[Reprinted from Proceedings of the IBM Scientific Computing Symposium on Control Theory and Applications, held on October 19-21, 1964, at the Thomas J. Watson Research Center, Yorktown Heights, N. Y.]

5

Applications of Liapunov Stability Theory to Control Systems[1]

J. P. LaSalle
Brown University

INTRODUCTION

Less than but close to 20 years ago, Liapunov's method for the study of stability was rediscovered in the Soviet Union. The method itself had lain dormant for half a century; this was the first break from linear approximation and linear feedback which dates back to Maxwell and Vyshnegradskii in the middle of the 19th century for the mathematical analysis of feedback control. During a period of over a decade following the Second World War, engineers and mathematicians in the Soviet Union enjoyed a virtual monopoly in the application and extension of Liapunov's method. They solved problems and information was gained about nonlinear control and about nonlinear systems in general that was beyond the algebraic, analytic, and geometric methods used to carry out linear analyses. During the next decade knowledge of the method became world-wide, new applications were found, and the theory itself was extended. Today the method is so well known that its usefulness should soon be decided. At least this is to be hoped for. One of the purposes of this paper is to indicate how this may come about, to point to some recent developments, and to suggest what are some important outstanding problems.

Another and more recent development in the theory of control systems was initiated by the solution of a simple optimal control problem in a Ph.D. dissertation in mathematics at Princeton University in 1952. At least it was this dissertation that brought the problem to the attention of mathematicians in this country and in the Soviet Union, many of whom

[1] This paper is based in part on research supported by the United States Air Force through the Air Force Office of Scientific Research under Grant AF-AFOSR-693-64, in part by the National Aeronautics and Space Administration under Grant NGR 40-002-015, and in part by the United States Army Research Office (Durham) under Contract DA-31-124-ARO-D-270.

are in attendance at this meeting. Today even a modest bibliography would not do credit to all who have contributed; in any case, this is not meant to be an historical account but simply some background to introduce what I have to say.

Shortly after the importance of Liapunov's theory of stability and the theory of optimal control was recognized, the two began to coalesce. This phenomenon was discovered by a number of people, and many attempts are being made to use the relationship between the two theories in designing control systems. A few simple examples will be given here to illustrate this. A more complete account which contains numerous references can be found in Geiss (1964).

The theories have had some successes, and they have certainly instilled, at least for a time, new life in a theory of control. But today progress seems slower, the problems become harder, and there have been for some time now no essentially new ideas.

The Liapunov theory for autonomous (stationary) systems would seem today to be quite complete, and now, as we will indicate in the following section of the paper, all of the major theorems on stability and instability of Liapunov type, and indeed some new ones, can be obtained from just two basic theorems. The first of these is Liapunov's original theorem on simple stability; the second is based on ideas of my own.

It is intuitively clear and it is easy to see why mathematically the study of the stability of nonautonomous (time-varying) systems should be difficult; and this is true even for linear equations with nonconstant coefficients. There are, however, some results which appear to be useful that are not quite as well known as they should be. There are now some indications that improved methods can be devised for analyzing the stability of nonautonomous systems, and this is discussed in the following section. Recent results on the stability of functional differential equations and stochastic differential equations which should be in the future of growing importance are also described briefly in this section.

We still have much to learn about how to use computers effectively to carry out stability analyses. The computer can be helpful in studying the stability of a particular system, and by that is meant one where all the parameters and functions are specified, but we know very little about how to use computers to generate Liapunov functions for a general class of systems. The nature of the problem suggests that it will be necessary to develop nonnumerical programs, and there may be some hope in this direction. However, a more realistic appraisal of the problem suggests that perhaps what we should do is to abandon the notion of classical Liapunov stability in favor of a more practical concept of stability. In this paper this point will arise from our discussion of the relationship between Liapunov's theory and optimal control (see final section).

STABILITY THEORY

Autonomous Systems

In LaSalle (1960) and LaSalle and Lefschetz (1961) an extension was given of Liapunov's basic theorem on asymptotic stability. The essential idea behind this extension was that the limit sets, which determine the asymptotic behavior of solutions of autonomous systems, can be located by a Liapunov function. What we want to show here is that Liapunov's first theorem on stability and a restatement of this result on limit sets contain all the major theorems of Liapunov type on stability and instability of autonomous systems. The autonomous system is represented by a system of differential equations:

$$\dot{x} = f(x), \tag{1}$$

where x is an n-vector (the state of the system), $\dot{x} = dx/dt$, and f is a function on R^n to R^n (n-dimensional Euclidean space). We assume throughout that all functions we introduce have continuous first partial derivations for all x in R^n. The vector field defined by f does not depend upon time, which means that the flow in state space is stationary.

The (positive) *limit set* of a solution $x(t)$ of (1) is the collection of points p with the property that there is a sequence of times t_n approaching ∞ such that $x(t_n) \to p$ as $n \to \infty$. Limit sets have the property that they are closed, connected, and invariant. If $x(t)$ is bounded in the future, they are, in addition, nonempty and compact (closed and bounded). The important property for us here is that they are invariant sets; that is, solutions that start in a limit set remain in that set for all time, $-\infty < t < \infty$. It is this property of limit sets of autonomous systems that accounts for the simplicity of the Liapunov theory of stability of autonomous systems.

Liapunov theory is concerned with a scalar function $V(x)$ (a function on R^n to R^1) and its rate of change along solutions

$$\dot{V}(x) = [\text{grad } V(x) \cdot f(x)] = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x).$$

Liapunov's first theorem states:

Theorem 1: If $V(x)$ is positive definite and $\dot{V}(x)$ is nonpositive (in a neighborhood of the origin), then the origin is stable.

The second basic theorem is:

Theorem 2: Let G be an arbitrary set of R^n, and let $V(x)$ be a function on R^n to R with the property that $\dot{V}(x)$ does not change sign on G. Define

$$E = \{x;\ \dot{V}(x) = 0,\ x \in \bar{G}\}$$

($\bar{G}$ is the closure of G), and let M be the largest invariant set in E. Then each solution which remains for all $t \geq 0$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$.[2]

This theorem remains valid with t replaced by $-t$, and hence all solutions which remain in G for $t \leq 0$ approach either ∞ or M as $t \to -\infty$. If M is bounded, the two possibilities are mutually exclusive. Hence, the theorem states that a suitable Liapunov function relative to a set G locates all of the possible limit sets of solutions which remain in G.

From these two theorems it is possible to obtain, as was asserted earlier, by simple arguments all of the fundamental theorems of Liapunov type on stability and instability and new ones in addition. For example, the following are immediate corollaries (M is as defined in theorem 2):

Corollary 1: If the set $G = \{x;\ V(x) < l\}$ is bounded and $\dot{V}(x) \leq 0$ on G, then each solution starting in G approaches M as $t \to \infty$.

Proof: Since G is bounded and no solution can leave G because V is non-increasing along solutions in G, this is a direct consequence of theorem 1.

If in the above corollary M is in the interior of G, then M is an attractor. If V is constant on the boundary of M, then it can be shown that M is also stable and hence asymptotically stable. Thus when M consists of a single point p in G, the point p is asymptotically stable and G is in its region of asymptotic stability. Note, however, that it is not required that V be positive definite, and there are examples which illustrate that information on asymptotic stability of an equilibrium state p can be obtained where p is not an isolated minimum of the Liapunov function.

Corollary 2 (Ĉetaev's instability theorem): Let G_0 be an open set of R^n, and let p be an equilibrium point on the boundary of G_0. If N is a neighborhood of p, $V\dot{V} > 0$ on $G = G_0 \cap N$, and $V = 0$ on that part of the boundary of G in N, then p is unstable.

Proof: We may assume $V > 0$ and $\dot{V} > 0$ on G. The conditions of the corollary imply that M (it may be empty) must lie on the boundary of G and that no solution starting in G can reach or approach boundary points of G inside N. Hence all solutions starting in G either leave N for some $t > 0$ or approach the boundary of N as $t \to \infty$. Since p is on the boundary of G inside N, this implies p is unstable.

In almost the same way we obtain a theorem on Lagrange instability (unboundedness) of solutions.

---

[2] This does not by itself exclude the possibility that $x(t)$ be unbounded and have finite limit points which will be in M.

Corollary 3: G is an arbitrary open set of R^n, $V\dot{V} > 0$ on G, and $V = 0$ on the boundary of G. Then every solution starting in G approaches infinity as $t \to \infty$.

Consider, as an example, the equation

$$\ddot{x} + \beta\dot{x} + ax + bx^2 = 0, \qquad \beta > 0,\ a > 0,\ b > 0,$$

and the equivalent system

$$\dot{x} = y,$$
$$\dot{y} = -ax - bx^2 - \beta y.$$

Let

$$V = \tfrac{1}{2}ax^2 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}y^2 - \tfrac{1}{6}\frac{a^3}{b^2}.$$

Define

$$G = \left\{(x, y);\ V < 0,\ x < -\frac{a}{b}\right\}.$$

$\dot{V} = -\beta y^2$. Hence,

$$E = \left\{(x, 0);\ x \leq -\frac{a}{b}\right\} \quad \text{and} \quad M = \left\{\left(-\frac{a}{b}, 0\right)\right\}.$$

It is clear that no solution starting inside G can leave G nor approach $(-a/b, 0)$ since $V = 0$ on the boundary of G and V decreases along solutions in G. Hence, by theorem 2, all solutions starting in G approach ∞ as $t \to \infty$.

Nonautonomous Systems

Results such as the above are not in general valid for nonautonomous (time-varying) systems. One can define for nonautonomous systems the concept of a limit set, but now they are not necessarily invariant sets. To illustrate this and the fact that theorem 2 does not hold for nonautonomous systems, consider the following simple linear example:

$$\ddot{x} + (2 + e^t)\dot{x} + x = 0.$$

For the equivalent system

$$\dot{x} = y,$$
$$\dot{y} = -x - (2 + e^t)y,$$

take $2V = x^2 + y^2$. Then $\dot{V} = -(2 + e^t)y^2$, and if theorem 2 were valid for nonautonomous systems, then the origin would be asymptotically stable in the large. However, it is not, since $1 + e^{-t}$ is a solution.

There are, however, a number of general classes of nonautonomous systems where the limit sets do have an invariance property and for which a result such as theorem 2 does hold. Periodic systems are one such class. The limit set of a solution of a periodic system of differential equations is invariant in the following sense: Through each point of the set there is a time t_0 such that the solution through that point at time t_0 remains for all time $-\infty < t < \infty$ within the set. Then, just as in LaSalle (1962), the following periodic version of theorem 2 can be proved:

The periodic system is

$$\dot{x} = f(t, x), \tag{2}$$

where $f(t + T, x) = f(t, x)$ for all x and $T > 0$. For $V(t, x)$ a function on R^{n+1} to R^n,

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x).$$

Let $x(t; t_0, x^0)$ denote the solution of (2) satisfying $x(t_0; t_0, x^0) = x^0$. Relative to a set E in R^{n+1}, the terminology "*M is the largest invariant set relative to E*" means that M is the union of all solutions $x(t)$, $-\infty < t < \infty$, with the property that $(t, x(t))$ is in E for all t, $-\infty < t < \infty$.

Theorem 3: Let G be an arbitrary set in R^n, and assume that

1. $V(t, x)$ is periodic with period T,
2. $\dot{V}(t, x) \leq 0$ for all $t \geq 0$ and all x in G.

Define $E = \{(t, x); \dot{V}(t, x) = 0, x \in \bar{G}, t \geq 0\}$, and let M denote the largest invariant set relative to E. Then each solution of (2) which remains in G for all $t \geq 0$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$.

Theorem 2 and its consequences have proved to be quite useful in studying the stability of autonomous systems, but their periodic version appears to be less well known and little used. For this reason we give below a simple example illustrating how theorem 3 can be used. Recently Miller (1964) has extended the results in LaSalle (1962) to almost periodic systems, and a similar version of theorem 3 holds for almost periodic systems.

Example:

$$\dot{x} = y,$$

$$\dot{y} = -(a + \cos t)x - \tfrac{1}{2}\beta y.$$

Take $2V = x^2 + (a + \cos t)^{-1}y^2$. Then

$$\dot{V} = -\tfrac{1}{2}(a + \cos t)^{-1}\left(\beta - \frac{\sin t}{a + \cos t}\right)y^2.$$

Hence, if $a > 1$ and $\beta\sqrt{a^2 - 1} > 1$, then $\dot{V} \leq 0$. Clearly, the origin is stable and solutions are bounded in the future because of the form of V. Set M is simply the origin. Hence, $a > 1$ and $\beta\sqrt{a^2 - 1} > 1$ imply that the origin is asymptotically stable in the large.

Markus (1956) and Opial (1960) have studied the limit sets of systems which can be called "asymptotically autonomous." They are of the form

$$\dot{x} = f(x) + f_1(t, x) + f_2(t, x), \tag{3}$$

where (1) $f_1(t, x)$ approaches zero as $t \to \infty$ uniformly for x in any compact set of R^n (Markus) and (2) $\int_0^\infty \|f_2(t, g(t))\|\, dt < \infty$ for all $g(t)$ continuous and bounded on $[0, \infty]$ to R^n (Opial).

They then show that the (positive) limit sets of solutions of (3) are invariant sets of $\dot{x} = f(x)$. Yoshizawa (1963) exploits this invariance, and for asymptotically autonomous systems, one obtains, following Yoshizawa,

Theorem 4: Let G be an arbitrary set in R^n, and assume that

1. $f_2(t, x)$ is bounded for all $t \geq 0$ and x in an arbitrary bounded subset of G,
2. $V(t, x)$ is nonnegative for all $t \geq 0$ and x in G,
3. $\dot{V}(t, x) \leq -W(x) \leq 0$ for all $t \geq 0$ and x in G.

Define $E = \{x;\, W(x) = 0,\, x \in \bar{G}\}$, and let M be the largest invariant set of $\dot{x} = f(x)$ in E. Then each solution of the asymptotically autonomous system (3) which remains in G for all $t \geq 0$ approaches $M^* = M \cup \{\infty\}$ as $t \to \infty$.

The fact that this result is for asymptotically autonomous systems would seem to limit its usefulness severely. However, as we will indicate in a moment by an example, the theorem can often be applied to study nonautonomous systems which are not asymptotically autonomous. To do this, we need the following result, also due to Yoshizawa (1963) and of interest in itself. The theorem is for a general nonautonomous system

$$\dot{x} = f(t, x). \tag{4}$$

We state the theorem in a somewhat different form and with less generality (remember that unless stated otherwise, our functions are assumed to be C^1).

Theorem 5: Assume that

1. $f(t, x)$ is bounded for $t \geq 0$ and x in an arbitrary compact set,

2. $V(t, x)$ is nonnegative for all $t \geq 0$,
3. $\dot{V}(t, x) \leq -W(x) \leq 0$, where $W(x)$ is continuous.

Define $E = \{x;\ W(x) = 0\}$. Then every solution bounded in the future approaches E as $t \to \infty$.

It is not difficult to show that condition 1 can be replaced by the condition:

4. W is assumed to be C^1, and $\dot{W}$ is bounded either from above or below for all $t \geq 0$ and x in an arbitrary compact set.

The following trivial example shows how theorem 5 can often be used to obtain information about the solutions of (4) which make it possible to construct a system which is asymptotically autonomous and to which theorem 4 can be applied. Example:

$$\dot{x} = y,$$

$$\dot{y} = -x - p(t)y, \qquad 0 < \delta < p(t) < m.$$

Take $2V = x^2 + y^2$. Then

$$\dot{V} = -p(t)y^2 \leq -\delta y^2.$$

Since all solutions are bounded ($V \to \infty$ as $x^2 + y^2 \to \infty$), it follows from theorem 5 that, for each solution $(x(t),\ y(t))$, $y(t) \to 0$ as $t \to \infty$. Now, for a given solution $(x(t),\ y(t))$, consider the system

$$\dot{\bar{x}} = \bar{y},$$

$$\dot{\bar{y}} = -\bar{x} - p(t)y(t),$$

for which $(x(t), y(t))$ is certainly a solution. It is asymptotically autonomous to $\dot{x} = y$, $\dot{y} = -x$; and hence, by theorem 4, $x(t) \to 0$. Therefore we can conclude asymptotic stability in the large.

Functional Differential Equations

It is becoming clearer all the time that functional differential equations will play an important role as a mathematical model in control theory, biology, viscoelasticity, economics, etc. One type of problem that illustrates this is that of stabilizing an unstable system when it is not possible to measure all of the state variables but only some function of the state variables. The system to be controlled may be represented by an ordinary differential equation, and one may attempt through a differential equation (indirect control) to generate a control law on the basis of the control error that can be observed. If can be shown by rather simple examples that it will not be possible to do this by an ordinary differential equation, whereas it is possible to do it if the ordinary differential equation generating the control law is replaced by a functional differential equation. This is not surprising physically since it says in some cases, where complete information is not available, that in order to stabilize the system it is necessary to use the information obtained over an interval of time. A simple example of this is one where the error can be measured but not the rate of change of error. However, by measuring the error over an interval of time, it then becomes possible to compute the rate of change of error and hence to generate a control law which will stabilize the system.

It is also difficult to imagine a system which will be adaptive in some meaningful sense unless it uses at all times all of the past information that is available; and a mathematical model for this is a system of functional differential equations. These are extensions of and include the classical difference differential equations (differential equations with delayed arguments). These equations have also been called "hereditary" differential equations. The state of such systems is now no longer a point in finite dimensional Euclidean space but is a function, and a state space is now a space of functions (an infinite dimensional space). This point of view seems to have been introduced by Krasovskii (1963) and enabled him to see how to extend to functional differential equations almost all of the classical Liapunov theory. However, from the point of view of applications, Krasovskii's results are not very useful. More recently, Hale (1965) has studied the limit sets of solutions of autonomous functional differential equations and shows that, properly interpreted, they are invariant sets. This then enabled him to extend to autonomous functional differential equations results similar to theorem 2. He then shows by numerous examples that this does give an effective method for studying the stability and instability of autonomous functional differential equations. Some of the problems that he is able to solve appear to be quite complex, and one has the impression that they are more difficult than the problems that have been solved for ordinary differential equations.

LIAPUNOV'S METHOD APPLIED TO THE DESIGN OF CONTROL SYSTEMS

Let the control system be represented by a system of ordinary differential equations

$$\dot{x} = f(x, u(x)), \tag{5}$$

where $\dot{x} = f(x, 0)$ is the system to be controlled, x is an n-vector and is the control error, and $u(x)$ is the control function (law) which, subject to being in some admissible set of controls, the designer can select.

A natural approach is then to pick a measure $V(x)$ of the error, and to be a reasonable measure, $V(x)$ will be positive definite; that is, $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$. Then select an admissible control function $u(x)$

so that $\dot{V}(x)$ is negative definite or at least nonpositive. If $\dot{V}(x)$ is negative definite, then by Liapunov's theorem on asymptotic stability the origin (zero error) is asymptotically stable, and the error in control goes to zero as V approaches infinity. This control is, in some sense at least, satisfactory. If $\dot{V}(x)$ is only nonpositive and the conditions of corollary 1 are satisfied with M just the origin, then again the origin is asymptotically stable. The existence of such a measure of the error V and an admissible control function u is then sufficient to establish that the system can be stabilized.

This immediately suggests that one should try to choose u to minimize $\dot{V}(x)$, because offhand it would seem that this control reduces the error to zero most rapidly. This is of course not true in general, since different control functions correspond in state space to moving the system along different trajectories. As was pointed out quite some time ago by Kalman (1961), the correct answer to what is minimized is contained in an elementary lemma due to Carathéodory. Assume then that the system can be stabilized and that there does exist an admissible control u^0 which minimizes $\dot{V}$. Then

$$Q(x, u(x)) = \nabla V \cdot f(x, u(x)) - \nabla V \cdot f(x, u^0(x)) \tag{6}$$

has the property that $Q(x, u(x)) \geq 0$ for $u(x) \not\equiv u^0(x)$, and the control u^0 minimizes the functional

$$\int_0^\infty Q(x, u(x))\, dt, \tag{7}$$

or, for those $u(x)$ that stabilize the system, an equivalent statement is that u^0 minimizes

$$-\int_0^\infty \nabla V \cdot f(x, u^0(x))\, dt.$$

The above then shows that if a control function u^0 can be found which minimizes $\dot{V}$, then it is an optimal control relative to the performance criterion (7). One can also try, given a performance criterion (7), to solve (6) or a modification of that equation, and hence by this means to derive an optimal control law. For a discussion of this and additional references see, for example, Lefferts (1965). The converse Liapunov theorems also suggest the conjecture that under suitable restrictions every control law which stabilizes a system is in some nontrivial sense an optimal control law.

Conversely, it is well known, and easy to see, that if the integrand of a performance criterion (7) is positive definite, then with optimal control this system will be asymptotically stable. It will in fact be asymptotically stable for any control function that makes (7) converge. One aspect of this that might be examined somewhat more carefully is that the size of the region of asymptotic stability depends upon the choice of the performance criterion. It is not too difficult to give sufficient conditions on the performance criterion to assure that optimal control will imply asymptotic stability in the large. The nature of these conditions indicates (and this is not too surprising) that if one wants to assure strong stability, then the performance criterion should depend upon the nature of the system to be controlled.

It was implicit in all that was said above that the control process continues over an infinite interval of time. This is never actually the case, but it is a useful idealization. However, there are certainly control processes which are definitely of finite duration. For satisfactory performance we must certainly expect that they have some stability under perturbations. This is something we know too little about, even though we can give examples of finite time control processes which possess very strong stability under perturbations. One such example, particular cases of which have been considered a number of times, is the following:

Example: Consider the control of a conservative dynamical system

$$\dot{x} = \frac{\partial H(x, y)}{\partial y} + u(x, y),$$

$$\dot{y} = -\frac{\partial H(x, y)}{\partial x} + v(x, y),$$

where u and v are the control functions subject to the constraint $||(u, v)|| \leq 1$. Assume that the region $G = \{(x, y); H(x, y) \leq H_0\}$ is bounded. Then

$$\dot{H} = u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}.$$

The control

$$(u, v) = -\nabla H/||\nabla H||$$

is defined except at equilibrium states of the uncontrolled system, and this control minimizes $\dot{H}$ with $\dot{H} = -||\nabla H||$. The control force is always in the direction in which H is decreasing most rapidly, and, by corollary 2, all solutions starting in G approach the set of equilibrium states in G. Suppose that in a neighborhood N of an isolated stable equilibrium state, which we take to be the origin with $H(0, 0) = 0$,

$$||\nabla H|| \geq \Phi(H) \geq 0,$$

where

$$\int_0^a \frac{dh}{\Phi(h)} \quad \text{converges.}$$

Define

$$V(x) = \int_0^{H(x)} \frac{dh}{\Phi(h)}.$$

Then

$$\dot{V}(x) \leq \frac{\dot{H}(x)}{\Phi(H(x))} \leq -1.$$

Thus within N this control brings the system to the origin in finite time, and, if $||\nabla H|| = \Phi(H)$, this is the control that does this in the shortest possible time (time optimal control). Let the perturbed system be

$$\dot{x} = \frac{\partial H}{\partial y} + u(x, y) + p(x, y),$$

$$\dot{y} = -\frac{\partial H}{\partial x} + v(x, y) + q(x, y).$$

Then, for this perturbed system,

$$\dot{V} \leq -1 + \frac{\partial V}{\partial x} \cdot p + \frac{\partial V}{\partial y} \cdot q.$$

If $||\partial V/\partial x|| \leq K$ and $||\partial V/\partial y|| \leq K$ within N, then we see that, if $||p|| + ||q|| \leq \delta < K^{-1}$, the system still retains the property that solutions in N go to the equilibrium state in finite time.

One might suspect that this can be generalized. Let us look for a moment at the difficulty. Suppose, for the control system

$$\dot{x} = f(x, u(x)) = F(x),$$

there is a control function $u(x)$ defined for all $x = 0$ such that F is C^1 for $x \neq 0$ and with the property that each solution approaches the origin in finite time; that is, there is a $T(x^0)$ such that $x(t, x^0) \to 0$ as $t \to T(x^0)$, $(x(0, x^0) = x^0, 0 \leq t < T(x^0))$. Then

$$T(x(t, x^0)) = T(x^0) - t$$

and $d/dt\, T(x(t, x^0)) = -1$. But, to establish stability under perturbations by an argument similar to that used in the above example, one needs to know that, for the perturbed system $\dot{x} = F(x) + p(t, x)$,

$$\frac{d}{dt} T(x(t, x^0)) = \nabla T(x(t, x^0)) \cdot [F(x(t, x^0)) + p(t, x(t, x^0))] = -1 + \nabla T \cdot p.$$

This would be true if we knew that $T(x)$ were C^1. Sufficient conditions for this can be obtained by using implicit function theorems. However, these conditions involve having information about the solutions, and this is hardly satisfactory.

Thus, even though we expect feedback controls acting for a finite time to have some sort of stability, this is a problem we know little about. This, and the close relationship that has now been established between stability and optimal control of infinite time processes, and the absence of such a relationship for finite time processes certainly suggest the need for a theory of stability over finite time.

With regard to the Liapunov method, which so far has been confined to infinite time stability, there has been a complete lack of success in utilizing computers to aid in applying the method to any general type of stability analysis. In the monograph (LaSalle and Lefschetz, 1961) a brief discussion was given of what is involved in a more realistic, or what was called there "practical," stability analysis. Since this presentation was originally given, Weiss and Infante (1965) have pursued the idea mentioned in the monograph and have developed some basic Liapunov-type theorems for finite time stability. A promising feature of this work is that it does appear to be more feasible to utilize computers for finite time stability analyses than it has been for the more classical Liapunov method for asymptotic stability.

Let us take a look at how a finite time analysis of stability might be carried out.

Suppose again that the control system is described by

$$\dot{x} = f(x, u(x)),$$

but let the problem be one of starting from some closed set of initial conditions B_0 and reaching in time T a closed target set B_1 of acceptable terminal states. Let the desired terminal state be the origin which is taken to be an interior point of B_1. Some error in reaching the desired state is allowed. One possible way of determining a control law to do this is to attempt to find, for a suitable positive function $\Phi(v)$, a positive scalar function $V(x)$ and an admissible control law $u(x)$ satisfying

$$\dot{V} = \nabla V \cdot f(x, u(x)) \leq -\Phi(V).$$

If $x^0 = x(0)$ and $x^1 = x(t_1)$, then, from the above inequality,

$$t_1 \leq \int_{V(x^1)}^{V(x^0)} \frac{dv}{\Phi(v)}.$$

Hence, if $V_0 = \max \{V(x); x \text{ in } B^0\}$ and $V_1 = \min \{V(x); x \text{ on bd } B_1\}$, then

$$T_1 \leq \int_{V_1}^{V_0} \frac{dv}{\Phi(v)},$$

where T_1 is the time to go from any point in B^0 to any point on the boundary of B_1. What is wanted is

$$\int_{V_1}^{V_0} \frac{dv}{\Phi(v)} \leq T.$$

The problem is then one of finding suitable functions Φ and V and a suitable control law $u(x)$ satisfying these inequalities. The effect of perturbations could also be taken into account. Assume that the perturbed system is

$$\dot{x} = f(x, u(x)) + p(t, x),$$

where $||p(t, x)|| \leq g(x)$. If the solutions are in some bounded region and if in this region $||\nabla V|| \cdot ||g(x)|| \leq M$, then the last inequality becomes

$$\int_{V_1}^{V_0} \frac{dv}{\Phi(v) - M} \leq T.$$

Undoubtedly easier said than done, but some such scheme might well be used to estimate for a system already designed the variations in initial conditions B_0 and the size of the perturbations $p(t, x)$ that could be allowed and still have the system carry out its mission of reaching B_1 in time T. Practical answers to questions of this type are of far more importance than producing optimal control laws, particularly when it is not known to what extent the performance criterion determines stability and when there is not a realistic motivation for selecting the criterion.

REFERENCES

GEISS, G. R. 1964. The analysis and design of nonlinear control systems via Liapunov's direct method. Technical documentary report no. RTD-TDR-63-4076. Dayton, Ohio: Wright-Patterson Air Force Base, Air Force Flight Dynamics Laboratory, Research and Technology Division.

HALE, J. 1965. Sufficient conditions for stability and instability of autonomous functional differential equations. J. Differential Equations, 1:452–82.

KALMAN, R. E. 1961. Contributions to theory of optimal control *in* Proc. of symposium internacional de ecuaciones diferenciales ordinarias, Nat. Univ. of Mexico and Mexican Math. Soc., Mexico, D. F., Mexico, 102–19.

KRASOVSKII, N. N. 1963. Stability of motion. English translation. Stanford, Calif.: Stanford Univ. Press. (Originally published 1959 in Moscow.)

LASALLE, J. P. 1960. Some extensions of Liapunov's second method. IEEE Trans. Circuit Theory, vol. CT-7:520–27.

———. 1962. Asymptotic stability criteria *in* Hydrodynamic instability (Proceedings of symposia in applied mathematics, vol. XIII), ed. G. BIRKHOFF *et al.* Providence, R. I.: Amer. Math. Soc., 299–307.

LASALLE, J. P., and S. LEFSCHETZ. 1961. Stability by Liapunov's direct method with applications. New York: Academic Press.

LEFFERTS, E. J. 1965. A guide to the application of the Liapunov direct method to flight control systems. Washington, D. C.: NASA Contractor Report, NASA CR-209.

Markus, L. 1956. Asymptotically autonomous differential equations *in* Contributions to the theory of nonlinear oscillations, vol. III (Ann. of Math. study no. 36), ed. S. Lefschetz. Princeton, N. J.: Princeton Univ. Press, 17–29.

Miller, R. 1964. On almost periodic differential equations. Bull. Amer. Math. Soc., 70:792–95.

Opial, Z. 1960. Sur la dépendence des solutions d'un système d'equations differentielles de leurs second membres. Application aux systèmes presque autonomes. Ann. Polon. Math., 8:75–89.

Weiss, L., and E. Infante. 1965. On the stability of systems defined over a finite time interval. Proc. Nat. Acad. Sci. U. S., 54:44–48.

Yoshizawa, T. 1963. Asymptotic behavior of solutions of a system of differential equations. Contributions to Differential Equations, 1:371-87.

DISCUSSION

D. C. Gazis: Have you or anyone else investigated the question of stability of systems involving a time-lag between the lefthand and righthand sides of the differential equations?

J. P. LaSalle: Yes, there has been considerable work on the stability of such equations. A good reference is Bellman and Cooke (1963). This book contains considerable information with many references.

DISCUSSION REFERENCE

Bellman, R., and K. L. Cooke. 1963. Difference-difference equations. New York: Academic Press.