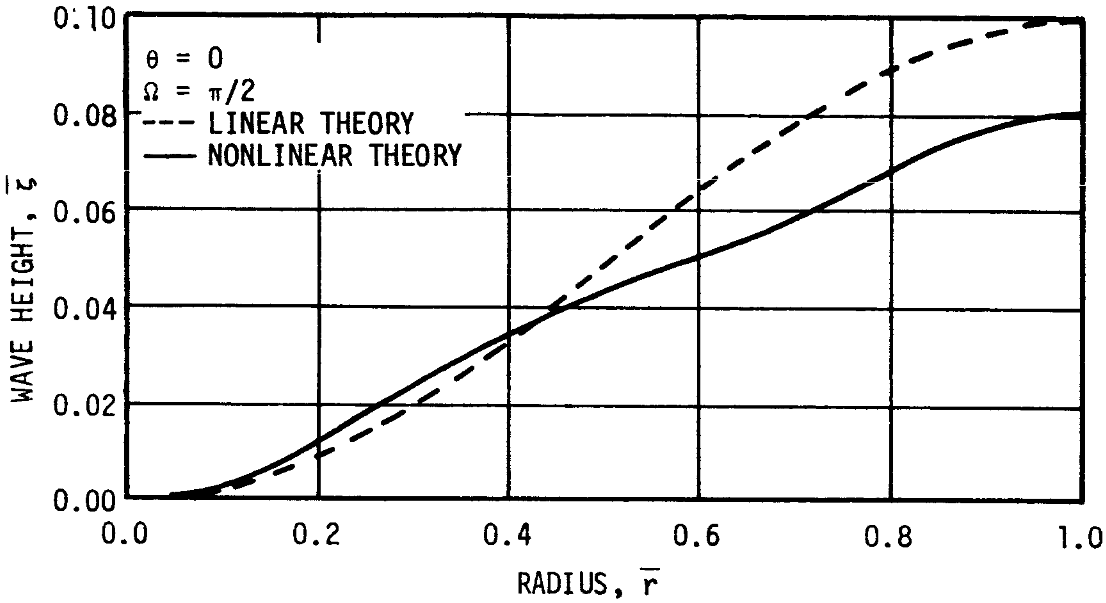

Figure 10. Wave Profile, $\bar{\epsilon}$ = 0.1, $\bar{h}$ = 1.0

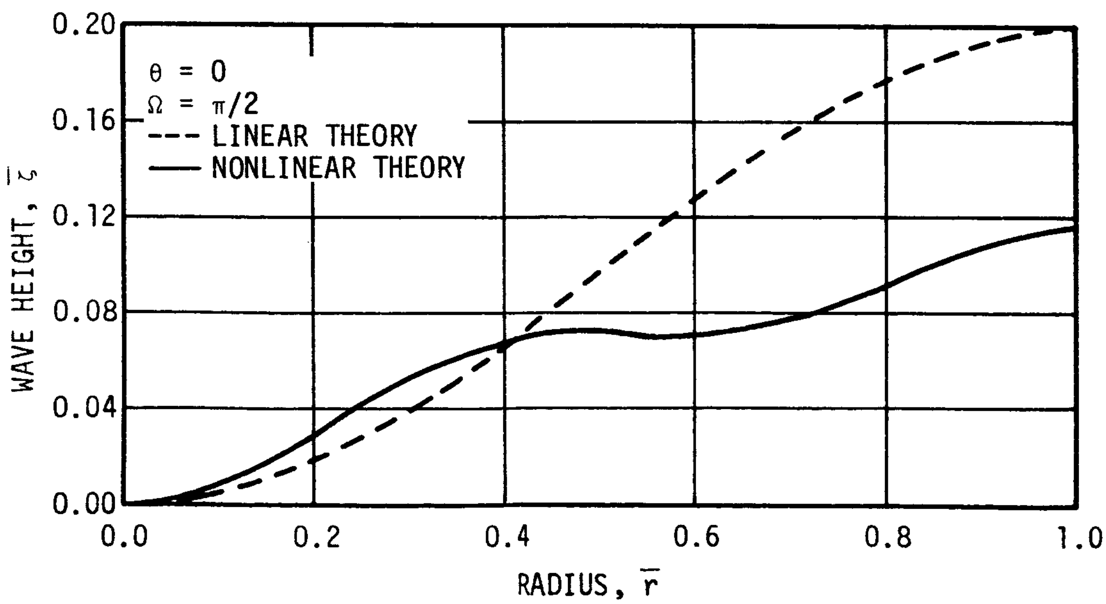

Figure 11. Wave Profile, $\bar{\epsilon}$ = 0.2, $\bar{h}$ = 1.0

CHAPTER VI

CONCLUSIONS

A solution through the third approximation for finite-amplitude free fluid oscillations of a partially filled cylindrical sector container has been presented. The solution is valid for general depth; however, it was found that some discrete depths must be excluded. The solution is found for a cylindrical sector tank with arbitrary angle π/α. Numerical results are presented for the case where $\alpha = 2$ which represents a 90° sector tank. The results for this tank indicate that the frequency is a function of the amplitude, decreasing for large depths and increasing for small depths. Several wave profiles are presented to compare linear and nonlinear theory.

To the author's knowledge, no experimental work exists which can be directly compared to these results. Verification of this work experimentally would seem to be a desirable extension.

APPENDIX A

IDENTITIES INVOLVING INTEGRALS OF PRODUCTS OF BESSEL FUNCTIONS

The number of integrals in the solution may be reduced by the use of some identities which are derived as follows.

Consider Bessel's equation

$$y'' + r^{-1} y' + (\lambda^2 - n^2 r^{-2}) y = 0 \qquad \text{(A-1)}$$

which has one solution $y = J_n(\lambda r)$ for n, an integer. Let u(r) be any function of r with continuous derivatives in the range $0 \leqq r \leqq a$. Multiplication of Equation A-1 by r u and integration from 0 to a yields

$$\int_0^a r y'' u \, dr + \int_0^a y' u \, dr + \lambda^2 \int_0^a r y u \, dr - n^2 \int_0^a r^{-1} y u \, dr = 0 \ . \qquad \text{(A-2)}$$

Integration by parts of the first term of Equation A-2 enables the equation to be written as

$$\int_0^a r y' u' \, de - \lambda^2 \int_0^a r y u \, dr + n^2 \int_0^a r^{-1} y u \, dr = 0 \qquad \text{(A-3)}$$

provided $y'(a) = J_n'(\lambda a) = 0$. It should be noted here that the notation ' in Equations A-1, A-2, and A-3 indicates $y' = \frac{d}{dr}(y)$ not $\frac{d}{d(\lambda r)}(y)$ as used in the main text.

Now by proper choice of y and u, several useful identities may be developed. For example, choosing $y = J_o(\lambda_{on} r)$ and $\mu = J_\alpha^2(\lambda_{\alpha o} r)$ Equation A-3 yields

$$2\,\lambda_{on}\,\lambda_{\alpha o} \int_0^a r\, J'_{on}\, J'_{\alpha o}\, J_{\alpha o}\, dr - \lambda_{on}^2 \int_0^a r\, J_{on}\, J_{\alpha o}^2\, dr = 0 \qquad \text{(A-4)}$$

where the notation now is that of the main text. If we now choose $y = J_\alpha(\lambda_{\alpha o} r)$ and $u = J_\alpha(\lambda_{\alpha o} r)\, J_o(\lambda_{on} r)$, Equation A-3 will give

$$\lambda_{\alpha o}^2 \int_0^a r\,(J'_{\alpha o})^2\, J_{on}\, dr + \lambda_{\alpha o}\,\lambda_{on} \int_0^a r\, J'_{\alpha o}\, J'_{on}\, J_{\alpha o}\, dr$$

$$- \lambda_{\alpha o}^2 \int_0^a r\, J_{\alpha o}^2\, J_{on}\, dr + \alpha^2 \int_0^a r^{-1}\, J_{\alpha o}^2\, J_{on}\, dr = 0 \quad . \qquad \text{(A-5)}$$

The combination of Equations A-4 and A-5 give the first identity which is

$$\lambda_{\alpha o}^2 \int_0^a r\,(J'_{\alpha o})^2\, J_{on}\, dr + \alpha^2 \int_0^a r^{-1}\, J_{\alpha o}^2\, J_{on}\, dr$$

$$= \frac{1}{2}\,(2\,\lambda_{\alpha o}^2 - \lambda_{on}^2) \int_0^a r\, J_{\alpha o}^2\, J_{on}\, dr \quad . \qquad \text{(A-6)}$$

By an identical procedure, the following identities can be established.

$$\lambda^2\, I\,[(J'_{\alpha o})^2\, J_{oj}] + \alpha^2\, I\,[r^{-2}\, J_{\alpha o}^2\, J_{oj}] = \frac{1}{2}\,(2\lambda^2 - \lambda_{oj}^2)\, I\,[J_{\alpha o}^2\, J_{oj}] \qquad \text{(A-7)}$$

$$\lambda^2 I[(J'_{\alpha o})^2 J_{2\alpha,j}] - \alpha^2 I[r^{-2} J^2_{\alpha o} J_{2\alpha,j}] = \frac{1}{2}(2\lambda^2 - \lambda_{2\alpha,j}) I[J^2_{\alpha o} J_{2\alpha,j}] \qquad \text{(A-8)}$$

$$-2\lambda\lambda_{oj} I[J'_{\alpha o} J_{\alpha n} J'_{oj}] + (\lambda^2 + \lambda^2_{oj} - \lambda^2_{\alpha n}) I[J_{oj} J_{\alpha o} J_{\alpha n}] = 0 \qquad \text{(A-9)}$$

$$-2\lambda_{2\alpha,j}\lambda I[J_{\alpha n} J'_{2\alpha,j} J'_{\alpha o}] + (\lambda^2_{2\alpha,j} + \lambda^2_{\alpha o} - \lambda^2_{\alpha n}) I[J_{\alpha n} J_{2\alpha,j} J_{\alpha o}]$$
$$- 4\alpha^2 I[r^{-2} J_{2\alpha,j} J_{\alpha o} J_{\alpha n}] = 0 \qquad \text{(A-10)}$$

$$-2\lambda_{2\alpha,j}\lambda I[J_{3\alpha,n} J'_{2\alpha,j} J'_{\alpha o}] + (\lambda^2_{2\alpha,j} + \lambda^2 - \lambda^2_{3\alpha,n}) I[J_{3\alpha,n} J_{2\alpha,j} J_{\alpha o}]$$
$$+ 4\alpha^2 I[r^{-2} J_{3\alpha,n} J_{2\alpha,j} J_{\alpha o}] = 0 \qquad \text{(A-11)}$$

$$6\lambda^2 I[(J'_{\alpha o})^2 J_{\alpha o} J_{\alpha n}] + 2\alpha^2 I[r^{-2} J^3_{\alpha o} J_{\alpha n}] = (3\lambda^2 - \lambda^2_{\alpha n}) I[J^3_{\alpha o} J_{\alpha n}] \qquad \text{(A-12)}$$

$$6\lambda^2 I[(J'_{\alpha o})^2 J_{\alpha o} J_{3\alpha,n}] + (\lambda^2_{3\alpha,n} - 3\lambda^2) I[J^3_{\alpha o} J_{3\alpha,n}]$$
$$- 6\alpha^2 I[r^{-2} J^3_{\alpha o} J_{3\alpha,n}] \qquad \text{(A-13)}$$

APPENDIX B

VALUES OF INTEGRALS OF PRODUCTS OF BESSEL FUNCTIONS

j	$I[J^2_{\alpha,o} J_{o,j}]$
j = 0	0.6759×10^{-1}
1	-0.1603×10^{-1}
2	0.2998×10^{-2}
3	-0.1639×10^{-3}
4	0.3646×10^{-4}
5	-0.1243×10^{-4}

j	$I[J^2_{\alpha,o} J_{2\alpha,j}]$
j = 0	0.2082×10^{-1}
1	0.2842×10^{-3}
2	-0.4713×10^{-4}
3	0.1449×10^{-4}
4	-0.5894×10^{-5}
5	$0(10^{-6})$

n	$I[(J'_{\alpha,o})^2 J''_{\alpha,o} J_{\alpha n}]$
n = 0	-0.2041×10^{-2}
1	-0.1729×10^{-2}
2	0.2912×10^{-3}
3	-0.1452×10^{-4}
4	0.3272×10^{-5}
5	$0(10^{-6})$

n	$I[(\bar{r})^{-4} J^3_{\alpha,o} J_{\alpha,n}]$
n = 0	0.3191×10^{-1}
1	0.1072×10^{-1}
2	-0.1538×10^{-2}
3	0.6793×10^{-3}
4	-0.3745×10^{-3}
5	0.2336×10^{-3}

$I[(\bar{r})^{-3} J^{2}_{\alpha,o} J'_{\alpha,o} J_{\alpha,n}]$

n	
n = 0	0.9810×10^{-2}
1	0.7763×10^{-2}
2	-0.5764×10^{-3}
3	0.2834×10^{-3}
4	-0.1596×10^{-3}
5	0.1004×10^{-3}

$I[(\bar{r})^{-2} J_{\alpha,o} (J'_{\alpha,o})^{2} J_{\alpha,n}]$

n	
n = 0	0.3775×10^{-2}
1	0.4252×10^{-2}
2	0.2539×10^{-3}
3	-0.3390×10^{-4}
4	0.9946×10^{-5}
5	$0(10^{-6})$

$I[J^{3}_{\alpha,o} J_{\alpha,n}]$

n	
n = 0	0.1251×10^{-1}
1	-0.2993×10^{-2}
2	0.4528×10^{-3}
3	-0.3274×10^{-4}
4	0.8529×10^{-5}
5	$0(10^{-6})$

$I[(J'_{\alpha,o})^{2} J''_{\alpha,o} J_{3\alpha,n}]$

n	
n = 0	-0.7340×10^{-3}
1	-0.1428×10^{-2}
2	-0.5163×10^{-3}
3	-0.2322×10^{-3}
4	-0.1207×10^{-3}
5	-0.6974×10^{-4}

$I[(\bar{r})^{-4} J^{3}_{\alpha,o} J_{3\alpha,n}]$

n	
n = 0	0.1514×10^{-1}
1	0.1201×10^{-1}
2	0.3905×10^{-2}
3	0.2495×10^{-2}
4	0.9945×10^{-3}
5	0.8325×10^{-3}

$I[(\bar{r})^{-3} J^{2}_{\alpha,o} J'_{\alpha,o} J_{3\alpha,n}]$

n	
n = 0	0.3691×10^{-2}
1	0.6195×10^{-2}
2	0.2470×10^{-2}
3	0.1482×10^{-2}
4	0.6699×10^{-3}
5	0.5022×10^{-3}

$I[(\bar{r})^{-2} J_{\alpha,o}(J'_{\alpha,o})^2 J_{3\alpha,n}]$

n = 0	0.1180×10^{-2}
1	0.2822×10^{-2}
2	0.1627×10^{-2}
3	0.8321×10^{-3}
4	0.4760×10^{-3}
5	0.2833×10^{-3}

$I[J^3_{\alpha,o} J_{3\alpha,n}]$

n = 0	0.7542×10^{-2}
1	0.2188×10^{-3}
2	-0.3768×10^{-4}
3	0.1236×10^{-4}
4	-0.5301×10^{-5}
5	$0(10^{-6})$

$I[J_{\alpha,o} J_{o,j} J_{\alpha,n}]$

n = 0

j = 0	0.6759×10^{-1}
1	-0.1603×10^{-1}
2	0.2998×10^{-2}
3	-0.1639×10^{-3}
4	0.3646×10^{-4}
5	-0.1243×10^{-4}

n = 1

j = 0	0.0000
1	0.1150×10^{-1}
2	-0.1058×10^{-1}
3	0.2985×10^{-2}
4	-0.7715×10^{-4}
5	0.1589×10^{-4}

n = 2

j = 0	0.0000
1	-0.5128×10^{-5}
2	0.6124×10^{-2}
3	-0.7433×10^{-2}
4	0.2484×10^{-2}
5	-0.4898×10^{-2}

n = 3

j = 0	0.0000
1	0.7971×10^{-6}
2	-0.1015×10^{-4}
3	0.4150×10^{-2}
4	0.5710×10^{-2}
5	0.2093×10^{-2}

$I[J_{\alpha,o} J_{o,j} J_{\alpha,n}]$

j	n = 4	n = 5
j = 0	0.0000	0.0000
1	-0.1756×10^{-6}	0.4868×10^{-7}
2	0.1217×10^{-6}	-0.2656×10^{-6}
3	-0.1118×10^{-4}	0.1362×10^{-4}
4	0.3123×10^{-2}	-0.1097×10^{-4}
5	-0.4632×10^{-2}	0.2499×10^{-2}

$I[J_{\alpha,o} J_{\alpha,n} J_{2\alpha,j}]$

j	n = 0	n = 1
j = 0	0.2082×10^{-1}	-0.5147×10^{-2}
1	0.2841×10^{-3}	0.1056×10^{-1}
2	-0.4713×10^{-4}	0.1032×10^{-3}
3	0.1449×10^{-4}	-0.1776×10^{-4}
4	-0.5894×10^{-5}	0.5726×10^{-5}
5	$0(10^{-6})$	$0(10^{-6})$

j	n = 2	n = 3
j = 0	0.8672×10^{-3}	-0.9622×10^{-4}
1	-0.5985×10^{-2}	0.1254×10^{-2}
2	0.6311×10^{-2}	-0.5094×10^{-2}
3	0.4739×10^{-4}	0.4394×10^{-2}
4	-0.9128×10^{-5}	0.2443×10^{-4}
5	$0(10^{-6})$	$0(10^{-6})$

$I[J_{\alpha,o} J_{\alpha,n} J_{2\alpha,j}]$

	n = 4		n = 5
j = 0	0.2687×10^{-4}	j = 0	-0.1047×10^{-4}
1	-0.6569×10^{-4}	1	0.1572×10^{-4}
2	0.1279×10^{-2}	2	-0.4861×10^{-4}
3	-0.4313×10^{-2}	3	0.1222×10^{-2}
4	0.3334×10^{-2}	4	-0.3708×10^{-2}
5	$0(10^{-6})$	5	0.2672×10^{-2}

$I[J_{\alpha,o} J_{2\alpha,j} J_{3\alpha,n}]$

	n = 0		n = 1
j = 0	0.1260×10^{-1}	j = 0	0.2134×10^{-3}
1	-0.2474×10^{-2}	1	0.8912×10^{-2}
2	0.3925×10^{-3}	2	-0.3774×10^{-2}
3	-0.6103×10^{-4}	3	0.6629×10^{-3}
4	0.1943×10^{-4}	4	-0.5100×10^{-4}
5	$0(10^{-6})$	5	$0(10^{-6})$

	n = 2		n = 3
j = 0	-0.3998×10^{-4}	j = 0	0.1344×10^{-4}
1	0.1199×10^{-3}	1	-0.2098×10^{-4}
2	0.5913×10^{-2}	2	0.6902×10^{-4}
3	-0.3646×10^{-2}	3	0.4311×10^{-2}
4	0.7581×10^{-3}	4	-0.3323×10^{-2}
5	$0(10^{-5})$	5	$0(10^{-5})$

$I[J_{\alpha,0} J_{2\alpha,j} J_{3\alpha,n}]$

	n = 4		n = 5
j = 0	-0.5824×10^{-5}	j = 0	0.2939×10^{-5}
1	0.6912×10^{-5}	1	-0.3001×10^{-5}
2	-0.1281×10^{-4}	2	0.4343×10^{-5}
3	0.4264×10^{-4}	3	-0.8465×10^{-5}
4	0.3348×10^{-2}	4	0.2775×10^{-4}
5	$0(10^{-5})$	5	$0(10^{-5})$

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13. ABSTRACT

This investigation is a study of finite-amplitude free oscillations of an inviscid incompressible fluid in a cylindrical sector container. The analysis is made for a standing wave whose motion to the first approximation is that of the first nonaxisymmetric mode. The effects of surface tension are not considered. The method of Krylov and Bogoliubov is used to satisfy the nonlinear boundary condition. Numerical results are presented for 90° sector tanks.

14. KEY WORDS

Nonlinear
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