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Prepared by
WYLE LABORATORIES
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SUMMARY

This report contains a collection of new results, both qualitative and quantitative, concerning the nature of propagation of waves of finite amplitude in thermoviscous media; with particular attention to piston-driven propagation. The vehicle for the analysis is Burgers' equation, for which an appropriate boundary value problem is solved. Several critical nonlinear quantities are defined and discussed; criteria for the appearance of shocks are given; a generalized analytic explanation of the nature of nonlinear diffusion is given. An explicit quantitative solution for piston-driven propagation, also given here, is shown to contain, as a special case, Fay's result (R.D. Fay, J. Acoust. Soc. Am. 3,222 (1931).

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FOREWORD

This Technical Report contains several new results of fundamental importance, concerning one dimensional viscous fluid flow in general, and piston-driven excitation in an infinite pipe in particular. Because of the limited time allotted to this research and because of the unusually large number of new results obtained it was impossible, in several cases, to delve deeply into the implications of the results achieved. However, the delicate line which had to be drawn therefore between fundamental result and its basic meaning on the one hand, and the particular application of this result on the other, was constructed in such a way, that the omissions of the former should fall into the realm of basic research, while the latter could be expanded by a modest and well-defined effort. In this Foreword we are including, therefore, those achievements which are immediately applicable in either a qualitative or quantitative sense, but we are reserving for the last section the enumeration of such aspects of our results as require further basic study.

It will be convenient to discuss the achievements contained in each section, in the order of their appearance. Thus, we shall go through the entire contents of the Report, underlining the parts which are both new and important.

1.0 This introductory section gives a general overview of the problems dealt with, their history and present knowledge concerning them.

2.0 We are giving here the derivation (in fact, two derivations) of the equation that is used as the analytical predictor. The approach selected points almost automatically to the next higher order equation which awaits a "third generation" solution. The question of lossless fluids is also touched upon.

3.0 An energy relationship is obtained in Section 3.0 for the analytical vehicle, which is Burgers' equation. An explicit and complete solution of this equation, together with a general discussion and the description of the initial value problem for the equation are also given.

4.0 Because of its great importance, at least two research workers tried recently to solve the boundary value problem for Burgers' equation. Section 4.0 discusses the reason why the approach of these two workers is entirely fallacious. A more detailed discussion of this is given in Appendix B.

5.0 Here an analytic description of the boundary value problem, connected with the motion of a piston in a fluid, is given. The discussion is restricted to at most sonic piston velocities. An approximation of the resulting complicated boundary condition is derived, together with a definite estimate of the order of this approximation.

6.0 Formula (38) of Section 6.0 gives the most general solution for the piston problem, under the approximations of the previous section. A new, distinctly nonlinear quantity (termed the "impulse of interacting energy") is defined in formula (41); it represents the nature of the nonlinear and diffusive interactions. Its basic building block is the "determinant of nonlinear diffusion", defined by (40). Some conclusions are also drawn at this point concerning the validity of assuming the constancy of the dissipation number δ. Other results of this section include the definition of three analytic conditions for the appearance of shocks; and the attainment of the very easily analyzable formula (47), derived from (38), which is the simplest quantitative description of piston-driven particle velocity in a thermoviscous medium.

7.0 Since one of the most important cornerstones in nonlinear acoustical theory was the classical paper of Fay of 1931, the reduction of our formula (47) to Fay's result was considered an important verification of our theory. Thus, Fay's solution is obtained, as a very special case of our results, in formula (50) of this Section.

8.0 According to linear diffusion theory any waveform will dissipate itself with time. While this is a highly gratifying result, there is no logical reason why time itself should act as a diffusing agent of some sort. This consideration is the one that can put in proper focus the importance of the discovery to which Section 8.0 is devoted: an analytical explanation of the nature of diffusion, represented by formula (54). The formula reduces, in a limiting case, to a description (a time - exponential); but since, in fact, every wave motion is of finite amplitude, (54) can be taken as a new physical law.

9.0 In order to gain a better insight of the complicated formula (38), Section 9.0 discusses the time independent solutions of (12). Two critical parameters, important for piston driven propagation, are defined and discussed.

10.0 This section is devoted to the presentation of some graphical data, supporting and clarifying the time independent solution.

APPENDICES

Appendix A constitutes the rigorous mathematical background for many of the results presented in this report. Appendix B, on the other hand, discusses in detail the reasons why some recent attempts at obtaining the type of results presented here are actually fallacious. Thus, this Appendix is an extension of Section 4.0.

1.0 INTRODUCTION

Two of the most important problems in the theoretical study of the propagation of waves of finite amplitude are the postulation of appropriate equations on the one hand, and the formulation of meaningful boundary and initial conditions for them, on the other.

In setting up, or deciding on the equations which are to describe the phenomena to be encountered, one has to keep constantly in mind that there are no so-called "exact" equations; every equation involves a certain degree of approximation. One has to compromise therefore : starting from basic principles, an equation, or a set of equations, has to be arrived at; such that it contains most of the significant features of the particular physical situation being analyzed and yet is not beyond present mathematical techniques. Thus, for instance, the equation designed to describe the propagation of finite amplitude waves, in air of variable temperature and variable composition (humidity), must at least be nonlinear and must also account for heat dissipation.

The formulation of a general, yet analyzable, boundary and/or initial value statement is equally important; for without it almost everything else has merely academic importance. To be sure, one can in many cases make general qualitative statements about the phenomena described, even under idealized boundary conditions. If one is interested, however, in more specific statements or in actual quantitative answers, then a way must be found to formulate boundary and/or initial conditions which are descriptions of physically realizable and experimentally feasible situations.

In 1948, as part of an attempt to explain the nature of turbulence, Burgers [1] introduced a time-dependent partial differential equation, which was both nonlinear and also contained a second order viscous term. The equation bore a striking inherent resemblence to the one dimensional Navier-Stokes equations; and, as it later turned out [2], [3], [4], [5], it was very significant from a statistical point of view also. Therefore, the equation seems to describe phenomena which can have both deterministic and probabilistic interpretations. An example would be the interaction of shocks in a thermoviscous medium.

The circumstance that made this equation truly significant was the discovery of its complete and explicit solution. This was accomplished by Hopf in 1950 [6] and by Cole in 1951 [7] , independently of each other. Hopf concentrated his efforts on the mathematical aspects of the solution, while Cole followed the solution up with its aerodynamical implications; although he regarded the equation more as an analogy, and not as the culmination of a sequence of approximations.

Soon, however, this void was filled also. Lagerstrom, Cole and Trilling had previously (in 1949) provided the analysis [8] , which reduced the set of equations of motion and conservation relationships to Burgers' equation. They did this for viscous perfect gases. In 1953, Mendousse [9] extended their analysis, to include viscous fluids of arbitrary equations of state. Lighthill provided the justification for the use of this equation in thermoviscous perfect gases [10] in the year 1956; this was extended by Hayes [11] in 1958 to thermoviscous fluids of arbitrary equation of state. Finally, in 1962, Gol'dberg justified the use of Burgers' equation for propagation in magnetically conductive thermoviscous perfect gases [12] .

As we shall see later, a completely accidental peculiarity in the form of the solution of Burgers' equation makes it very natural to consider an initial value problem. This would correspond to the description of how an already excited waveform propagates and dissipates itself, under the mechanism of internal diffusion but with no outside forces acting on it. Because of this peculiarity and because of the lack of a solution for any true boundary value problem for Burgers' equation, each one of the authors mentioned previously - and as far as we could determine, everyone else using this equation - considered the initial value problem only. However, to overcome this obvious defect, several of them transformed their equations into a moving frame of reference, with a space-like coordinate taking the place of the physical one. Mendousse [9] went even a step farther than this. He provided a rational approximation in which the form of Burgers' equation remained essentially invariant, but where the physical coordinate replaced the time variable, while a time-like quantity (retarded time) took the place of the former. Based on this work, Blackstock [13] gave a solution for a quasi-boundary value problem; and also obtained a very elegant generalization of the classical solution of Fay [14] , who had considered propagation in a dissipative medium and obtained his solution for unspecified boundary conditions, through a series of approximations, starting from a completely different equation.

While the qualitative agreement of these two solutions is quite remarkable, it is nevertheless subject to some suspicion. The reason for this is that both the equation used by Fay, and Burgers' equation, have a strong connection with the linear heat equation. Rational approximations, therefore, would tend to go in a direction where the simplified expressions can be handled on better-known mathematical ground; in this case, the realm of the solutions of the heat equation. Specifically, Blackstock had utilized certain properties of the Jacobian Theta and Zeta functions (first connected to Burgers' equation by Cole [7]). As is well known, the former are solutions of the heat equation, while the Zeta functions are just the logarithmic derivatives of the Theta functions. On the other hand, the basic linearization procedure employed by Fay led to classes of solutions which, for certain ranges of the variables, are also solutions of the heat equation. One would conclude, therefore, that the intersection of the two sets of solutions is the place to which the path of least resistance leads. This does not mean, however, that the set composed of the union less the intersection of the two solution sets is empty, or even small; indeed, the contrary is true.

In the study here presented, however, a true boundary value problem is treated and analyzed.

One of the vexing problems of this field of study is the specification of physically meaningful boundary conditions. On occasion this is quite straightforward: for instance, the fact that a rigid wall is a barrier for a fluid; or that in a viscous flow inside a pipe of finite diameter the flow velocity along the pipewalls vanishes, are both easily translatable conditions. On the other hand, suppose that a kind of excitation exists at a certain point in the fluid. For instance, suppose we have a "pipe" of infinite length and infinite diameter, and that a piston, of similar dimensions, excites the fluid. The question that arises is the following: just where is the locus of the excitation ? A clearly related problem is one where the fluid impinges on a "rigid" wall. The fact is that no wall is really rigid; and given a strong enough excitation, together with a wall of not too great strength, the question of where to prescribe a condition of zero velocity becomes one of great practical importance. Furthermore, there are occasions where the boundary conditions are well known, but cannot be used; since they are not applicable (mathematically) to the equation that has been decided upon. The paper of Blackstock [13] is an example here: for a detailed critique, see Appendix B.

Here we shall be interested in the classical piston problem, as described in the preceding paragraph. We shall not solve the exact problem of propagation by a piston; this involves the question of how to apply boundary conditions, in one-dimensional flow, on an arbitrary curved boundary [15], [16], [17]; a question beyond present mathematical techniques. We shall, however, give a second order approximation of the solution, valid for piston motion of arbitrary frequency and moderate amplitude. This itself represents an advance in the present state of the art.

In classical analysis the occurrence of a (physical) shock corresponds to the existence of a (mathematical) characteristic. It is not surprising, therefore, that heretofore shock problems were considered only for first order flows, whose descriptors are essentially hyperbolic conservation laws. On physical grounds, however, one would want to say that even in a medium with thermoviscous properties, the appearance of shocks is a possibility; indeed, it is an observed phenomenon.

Because of this consideration, we shall discuss the following problem: what type of excitation is necessary in thermally conducting and viscous media, in order to produce shocks ?

Finally, as a by-product of our analysis, we shall discuss a unique feature of the solution. It is essentially a qualitative and comparative evaluation of the thermoviscous mechanism, as measured against the type of excitation applied. We shall discuss this on general grounds and also in graphical form, to observe several significant features. One of the conclusions is the verification of an observation [1], [10], [13], that the assumption of a constant value for the description of thermoviscous effects is indeed a weak one.

2.0 THE GOVERNING EQUATION

It is not the intention of this report to give the rational approximations leading to Burgers' equation under the various physically important assumptions; this task has been accomplished, as we pointed out in the Introduction, for many kinds of fluids of interest. However, in order to remain in one fixed frame of reference, so that it will be possible to draw the appropriate physical consequences of the forthcoming analysis, we elected to treat thermoviscous perfect gases. This implies of course that our starting point is Lighthill's analysis [10] ; but our results can be applied with equal ease to fluids obeying other laws, as mentioned in the Introduction.

Let us start with equation (30) of Lighthill, at the stage where the governing equations have already been reduced to a pair of one dimensional equations, which are

$$\bar{v}_t + \bar{v}\bar{v}_x + \frac{2}{\gamma - 1} a a_x = \epsilon \bar{v}_{xx} \tag{1}$$

$$a_t + \bar{v} a_x + \frac{\gamma - 1}{2} a \bar{v}_x = 0 . \tag{2}$$

We shall use the subscript notation throughout for partial derivatives. In equations (1) and (2) $\bar{v} = \bar{v}(x, t)$ is the particle velocity function, $a = a(x, t)$ is the speed of an isentropic small-amplitude sound wave, γ is the ratio of specific heats, and ϵ is a positive constant representing the strength of the thermoviscous mechanism. In particular,

$$\epsilon = \nu \left[2 + \frac{\eta'}{\eta} + \frac{\gamma - 1}{Pr} \right]. \tag{3}$$

In this formula ν is the kinematic viscosity, η and η' are shear and dilational viscosity numbers, respectively, while Pr is the Prandtl number. Dimensionally, the units of ϵ are those of (length) (velocity) $= ft^2/sec$, in the English system of units. The quantity a is connected to the density ρ of the fluid by means of the formula

(4) $$a = a_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma - 1}{2}}$$

where the subscripts 0 denote values of the variables in the undisturbed fluid. Let us point out here that the classical analyses of Riemann and of Earnshaw [18], [19] were performed on the left-hand sides of equations (1) and (2); which are the exact equations for the propagation of sound waves of finite amplitude in a lossless medium. Riemann, in his analysis, transformed equations (1) and (2) to the frame of reference of characteristic coordinates r, s, defined by

(5) $$r = \frac{a}{\gamma - 1} + \frac{\bar{v}}{2}, \quad s = \frac{a}{\gamma - 1} - \frac{\bar{v}}{2} .$$

These coordinates are invariant in the sense that r is constant along wavelets such that

$$\frac{dx}{dt} = a + \bar{v} ;$$

while s is constant along

$$\frac{dx}{dt} = - a + \bar{v} .$$

Let us note, that it is not the velocity in general of the wavelets, which is constant, but rather their speed equals that of the local speed of sound with respect to the fluid velocity $\bar{v}$ [10].

If we transform equations (1) and (2) to the coordinates defined by (5), we obtain

(6) $$r_t + \left(\frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s\right) r_x = \frac{\epsilon}{2}\left(r_{xx} - s_{xx}\right)$$

(7) $$s_t - \left(\frac{\gamma - 3}{2} r + \frac{\gamma + 1}{2} s\right) s_x = \frac{\epsilon}{2}\left(s_{xx} - r_{xx}\right),$$

where the parenthesis on the left hand side of (6) contains $(a + \bar{v})$, and that on the left hand side of (7) is equal to $(a - \bar{v})$.

To quote Lighthill: he observes that the variations of s are of second order in general; while we obtain from the Rankine-Hugoniot shock conditions the fact that across a shock wave the variations of s are only of third order. This will constitute our justification for taking $s = s_0$ a constant. Then, disregarding equation (7), we obtain

(8) $$r_t + \left(\frac{r + 1}{2} r + \frac{\gamma - 3}{2} s_0\right) r_x = \frac{\epsilon}{2} r_{xx} .$$

As we pointed out previously, the following approximate equality holds:

(9) $$\frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s_0 \approx a + \bar{v} .$$

Thus, if we let

(10) $$v = \frac{\gamma + 1}{2} r + \frac{\gamma - 3}{2} s_0 ,$$

then v is a certain excess velocity, such that the mainstream flow M can be decomposed into

$$M = C + v ,$$

where C is constant. Under special circumstances this C is the local sound speed. At any rate, it is clear that all nonlinear and dissipational effects are borne by v , which, therefore, is the significant quantity here. Thus, using (10) and letting

$$\frac{\epsilon}{2} = \delta , \tag{11}$$

we arrive at

$$v_t + v v_x = \delta v_{xx} ; \tag{12}$$

the equation on which our analysis is based.

As the final remark of this section, it might be worthwhile to point out that while we elected to quote Lighthill's derivation in arriving at Burgers' equation, we could just as well have used, as our starting point, the following set of equations :

$$\begin{array}{ll} \mathrm{Co} : & \left\{ \begin{array}{l} \rho_t + (\rho v)_x = 0 \\ v_t + v v_x + \frac{1}{\rho} P_x = \text{viscosity term} \\ P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} \end{array} \right. \\ \mathrm{NS} : & \\ \mathrm{St} : & \end{array} .$$

Here Co is an abbreviation for the equation of continuity; NS stands for the appropriately modified form of a Navier-Stokes equation, while St is the adiabatic equation of state. The letter P here means pressure ; all other symbols have their previous meanings.

The fact to observe here is that if in the NS equation above we take the viscosity term on the right-hand side the same as the right-hand side of (1), then the substitution of equation St into Co and into NS, coupled with a transformation similar to that defined by (4), leads us immediately to the set (1) - (2).

3.0 BURGERS' EQUATION; GENERAL DISCUSSION

In the preceding section we derived equation (12),

$$(12) \qquad v_t + v v_x = \delta v_{xx}$$

as the model on which our discussion will be based. Let us now examine this equation somewhat more closely, and also obtain its general solution.

First of all let us try to gain an insight into the "content" of (12). If we integrate this equation with respect to x from some point x_1 to another point x_2, then integration by parts and some rearrangements yield

$$(13) \qquad \frac{1}{2}\int_{x_1}^{x_2} (v^2)_t \, dx + \frac{1}{3}\left\{ v^3(x_2,t) - v^3(x_1,t) \right\} =$$

$$= \delta \left\{ v(x_2,t)\, v_x(x_2,t) - v(x_1,t)\, v_x(x_1,t) \right\} - \delta \int_{x_1}^{x_2} (v_x)^2 \, dx \; .$$

This, clearly, is a conservation of energy relationship. We must discuss it, because in the course of the approximations that yielded (12), the original equation for the conservation of energy was simply dropped. [10] .

If we go in (13) in order from term to term, we can interpret this equation as one stating that the total rate of change of kinetic energy in the system, plus the net flux of this energy out across the boundaries, exactly balances the rate at which work is done at the boundaries, less the total dissipation present. [7] .

Such a statement is, of course, quite acceptable; in particular, if we note that (13) allows a steady state also, on any interval. It is obtainable by setting the right-hand side equal to 0.

Let us now introduce a stream-function-like quantity ϕ into (12), defined by

$$(14) \qquad v = \phi_x \; ; \quad \phi = \phi(x, t) \; .$$

We then obtain

$$(15) \qquad \phi_{xt} + \phi_x \phi_{xx} = \delta \phi_{xxx} \; ,$$

a relation that can be integrated readily, and which yields the nonlinear diffusion equation in its classical form:

$$(16) \qquad \phi_t + \frac{1}{2}\left(\phi_x\right)^2 = \delta \phi_{xx} \; .$$

Let us now assume that (16) has a solution of the form

$$(17) \qquad \phi = F(h) \; ,$$

where F is a function to be determined and $h = h(x, t)$ is a solution of the linear heat equation,

$$(18) \qquad h_t = \delta h_{xx} \; ,$$

with the same δ as in (12). Substitution of (17) into (16) yields

$$(19) \qquad F' h_t + \frac{1}{2}\left(F' h_x\right)^2 = \delta \left[F' h_{xx} + F'' h_x^2 \right] \; .$$

Here the primes denote derivatives with respect to h . We can rearrange (19) and obtain

(20) $$F'\left(h_t - \delta h_{xx}\right) + h_x^2\left(\frac{1}{2}(F')^2 - \delta F''\right) = 0.$$

The first term on the left-hand side vanishes by virtue of (18). In the second term, since we are not interested in trivial solutions of the heat equation, we must have

(21) $$\frac{1}{2}(F')^2 - \delta F'' = 0 .$$

This ordinary nonlinear differential equation can be solved quite easily. Its solution is

(22) $$F = -2\delta \ln\left[h + c_1\right] + c_2 ,$$

with c_1 and c_2 arbitrary constants.

Let us note here that the appearance of c_1 in (22) is redundant; since if $\bar{h}$ is a solution of (18), then so is $\bar{h} + k$, for any constant k. We then find, however, an important limitation on h, from (22). From a physical point of view, this limitation is quite important; it was first pointed out by Rodin in [20] ; although it was taken into consideration - mathematically - by Hopf [6] . Cole, on the other hand, seems not to have noticed it. This limitation is a restriction of the solutions h of (18) to those which are positive for all values of the variables,

(23) $$h(x, t) > 0, \quad -\infty < x < \infty, \quad t > 0.$$

Note that $h > 0$ at $t = 0$ also. We shall return to the implications of this positivity later.

Combining now (14) and (22), we obtain the solution of (12) as

(24) $$v = -2\delta \frac{h_x}{h} ,$$

where h is any function satisfying (18) and (23) .

It is to be noted that this entire derivation was merely a formal procedure. What we have shown here is that if Burgers' equation has any solutions at all, then some have the form (24). That this equation actually does have solutions, that they are all of the form (24), and that, under appropriate initial conditions, they are unique, was shown by Hopf [6] . The corresponding question of existence and uniqueness for the boundary value problem was solved by Rodin; see Appendix A.

A question of great importance is the following: suppose we are given initial and/or boundary conditions for v. Then, since our solution will be obtained in terms of the solution of an associated linear heat equation (18), how do we "translate" the conditions given for v to h? The answer to this question, with respect to a one-point boundary value problem, is not immediate. We shall give it later; it is also discussed, more extensively, in Appendix A. Because of the peculiar form of the solution (24), however, we can translate initial values from the v - plane to the h - plane quite easily. For integration and rearrangement of (24) yields

(25) $$h(x, t) = G(t) \exp\left[-\frac{1}{2\delta} \int_{x_0}^{x} v(y, t)\, dy \right] ,$$

where G and x_0 are quite arbitrary; except that $G > 0$ and $x_0 < x$. In fact, in forming the quotient (24), both of these cancel out.

Now, however, if we specify an initial condition of the form

(26) $$v(x, t) = v_0(x), \text{ as } t \longrightarrow 0^+ , \quad -\infty < x < \infty$$

for v, we obtain the corresponding initial condition for h from (25) as

(27) $$h(x, 0) = \exp\left[-\frac{1}{2\delta}\int_0^x v_0(y)\,dy\right], \quad -\infty < x < \infty$$

where we combined, and then normalized, the inessential constants $G(0)$ and x_0.

4.0 ON THE INADMISSIBILITY OF CERTAIN CONDITIONS

While solutions as explicit as that of Burgers' equation are as yet unavailable for the pair of equations (1) and (2), it is nevertheless well known that the limiting case $\epsilon \longrightarrow 0$ introduces discontinuities not only in the solution itself [6] , [21] , but also in the number of boundary conditions needed in order to obtain a unique solution [22] , [23] . It is natural to expect, therefore, that while the pair of equations (1) and (2), with $\epsilon = 0$ needs only two boundary conditions [22] , [24] , the single equation (12), with $\epsilon > 0$, needs also two of them [25] , [26] . The pair (1) and (2), we observe, should - and does - yield unique solutions (in the small, to be sure) for conditions of the type

$$v(x_0, t) = v_0(t), \quad a(x, t_0) = a_0(x) \quad , \tag{28}$$

provided we are seeking solutions for the case when $\epsilon = 0$. Similarly, the single equation

$$v_t + v v_x = 0 \quad , \tag{29}$$

which is the left hand side of (12) , also possesses unique solutions of the boundary value problem defined by

$$v(x_0, t) = v_0(t) \quad . \tag{30}$$

It is, therefore, perhaps not too surprising that, partly because of a certain inertia which developed in the course of using, for about a century, equations such as (29) with conditions of the type (30), there would be some who would attempt to use (30) with (12) also. Such an attempt is not a case of ignorance necessarily; for one has to consider the fact that usually it is not known, from the physical considerations, what the necessary boundary conditions are. However, if one reflects on the problem for a while, one recognizes that the case of a flow in a lossless medium is quite different - mathematically even - from that in a viscous one: for one condition in the former should suffice to describe the nature of the excitation applied to the system, but one condition cannot conceivably incorporate the effects of both excitation and interaction of the (now viscous) medium with this excitation.

One would expect, therefore, that for equation (12), for instance, two conditions of the type

$$v(x_0, t) = g(t), \qquad v(x, 0) = f(x) \tag{31}$$

should be both necessary and sufficient. These two conditions describe the type of excitation applied to the system at the point x_0 at time t, and the initial state of the system.

Conditions (31), mathematically speaking, are very nice. On physical grounds, however, two things are clear: First, the position of x_0 should make no essential difference in the solution; and, second, these conditions (the first one is really problematic) should be physically realizable.

Let us see now what type of treatment Burgers' equation received in the literature so far. As we pointed out somewhat earlier, most of the investigators discussed initial value problems, defined by

$$v(x, 0) = f(x) \ . \tag{32}$$

The use of such a condition can answer the question of what happens in a given system, whose initial state is known and on which no outside forces act. This is clearly an important problem, but gives no answers for situations where one has to contend with an excitation that goes on in time. In order to overcome this objection somewhat, several of the authors mentioned earlier considered equation (12) as one valid in a moving frame of reference. This alleviated the insufficiency in the mathematical model to a certain extent. The most radical step, however, was taken by Mendousse [9] , who derived the equation which, in normalized form, can be written as

$$\bar{v}_x + \bar{v}\bar{v}_\tau = \delta \bar{v}_{\tau\tau} \tag{12'}$$

and where x is the physical coordinate, while τ is retarded time. This equation, as we mentioned at some length in the Introduction, formed the basis for Blackstock's analysis of the boundary value problem defined for $\bar{v} = \bar{v}(\tau, x)$ by

(26') $$\bar{v}(\tau, x) = \bar{v}_0(\tau) \text{ as } x \longrightarrow 0^+, \quad -\infty < \tau < \infty .$$

Now, according to (24), the solution of (12') is

(24') $$\bar{v} = -2\delta \frac{\bar{h}_\tau}{\bar{h}} ,$$

where

(23') $$\bar{h} = \bar{h}(\tau, x) > 0, \quad -\infty < \tau < \infty, \quad x \geq 0$$

and where $\bar{h}$ satisfies a linear heat equation,

(18') $$\bar{h}_x = \delta \bar{h}_{\tau\tau} .$$

Furthermore, the boundary values, from (26'), are related by

(27') $$\bar{h}(\tau, 0) = \exp\left[-\frac{1}{2\delta}\int_0^\tau \bar{v}_0(T)\, dT\right], \quad -\infty < \tau < \infty .$$

We shall list now four objections, based on the preceding general discussion and some additional facts, to the statement of the problem and to its solution, as defined by the mathematical formulation containing primes (') .

1.) Physical intuition dictates (see preceding comments) that more than one condition be prescribed for a boundary (as opposed to an initial) value problem.

2.) It is impossible to realize, physically, a situation where the excitation is at one given point, say at $x = 0$. For, clearly, the exciting agent will be moving also, and it will have no fixed position; except perhaps in an average sense.

3.) As is well known, [26] , [27] , [28] , the heat equation (18') has unique and continuous solutions (ones we must have in order for them to serve as a building block in obtaining (24') for conditions prescribed at some point $x = x_0$, only if $x_0 = 0$; or, in the most general case, when $x_0 \geq 0$. This, however, seems to be an unacceptable limitation; for, as we mentioned previously, the location of the boundary point should make no difference; its prescription at $x = 0$ should be merely a convenience.

4.) The last, and probably most serious, objection that we shall raise here is based on the fact [28] , [29] , [30] that if one considers the solution $\bar{h}$ of (18'), (27') at the point (τ_0 , x_0) in time-space, then $\bar{h}(\tau_0 , x_0)$ depends on <u>all</u> boundary values. This, however, is a preposterous proposition: we clearly cannot accept as fact a statement that the <u>present</u> of a physical situation depends, in whatever measure, on its <u>future</u>.

Thus, what we need is the following: to formulate a <u>true</u> boundary value problem, not subject to the criticism offered above, such that the analytical tools now available will be sufficient for its application to Burgers' equation (12). This we shall do in the next section.

5.0 PISTON DRIVEN PROPAGATION

The physical framework for which we shall formulate our boundary condition is a pipe of arbitrarily large radius. One end of the pipe contains a piston, of a dimension appropriate to the pipe, while the other end extends to infinity. We prescribe the motion of the piston by noting that its position is a function of time: therefore, we have

$$x = g(t) \quad . \tag{33}$$

We also stipulate that the piston moves with at most sonic velocity, so that the fluid adheres to the piston and it has the same velocity as the piston, at all points of contact. If, therefore, we let $v = v(x, t)$ denote the fluid velocity function, we can formulate our boundary statement as

$$v(g(t), t) = g'(t) \quad . \tag{34}$$

This then is an exact boundary condition. Let us point out here that it is not known whether the single, although complicated, condition (34) is sufficient for obtaining a unique continuous solution for (12). The answer is probably in the negative; for, as pointed out in connection with conditions (31), *one ought to specify the initial* state of the system also. Since, however, (34) is a condition on a curved boundary, it is very difficult to apply. At any rate, the two published attempts at its utilization [15], [17] report very little success. We shall, therefore, resort to an approximation: by expanding (34) in a Taylor series and using only the first two terms. While we chose to effect this expansion about the point $x = 0$, let us hasten to point out that both here, and in all subsequent developments, this choice is merely a convenience; <u>any</u> other point could have been chosen. We shall write the expansion in the following two, essentially equivalent, forms:

$$v\Big(g(t),t\Big) = v(0,t) + g(t)\, v_x(0,t) + \sum_{n=2}^{\infty} \left\{ \frac{\partial^n}{\partial x^n} v(0,t) \frac{g^n(t)}{n!} \right\} = g'(t) \tag{35a}$$

$$\text{(35b)}\quad v\Big(g(t),t\Big) = v(0,t) + g(t)\, v_x(0,t) + \frac{1}{2}\int_0^{g(t)} y^2\, v_{xxx}\Big(g(t)-y,\, t\Big)\,dy = g'(t)\ .$$

Formula (35a) is the classical Taylor expansion, while (35b) is an expansion with integral remainder. Note that for (35a) we essentially had to assume that v is analytic in x. However, we could avoid even the tacit assumptions inherent in (35b) concerning the third derivative of v, by writing it in the form

$$\text{(35c)}\qquad v\Big(g(t),t\Big) = v(0,t) + \int_0^{g(t)} y\, v_{xx}\Big(g(t)-y,\, t\Big)\, dy = g'(t)\ .$$

One can obtain an estimate of the size of the integral in (35c) by writing it in the form

$$\text{(36)}\quad \left|\int_0^{g(t)} y\, v_{xx}\Big(g(t)-y,t\Big)dy\right| = \left|g(t)\Big(g(t)-g_1(t)\Big)v_{xx}\Big(g_1(t),t\Big)\right| \leq g^2(t)\left|v_{xx}\Big(g_1(t),t\Big)\right|$$

where $g_1(t)$ is some function lying in the band between $x = g(t)$ and $x = 0$. Since we shall have a v such that v_{xx} is continuous, and therefore bounded, formula (36) tells us that for piston oscillations of sufficiently small amplitude the integral in (35c) may be safely neglected. We can, however, get an even better result from (35b); through it, we can allow piston oscillations which are somewhat larger: for in an approximation of the type (36), applied to (35b), we would have the third power of $g(t)$. This, on the other hand, would force us to assume third order continuity for v. That we shall pursue the latter alternative is not as much a choice as it is a necessity, dictated by Burgers' equation. For, as is shown in Appendix A, the prescription of the two conditions

(37) $$v(0, t) = a(t), \quad v_x(0, t) = b(t)$$

is both necessary and sufficient for the attainment of a unique solution, provided $a(t)$ and $b(t)$ satisfy certain regularity conditions. The reason that this mathematical necessity also constitutes physical sufficiency lies in formulas (35b) and (36), together with our remarks in Section 2.0, preceding (8). The former two formulas show that assumptions (37) introduce an approximation of the third order; a fact which is analogous to the approximation used in obtaining (8) from (7).

6.0 GENERAL SOLUTION OF THE PISTON PROBLEM

Under the expansion derived in the preceding section, we can now obtain the solution of

(12) $$v_t + v v_x = \delta v_{xx} \quad , \quad \delta > 0$$

with the boundary conditions

(37) $$v(0, t) = a(t) \quad , \quad v_x(0, t) = b(t) \quad .$$

This solution (see Appendix A) can be written as

(38) $$v(x,t) = \frac{-2\delta \sum_{n=1}^{\infty} \frac{K^{(n)}(t)}{\delta^n} \frac{x^{2n-1}}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{[a(t)\, K(t)]^{(n)}}{\delta^n} \frac{x^{2n}}{2n!}}{\sum_{n=0}^{\infty} \frac{K^{(n)}(t)}{\delta^n} \cdot \frac{x^{2n}}{2n!} - \frac{1}{2\delta} \sum_{n=0}^{\infty} \frac{[a(t)\, K(t)]^{(n)}}{\delta^n} \frac{x^{2n+1}}{(2n+1)!}} ,$$

where

(39) $$K(t) = \exp\left[\frac{1}{2\delta} \int D(t)\, dt\right]$$

and where

(40) $$D(t;\delta) = D(t) = \begin{vmatrix} a(t) & 2\delta \\ b(t) & a(t) \end{vmatrix}$$

We might call (40) the "determinant of nonlinear diffusion". The superscripts in parentheses in (38) denote differentiations with respect to t ; in (39) an indefinite integral is intended; and, in (40), the vertical bars designate a determinant.

Let us make a few observations about this quite general solution. In the first place, we note that dissipation with time is still an exponential phenomenon, as in the linear case; except that here it is far from a simple exponential. As a result, there is a great deal of nonlinear interaction between the various harmonics; a fact which is well known already [1] , [7] , [12] , [13] . One can make, however, an important observation which was previously unnoticed: nonlinearity is <u>not</u> a result of the exponential nature of dissipation. This is significant, for it is an observed fact that in transmission of waves in only slightly elastic media (e.g., ground waves resulting from an explosion), dissipation behaves in an inverse - power manner.

Before we comment on formula (40), let us observe that it essentially represents energy: that applied to the system and the one already in it (viscous mechanism). This we can see if we note that the dimension of D is that of $(\text{velocity})^2$; for

$$D(t) = a^2(t) - 2\delta b(t) = \left(\frac{f(t)}{\sec}\right)^2 - \frac{(ft)^2}{\sec}\,\frac{1}{\sec} = (\text{velocity})^2 .$$

(We already noted, in connection with (3), that the dissipation constant δ has dimensions $ft^2/\sec$.) Now $K(t)$ in (39) is a dimensionless quantity; however, we might term the integral appearing in it as the quantity measuring the impulse of interacting energy,

$$IIE = \int D(t)\, dt \quad .$$

This is a purely nonlinear quantity; it is a measure of the total impact of a force imposed on a viscous system.

It is probably quite remarkable, that the solution of the nonlinear second order equation (12) depends so explicitly on the second order quantity (40). However, an even more striking feature of (40) is the direct coupling between the second order condition $b(t)$ and the dissipation number δ . From a priori considerations one could expect a coupling of this kind and, also, to find no direct connection between $a(t)$ and δ.

But it can be ascribed only to a very fortunate circumstance that these relations are explicit and simple.

We mentioned earlier the speculation concerning the validity of assuming that δ is a constant. If (40) is any indication, as it probably is, we would conclude that here we have yet another motivation for trying to obtain solutions of (12) with a variable δ; at least one which depends on time.

Before we can continue our discussion of the solution (38), we must list the conditions under which it is valid. These are restrictions on the functions $a(t)$ and $b(t)$, in order for (12) to have a unique and twice continuous solution. We list them in an order which is the reverse of that in Appendix A.

Condition 1: each of $a(t)$ and $D(t)$ is analytic for $t > 0$. This condition merely ensures the existence of appropriate solutions for the linear heat equation, from which solutions of Burgers' equations are constructed.

Condition 2: $$\lim_{t \to 0^+} \int_0^t \sqrt{\frac{t}{t-r}}\; a(r)\, K(r)\, dr = 0.$$

Here we have a continuity requirement. Essentially, it states that the system contains no shocks initially.

Condition 3: For any positive number a and at all times past the initial moment, the two expressions

$$\frac{1}{\sqrt{t}}\left[-a + K\left(\frac{1}{t}\right) \pm \frac{1}{2\delta\sqrt{\pi}} \int_0^{1/t} \sqrt{\frac{t}{1-rt}}\; a(r)\, K(r)\, dr\right]$$

must be completely monotonic. (We have two expressions here, because of the $\pm$ signs.) The content of this condition is that, somewhat loosely speaking, K must be a positive function with positive derivatives and that it has to dominate the rest of the terms in the bracket. (A function $P(t)$ is completely monotonic if $(-1)^n\, P^{(n)}(t) \geq 0$ for $n = 0, 1, 2, \ldots$).

Our task is to specify from (35) the $v(0, t)$ and $v_x(0, t)$ that we intend to use in (37). As we remarked earlier, we shall use formula (35b). Neglecting the integral appearing there, we obtain a radiation - type condition:

$$(42) \qquad v(0, t) + g(t)\, v_x(0, t) = g'(t) \quad .$$

We recall that $g(t)$ and $g'(t)$ are the piston displacement and the piston velocity, respectively. Because of the approximation employed, we are in fact free to choose either $v(0, t)$ or $v_x(0, t)$ almost arbitrarily. The choice probably makes a difference in the final results; however, it is not known yet what the full implications are. The meaning of the qualifying adjective "almost" is that the degree to which either $v(0, t)$ or $v_x(0, t)$ can be chosen arbitrarily is limited by the requirements in conditions 1 - 3. The choice we make is taking a vanishing $v(0, t)$:

$$(43) \qquad v(0, t) = a(t) = 0 \; .$$

This then implies that we must have

$$(44) \qquad v_x(0, t) = b(t) = \frac{g'(t)}{g(t)} \; .$$

Then, from (39) and from (40) ,

$$(45) \qquad D(t) = a^2(t) - 2\delta\, b(t) = -\, 2\delta\, \frac{g'(t)}{g(t)} \; ,$$

and

$$(46) \qquad K(t) = \exp\left[\frac{1}{2\delta}\int D(t)\, dt\right] = \frac{1}{g(t)} \quad .$$

That Condition 2 is satisfied and that Condition 3 is much simplified is quite immediate. However, we shall discuss this point more fully in the next section. For now, let us note that the solution, (38), also became simpler: we now have only two summations, one in the numerator and one in the denominator:

$$(47) \qquad v(x,\ t) = -2\delta \frac{\displaystyle\sum_{n=1}^{\infty} \frac{[g^{-1}(t)]^{(n)}}{\delta^n} \frac{x^{2n-1}}{(2n-1)!}}{\displaystyle g^{-1}(t) + \sum_{n=1}^{\infty} \frac{[g^{-1}(t)]^{(n)}}{\delta^n} \frac{x^{2n}}{(2n)!}}$$

Let us note that in assuming (in 43) that $v(0,\ t) = a(t) = 0$ we gained analytical simplicity, but we lost the possibility of a piston motion which passes through $x = 0$. This, however, is immaterial; the piston could be located anywhere. If, for some reason, piston motion with 0 values has to be considered, one simply does not use assumption (43); but for most cases of interest, (47) is just as general as (38); while, at the same time, much more easily analyzable. As a last remark concerning the relative importance of assuming piston motion which never passes through $x = 0$, let us observe that an object traversing the air (projectile, spaceship, etc.) can also be regarded as a "piston"; with motion that is not periodic, but rather monotonic.

Finally, let us raise the following question: what happens if any one of Conditions 1 - 3 is not fulfilled?

In the case of Condition 1 the answer is not yet known. This is because Condition 1 is a sufficient condition; it may not be necessary at all. The probable implication, if it is not fulfilled, is that the series used in (38) may not have meaning, for K may not be differentiable infinitely many times.

Passing on to Condition 2: should it not be satisfied, we would conclude that the state of the medium at time $t = 0$ is one which sustains shocks; and, further, that these shocks emenate from the type of piston excitation applied.

Finally, possibly the most important case that one has to consider is when Condition 3 is violated. For this means that in the general solution,

$$v(x, t) = -2\delta \frac{h_x(x, t)}{h(x, t)} \quad ,$$

the denominator is not positive for all (x, t). Thus, at vanishing values of the denominator shocks appear in the medium.

We have, therefore, a characterization - although only implicit - of those types of piston behavior (described by the boundary functions $a(t)$, $b(t)$) which are shock producing. While an explicit statement may also be possible, its attainment seems to be quite difficult: it would entail solving the two conditions in Condition 3 for classes of functions $\{a(t)\}$, $\{b(t)\}$ which will not yield complete monotonicity.

7.0 THE REDUCTION TO FAY'S SOLUTION

The work of Fay [14] , to which we had referred already, certainly forms one of the most important milestones in the history of the propagation of waves of finite amplitude. However, since the appearance of his classical paper in 1931 no real advance has been made in modifying or in generalizing his results both qualitatively and quantitatively; although some important partial results were obtained ; for example [31] .

In his paper, Fay was not concerned with piston motion, or indeed with any particular boundary value problem. Rather, he sought to determine the periodic elements of the most stable waveform for propagation of the type that we are considering. In the Lagrangian frame of reference that he had cast his equation in, his results could be expressed in the form

$$P = a_0 \sum_{n=1}^{\infty} \frac{\sin n X}{\sinh n \tau} . \tag{48}$$

Here P is pressure (because of the Lagrangian formulation), a_0 a group of constants, τ is a time-like quantity and X is the physical variable.

The only attempt at generalizing this result with a start from Burgers' equation was made by Blackstock; this, however, was not only fallacious - as we pointed out earlier - but also of a very limited scope. We deem it important enough, therefore, to demonstrate that the results achieved by Fay are but a special case of the solutions here presented. Because of the approximations that he employed and the very indeterminate nature of his boundary condition, Fay's results can be expected to be a special case of ours only for a very special type of piston motion. Furthermore, since his results were given in terms of pressure, while ours are in terms of particle velocity, the agreement is merely qualitative. This, however, is sufficient; because both of these functions are significant flow parameters.

We can obtain Fay's result from our solution (47) by assuming a piston motion g described by

$$g(t) = \left[1 + 2 \sum_{k=1}^{\infty} e^{-k^2 t} \right]^{-1} . \tag{49}$$

Then (47) becomes, in successive steps, Fay's solution (48):

$$(50) \qquad v(x,t) = -2\delta \frac{\partial}{\partial x} \ln \left\{ \sum_{n=0}^{\infty} \left[\frac{1}{\delta^n} \left(1 + 2 \sum_{k=1}^{\infty} e^{-k^2 t} \right)^{(n)} \frac{x^{2n}}{2n!} \right] \right\}$$

$$= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + \sum_{n=0}^{\infty} \left[\frac{2}{\delta^n} \left(\sum_{k=1}^{\infty} e^{-k^2 t} \right)^{(n)} \frac{x^{2n}}{2n!} \right] \right\}$$

$$= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + 2 \sum_{n=0}^{\infty} \left[\frac{1}{\delta^n} \left(\sum_{k=1}^{\infty} (-1)^n k^{2n} e^{-k^2 t} \right) \frac{x^{2n}}{2n!} \right] \right\}$$

$$= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + 2 \sum_{k=1}^{\infty} \left[e^{-k^2 t} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(xk/\sqrt{\delta})^{2n}}{2n!} \right) \right] \right\}$$

$$= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + 2 \sum_{k=1}^{\infty} \left[e^{-k^2 t} \cos\left(\frac{xk}{\sqrt{\delta}} \right) \right] \right\}$$

$$= -2\delta \frac{\partial}{\partial x} \ln \left\{ \theta_3 \left(\frac{x}{2\sqrt{\delta}}, \; e^{-t} \right) \right\}$$

$$= -4\delta \sum_{n=1}^{\infty} (-1)^n \frac{\sin\left(\frac{nx}{\sqrt{\delta}}\right)}{\sinh nt}$$

Observe that the last formula here is not quite identical with that of Fay; the discrepancy is the factor $(-1)^n$ inside the summation. We could have obtained Fay's result exactly, had we inserted this factor in the summation of (49); the reason we chose not to do so is our intention to stress the fact that what we have here is indeed a generalized qualitative and quantitative result.

A few words will be necessary to justify the development of (50). We start out with formula (47), written in a compact form, with $g(t)$ given by (49). The first step is a slight rearrangement of this series, which, let us observe, is absolutely and uniformly convergent on $t \geq 0$; and so are its derivatives. This allows us, in the next step, to perform the indicated differentiations term by term; and, after this, to change the order of summations. In this manner we obtain the cosine factor, and reduce the expression to a single sum. Then, using the terminology of [32] , we write the sum as the Jacobian θ_3 function which it defines; finally, from the same source [32] , we obtain the final series expression.

8.0 THE NATURE OF NONLINEAR DISSIPATION WITH TIME

It is well known that, in the small amplitude theory, dissipation with time occurs in an exponential manner. Cole [7] and others found, to a first approximation, that this is essentially what happens in the finite amplitude case also. We will now show that the first approximation employed by these authors is indeed insufficient and that the nature of nonlinear dissipation with time is radically different from that in the small amplitude case.

We shall base our discussion on formula (38), which is the general solution. Our boundary functions we shall obtain from (35b), by assuming that the integral there is very small; so that we shall use

(35b') $$v(0, t) + g(t)\, v_x(0, t) = g'(t) \quad .$$

Here, we recall, $g(t)$ is the path of the piston and $g'(t)$ is velocity; also, we denote, as before, $v(0, t) = a(t)$ and $v_x(0, t) = b(t)$.

In order to obtain our results as rapidly and simply as possible, we shall employ techniques from operational theories. This will commence by a statement of definitions of certain irrational and transcendental differential operators; details concerning these can be found in textbooks on the subjects of operational calculus in general, and fractional integral operators in particular.

(51) Definitions 1.) For $c > 0$,

$$D^{-c}\, q(t) \equiv \int_0^t \frac{(t-r)^{c-1}}{\Gamma(c)}\, q(r)\, dr \quad ;$$

$$D^{c}\, q(t) \equiv D^{n} D^{c-n}\, q(t) \quad ;$$

where $(n - 1)$ is the largest integer contained in c and where D is the ordinary differential operator.

2.) $\cosh x\sqrt{D}\,q(t) \equiv \sum_{k=0}^{\infty} q^{(k)}(t)\,\frac{x^{2k}}{(2k)!}$

3.) $\sinh x\sqrt{D}\,q(t) \equiv \sum_{k=0}^{\infty} Q^{(k)}(t)\,\frac{x^{2k+1}}{(2k+1)!}$;

where $Q(t) = \sqrt{D}\,q(t)$

4.) $e^{x\sqrt{D}} f(t) \equiv \cosh x\sqrt{D}\,f(t) + \sinh x\sqrt{D}\,f(t)$.

Let us now assume the following relationship :

(52) $$D^{-1/2}\Big(a(t)\,K(t)\Big) = -\,2\delta\,K(t) \qquad .$$

We can write this out as

$$-\frac{2\delta}{\sqrt{\pi}}\int_0^t \frac{a(r)\exp\left[\int^r\Big(a^2(t) - 2\delta\,b(t)\Big)dt\right]}{\sqrt{t-r}}\,dr = \exp\left[\int^t\Big(a^2(t) - 2\delta\,b(t)\Big)dt\right] ,$$

which is a complicated integral equation. While nothing has been established concerning its solutions, we shall assume that they exist, even when subject to the restraints in the previous three Conditions and to (35b').

Under such a sweeping assumption, however, the entire expression (38) reduces to

(53) $$v(x_0, t) = -2\delta \frac{\partial}{\partial x}\left[\ln\left(e^{x\sqrt{D}} K(t)\right)\right] =$$

$$= -2\delta\sqrt{D}\, K(t) = a(t)\, K(t) \quad ,$$

where the last equality was obtained by utilizing (52). Let us repeat the first and last expressions here, written out as

(54) $$v(x_0, t) = a(t) \exp\left[\int\left[a^2(t) - 2\delta\, b(t)\right] dt \quad ,\right.$$

where, as before,

$$a(t) = v(0, t)$$

$$b(t) = v_x(0, t) \quad .$$

We shall confine outselves to some brief conclusions in connection with (54). Let us note, first of all, that while the nature of the diffusion with time is still exponential, it is nevertheless radically different from what we have in the linear case. There, time itself, as it were, was the diffusing agent, but here it is the total energy content of the imposed forces that is responsible for dissipation. This is because, essentially, the integrand in (54) is the square of the piston velocity. We also note that in the absence of some viscous mechanism dissipation becomes impossible; for then we would have an ever-increasing velocity. It should be pointed out, that such a conclusion is evidently much more reasonable than the one in the linear case, where we have an effect without a cause.

Formula (54) shows also, from the factor $a(t)$, that the basic course that dissipation takes depends on the piston velocity, whose amplitude is modulated by the effect of the total energy content of the system.

Finally, let us note that (54) is reducible to theories of linear dissipation; for there one assumes that $a(t)$ is very small, while $b(t)$ is constant. Such an assumption reduces our formula to ordinary exponential dissipation with time.

9.0 THE TIME INDEPENDENT CASE

The time independent solution of (12) is always of interest. Of even more interest, however, is that case of the time independent solution which arises from constant boundary conditions. The reason for this is the following. In the linear formulation of physical problems it is relatively easy to spot the physically significant parameters (Reynolds numbers, etc.). However, this task can be very difficult, if not impossible, in the nonlinear case, because of the complicated interaction of the harmonics, which is manifested in the complicated appearance of the various parameters. It is, however, possible to get some notion of the significance of the various parameters by analyzing the relatively simpler time independent situation.

If we let $v_t = 0$ in (12), the equation becomes

$$vv_x = \delta v_{xx} , \tag{55}$$

an expression that can be integrated directly. The solution is

$$v = k_1 \tan\left[\frac{k_1}{2\delta}\left(x + k_2\right)\right] , \tag{56}$$

where k_1 and k_2 are arbitrary real or complex constants. Expression (56) is the most general time independent solution of (12).

Let us now assume that, in formula (40), we are taking

$$a(t) = a_0 , \quad b(t) = b_0 , \tag{57}$$

where a_0 and b_0 is each a constant. Then

(58) $$D(t;\delta) = D(t) = a_0^2 - 2\delta b_0 = D_0 ,$$

so that, from (39) ,

(59) $$K(t) = \exp\left[\frac{1}{2\delta}\int D(t)\,dt\right] = \exp\left[\frac{D_0}{2\delta}\,t\right] .$$

This is the function we shall have to insert in (38) ; to prepare it for such use, let us form

(60) $$K^{(n)}(t) = \left(\frac{D_0}{2\delta}\right)^n \exp\left[\frac{D_0}{2\delta}\,t\right] .$$

Writing (38) in the somewhat simpler form of a logarithmic derivative, and using (60), we get the following:

(61) $$v = -2\delta\frac{\partial}{\partial x}\left\{\ln\left[\sum_{n=0}^{\infty}\left\{\left(\frac{D_0}{2}\right)^n\left(\frac{x}{\delta}\right)^{2n}\frac{\exp\left[\frac{D_0}{2\delta}\,t\right]}{(2n)!}\right\} - \right.\right.$$

$$\left.\left. - \frac{a_0}{2}\sum_{n=0}^{\infty}\left\{\left(\frac{D_0}{2}\right)^n\left(\frac{x}{\delta}\right)^{2n+1}\frac{\exp\left[\frac{D_0}{2\delta}\,t\right]}{(2n+1)!}\right\}\right]\right\} =$$

$$= -2\delta \frac{\partial}{\partial x}\left\{\ln\left[\sum_{n=0}^{\infty}\left\{\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right)^{2n} \frac{1}{(2n)!}\right\} - \right.\right.$$

$$\left.\left. - \frac{a_0}{\sqrt{2D_0}} \sum_{n=0}^{\infty}\left\{\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right)^{2n+1} \frac{1}{(2n+1)!}\right\}\right]\right\} =$$

$$= -2\delta \frac{\partial}{\partial x}\left\{\ln\left[\cosh\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right) - \frac{a_0}{\sqrt{2D_0}} \sinh\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right)\right]\right\} =$$

$$= \sqrt{2D_0}\, \frac{-a_0 \cosh\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right) + \sqrt{2D_0}\sinh\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right)}{-\sqrt{2D_0}\cosh\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right) + a_0 \sinh\left(\frac{x}{\delta}\sqrt{\frac{D_0}{2}}\right)}$$

Let us introduce now two characteristic constants c_1 , c_2 , defined by

(62) $$c_1 = -2\delta \frac{c_2}{a_0}, \quad c_2 = \sqrt{\frac{D_0}{2\delta^2}} .$$

Then the last formula of (61) can be written as

(63) $$v = 2c_2\delta \frac{c_1 \sinh c_2 x + \cosh c_2 x}{\sinh c_2 x + c_1 \cosh c_2 x}$$

The nature of the constants c_1, c_2, but particularly the latter, determines the type of time independent solution that we can have.

Before we can proceed, however, it is necessary to establish the connection between c_1, c_2 on the one hand, and k_1, k_2 of formula (56) on the other.

To this end, let us rewrite (56) in the following form

(64) $$v = k_1 \tan\left[\frac{k_1}{2\delta}\left(x + k_2\right)\right] = k_1 \frac{\tan\left(\frac{k_1}{2\delta} x\right) + \tan\left(\frac{k_1 k_2}{2\delta}\right)}{1 - \tan\left(\frac{k_1}{2\delta} x\right)\tan\left(\frac{k_1 k_2}{2\delta}\right)} =$$

$$= k_1 \frac{\sin\left(\frac{k_1}{2\delta} x\right) + \tan\left(\frac{k_1 k_2}{2\delta}\right)\cos\left(\frac{k_1}{2\delta} x\right)}{\cos\left(\frac{k_1}{2\delta} x\right) - \tan\left(\frac{k_1 k_2}{2\delta}\right)\sin\left(\frac{k_1}{2\delta} x\right)} =$$

$$= k_1 \frac{- i \sinh\left(\frac{k_1}{2\delta} x\, i\right) + \tan\left(\frac{k_1 k_2}{2\delta}\right)\cosh\left(\frac{k_1}{2\delta} x\, i\right)}{\cosh\left(\frac{k_1}{2\delta} x\, i\right) + i \tan\left(\frac{k_1 k_2}{2\delta}\right)\sinh\left(\frac{k_1}{2\delta} x\, i\right)}$$

From (64) we obtain the desired identifications:

(65) $$i\, c_1 = \operatorname{ctn}\left(\frac{k_1 k_2}{2\delta}\right), \quad c_2 = i\, \frac{k_1}{2\delta} .$$

Thus, we can take k_1 as completely arbitrary, but, because of (62), k_2 is given by

$$k_2 = \frac{2\delta}{k_1} \text{ Arcctn } \frac{k_1}{a_0} \quad . \tag{66}$$

This is indeed to be expected; for the time independent solution (63) is but a special case of the most general one, given by (56).

Returning now to (62): we can see from it yet another manifestation of the importance of

$$D_0 = a_0^2 - 2\delta b_0 \quad , \tag{67}$$

as a critical quantity; for, clearly, its parity changes the nature of the solution (63) entirely.

Expression (63) can be utilized in yet another way; through a parametric study, it can give us a qualitative estimate of the relative significance of the various parameters: *changes in the value of the viscosity number* δ *or the piston force input,* characterized by the constants a_0 and b_0. We shall give some simple instances of such a study in the next section.

10.0 GRAPHICAL RESULTS

The graphs that are given in this section are intended to convey qualitative conclusions only; they are not to be taken quantitatively at all. Our principal reason for reproducing them here is to give some limited graphical confirmation to some of our assertions in the report. Accordingly, we chose to illustrate such aspects as was possible to graph within the bounds of minimal computer use.

Each of the graphs is obtained from formula (63), by use of (62) and (67). Thus, the graphs are those of time independent particle velocity. The actual method of graphing was somewhat unusual; the computer divided the vertical axis into uniform portions, different for each case, in accordance with the maximum and minimum values of (63) that it obtained for the domain in question.

A description of the graphs, and some of the conclusions that can be drawn from them, are as follows:

Figures 1a,b,c. The first three figures depict the variations introduced in the time independent solution (63) by changes in the viscosity number δ, for different values of the excitation a_0 b_0. Note that in all three cases linear increase in δ produces a linear increase in v; an effect which is totally unrealistic. However, it supports our contention that δ ought to be regarded as a function of time.

Figures 2a, b. By substituting known values for viscosity number etc., into the relationships (3), (11) defining δ, and considering the approximations used in obtaining our solutions, it can be shown that our results are valid for a range of δ which is approximately $10^{-3} \le \delta \le 2 \times 10^{-3}$. Thus, these figures show that under general excitation ($a_0 \neq 0$, $b_0 \neq 0$) the solution is beginning to become meaningless on the two boundaries of this range; or, perhaps, they are significant only from a statistical point of view.

Figures 3a, b. The only difference between these two figures and the preceding pair is that here we are taking $b_0 = 0$. Since b represents a second order effect, the change introduced into our solution is quite remarkable; it now exhibits once again the type of deterministic behavior that one would expect. (We should note here that the discontinuities on these graphs are effects introduced by the computer, in changing from real to imaginary numbers.)

Figures 4a, b. These two figures intend to illustrate the effect of considering a viscosity number which is too large. As we pointed out previously, the upper limit for δ is 2×10^{-3}. This is given on 4a. If, however, we increase this number by one half of an order of magnitude, the result deteriorates dramatically, as shown on 4b.

Figure 5. If an unreasonably large value is given for δ, then our results can have only a statistical interpretation. This figure is an extreme illustration of this observation.

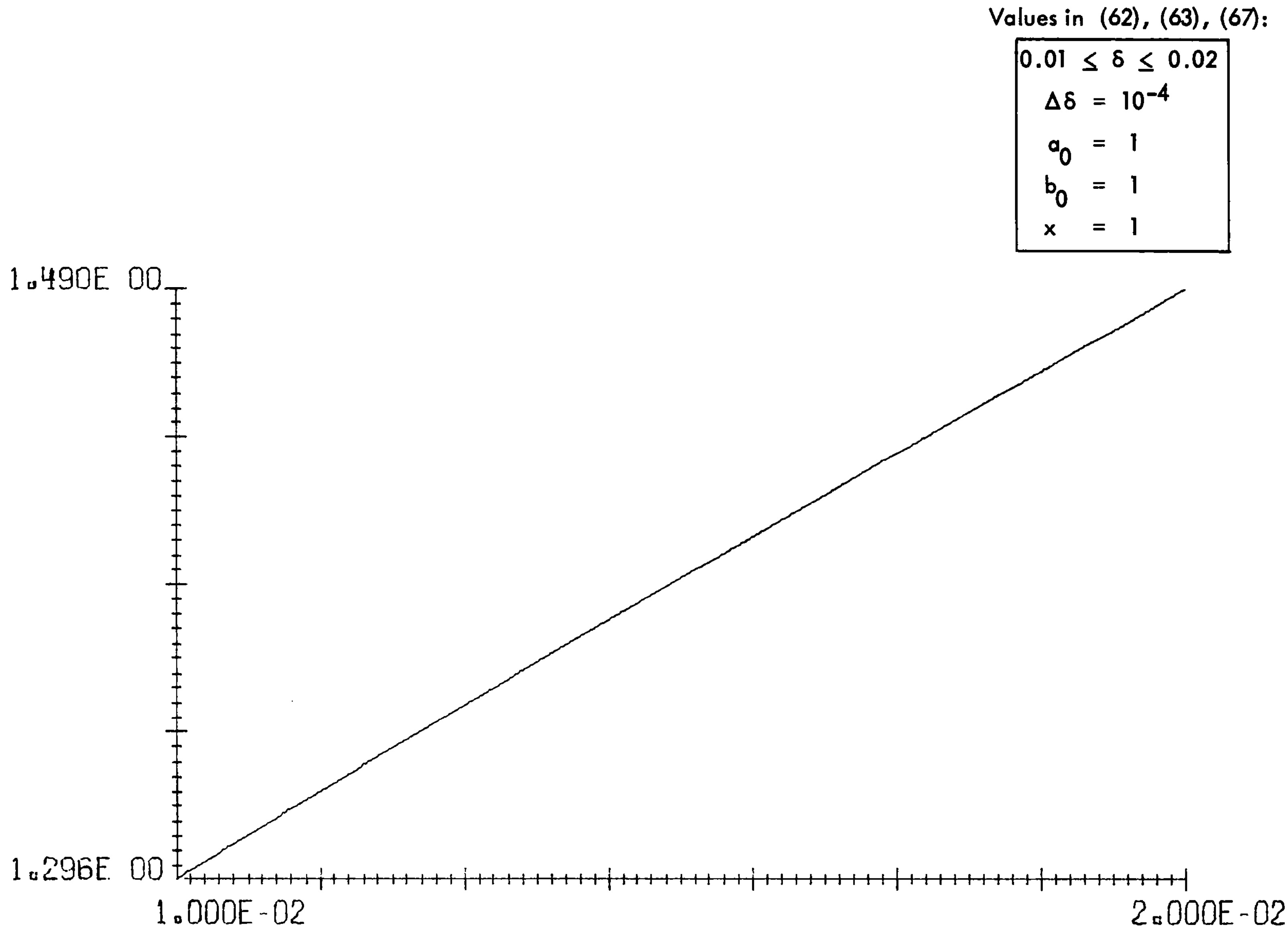

Figure 1a

(For explanation see page 39)

Values in (62), (63), (67):

$0.01 \leq \delta \leq 0.02$
$\Delta\delta = 10^{-4}$
$a_0 = 1$
$b_0 = 0$
$x = 1$

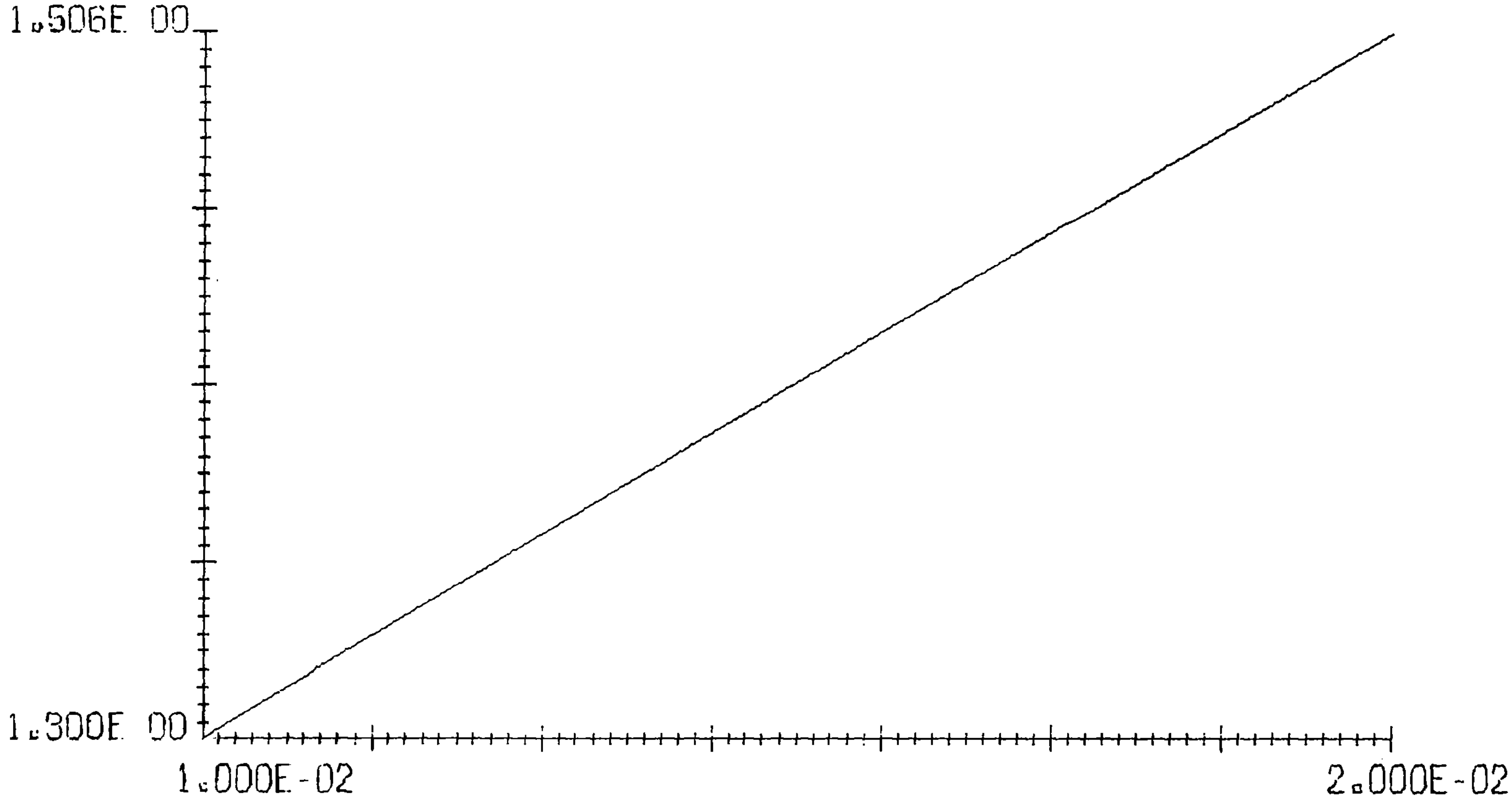

Figure 1b

(For explanation see page 39)

Values in (62), (63), (67):

$0.01 \le \delta \le 0.02$		
$\Delta\delta$	=	10^{-4}
a_0	=	1
b_0	=	- 1
x	=	1

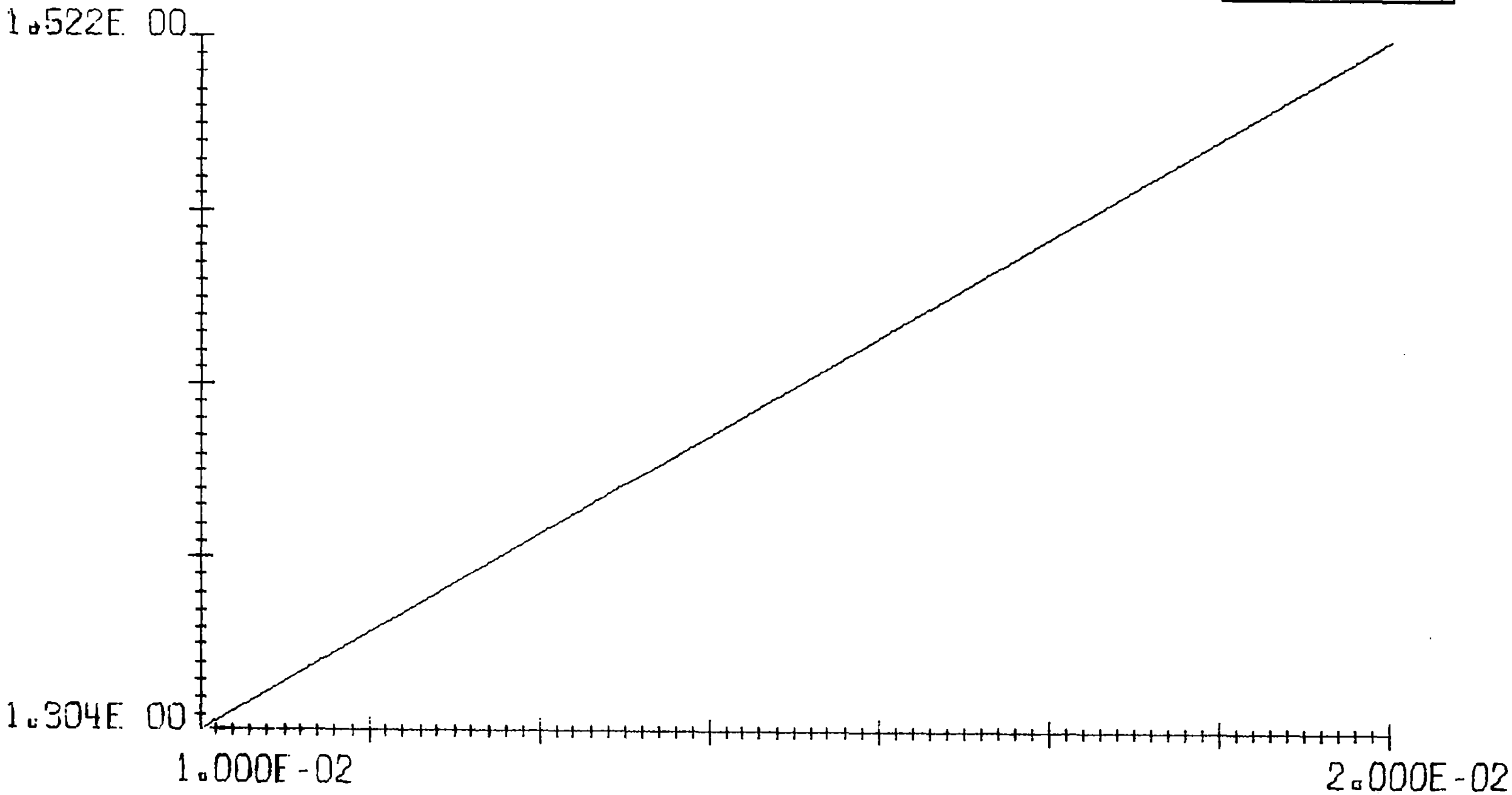

Figure 1c

(For explanation see page 39)

Values in (62), (63), (67):

$-5 \leq a_0 \leq 3$		
Δa_0	=	0.05
b_0	=	1
x	=	1
δ	=	10^{-3}

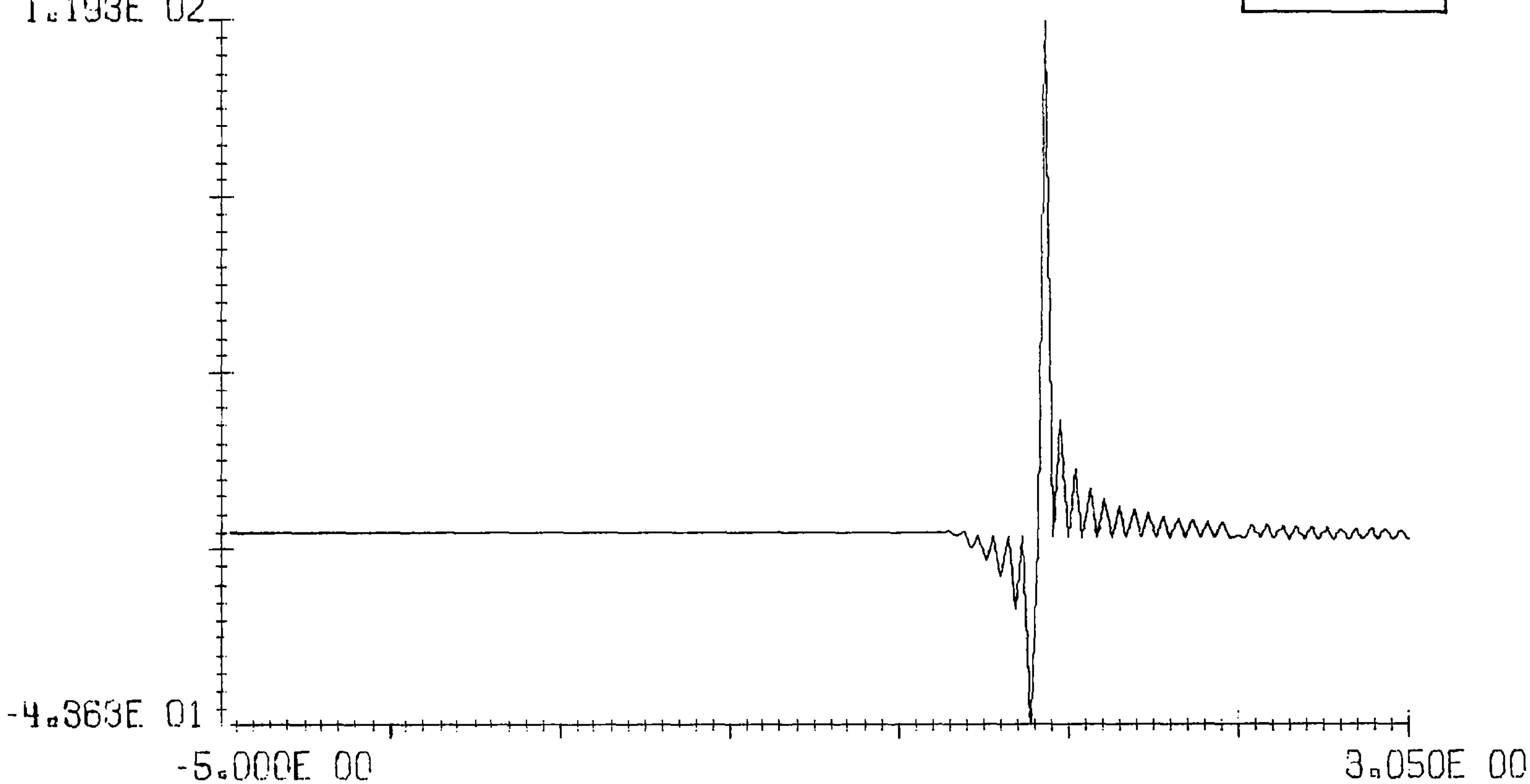

Figure 2a

(For explanation see page 39)

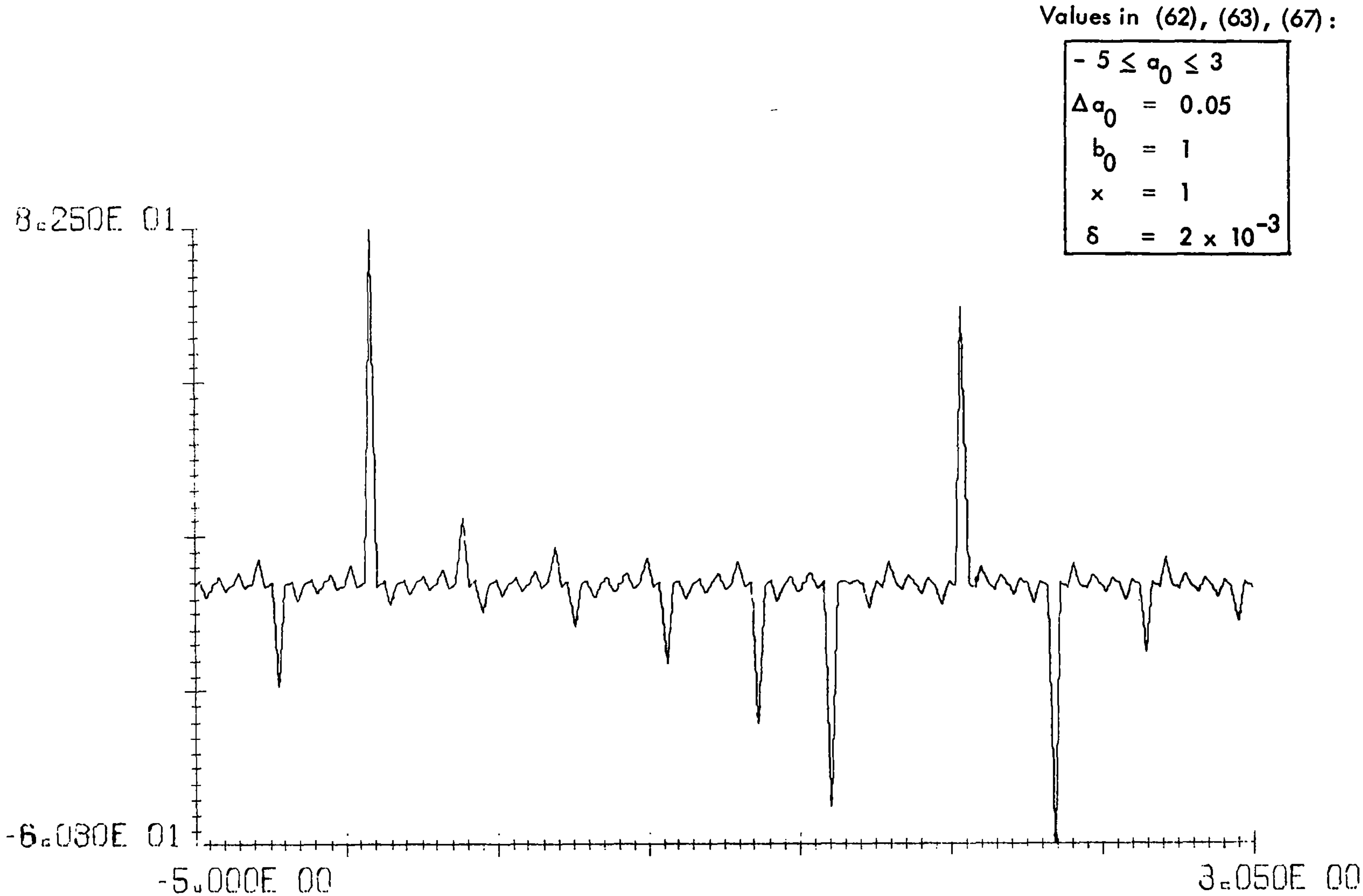

Figure 2b

(For explanation see page 39)

Values in (62), (63), (67):

$-5 \leq \Delta a_0 \leq 3$		
Δa_0	=	0.05
b_0	=	0
x	=	1
δ	=	10^{-3}

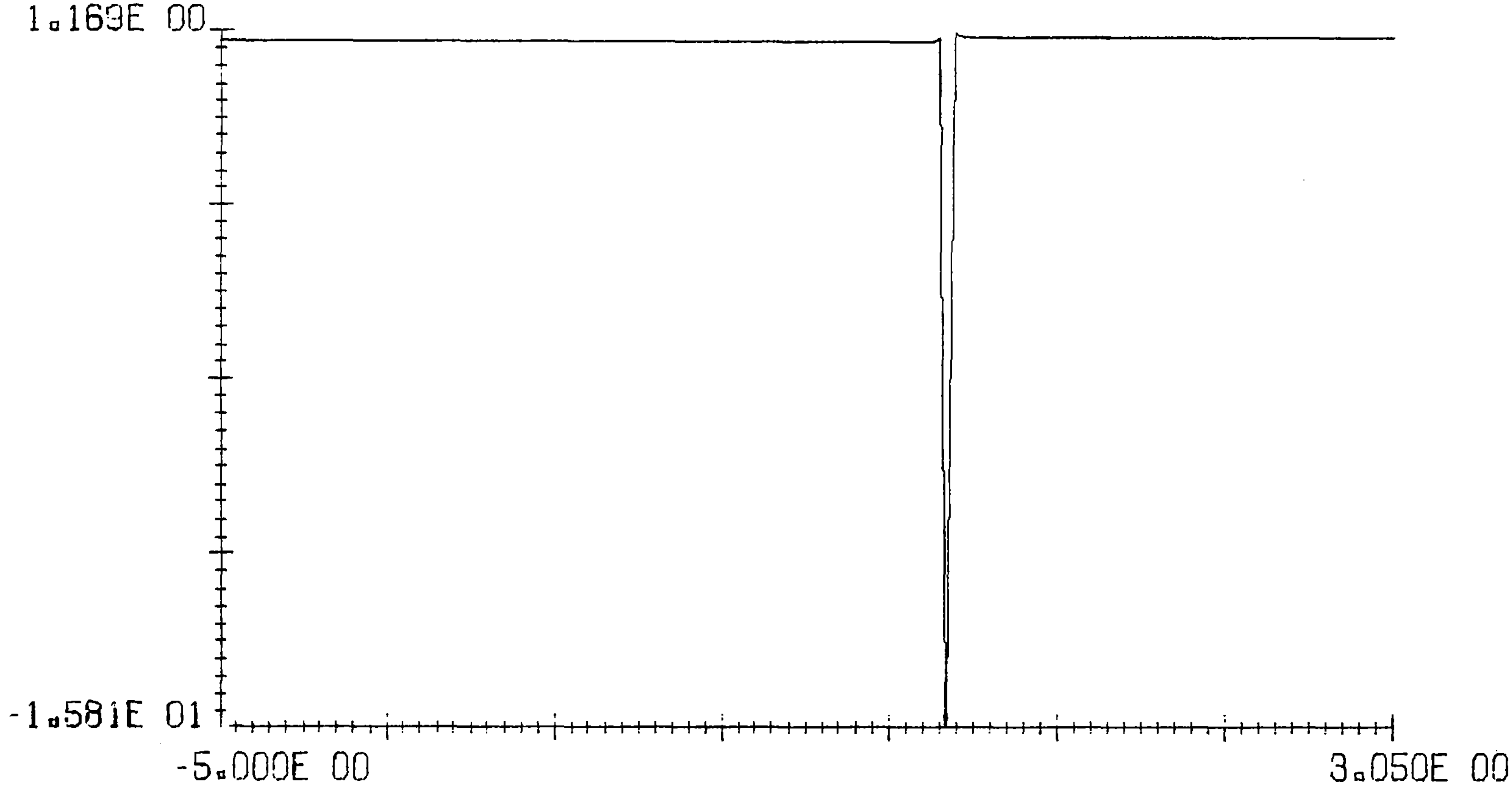

Figure 3a

(For explanation see page 39)

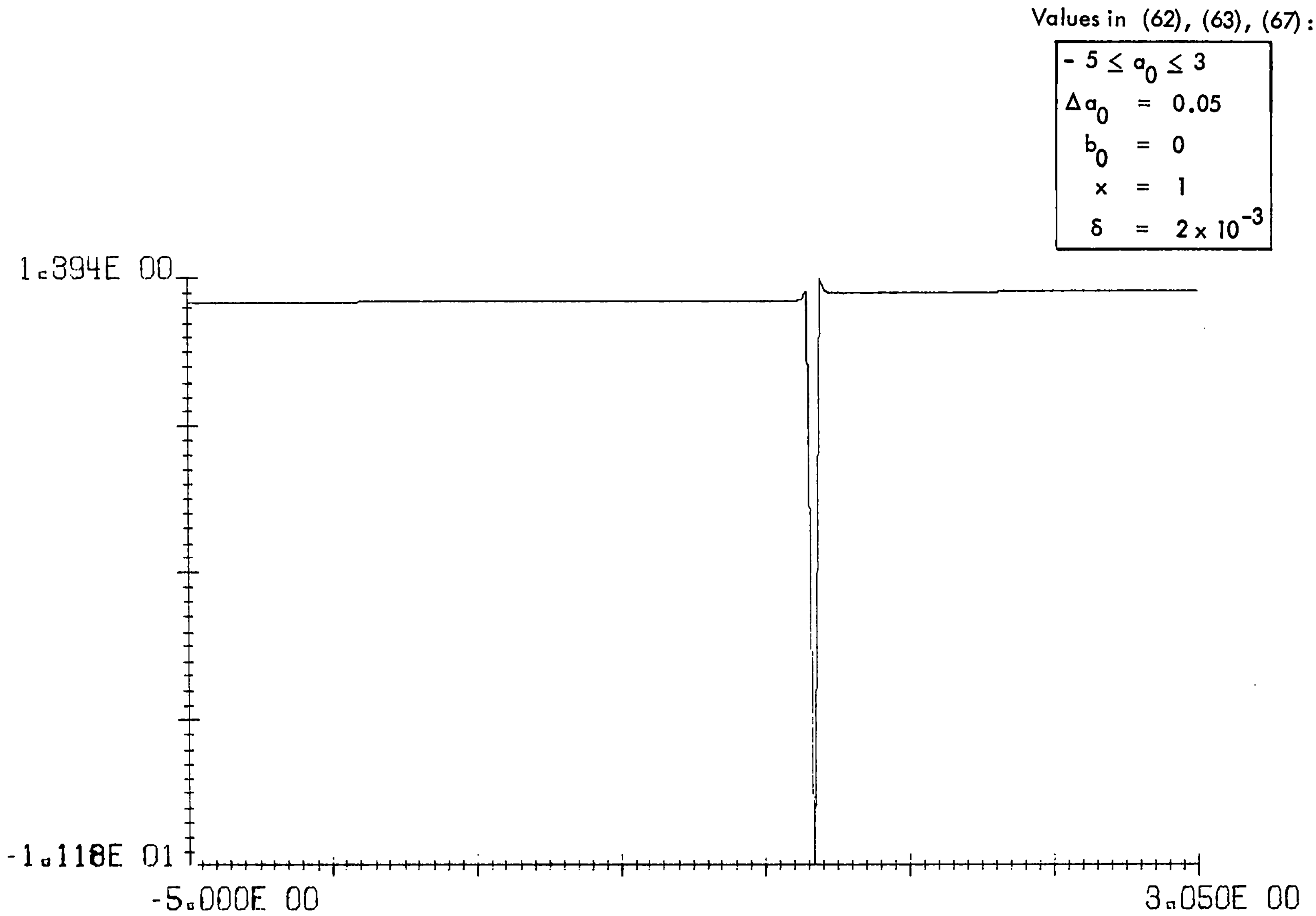

Figure 3b

(For explanation see page 39)

Values in (62), (63), (67) :

$-5 \leq a_0 \leq 3$		
Δa_0	=	0.05
b	=	- 1
x	=	1
δ	=	2×10^{-3}

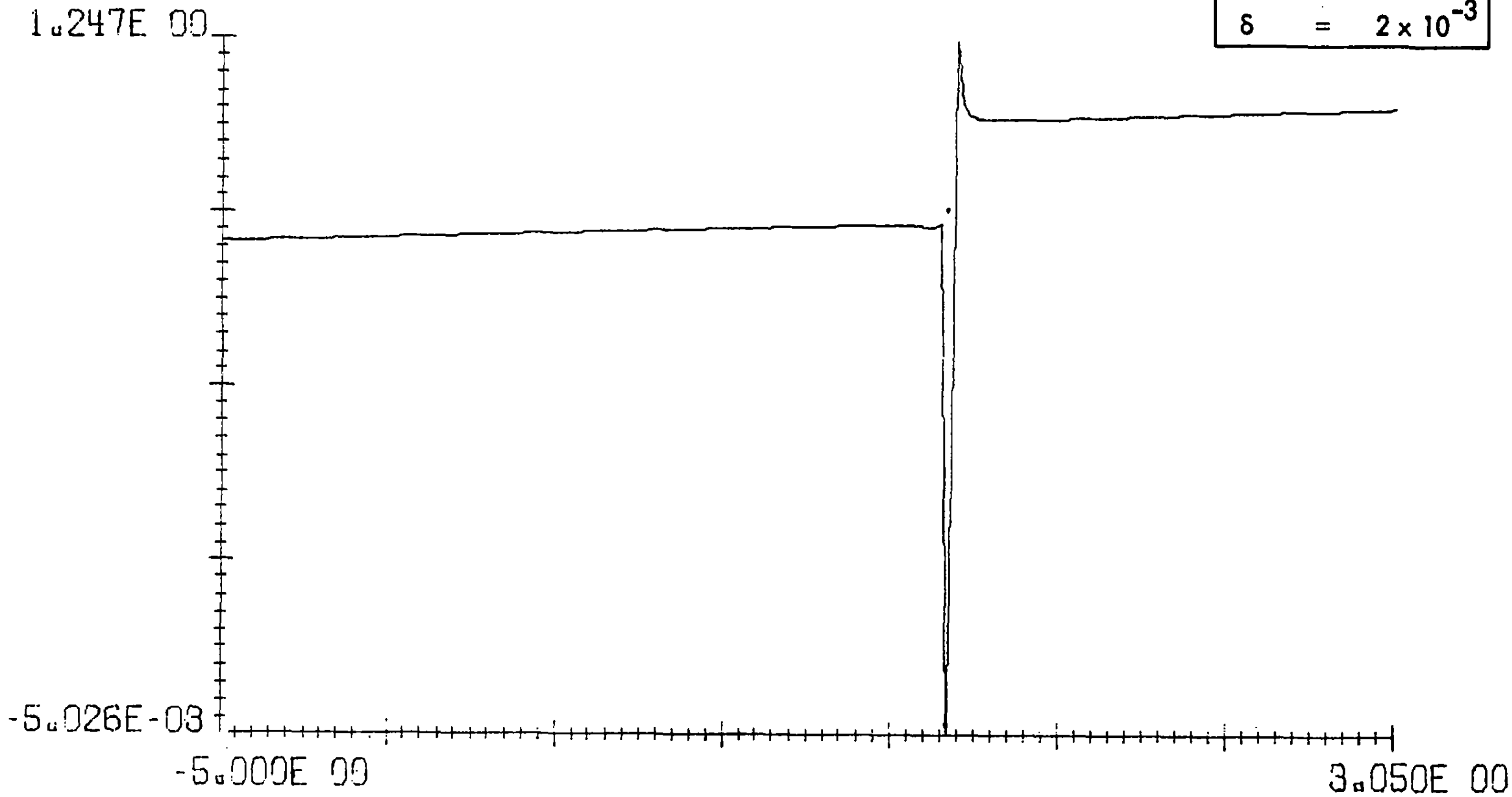

Figure 4a

(For explanation see page 39)

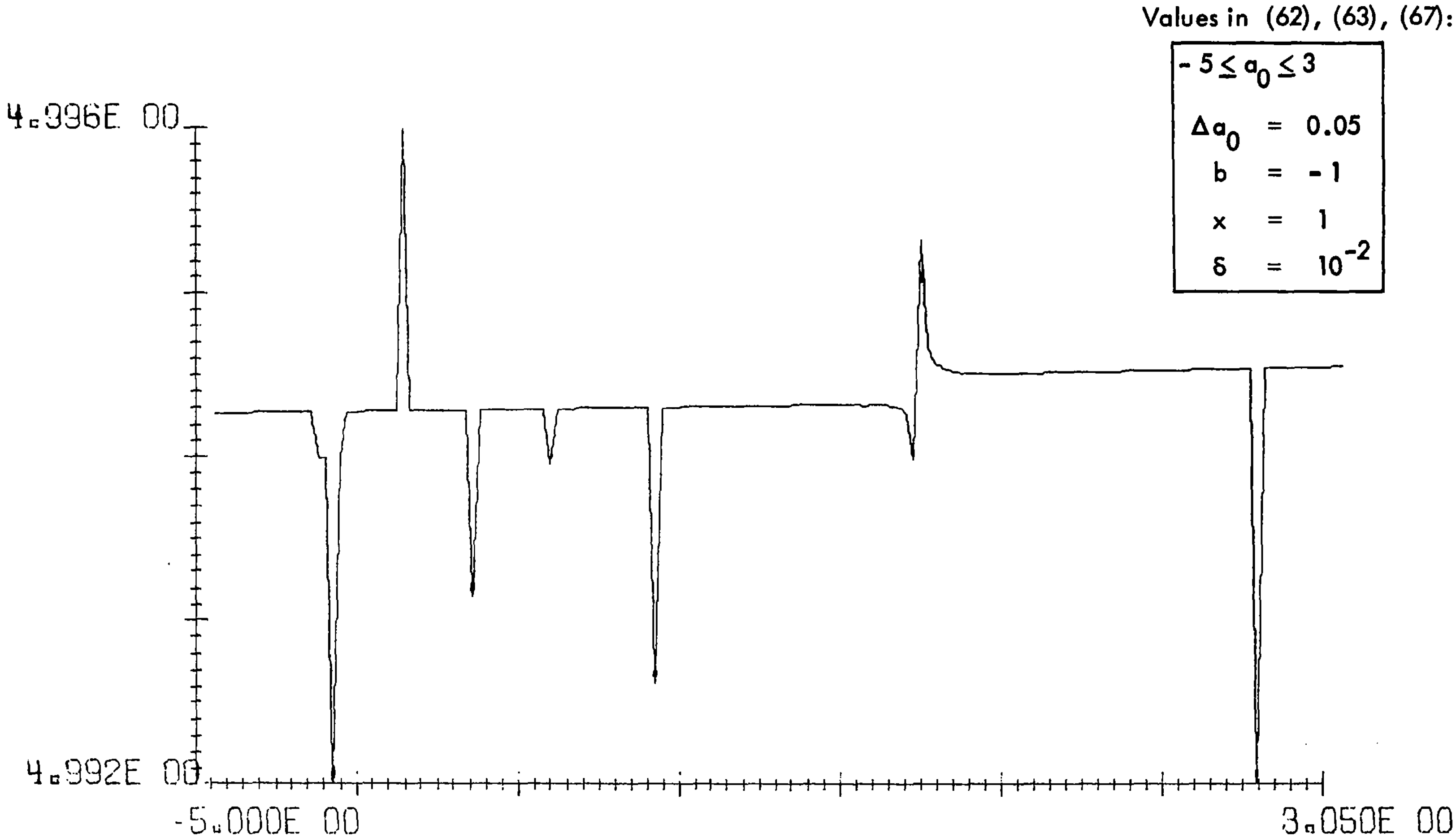

Figure 4b

(For explanation see page 39)

Values in (62), (63), (67):

$-5 \leq a_0 \leq 3$

$\Delta a_0 = 0.05$

$b_0 = -1$

$x = 1$

$\delta = 5$

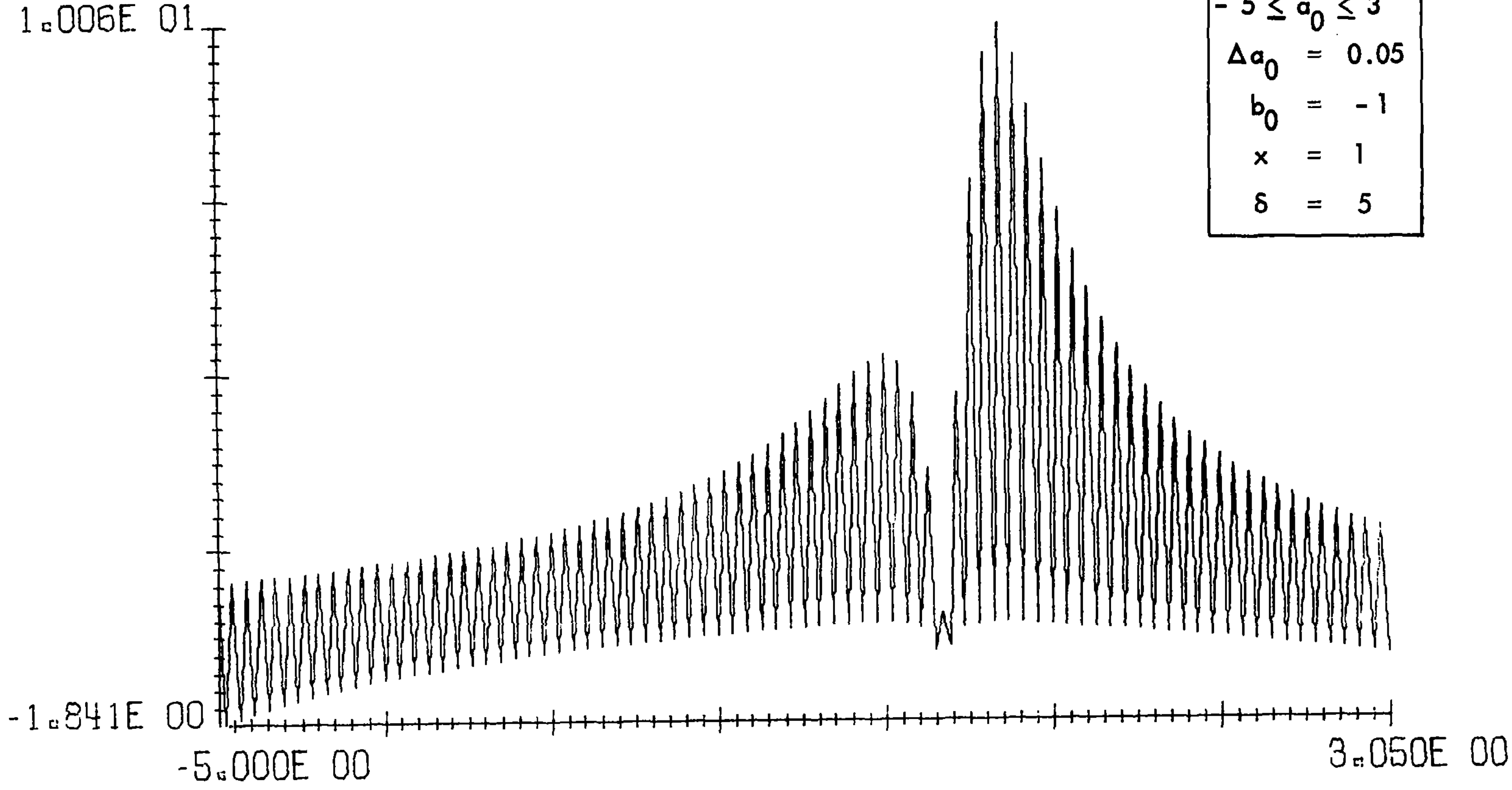

Figure 5

(For explanation see page 39)

DIRECTIONS FOR FUTURE RESEARCH

Because of the wealth of new results presented in this Report it is quite difficult to make an immediate assessment as to which of these can be followed up most fruitfully, and what new directions of research are indicated in general. It seems clear, however, that an effort should be made to solve exactly the piston problem, as defined by the boundary condition (34). A somewhat less ambitious, but nevertheless very important, task would be to solve Burgers' equation with conditions (37), with an arbitrary initial wave shape.

Another avenue of investigation that should be followed up lies in the basic result (38). This very important formula should be analyzed in an attempt to obtain a basic explicit description of the nonlinear interactions of various harmonics. Many approximate results are already available here, particularly in the Russian literature; but they all discuss the first two or three harmonics only.

The fact that we were able to give an analytic description, by means of three conditions, of the eventuality of shocks, should definitely be made more precise. This would involve the task of obtaining that class of functions from these conditions, which are shock-producing.

One of the most important results here is given by formula (54). A basic examination of this formula, together with its physical implications, would probably be a very important contribution towards an explanation of the nature of diffusion.

As a last general possibility we ought to mention here, that while our results have demonstrable physical significance, their extension to a three dimensional framework would be most desirable. Furthermore, the ways of achieving such an extension are quite clear and could be followed up very directly.

APPENDIX A

A BOUNDARY VALUE PROBLEM FOR BURGERS' EQUATION

1. The nonlinear partial differential equation for $v = v(x,t)$

(A1) $$v_t + v\,v_x = \delta v_{xx} \quad , \ \delta > 0$$

first introduced as a model equation for certain aspects of one dimensional viscous flow by Burgers [1;171] and often called Burgers' equation, is one of the very few with the property that it is both closely related to the Navier-Stokes equations and its complete solution is known. This solution, obtained independently by J. Cole and by E. Hopf, (see Bibliography) is a function of an associated linear heat equation

(A2) $$\theta_t = \delta\,\theta_{xx} \quad ,$$

where the δ of (A1) is the same as that of (A2). In particular,

(A3) $$v = -2\delta \frac{\theta_x}{\theta}$$

is a solution of (A1), if θ is a nonnegative solution of (A2), provided certain continuity conditions are met by these functions.

The function that transforms (A1) to (A2) is an exponential one, involving an integration with respect to x. The necessity for positiveness in the associated heat equation is explained by the exponential nature of the transformation. The fact that this transformation also involves a partial integration with respect to x turns out to be important in treating the boundary and/or initial value problems. Thus, it becomes very natural to consider an initial value problem of the type

$$v(x,t) = f(x) \quad \text{as} \quad t \rightarrow 0^{+} .$$

This was discussed extensively both by Cole and by Hopf. In proving the uniqueness of the solution thus found, Hopf made use of a theorem about nonnegative solutions of the heat equation, published by Widder in 1944 [5;85] .

A question of importance which arises in connection with (A1) is when the conditions on v are given on a line $x = x_o$, and which are of the type

$$\text{(A4)} \qquad v(x_o,t) = a(t) \quad , \quad v_x(x_o,t) = b(t) \quad .$$

The boundary value problem for (A2), as defined by (A4), is not a well posed one in the sense of Hadamard. Nevertheless, (A1) and (A4) together define a problem which, besides the intrinsic mathematical interest in it, is applicable both in magnetogasdynamics and in the statistical interpretation of (A1).

To solve this problem, we shall make use of characterizations of positive solutions of the heat equation, given by Widder, which are different from the ones used by Hopf. Thus, the conditions for a unique solution v of (A1) under (A4), such that $v \in C^2(x)$ in some region R of the x,t plane, will be formulated in terms of the concept of complete monotonicity.

It will also be shown, that the crucial quantity in the solution of the second order equation (A1) is a second degree combination D of the boundary functions a and b ,

$$D(t,\delta) = D(t) = \begin{vmatrix} a(t) & 2\delta \\ b(t) & a(t) \end{vmatrix} \tag{A5}$$

In the light of what is known about the viscous solutions of the Navier Stokes equations this form is not surprising. Therefore, some connection between δ (which is positive) and the second order condition $v_x(x_o, t) = b(t)$ is to be expected. Nevertheless, it is interesting to see it exhibited in as explicit a form as given by (A5). However, the particular relevance of this relationship appears only when one considers the solution as $\delta \rightarrow 0$; a subject we shall not discuss here.

2. We shall require four classes of functions in the sequel. It will be convenient to define them at this point.

<u>Definition 1.</u> $u(x,t) \in \bar{H}$ if and only if u satisfies the linear heat equation,

$$u_t = \delta u_{xx} \quad ,$$

in a region R of the x,t plane, such that $u \in C^3(x)$ $\Big($and therefore at least $\in C'(t)\Big)$ there.

Note that contrary to the customary assumption of $C^2(x)$ continuity, we require $C^3(x)$ here. The bar on $\bar{H}$ is a reminder of this; whenever only $C^2(x)$ continuity will be needed, we shall employ the class H.

<u>Definition 2.</u> $u(x, t) \in \bar{H}^+$ if and only if $u \in \bar{H}$ and $u(x,t) \geq 0$ for

$(x,t) \in R$; similarly for the class H^+.

Clearly, $\bar{H}^+$ is a proper subset of $\bar{H}$, and $H^+ \subset H$ also.

<u>Definition 3.</u> $f(y) \in A\,M$ if and only if f is absolutely monotonic on an interval (c, d),

$$f^{(n)}(y) \geq 0\,, \quad \text{for } y \in (c,d) \text{ and } n = 0,1,2,\ldots\ldots.$$

<u>Definition 4.</u> $f(y) \in C\,M$ if and only if f is completely monotonic on an interval (c,d); that is, if $f(-y)$ is absolutely monotonic on $(-d, -c)$.

This means, of course, that in the region of its complete monotonicity, a function f satisfies the inequality

$$(-1)^n \; f^{(n)}(y) \geq 0$$

for each n.

After these preliminaries, we can state the basic result of Hopf in the following form:

Theorem 1. (Hopf, [2; 204]) The following two statements are equivalent:

A) i) v is a solution of (1) in an open rectangle R of the x,t plane
   ii) $v \in C^2(x)$ in R

B) i) For $t \geq 0$, there exists a function Φ of the form
$$\Phi = C(t)\ \exp\left[-\frac{1}{2\delta}\int v(x,t)\,dx\right], \text{ such that}$$
   ii) $\Phi \in \bar{H}^+$

The proof of these statements can be easily given by subjecting (1) to the transformation defined by

$$\Phi = \exp\left[-\frac{1}{2\delta}\int v(x,t)\,dx\right].$$

Part B of Theorem 1. gives the explicit form of the exponential transformation, referred to earlier. On examining it, one sees immediately how the condition $v(x,0^+) = f(x)$ is "translated" to θ. Our next theorem will contain in it that form of the solution of the heat equation, which lends itself to the problem defined by (A4). We shall also fix x_o, as $x_o = 0$; this will entail no essential loss of generality.

Theorem 2. Let $p(t)$ and $q(t)$ be analytic functions for $t > 0$. Then the conditions

i) $\theta(x,t) \in H$ on $-\infty < x < \infty$, $t > 0$

ii) $\theta_x(0,t) = p(t)\ \theta(0,t)$ and $\theta_{xx}(0,t) = q(t)\ \theta(0,t)$

are necessary and sufficient for θ to have the representation

$$\text{(A6)}\quad \theta(x,t) = C\left\{\sum_{n=o}^{\infty} \frac{\{\exp[\delta\int q\,dt]\}^{(n)}}{\delta^n}\frac{x^{2n}}{2n!} + \sum_{n=0}^{\infty}\frac{\{p\exp[\delta\int q\,dt]\}^{(n)}}{\delta^n}\frac{x^{2n+1}}{(2n+1)!}\right\}$$

where C is an arbitrary constant. Here the superscripts denote derivatives with respect to t.

To prove this, let $w(t) \in C^{\infty}$ be an arbitrary function, positive for each $t \geq 0$. Then, since solutions of the heat equation are of C^{∞} in t for fixed x, we can write in place of ii) above:

$$\text{(A7)}\qquad \text{ii)}\left\{\begin{aligned}\theta(0,t) &= w(t)\\ \theta_x(0,t) &= p(t)\ w(t)\\ \theta_{xx}(0,t) &= q(t)\ w(t)\end{aligned}\right.$$

For a particular w, the heat equation, together with the first two conditions of (A7), has a unique solution (see D.V. Widder, [4; 292]), of the form

$$\text{(A8)} \quad \theta(x,t) = \sum_{n=0}^{\infty} \frac{w^{(n)}}{\delta^n} \frac{x^{2n}}{2n\,!} + \sum_{n=0}^{\infty} \frac{(pw)^{(n)}}{\delta^n} \frac{x^{2n+1}}{(2n+1)\,!}$$

which, for any fixed real x is an analytic function of t in $t > 0$, and, for fixed $t > 0$, is entire in x. Therefore, term by term differentiation is justified and we obtain from (A8)

$$\text{(A9)} \qquad \begin{aligned} \theta\,(0,t) &= w(t) \\ \theta_x\,(0,t) &= p(t)\,w(t) \\ \theta_{xx}\,(0,t) &= \frac{1}{\delta}\,w'(t). \end{aligned}$$

The first two entries of (A9) coincide, of course, with those of (A7). To complete the agreement in the third lines also, we must have

$$\text{(A10)} \qquad w(t) = C \exp\,[\,\delta \int q(t)\,d\,t\,]\,,$$

where C is an arbitrary (positive) constant. Using (A10) in (A8) we obtain (A6), which then proves the necessity.

The proof of sufficiency is immediate. Clearly, (A6) is $C^2(x)$ and it satisfies the heat equation for $-\infty < x < \infty$, $t > 0$. This can be checked by term-by-term differentiation, justified above. We similarly obtain the conclusions that the conditions ii) are also satisfied.

3. Our solution v will be constructed of a function similar to (A6). It is now

necessary to ensure that it be positive. To do that, we shall employ a characterization of nonnegative solutions of the heat equation, satisfying the conditions $\theta(0,t) = f(t)$, $\theta_x(0,t) = g(t)$, as given by Widder [4;279] and mentioned previously. Eventually, this will yield the entire $C^2(x)$ solution set of (A1) satisfying $v(0,t) = a(t)$, $v_x(0,t) = b(t)$, and no other functions. We state this characterization as

Theorem 3. Let $\theta(x,t) \in H$ on $-\infty < x < \infty$ and $t > 0$, such that $\theta(0,t) = f(t)$, $\theta_x(0,t) = g(t)$. Then the three conditions

$$A : \left\{ \left[f\left(\frac{1}{t}\right) \pm \frac{1}{\sqrt{\pi}} \int_0^{1/t} \sqrt{\frac{t}{1-rt}} \; g(r)\,dr \right] \frac{1}{\sqrt{t}} \right\} \in CM \text{ for } t > 0$$

$$B : \lim_{t \to 0^+} \int_0^t \sqrt{\frac{t}{t-r}} \; g(r)\,dr = 0$$

are necessary and sufficient for $\theta(x,t) \in H^+$ on this range.

A trivial example satisfying conditions A, B is one involving the functions $g(t) = 0$ and $f(t) = t^{-1/2}$. However, it is worthwhile to look at the example where

$$g(t) = e^{at} \qquad , \; a \neq 0$$

with f as yet unspecified. We shall first take the case $a > 0$. Then part B of Theorem 3 becomes

$$\lim_{t\to 0^+} \sqrt{t}\int_0^t \frac{1}{\sqrt{t-r}} e^{ar} dr = \lim_{t\to 0^+} \sqrt{t}\left[\frac{1}{\sqrt{t}} * e^{at}\right] =$$

$$= \lim_{t\to 0^+} \sqrt{t}\left\{ L^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}(s-a)}\right]\right\} = \lim_{t\to 0^+} \sqrt{\frac{\pi t}{a}}\; e^{at} \operatorname{erf}\left(\sqrt{at}\right) = 0 ,$$

so that this part is satisfied. The $*$ in this notation means convolution and L^{-1} designates an inverse Laplace transform. An examination of this evaluation shows, however, that while it is possible to satisfy part A of the theorem also, it would involve taking for f functions that are much too complicated for the purposes of an example. We pass on therefore, to the case $a < 0$. One then obtains

$$\lim_{t\to 0^+} \sqrt{t}\int_0^t \frac{1}{\sqrt{t-r}} e^{ar} dr = \lim_{t\to 0^+} \sqrt{t}\left[\frac{1}{\sqrt{t}} * e^{at}\right] =$$

$$= \lim_{t\to 0^+} \sqrt{t}\left\{ L^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}(s-a)}\right]\right\} = \lim_{t\to 0^+} 2\left\{\sqrt{\frac{t}{-a}}\; e^{-at}\int_0^{\sqrt{-at}} e^{r^2} dr\right\} = 0$$

The integral in part A becomes

$$\left[\frac{2}{\sqrt{-\pi a}} e^{-au}\int_0^{\sqrt{-au}} e^{r^2} dr\right]_{u=\frac{1}{t}}$$

If we note now that this expression is bounded, and at $u = t$ it is absolutely monotonic, together with the following facts

i) $t^{-1/2} \in CM$, and $t^{-1} \in CM$, for $t > 0$

ii) If $f_1 \in AM$ on D_1 and $f_2 \in CM$ on D_2, then $f_1(f_2) \in CM$ on $D_2 \cap R_1$, where R_1 is the range of f_1.

iii) $f_1 \in AM$ and $f_2 \in AM$ imply $(f_1 + f_2) \in AM$; similarly for CM; the same holds for products.

then we see that our choice for an appropriate f is much easier. In fact, it has to be absolutely monotonic and bounded below in a certain manner.

4. Let us now consider a solution $v = v(x,t)$ of (1), satisfying conditions A of Theorem 1, such that $v(0,t) = a(t)$ and $v_x(0,t) = b(t)$. This v defines a function $\Phi = \Phi(x,t)$, according to the second part of that theorem, such that

(11A) $$\Phi(x,t) = C(t) \exp\left[-\frac{1}{2\delta} \int v(x,t)\, dx \right] .$$

Denoting now

$$\exp\left[\frac{1}{2\delta} \int v(x,t)\, dx \right] \Bigg|_{x=0} = v_o(t) ,$$

we can write

(12A) $$\Phi(0,t) = C(t)\, v_o(t)$$

and we know that $\Phi(0,t) > 0$ for $t > 0$. Furthermore, we can differentiate (11A) to obtain

(13A) $$\Phi_x(0,t) = -\frac{1}{2\delta}\, a(t)\, C(t)\, v_o(t)$$

and

$$\text{(14A)} \qquad \Phi_{xx}(0,t) = \frac{1}{4\delta^2} C(t)\, v_o(t)\, [a^2(t) - 2\delta\, b(t)] = \frac{1}{4\delta^2} C(t)\, v_o(t)\, D(t)$$

Combining (12A), (13A) and (14A), we have

$$\text{(15A)} \qquad \begin{aligned} \Phi_x(0,t) &= -\frac{1}{2\delta} a(t)\, \Phi(0,t) \\ \Phi_{xx}(0,t) &= \frac{1}{4\delta^2} D(t)\, \Phi(0,t) \end{aligned}$$

If now each of $a(t)$ and $D(t)$ is analytic for $t > 0$, we have, by Theorem 2, that

$$\text{(16A)} \qquad \Phi(x,t) = C \left\{ \sum_{n=0}^{\infty} \frac{\{\exp[\frac{1}{2\delta}\int D\,dt]\}^{(n)}}{\delta^n} \frac{x^{2n}}{(2n)!} - \right.$$

$$\left. - \frac{1}{2\delta} \sum_{n=0}^{\infty} \frac{\{a \exp[\frac{1}{2\delta}\int D\,dt]\}^{(n)}}{\delta^n} \frac{x^{2n+1}}{(2n+1)!} \right.$$

with C arbitrary $\neq 0$.

Recalling now that

$$v = -2\delta \frac{\Phi x}{\Phi}$$

and writing this in the form

$$v = -2\delta \frac{(\Phi - \epsilon)_x}{(\Phi - \epsilon) + \epsilon}$$

where ϵ is an arbitrary positive constant, we shall utilize Theorem 3 in connection with the function $F = (\Phi - \epsilon)$. The boundary values for F_x and Φ_x are the same · for F and Φ the relation is

$$F(0,t) = \Phi(0,t) - \epsilon$$

Letting now

$$K(t) = \exp\left[\frac{1}{2\delta}\int D(t)\,dt\right],$$

where $D(t)$ is given by (5A), we state our main result as

Theorem 4. Suppose that, for some $\epsilon > 0$,

i) $$\left[-\epsilon + K\left(\frac{1}{t}\right) \pm \frac{1}{2\delta\sqrt{\pi}}\int_0^{1/t} \sqrt{\frac{t}{1-rt}}\; a(r)K(r)\,dr\right]\frac{1}{\sqrt{t}} \in CM \text{ for } t > 0$$

ii) $$\lim_{t \to 0^+}\int_0^t \sqrt{\frac{t}{t-r}}\; a(r)K(r)\,dr = 0$$

iii) each of a and D is analytic for $t > 0$.

Then

$$\text{(17A)}\quad v = \frac{-2\delta\displaystyle\sum_{n=1}^{\infty}\frac{K^{(n)}(t)}{\delta^n}\frac{x^{2n-1}}{(2n-1)!} + \sum_{n=0}^{\infty}\frac{[a(t)\,K(t)]^{(n)}}{\delta^n}\frac{x^{2n}}{2n!}}{\displaystyle\sum_{n=0}^{\infty}\frac{K^{(n)}(t)}{\delta^n}\frac{x^{2n}}{2n!} - \frac{1}{2\delta}\sum_{n=0}^{\infty}\frac{[a(t)\,K(t)]^{(n)}}{\delta^n}\frac{x^{2n+1}}{(2n+1)!}}$$

is a solution of (1) such that, for $t > 0$, $-\infty < x < \infty$

(a) $v \in C^{(2)}(x)$

(b) $v(0,t) = a(t)$, $v_x(0,t) = b(t)$

(c) v is unique.

That v is unique, we can see from the fact that Φ is unique under the stated conditions, up to a nonzero multiplicative constant. This, however, cancels out in the ratio, rendering v completely unique.

5. Examples. Solutions of nonlinear partial differential equations which are independent of time are always of some interest. Thus, in the case of (1), we obtain the simplest such solution from the time-independent solution of the heat equation. It is of the form

$$v_1 = \frac{-2\delta\, c_1}{c_1 x + c_2} \tag{18A}$$

where c_1 and c_2 are arbitrary constants. Another solution of this type is obtainable by solving (1) directly, with $v_t = 0$. This will have the form

$$v_2 = k_1 \tan\left[\frac{k_1}{2\delta}(x + k_2)\right]. \tag{19A}$$

Here, also, k_1 and k_2 are arbitrary (real or complex) constants.

An interesting form of the time independent solution (19A) can be obtained from (17A). One way of doing it is assuming that each of $a(t)$ and $b(t)$ is a constant,

$$a(t) = a_o , \quad b(t) = b_o .$$

Then

$$D_o = D(t) = a^2(t) - 2\delta\, b(t) = a_o^2 - 2\delta\, b_o .$$

In the general case $D_o \neq 0$, so that

$$K(t) = \exp\left[\frac{1}{2\delta}\int D(t)\,dt\right] = \exp\left[\frac{D_o}{2\delta}\, t + k\right]$$

where k is arbitrary. Using this value of K in (17A), simplifying and writing the four series in terms of the appropriate hyperbolic functions we obtain

$$\text{(20A)} \qquad v_3 = a_o c_1 \frac{c_1 \sinh c_2 x + \cosh c_2 x}{\sinh c_2 x + c_1 \cosh c_2 x}$$

where

$$c_1 = -2\delta \frac{c_2}{a_0} , \qquad c_2 = \sqrt{\frac{D_o}{\delta}}$$

This, then is the third type of stationary solution. It is, of course, reducible to (19A) ; in fact, letting

$$i c_1 = \operatorname{ctn}\left(\frac{k_1 k_2}{2\delta}\right) , \quad c_2 = i \frac{k_1}{2\delta}$$

in (19A) we obtain (20A).

As a final note, it will be worthwhile to point out the importance of the positive constant

ϵ in Theorem 4. Let us suppose that we are given the boundary conditions $v(0,t) = 0 = v_x(0,t)$. Then, of course, we would expect to obtain the solution $v \equiv 0$.

Under these zero initial conditions $D(t) = 0$ itself, so that $K(t) = K_o$, an arbitrary constant. That it is not completely arbitrary, however, we can see from condition i) of Theorem 4. ; in particular, it tells us that K_o must be positive. We get this because of the presence of ϵ there; otherwise a nonnegative K_o would do also. However, an examination of (17A) shows that only $K_o > 0$ will give the expected $v = 0$ solution.

6. Acknowledgements. It is a pleasure to acknowledge my indebtedness to Professor D.V. Widder of Harvard University, for his critical comments on the first draft of this paper.

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APPENDIX B

ON A PHENOMENOLOGICAL FAUX PAS OF FINITE AMPLITUDE

In recent years a very fruitful method of analyzing propagation of waves of finite amplitude in viscous media came to the fore. It employs, as the descriptor of such phenomena, a time dependent, one dimensional nonlinear partial differential equation of the second order, first introduced by Burgers[1] in 1948 as a model for the description of turbulence. This equation bears a very strong resemblence to the one dimensional Navier-Stokes equations with linear dissipation. Its most important virtue, however, is that its complete and explicit solution is known. This was published in 1950 and 1951, by E. Hopf[2] and J.D. Cole[3], respectively, who obtained their results independently of each other.

Since the number of physically significant nonlinear partial differential equations, for which explicit solutions are known, is very small[4], and since the solution of Burgers' equation represented an important step forward in the field of continuum mechanics, workers in many fields attempted to make use of it. This was and is being done in several ways: by using the equation in its original form, and solving boundary and/or initial value problems for it[5]; by a reinterpretation of the dependent and/or independent variables, so that the equation will be the culmination of a sequence of rational approximations of a higher order equation or system of equations[6]; and by the generalization of the equation itself[7].

The history of modern science is replete with examples of scientists possessing great physical insight, coupled with a lesser degree of mathematical acuity. In fact, many results, now deemed classical, had very little mathematical justification; or, indeed, were based on erroneous arguments. In such cases the historical course of events seems to have supplied, at a later time, the necessary mathematical arguments which vindicated the physical results from a formal point of view also. Thus, the critique we shall here give ought to be so viewed; particularly since the conclusions of those who based their work on assumptions we shall show to be insupportable, are certainly interesting.

I. BURGERS' EQUATION

Burgers' equation, as originally given, can be written as

$$\text{(B1)} \qquad v_t + v v_x = \delta v_{xx}$$

Here $v = v(x, t)$ is a certain excess velocity (turbulence), t is time, x the physical coordinate and δ a positive constant, being a measure of dissipation[8]. The subscripts denote partial derivatives with respect ot the symbol involved. The solution of (B1) is given by[2, 3]

$$\text{(B2)} \qquad v(x, t) = - 2\delta \frac{h_x(x, t)}{h(x, t)}$$

where h itself is any positive solution of the associated linear heat equation

(B3) $$h_t = \delta h_{xx} .$$

Because of the peculiar form of the solution (B2) - which, incidentally, is a purely accidental feature - the most "natural" problem to consider is an initial value one, defined by

(B4) $$v(x, 0) = f(x) .$$

The problem defined by (B1) and (B4) could be construed, for instance, as one describing the decay of a wave in a pipe of infinite length, with no external excitation, given the shape of the wave at time $t = 0$.

In order to obtain the value of v from (B2), we must first "translate" the condition (B4) to the heat equation (B3). This can be done by integrating relation (B2) and exponentiating it, to obtain

(B5) $$h(x, t) = \exp -\left[\frac{1}{2\delta}\int^{x} v(y, t)\, dy\right]$$

Then, from (B4) and (B5) we obtain the initial condition $h = h(x, t)$ must satisfy, in order for it to serve as a building block in the desired solution (B2); it is

$$(B6) \qquad h(x, 0) = \exp\left[- \frac{1}{2\delta} \int^{x} f(y)\, dy \right] = F(x)$$

Those solutions of the heat equation (B3) which we shall need here can all be written in the form of the classical Poisson integral,

$$(B7) \qquad h(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\, F(y)\, dy \quad .$$

Here $k(x, t)$ is the source solution of (B3), while F comes from (B6). Formulas (B2), (B6) and (B7) constitute the complete "recipe" from which one can obtain the desired solution.

II. TIME DEPENDENT EXCITATION

A problem which, from a practical point of view, is more important than that defined by (B4) is a boundary value problem. For instance, we might inquire what the flow regime will be if, at some point $x = x_o$ in our infinite pipe, a piston is set into motion. If the piston motion is described by $x = g(t)$, we could make appropriate.assumptions to claim that at a certain distance from the piston, at $x = x_1$, the fluid particles behave according to the rule $v(x_1, t) = g(t)$. This then could formulate a boundary value problem.

In a 1953 paper[9], in an attempt to explain the nature of nonlinear dissipative distortion of sound waves of finite amplitude, J.S. Mendousse introduced a "universal" equation for the particle displacement u . Having eliminated the dimensional constants and having performed a slight transformation on the variables, he gave the equation as

(B8) $$u_{\tau\tau} - 2u_{X\tau} + 4u_X u_{XX} - 2u_{\tau XX} + 2u_{\tau XXX} = 0$$

For our purposes it will be sufficient to note only that τ here is a time-like and X a space-like variable. Mendousse then proceeded to simplify this equation in a very ingenious - and physically impeccable - manner, to obtain the two equations

(B9a) $$\theta_\tau - 2\theta\theta_X = \theta_{XX}$$

and

(B9b) $$\theta_X + 2\theta\theta_\tau = \theta_{\tau\tau} \quad .$$

The quantity θ in these is defined by $u_x = \theta$. He then stated that (B9a) is suitable for shock-wave problems, with an initial condition of the type

(B10a) $$\theta(X, 0) = F(X) \ ;$$

while (B9b) should serve as a descriptor of phenomena in which there is "motion without beginning not end", for boundary conditions at $X = 0$, of the type

(B10b) $\theta(0, \tau) = g(\tau)$.

Mendousse then gave the solution of the problem defined by (B9b) and (B10b). He also seems to have implied that this solution is similar to, or is an extension of, the celebrated results that R.D. Fay[10] had obtained in 1931.

In 1964 D.T. Blackstock[11] presented a thorough-going analysis of the groundwork that Mendousse had laid, also based on (B9b) and B10b). In fact, he presented explicit - although only asymptotic - solutions, which are certainly more general than that obtained by Fay.

It is our contention that these results of Mendousse and of Blackstock, qua solutions of (B9b) and (B10b), are physically meaningless.

III. THE PHENOMENOLOGICAL CONTRADICTION

The philosophical foundation on which we are basing our rejection of the solutions of Mendousse and of Blackstock is the principle of determinacy in physical events. By this we mean that in order to work in the customary framework of science, one must assume that the cause of present events lies in the past; and that the future has no effect on the present.

If we assume that (B9b) and (B10b) constitute the valid statement of a problem, then, as has been shown by Hopf[2], the only solution is given by (B2), (B3), (B6) and (B7). In particular, we shall have that

$$\theta(X, \tau) = -\frac{h_\tau(X, \tau)}{h(X, \tau)} \tag{B2'}$$

where

$$h_X = h_{\tau\tau} , \tag{B3'}$$

and where

$$h(0, \tau) = \exp\left[-\int^\tau g(T)\, d(T)\right] = G(\tau) , \tag{B6'}$$

with h given by

$$h(X, \tau) = \int_{-\infty}^{\infty} k(\tau - T, X)\, G(T)\, dT . \tag{B7'}$$

This is, however, totally unacceptable. For, since τ is a time-like variable here, the function $g(\tau)$ - and ipso facto, $G(\tau)$ - contains in it the time history, past present and future, of the excitation (piston) applied to the medium. By integrating this, as we have to in (B7'), we are essentially summing up the contributions of this excitation for all time.

If we now take a fixed point (X_o, τ_o) in space-time, and consider $\theta(X_o, \tau_o)$, we see that the value of the function depends not only on what had happened previously (times $\tau_1 < \tau_o$), but also on the entire future ($\tau_2 > \tau_o$). On the grounds that we mentioned before, we must reject such a possibility[12].

It is perhaps necessary to point out why, on the other hand, the formulation given by (B9a) and (B10a) can be considered reasonable. If we apply the argument given above to these, we come to the conclusion that $\theta(X_o, \tau_o)$ depends on the values of the initial function f at every point of the X axis. This implies the instantaneous transmission of effects from point to point; a conclusion which, while unrealistic, is nevertheless not repugnant[13].

IV. CONCLUSIONS

We demonstrated that it is impossible to accept the framework given by Mendousse and utilized by Blackstock in obtaining their results. However, the remarkable qualitative agreement of their solutions with that of Fay cannot be lightly dismissed.

To be sure, this agreement is subject to some suspicion. The reason for this is that both the equation used by Fay and Burgers' equation, have a strong connection with the linear heat equation (B3). Rational approximations, employed both by Fay and by Blackstock, would therefore tend to go in a direction where the simplified expressions can be handled on better-known mathematical ground; in this case, the realm of the solutions of the heat equation. Specifically, Blackstock had utilized certain properties of the Jacobian Theta and Zeta functions (first connected to Burgers' equation by Cole[3]). As is well known, the former are solutions of (B3), while the Zeta functions are the logarithmic derivatives of the Theta functions. On the other hand, the basic linearization procedure employed by Fay, and the fact that he specified no particular conditions but only sought the periodic terms of a stable wave form, led to classes of solutions which, in certain limiting cases, are also solutions of the heat equation. One would conclude therefore that the intersection of the two sets of solutions is the place to which the path of least resistance leads. This does not mean, however, that the set composed of the union less the intersection of the two solution sets is empty, or even small; indeed, the contrary is true.

Thus, a new approach to Burgers' equation is indicated; one which will treat a <u>true</u> boundary value problem, without using the artifice employed by Mendousse. One has to keep in mind, however, that much as the solutions change discontinuously as one passes from the lossless case (described, for instance, by $v_t + v v_x = 0$) to the viscous alternative (B2), the <u>number of necessary boundary conditions also changes from one to two</u>. This of course is quite reasonable. In a lossless medium one condition should suffice to describe the nature of the excitation; but such a single condition cannot conceivably incorporate the effects of both excitation and interaction of excitation with the viscous mechanism.

What Blackstock and Mendousse were nevertheless able to show was that if a true boundary value problem is solved for Burgers' equation, it should be similar to Fay's result. Because of some recent results obtained by Rodin[14], this view is in fact vindicated. We shall not give the details here; they will appear, in all their generality, in a different framework.

Fay's solution can be written as

(B11)
$$u_{\text{stable wave}} = C_o \sum_{n=1}^{\infty} \frac{\sin n\,(\omega t - (\omega x)/S_o)}{\sinh n\,(\alpha_o + \alpha x)}$$

Here C_o is a group of constants, ω is a frequency number, S_o the ambient sound speed and α_o, α_1 are phase constants. The general solution we referred to above, on the other hand, can be written in the form

(B12)
$$v(x, t) = -2\delta \frac{\partial}{\partial x} \ln \left\{ \sum_{n=0}^{\infty} \frac{[P^{-1}(t)]^{(n)}}{\delta^n} \frac{x^{2n}}{(2n)!} \right\}$$

Here $P(t)$ is the path of the piston; the superscripts in parentheses indicate derivatives. (B12) will reduce to an expression similar to (B11) if we take

(B13)
$$P(t) = \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n \, e^{-n^2 t} \right]^{-1} \quad ;$$

for then it becomes[15]

(B14)
$$v(x, t) = -4\delta \sum_{n=1}^{\infty} \frac{\sin (nx/\delta^{1/2})}{\sinh nt}$$

Thus, expression (B12) is the one that can be considered the analogue of Fay's result (B11); it differs from the latter in that it is exact and by virtue of the fact that it is a descriptor not only of the stable wave forms. Formulas (B13) and (B14), on the other hand, show that it is not sinusoidal excitation that leads to a sawtooth-like curve; but rather an excitation of monotonic type. The final conclusion is that by consideration of a properly posed problem one obtains a generalization of much greater scope than that available through an artifice.

FOOTNOTES
(to Appendix B)

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3. J.D. Cole, Quart. Appl. Math. 9, 225 (1951)
4. W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press (1965)
5. Because of the very extensive literature on Burgers' equation and on its application, we shall cite in this footnote (and in the next two also) only representative samples of the directions in which work has been done. Thus, the initial value problem was treated in the papers of Hopf and of Cole. A boundary value problem for curved boundary is discussed by J. Burgers in Nonlinear Problems of Engineering, edited by W.F. Ames, Academic Press (1964), pp. 123-137.
6. M.J. Lighthill, Surveys in Mechanics, edited by G.K. Batchelor and R.M. Davies, Cambridge University Press, Cambridge, England (1956) pp. 250-351.
7. W.F. Ames, Ref. 4, discusses several examples.
8. Actually, it has been shown that δ measures the effect of dissipation versus nonlinearity. See for example D.T. Blackstock, J. Acoust. Soc. Am. 36, 534 (1964), and Z.A. Gol'dberg, Akust. Zh. 2,235 (1956) and 3,329 (1957) [English transl.: Soviet Phys. - Acoust. 2,346 (1956) and 3,329 (1957)] .

9. J.S. Mendousse, J.Acoust. Soc. Am. 25,51 (1953)

10. R.D. Fay, J. Acoust. Soc. Am. 3,222 (1931)

11. D.T. Blackstock, J.Acoust. Soc. Am. 36, 534 (1964)

12. Actually, one could mention several other reasons which make the formulation unacceptable. For instance, in a boundary value problem of this type the location of the boundary should make no essential difference; the choice of $x = 0$ ought to be merely a convenience. It is well known, however, that (B9b) and (B10b) cannot be solved for a boundary condition at $x_1 < 0$.

13. Because of its intimate connection with the linear heat equation, Burgers' equation actually "inherited" this property. A discussion of the question of the instantanwous transmission of effects in the case of the heat equation can be found, for example, in R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, Interscience Publishers (1962).

14. E.Y. Rodin, A Boundary Value Problem for Burgers' Equation, to be published.

15. Cole obtained essentially the same series as the one given in (B14). The important difference, however, is that in the case of Cole the exact result of an initial value problem was approximated; while (B14) is an exact solution (of the boundary value problem) as it stands.

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