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UPPER AND LOWER BOUNDS FOR THE EIGENVALUES OF VIBRATING BEAMS WITH LINEARLY VARYING AXIAL LOAD

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ABSTRACT

Previous investigations have demonstrated the importance of the effect of linearly varying axial or in-plane loading on the vibration characteristics of beams and flat plates. It has already been established that the problem reduces to solving for the eigenvalues of a fourth order, variable coefficient differential equation that can not be solved in closed form. Beginning with a variational representation of the eigenvalue problem, methods are discussed by which both upper and lower bounds for the eigenvalues may be formed. The true eigenvalues may thus be estimated as being bracketed by the upper and lower bounds which are shown to approach each other. The bounds for the eigenvalues may also be estimated by an averaging procedure which may or may not compare favorably with the true values depending on the values of the loading parameters. Finally, numerical values for upper bounds, lower bounds, and average lumped end-load eigenvalues are computed on an IBM 7090 Computer.

NOMENCLATURE

A	Differential operator of loaded beam
c	Eigenvectors
c_i	Constants
E	Modulus of Elasticity
f	Natural frequency of vibrating beam
I	Moment of inertia
K_b	Class of admissible functions in elastic stability problems
K_v	Class of admissible functions in vibration problems
L	Length of the beam
P_1, P_2	Constant end loads
u	Function
v	Function
x	Axial coordinate
α	Distributed axial load parameter
α_c	Critical axial load
β	Ratio of end load to total distributed load
γ^4	Separation constant
ϕ	Function
λ	Eigenvalue
$\tilde{\lambda}$	Upper bound
ξ	Nondimensional axial variable
ζ	Density per unit of length
ψ	Mode shape, dependent deflection variable

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I. INTRODUCTION

In recent years much attention has been given to the effect of linearly varying axial or in-plane loads on the vibrational characteristics of beams and plates. This topic is of particular interest in aerospace applications where inertia and friction drag forces manifest themselves as axial or in-plane loads. A detailed formulation of the problem is the subject of a prior NASA report by authors [1] and is the subject of considerable literature (see ref. 2 through 16).

Formulation of the Problem

As described in references (1) and (2), the eigenvalue problem for both the beam and the rectangular plate may be resolved, under certain restrictions, to a solution of the ordinary differential equation

$$\frac{d^4\psi}{d\xi^4} + \alpha\frac{d}{d\xi}\left\{(\beta+\xi)\frac{d\psi}{d\xi}\right\} - \lambda\psi = 0 \qquad (1)$$

and the boundary conditions

$$\frac{d^2\psi}{d\xi^2} = 0 \quad , \quad \frac{d^3\psi}{d\xi^3} + \alpha(\beta+\xi)\frac{d\psi}{d\xi} = 0 \qquad \text{at a free end}$$

$$\psi = 0 \quad , \quad \frac{d^2\psi}{d\xi^2} = 0 \qquad \text{at a simply supported end} \qquad (2)$$

$$\psi = 0 \quad , \quad \frac{d\psi}{d\xi} = 0 \qquad \text{at a clamped end.}$$

where

$$\xi = \frac{x}{L} \qquad \alpha = \frac{wL^3}{EI} \quad , \quad \beta = \frac{P}{wL} \quad \text{and} \quad \lambda = \frac{\gamma^4 \rho L^4}{EI} \qquad (3)$$

In view of the definition of the parameter β, it is clear that for a given compressive distributed load w, the following cases may occur:

1) $\beta > 0$, the beam is entirely in compression

2) $0 > \beta > -1$, the beam is partly in tension and partly in compression

3) $-1 \geqslant \beta$, the beam is entirely in tension since the tensile and load P_1 is larger than the total distributed load L.

In the last case, the problem of elastic stability does not exist.

The determination of mode shapes and natural frequencies involves the solution of the differential eigenvalue problem defined by eqs. (1) and (2). Variational techniques (1) (2) finally resolve this to obtaining solutions to the variational principle

$$\lambda_1 = \min_{u \in K} \frac{\langle Au, u \rangle}{\langle u, u \rangle} \quad * \tag{4}$$

where K is the class of functions constituting the domain of definition of the operator A, and, hence, satisfying both the prescribed and the natural boundary conditions, and $\langle u, v \rangle$ denotes the inner product between two functions u,v , where

$$\langle u, v \rangle = \int_0^1 u\, v\, d\xi \quad \text{and} \quad A = \frac{d^4}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{(\beta+\xi)\frac{d}{d\xi}\right\} \tag{5}$$

Equation (4) may be characterized by Courant's maximum-minimum characterization (ref. 18, Chap. III) given by

$$\lambda_j = \max_{\{u_i\}} \left\{ \min_{\langle \phi, u_i \rangle = 0} \frac{\int_0^1 \left[\left(\frac{d^2\phi}{d\xi^2} \right)^2 - \alpha(\beta+\xi)\left(\frac{d\phi}{d\xi} \right)^2 \right] d\xi}{\int_0^1 \phi^2\, d\xi} \right\}_{i=1 \text{ to } j-1.} \tag{6}$$

where ϕ and μ_i belong to K_v, where K_v is the class of admissible functions required to satisfy only the prescribed boundary conditions.

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* This functional is known as Rayleigh's quotient.

In resume, the situation is as follows: if β is such that buckling may occur, there exists for the given falue of β a critical value of this distributed axial load parameter, α_c , for which the beam is unstable and the potential energy is equal to zero. For any value of α less than α_c , the potential energy is positive, and the beam has discrete natural frequencies whose square are proportional to the eigenvalues of the operator A,

where

$$A = \frac{d^4}{d\xi^4} + \alpha \frac{d}{d\xi}\left\{(\beta+\xi)\frac{d}{d\xi}\right\} \tag{7}$$

These eigenvalues are assumed to be ordered in the non-decreasing sequences

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots\ldots$$

The eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal, and correspond to the mode shapes of the beam. For a given value of β , as α increases, the numerator of the Rayleigh quotient decreases and the eigenvalues decrease. Buckling occurs when α becomes equal to α_c , for which the first eigenvalue goes to zero.

In the next section, we review the methods used in this work to obtain approximate solutions.

II. BOUNDS FOR EIGENVALUES

There appear in the literature many methods for finding bounds for eigenvalues. Upper bounds are usually found without too many difficulties by the Rayleigh-Ritz method. Lower bounds present considerably more difficulties, and it can be said that no method having the generality, simplicity, and success of the Rayleigh-Ritz method exists for the computation of lower bounds. The most suitable method usually depends on the problem at hand.

In this section, we review briefly the methods used in this work in the calculations of approximations to eigenvalues. They are the Rayleigh-Ritz method, the method of Kato, and the method of intermediate problems of Weinstein and Aronszajn, with some modifications introduced by Bazley and Fox.

A. The Rayleigh-Ritz Method

The Rayleigh-Ritz method for numerical computations of approximations to eigenvalues has been used extensively and with great success in the literature.* Consequently, it will only be outlined briefly here.

The basic idea of the method consists in determining the stationary values of the Rayleigh quotient, not over all admissible functions u, but only over the linear manifold spanned by an arbitrary set of n linearly independent functions $\{u_i\}$ * satisfying the boundary conditions of the operator A. The problem then consists in finding the functions u of the form

$$u = \sum_{i=1}^{n} c_i u_i \tag{8}$$

i.e., in finding the constants C_i, making the Rayleigh quotient stationary, and the stationary value of the quotient. Substitution of Equation (8) into Rayleigh's quotient yields

$$\frac{\langle u, Au \rangle}{\langle u, u \rangle} = \frac{\sum_{i,j=1}^{n} c_i c_j \langle u_i, Au_j \rangle}{\sum_{i,j=1}^{n} c_i c_j \langle u_i, u_j \rangle} \tag{9}$$

which is the ratio of two quadratic forms in the n real variables C_1, C_2, ...C_n. Its stationary values can be obtained by finding, for instance, the stationary values of the quadratic form in the numerator, subject to the auxiliary condition that the denominator be equal to one, and using the method of the Lagrange undetermined multiplier. The result is the general matrix eigenvalue problem

$$\left[\langle u_i, Au_j \rangle\right]\left[c_j\right] = \tilde{\lambda}\left[\langle u_i, u_j \rangle\right]\left[c_j\right] \tag{10}$$

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* See, for instance, references 17, 18, and 19.

* The functions u_i are often called coordinate functions.

Since the class of admissible functions was restricted to the finite dimensional manifold, it follows that the eigenvalues $\tilde{\lambda}_j$ are upper-bounds for the eigenvalues of A, i.e.,

$$\lambda_j \le \tilde{\lambda}_j \qquad , j=1,2,\ldots n \tag{11}$$

Furthermore, it follows that as n increases, the upper bounds are improved, or at least, not worsened.

From a computational standpoint, it is advantageous to choose mutually orthogonal coordinate functions to avoid the solution of a general eigenvalue problem. Also, Equation(9)may be written as in Equation(6)with the functions $\{u_i\}$ required to satisfy only the prescribed boundary conditions. This point is discussed in detail in references 18 and 19. The coordinate functions utilized in this work satisfy both the prescribed and the natural boundary conditions, as will be seen later.

B. The Method of Kato

The Rayleigh-Ritz method described above furnishes upper bounds for eigenvalues. The results, particularly for the first eigenvalue, are usually in agreement with the exact eigenvalues for the cases where the latter can be obtained. However, in general, the question regarding the closeness of these bounds to the true values remains unanswered, although in some instances, an estimate of the error is possible.* One way of determining how good the approximations are is to compute also lower bounds. If these turn out close to the upper bounds, the question is essentially answered. The method of Kato[22], which is an extension of Temple's method, furnishes lower bounds, provided rough estimates of the sought eigenvalues are known. This is outlined by Freidman [22, p. 212].

C. The Method of Intermediate Problems

The methods described in the preceding two sections furnish upper and lower bounds for eigenvalues. In both methods, the quality of the results

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* See, for instance, reference 21, p. 336.

depends strongly on how well the trial functions approximate the eigenvectors of the operator. Hence, both methods may require considerable ingenuity in the selection of the trial functions. Furthermore, for different sets of trial functions, there is little prior knowledge of which set will give the best results. For these reasons, it is in order to consider also another method for the computation of the lower bounds. The method used here is the method of intermediate problem, which presents the advantage that the bounds can be improved.

Quite a few years back, Weinstein [24] introduced the method of intermediate problems, which gives improvable lower bounds by changing the boundary conditions of differential operators. Briefly, the method consists in relaxing the boundary conditions to obtain a solvable problem, the base problem, whose eigenvalues give rough lower bounds for the eigenvalues of the given problem. A sequence of intermediate problems linking the base problem to the given problem is then introduced. These are such that they can be solved in terms of the base problem, and that they give improved lower bounds. The details of the procedure are exposed in references 17 and 25.

In 1951, Aronszajn [26] pointed out that a base problem can be obtained by changing the differential operator, and indicated the method of construction of the intermediate problems. The solution of these intermediate problems requires the determination of the poles and the zeroes of a meromorphic function given in its partial fractions representations. From a computational standpoint, the determination of the zeroes present many difficulties which have been removed in a dissertation by Bazley [27], and in a series of recent papers by Bazley and Fox [28-33]. These authors have applied their method to the determination of the eigenvalues of Schrodinger's equation and Mathieu's equation with excellent results.

A more detailed resume of the Method of Kato and the Method of Intermediate Problems is given in Reference (2). Reference (2) also describes specific application to the simply supported beam and the beam with built-in ends. These procedures are not particularly difficult in principle, but the calculations involved are somewhat laborious.

D. Lumped Constant End Load Approximation

An approximation to the response of beams with distributed axial load may be accomplished by replacing the distributed load and its reaction with equal and opposite average end loads. This results in an ordinary linear differential equation with constant coefficient which may be solved exactly in terms of trigonometric functions. A comparison of the eigenvalues calculated in this manner is made with the upper and lower bounds in the section on Results and Discussion.

III. RESULTS AND DISCUSSION

Following the methods described above, upper and lower bounds for the eigenvalues of the simply supported and clamped beam were calculated on an IBM 7090 Computer in the Computation and Data Processing Center of the University of Pittsburgh. The results are displayed in Tables I and II and Figures 1, 2, and 3. Upper bounds, lower bounds and lumped end-load eigenvalues are displayed for a wide range of loading parameters α and β.

A. Simply Supported Beam

The bounds for the first five eigenvalues of the simply supported beam are presented in Table I. To facilitate the comparison between the Rayleigh-Ritz upper bounds and the lower bounds by the method of intermediate problems, the ratio of their difference to their average has been computed and is also presented in Table I. Since the eigenvalues of a simply supported beam are easy to obtain, it is interesting to compare the upper and lower bounds of the eigenvalues obtained by lumping half of the total distributed load as a constant load at each end. These results are also included in Table I.

Analysis of the results in Table I indicates that the Rayleigh-Ritz upper bounds and the lower bounds by the method of intermediate problems remain close over the whole range of axial loadings. This is particularly true for the first eigenvalue. Only when the beam is extremely close to buckling does the relative error increase greatly as a result of the smallness of the eigenvalues. For eigenvalues of order higher than one, the error is slightly higher, but, if necessary, it could be reduced by considering higher intermediate problems.

The lower bounds for the first eigenvalue by the method of Kato remain close to the upper bounds for moderate loading, but drop off considerably at the loading increases. Perhaps, this effect might be attributed to the fact that as the first eigenvalue approaches zero, the choice arbitrary trial variations becomes more and more critical. For higher eigenvalues, this selection is not as critical, and consequently, the lower bounds remain close to the upper bounds. However, in the cases where the beam can not become elastically unstable, the Kato lower bounds eventually decrease as the loading becomes very large, and no explanation for this behavior can be offered.

The eigenvalues of the beam with lumped constant end load are remarkably close to those of the beam with distributed load for compressive end thrusts, i.e., for $\beta > 0$. For negative β, the results are quite far apart. In particular, for $\beta = -5$, the beam with distributed axial load may become elastically unstable, while the beam with lumped load can not buckle, because its net thrust is zero. Consequently, extreme care should be exercised in the lumping of the loads when they are of opposite signs.

The effect of the axial loads on the first frequency of the simply supported beam is shown in Figures 1 and 2. Figure 1 represents the ratio of the first frequency of the loaded beam to that of the unloaded beam as a function of the distributed load parameter α , as obtained by Kato's method and the Rayleigh-Ritz method. The lower bounds of the method of intermediate problems are not shown because their curve practically coincides with the Rayleigh-Ritz curve for the scale used. The curves correspond to $\beta = 0$. Figure 2 also represents the ratio of the fundamental frequency of the loaded beam to that of the unloaded beam as a function of α for various values of β . The curves were obtained by using the average of the upper bounds and lower bounds by the method of intermediate problems.

The values of the critical axial load α_c are given at the intersection of the frequency ratio curve with the horizontal axis. The buckling loads obtained from graphs having a larger scale than that of Figure 2 compare favorably with the exact results of Tyler and Rouleau [11]. For $\beta = 0$, the graphs indicate that $\alpha_c \simeq 18.7\, EI/L^3$ while Tyler and Rouleau's result is $\alpha_c = 18.763\, EI/L^3$ For $\beta = 1.0$, we obtain $\alpha_c \simeq 6.5\; EI/L^3$ while the exact answer is $\alpha_c = 6.519\, EI/L^3$ and for $\beta = -.50$ we have $\alpha_c \simeq 83\, EI/L^3$ against the exact result of $82.8819\; EI/L^3$. The approximate values are certainly close enough for engineering application.

B. Clamped Beam

The bounds for the first four eigenvalues of the clamped beam are presented in Table 2. The ratio of the difference between the upper bound and the corresponding lower bound by the method of intermediate problems to their average has also been computed. The eigenvalues of the clamped beam carrying a constant end load equal to half the total distributed load and the constant end load are also presented in Table II to indicate for what values of the loading parameters this lumping is acceptable.

Examination of the results indicate the following:

i) The lower bounds by the method of intermediate problems are very close to the Rayleigh-Ritz upper bounds for all eigenvalues and for the whole range of the loading parameters.

ii) The lower bounds by the method of Kato present the same features demonstrated in the simply supported beam calculations: whenever the loading is small, the bounds are fairly good but become worse as the loads increase.

iii) The eigenvalues of the beam with lumped end load are fairly close to the upper bounds for moderate loading, particularly for $\beta > 0$. For negative values of β , they can be quite remote from the upper bounds, particularly for β for which the beam with distributed axial load may buckle while the beam with lumped end load can not.

The effect of the axial loads on the first frequency of the clamped beam are shown in Figure 3, which represents the ratio of the first frequency of the loaded beam to that of the unloaded beam as a function of the axial load parameter α for various values of β .

IV. CONCLUDING REMARKS

Bounds for the eigenvalues of a simply supported and a clamped beam carrying linearly distributed axial loads have been presented. The main difficulty in problems of this nature arises from the fact that the governing differential equation has a varying coefficient which usually prevents one from obtaining exact solutions. Upper bounds were easily obtained by the Rayleigh-Ritz method. Lower bounds by the method of Kato were also easy to obtain. In both methods, the closeness of the results to the true eigenvalues depends on the quality of the coordinate functions. It appears that for moderate loading, the eigenfunctions of the unloaded beams were good coordinate functions, as our results indicate.

The lower bounds computed by the method of intermediate problems were very close to the upper bounds, both for the simply supported and the clamped beam. The modifications introduced by Bazley and Fox eliminate the computational difficulties which prevented extensive use of the method of intermediate problems.

For engineering applications, it appears that lumping the axial loads gives eigenvalues that are larger than the true eigenvalues, and that care must be exercixed whenever the distributed load and the constant end thrust are of opposite signs. In this case, the buckling loads predicted by the lumped end load problem can be quite remote from the actual critical loads.

The present research could be extended to the consideration of beams with other boundary conditions, closer determinations of the buckling loads, and the methods used here can be applied to other problems giving rise to differential equations with variable coefficients, such as in the problems of the determination of natural frequencies and buckling loads of beams of varying cross sections, plates with varying in-plane loads, and plates of non-uniform thickness, to mention a few. Information of this nature would be valuable to designers, particularly in the Aerospace industry.

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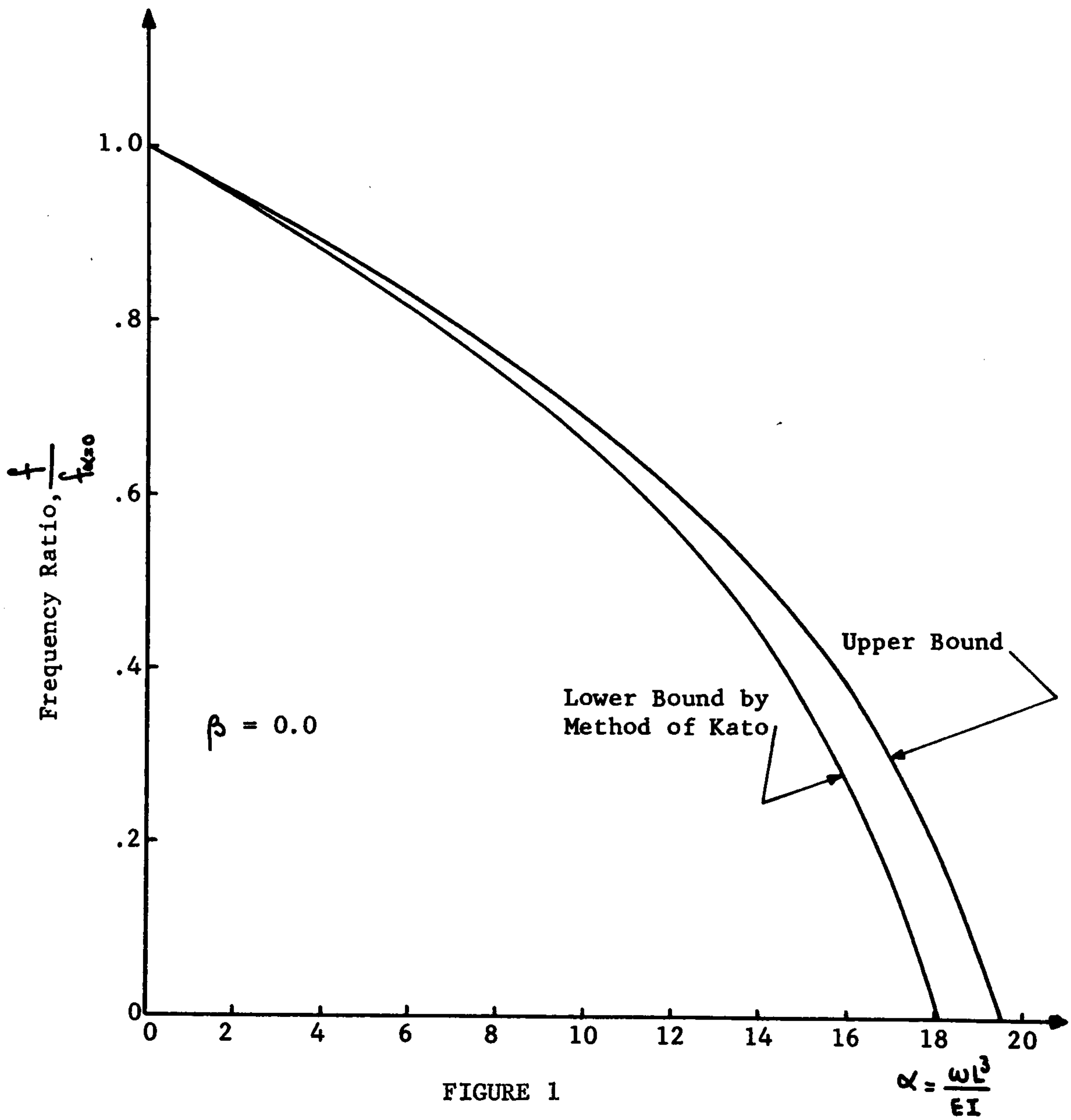

FIGURE 1

Effect of Distributed Axial Load on the First Frequency of the Simply Supported Beam

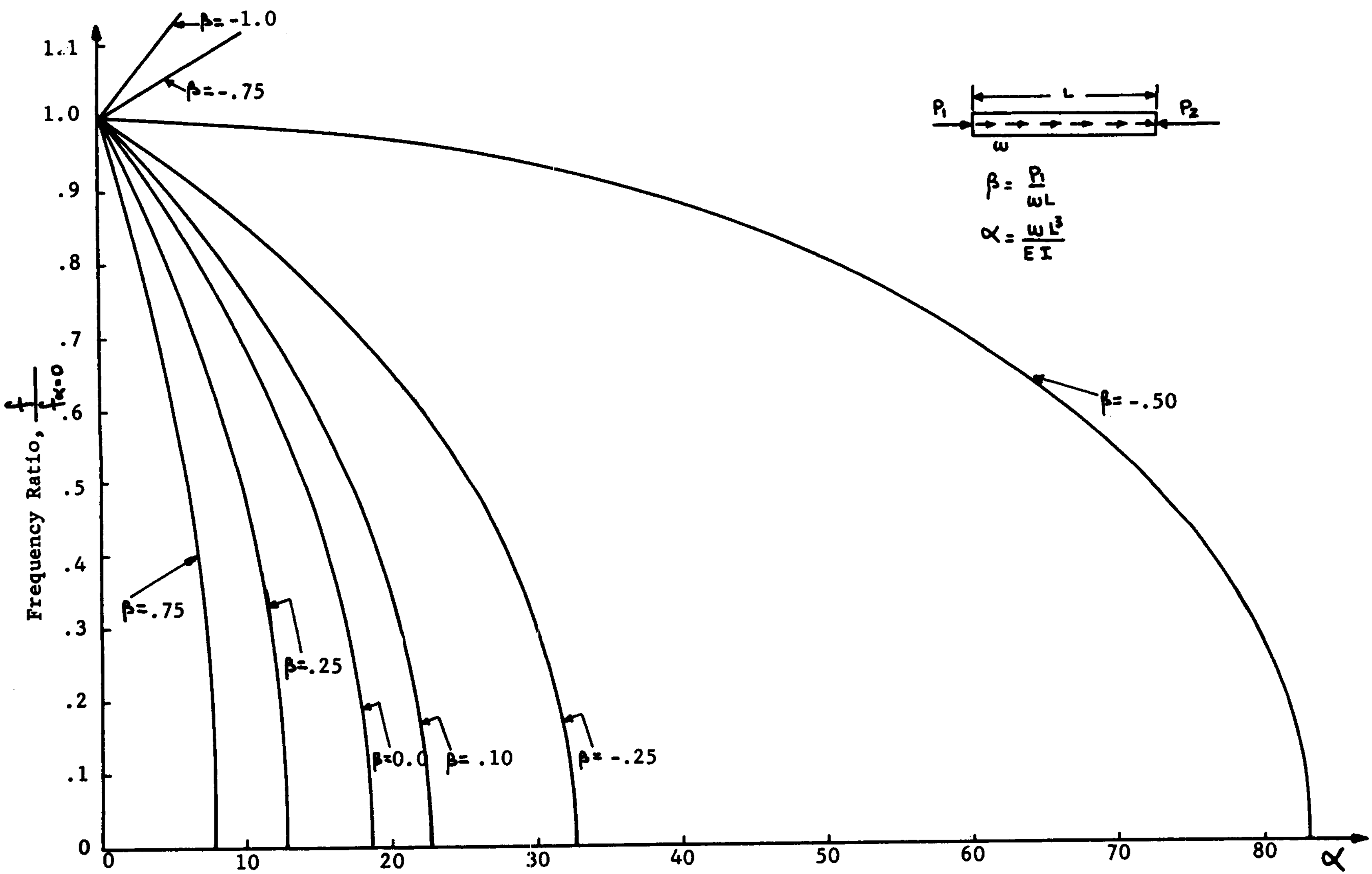

FIGURE 2

Effect of Axial Loads on the Fundamental Frequency of a Simply Supported Beam

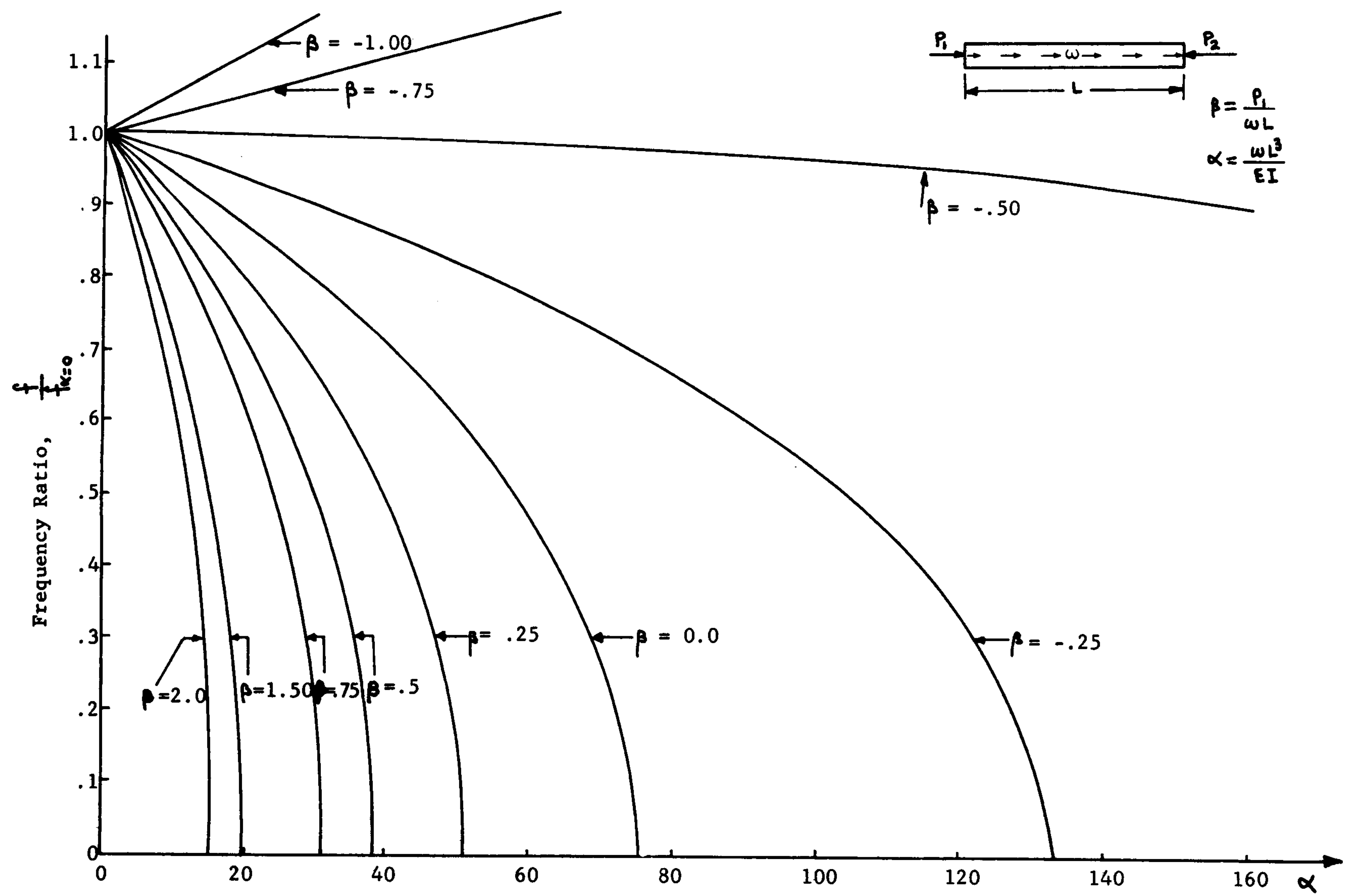

FIGURE 3

Effect of Axial Loads on the Fundamental Frequency of a Clamped Beam

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.0$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
0.00	1	97.40909	97.40909	97.40909	0.0	97.40909
	2	1 518.545	1 558.545	1 558.545	0.0	1 558.545
	3	7 890.136	7 890.136	7 890.136	0.0	7 890.136
	4	24 936.73	24 936.73	24 936.73	0.0	24 936.73
	5	60 880.68	60 880.68	60 880.68	0.0	60 880.68
2.00	1	87.48401	86.82118	87.47622	0.009	87.5395
	2	1 519.022	1 509.258	1 518.916	0.007	1 519.067
	3	7 801.265	7 759.954	7 798.981	0.029	7 801.309
	4	24 778.77	24 668.74	24 740.64	0.154	24 778.81
	5	60 633.89	60 403.19	60 441.32	0.318	60 633.94
4.00	1	77.44337	76.09688	77.42758	0.020	77.6699
	2	1 479.411	1 459.780	1 479.201	0.014	1 479.588
	3	7 712.310	7 629.280	7 707.814	0.058	7 712.483
	4	24 620.72	24 399.70	24 544.67	0.309	24 620.89
	5	60 387.02	59 923.86	60 001.91	0.640	60 387.20
6.00	1	67.27963	65.20801	67.25670	0.034	67.8003
	2	1 439.716	1 410.094	1 439.409	0.021	1 440.110
	3	7 623.273	7 498.086	7 616.639	0.087	7 623.656
	4	24 462.60	24 120.60	24 348.82	0.466	24 462.98
	5	60 140.07	59 442.65	59 562.45	0.965	60 140.46

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.0$

α	Order	Upper Bound by Rayleigh-Ritz Method	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
8.00	1	56.98461	54.11895	56.95145	0.058	57.9307
	2	1 399.940	1 360.181	1 399.538	0.029	1 400.631
	3	7 534.153	7 366.345	7 525.454	0.116	7 534.830
	4	24 304.39	23 858.37	24 153.09	0.624	24 305.07
	5	59 893.04	58 959.51	59 122.92	1.294	59 893.71
10.00	1	46.54933	42.78417	46.51453	0.075	48.0611
	2	1 360.085	1 310.021	1 359.597	0.036	1 361.153
	3	7 444.952	7 234.029	7 434.263	0.144	7 446.003
	4	24 146.10	23 585.99	23 957.49	0.784	24 147.15
	5	59 645.92	58 474.38	58 683.34	1.627	59 646.98
12.00	1	35.96395	31.14474	35.92827	0.099	38.1915
	2	1 320.157	1 259.591	1 319.593	0.043	1 321.674
	3	7 355.672	7 101.108	7 343.061	0.172	7 357.177
	4	23 987.74	23 312.42	23 762.01	0.945	23 989.24
	5	59 398.72	57 987.22	58 243.69	1.964	59 400.23
14.00	1	25.21769	19.12343	25.17796	0.158	28.3219
	2	1 280.161	1 208.868	1 279.623	0.050	1 282.196
	3	7 266.316	6 967.549	7 251.870	0.199	7 268.350
	4	23 829.29	23 037.61	23 566.66	1.109	23 831.33
	5	59 151.45	57 497.99	57 803.98	2.304	59 153.49

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.0$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
16.00	1	14.29866	6.61749	14.25562	0.301	18.4523
	2	1 240.103	1 157.825	1 239.414	0.056	1 242.718
	3	7 175.883	6 833.320	7 160.670	0.226	7 179.524
	4	23 670.76	22 761.53	23 371.45	1.273	23 673.41
	5	58 904.09	57 006.62	57 364.20	2.649	58 906.75
18.00	1	3.193809	----	3.148092	1.442	8.5827
	2	1 199.990	1 106.434	1 199.233	0.063	1 203.239
	3	7 087.378	6 698.383	7 069.476	0.253	7 090.697
	4	23 512.15	22 484.12	23 176.36	1.438	23 515.50
	5	58 656.65	56 513.06	56 924.36	2.998	58 660.02
18.50	1	.386901	----	.341591	12.448	6.1152
	2	1 189.955	1 093.528	1 189.187	0.065	1 193.370
	3	7 064.990	6 664.534	7 046.678	0.260	7 068.491
	4	23 472.49	22 414.56	23 127.62	1.480	23 476.02
	5	58 594.78	56 389.33	56 814.39	3.085	58 598.33

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	82.54863	81.91434	82.54596	0.003	82.5047
	2	1 499.283	1 489.607	1 499.177	0.007	1 499.327
2.00	3	7 756.852	7 715.692	7 754.567	0.029	7 756.896
	4	24 699.80	24 589.99	24 661.68	0.357	24 699.85
	5	60 510.51	60 280.09	60 317.95	0.319	60 510.57
	1	67.56889	66.34465	67.55165	0.026	67.8003
	2	1 439.935	1 420.664	1 439.726	0.015	1 440.110
4.00	3	7 523.485	7 541.064	7 618.987	0.059	7 623.656
	4	24 462.81	24 242.65	24 386.76	0.311	24 462.98
	5	60 140.28	59 678.24	59 755.17	0.642	60 140.46
	1	52.45787	50.68089	52.43477	0.044	52.9959
	2	1 380.505	1 351.718	1 380.197	0.022	1 380.892
6.00	3	7 490.036	7 366.244	7 483.402	0.089	7 490.416
	4	24 225.73	23 894.68	24 111.95	0.471	24 226.11
	5	59 769.96	59 075.06	59 192.34	0.971	59 770.35
	1	37.20190	34.89742	37.16991	0.086	38.1915
	2	1 321.000	1 282.770	1 320.595	0.031	1 321.674
8.00	3	7 356.507	7 191.216	7 347.814	0.118	7 357.177
	4	23 988.57	23 546.06	23 837.28	0.633	23 989.24
	5	59 399.56	58 470.54	58 629.44	1.305	59 400.23

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
10.00	1	21.78530	18.95922	21.74532	0.184	23.3871
	2	1 261.426	1 213.824	1 260.936	0.039	1 262.457
	3	7 222.901	7 015.972	7 212.221	0.148	7 223.937
	4	23 751.33	23 196.75	23 562.72	0.797	23 752.37
	5	59 029.08	57 864.62	58 066.49	1.644	59 030.12
12.00	1	6.190051	2.817389	6.1546701	0.573	8.5827
	2	1 201.793	1 144.882	1 201.233	0.047	1 203.239
	3	7 089.221	6 840.497	7 076.629	0.178	7 090.697
	4	23 514.01	22 846.74	23 288.30	0.965	23 515.01
	5	58 658.51	57 257.28	57 503.46	1.989	58 660.01
12.50	1	2.260952	----	2.224899	1.607	4.8815
	2	1 186.877	1 127.648	1 186.297	0.049	1 188.435
	3	7 055.789	6 796.590	7 042.739	0.185	7 057.387
	4	23 454.67	22 759.13	23 219.72	1.007	23 456.28
	5	58 565.86	57 105.22	57 362.69	2.076	58 567.49

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bounds by Kato's Method	Lower Bounds by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
1.00	1	87.52560	87.20957	87.51748	0.009	87.5395
	2	1 519.055	1 514.228	1 519.001	0.004	1 519.067
	3	7 801.297	7 780.766	7 800.157	0.015	7 801.309
	4	24 778.79	24 724.01	24.759.71	0.078	24 778.81
	5	60 633.91	60 518.92	60 537.63	0.159	60 633.94
3.00	1	67.67012	66.80439	67.65601	0.021	67.8003
	2	1 440.011	1 425.730	1 439.847	0.011	1 440.110
	3	7 623.559	7 562.125	7 620.156	0.045	7 623.656
	4	24 462.88	24 298.45	24 405.78	0.234	24 462.98
	5	60 140.35	59 794.90	59 851.52	0.481	60 140.46
5.00	1	47.68328	46.38864	47.66803	0.032	48.0611
	2	1 360.886	1 337.431	1 360.623	0.019	1 361.153
	3	7 445.739	7 343.619	7 440.165	0.075	7 446.003
	4	24 146.89	23 872.72	24 051.95	0.394	24 147.15
	5	59 646.70	59 070.15	59 165.95	0.809	59 646.98
7.00	1	27.54646	25.95841	27.51930	0.099	28.3219
	2	1 281.687	1 249.359	1 281.329	0.028	1 282.196
	3	7 267.840	7 125.257	7 260.169	0.106	7 268.350
	4	23 830.81	23 446.81	23 698.26	0.558	23 831.33
	5	59 152.97	58 344.64	58 479.11	1.146	59 153.49

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 0.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
8.00	1	17.41498	15.73609	17.38809	0.155	18.4523
	2	1 242.063	1 205.418	1 241.659	0.033	1 242.718
	3	7 178.863	7 016.133	7 170.170	0.121	7 179.524
	4	23 672.75	23 233.79	23 521.46	0.641	23 573.41
	5	58 906.08	57 981.60	58 135.96	1.316	58 906.75
9.00	1	7.237294	5.507694	7.2031238	0.473	8.5827
	2	1 202.425	1 161.547	1 201.981	0.037	1 203.239
	3	7 089.866	6 907.049	7 080.177	0.137	7 090.698
	4	23 514.66	23 020.71	23 344.70	0.725	23 515.50
	5	58 659.17	57 918.37	57 792.81	1.488	58 660.02
9.60	1	1.107041	----	1.070184	3.386	2.6609
	2	1 178.637	1 135.259	1 178.171	0.040	1 179.552
	3	7 036.460	6 841.618	7 026.174	0.146	7 037.401
	4	23 419.80	22 892.85	23 238.65	0.776	23 420.75
	5	58 511.01	57 400.33	57 586.90	1.592	58 511.97

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = .75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	85.05812	84.74914	85.05647	0.002	85.0721
	2	1 509.186	1 504.381	1 509.134	0.003	1 509.197
1.00	3	7 779.090	7 758.597	7 777.946	0.015	7 779.102
	4	24 739.31	24 684.58	24 720.24	0.077	24 739.33
	5	60 572.23	60 457.30	60 475.94	0.159	60 572.25
	1	72.67783	72.10252	72.66977	0.011	72.7351
	2	1 459.805	1 450.306	1 459.697	0.007	1 459.849
2.00	3	7 668.026	7 627.168	7 665.747	0.030	7 668.069
	4	24 541.89	24 432.50	24 503.77	0.155	24 541.94
	5	60 263.77	60 033.91	60 071.21	0.320	60 263.83
	1	60.26579	59.47118	60.25156	0.024	60.3981
	2	1 410.403	1 396.327	1 410.242	0.011	1 410.501
3.00	3	7 556.940	7 495.851	7 553.539	0.005	7 557.036
	4	24 344.45	24 180.50	24 287.33	0.235	24 344.55
	5	59 995.29	59 610.47	59 666.46	0.483	59 955.40
	1	47.81929	46.85748	47.80695	0.026	48.0611
	2	1 360.982	1 342.451	1 360.769	0.016	1 361.153
4.00	3	7 445.834	7 364.650	7 441.338	0.060	7 446.003
	4	24 146.98	23 928.56	24 070.93	0.315	24 147.16
	5	59 646.80	59 187.00	59 261.69	0.065	59 646.98

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = .75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	35.33536	34.26425	35.31343	0.062	35.7241
	2	1 311.542	1 288.684	1 311.276	0.020	1 311.805
5.00	3	7 334.708	7 233.570	7 329.137	0.076	7 334.970
	4	23 949.50	23 676.70	23 859.56	0.376	23 949.76
	5	59 338.28	58 763.48	58 856.91	0.815	59 338.55
	1	22.81063	21.69496	22.78719	0.103	23.3871
	2	1 262.085	1 235.035	1 261.775	0.025	1 262.457
6.00	3	7 223.563	7 102.614	7 216.936	0.092	7 223.937
	4	23 751.99	23 424.91	23 638.22	0.480	23 752.37
	5	59 029.74	58 339.92	58 452.12	0.983	59 030.12
	1	10.24133	9.153818	10.21664	0.241	11.0501
	2	1 212.514	1 181.513	1 212.262	0.029	1 213.109
7.00	3	7 112.400	6 971.787	7 104.729	0.108	7 112.904
	4	23 554.47	2 317.319	23 421.92	0.564	23 554.98
	5	58 721.18	57 916.33	58 047.31	1.154	58 721.70
	1	3.938669	2.895383	3.907605	0.792	4.8815
	2	1 187.873	1 154.802	1 187.496	0.032	1 188.435
7.50	3	7 056.811	6 906.424	7 048.634	0.116	7 057.388
	4	23 455.70	23 047.36	23 313.78	0.607	23 456.28
	5	58 566.89	57 704.51	57 844.89	1.240	58 567.49

TABLE I. BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = .75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	0.7825819	----	0.7542064	3.693	1.7973
	2	1 175.502	1 141.460	1 175.109	0.033	1 176.098
7.75	3	7 029.015	6 873.755	7 020.583	0.120	7 029.629
	4	23 406.31	22 984.45	23 259.71	0.628	23 406.94
	5	58 489.75	57 598.60	57 743.69	1.284	58 490.38

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	82.59065	82.28879	82.58765	0.004	82.6047
	2	1 499.316	1 494.533	1 499.263	0.004	1 499.327
1.00	3	7 756.884	7 736.428	7 755.743	0.015	7 756.896
	4	24 699.84	24 645.15	24 680.75	0.077	24 699.85
	5	60 510.55	60 395.69	60 414.26	0.159	60 510.57
	1	52.86138	52.14043	52.85093	0.020	52.9959
	2	1 380.795	1 366.927	1 380.634	0.012	1 380.892
3.00	3	7 490.320	7 429.579	7 486.927	0.045	7 490.416
	4	24 226.01	24 062.55	24 158.90	0.236	24 226.11
	5	59 770.24	59 426.05	59 481.41	0.484	59 770.35
	1	22.98678	22.15408	22.96599	0.090	23.3871
	2	1 262.199	1 239.950	1 261.929	0.021	1 262.457
5.00	3	7 223.677	7 123.531	7 218.113	0.077	7 223.937
	4	23 752.11	23 480.69	23 657.17	0.401	23 752.37
	5	59 029.85	58 456.83	58 548.49	0.819	59 030.13
	1	7.984866	7.241968	7.961275	0.296	8.5827
	2	1 202.877	1 176.730	1 202.567	0.026	1 203.239
6.00	3	7 090.327	6 970.827	7 083.696	0.094	7 090.698
	4	23 515.12	23 190.04	23 401.35	0.485	23 515.5
	5	58 659.63	57 972.37	58 082.01	0.990	58 660.02

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	4.22 834	3.524225	4.197706	0.692	4.8816
	2	1 188.045	1 160.955	1 187.721	0.027	1 188.435
6.25	3	7 056.987	6 932.686	7 050.098	0.098	7 057.388
	4	23 455.87	23 117.41	23 337.41	0.506	23 456.28
	5	58 567.07	57 851.28	57 965.38	1.033	58 567.49
	1	.4656114	----	.4387692	5.938	1.1804
	2	1 173.212	1 145.193	1 172.879	0.028	1 173.630
6.50	3	7 023.645	6 894.559	7 016.496	0.102	7 024.078
	4	23 396.62	23 044.79	23 273.46	0.528	23 397.06
	5	58 474.51	57 730.19	57 848.76	1.076	58 474.96

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 1.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	77.65570	77.36830	77.65391	0.002	77.6699
	2	1 479.577	1 474.838	1 479.628	0.003	1 479.588
1.00	3	7 712.471	7 692.090	7 711.329	0.015	7 712.483
	4	24 620.88	24 566.31	24 601.80	0.078	24 620.89
	5	60 387.17	60 272.46	60 290.89	0.160	60 387.20
	1	57.871	57.38978	57.86322	0.014	57.9307
	2	1 400.588	1 391.361	1 400.479	0.008	1 400.631
2.00	3	7 534.786	7 494.387	7 532.506	0.030	7 534.830
	4	24 305.02	24 196.28	24 266.90	0.157	24 305.07
	5	59 893.66	59 664.63	59 701.10	0.322	59 893.71
	1	38.05234	37.48683	38.04164	0.028	38.1915
	2	1 321.579	1 308.134	1 321.421	0.072	1 321.674
3.00	3	7 357.081	7 297.041	7 353.682	0.046	7 357.177
	4	23 989.14	23 826.66	23 932.03	0.238	23 989.24
	5	59 400.13	59 057.20	59 111.29	0.487	59 400.23
	1	18.19300	17.67678	18.18 21	0.070	18.4523
	2	1 242.554	1 225.180	1 242.339	0.017	1 242.718
4.00	3	7 179.357	7 100.068	7 174.870	0.063	7 179.524
	4	23 673.24	23 457.46	23 597.19	0.322	23 673.41
	5	58 906.58	58 450.16	58 521.47	0.656	58 906.75

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 1.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
4.80	1	2.272578	1.910738	2.251292	0.941	2.6609
	2	1 179.323	1 159.031	1 179.062	0.022	1 179.552
	3	7 037.165	6 942.770	7 031.815	0.076	7 037.402
	4	23 420.41	23 162.41	23 329.35	0.390	23 420.75
	5	58 511.72	57 964.83	58 049.61	0.793	58 511.97
4.90	1	.2892582	----	.2603751	7.358	.686968
	2	1 171.419	1 150.776	1 171.164	0.022	1 171.657
	3	7 019.390	6 923.126	7 013.928	0.078	7 019.636
	4	23 388.92	23 125.55	23 295.88	0.399	23 389.17
	5	58 462.36	57 904.18	57 990.62	0.810	58 462.62

TABLE I - BOUNDS FOR THE EIGENVLAUES OF THE SIMPLY SUPPORTED BEAM $\beta = 2.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
1.00	1	72.72075	72.44813	72.71235	0.012	72.7351
	2	1 459.838	1 455.143	1 459.786	0.004	1459.849
	3	7 668.058	7 647.753	7 666.911	0.015	7 668.069
	4	24 541.92	24 487.45	24 522.84	0.045	24 541.94
	5	60 263.81	60 149.23	60 167.52	0.150	60 263.83
2.00	1	48.00062	47.58488	47.99471	0.012	48.0611
	2	1 361.110	1 352.067	1 360.999	0.008	1 361.153
	3	7 445.960	7 405.870	7 443.674	0.031	7 445.003
	4	24 147.11	24 038.80	24 108.99	0.158	24 147.15
	5	59 646.92	59 418.45	59 454.36	0.323	59 646.98
3.00	1	23.24296	22.84481	23.232020	0.047	23.3871
	2	1 262.364	1 249.352	1 262.204	0.013	1 262.457
	3	7 223.842	7 164.512	7 220.445	0.097	7 223.937
	4	23 752.27	23 590.78	23 695.16	0.241	23 752.37
	5	59 030.02	58 688.35	58 741.18	0.491	59 030.13
3.75	1	4.645885	4.391169	4.629351	0.357	4.88155
	2	1 188.294	1 172.574	1 188.087	0.017	1 188.435
	3	7 057.242	6 983.855	7 053.022	0.050	7 057.388
	4	23 456.13	23 255.20	23 384.81	0.305	23 456.28
	5	58 567.33	58 141.24	58 206.29	0.618	58 567.49

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = 2.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	2.164188	1.938379	2.150358	0.641	2.41419
	2	1 178.418	1 162.355	1 178.210	0.018	1 178.565
3.85	3	7 035.028	6 959.792	7 030.706	0.061	7 035.181
	4	23 416.64	23 210.48	23 343.44	0.313	23 416.81
	5	58 505.64	58 068.32	58 134.96	9.636	58 505.81

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.10$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
1.00	1	93.44753	93.11508	93.44343	0.004	93.5612
	2	1 542.742	1 537.864	1 542.688	0.004	1 542.754
	3	7 854.593	7 833.972	7 853.450	0.015	7 854.605
	4	24 873.54	24 818.62	24 854.46	0.077	24 873.56
	5	60 781.96	60 666.80	60 685.67	0.159	60 781.98
4.00	1	81.39310	80.01163	81.37477	0.023	81.6177
	2	1 495.202	1 475.431	1 494.993	0.014	1 495.380
	3	7 747.841	7 664.568	7 743.345	0.058	7 748.013
	4	24 683.89	24 462.53	24 607.84	0.309	24 684.06
	5	60 485.71	60 022.11	60 100.61	0.639	60 485.89
8.00	1	64.89664	61.93527	64.87003	0.041	65.8264
	2	1 431.516	1 391.182	1 431.118	0.028	1 432.214
	3	7 605.211	7 436.416	7 596.508	0.115	7 605.891
	4	24 430.72	23 983.31	24 279.42	0.621	24 431.40
	5	60 090.42	59 155.1	59 320.31	1.290	60 091.11
12.00	1	47.86730	42.97865	47.83155	0.075	50.0350
	2	1 367.508	1 305.609	1 366.943	0.041	1 369.049
	3	7 462.254	7 205.422	7 449.643	0.169	7 463.769
	4	24 117.23	23 498.74	23 951.47	0.938	24 178.74
	5	59 694.81	58 279.23	58 539.68	1.954	59 696.32

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.10$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
15.00	1	34.70939	28.00187	34.66177	0.137	38.1915
	2	1 319.305	1 240.427	1 318.638	0.051	1 321.675
	3	7 354.827	7 030.143	7 339.476	0.209	7 357.177
	4	23 986.98	23 131.58	23 705.87	1.178	23 989.24
	5	59 397.88	57 616.27	57 954.20	2.460	59 400.24
19.00	1	16.59538	6.639269	16.55101	0.268	22.4001
	2	1 254.801	1 151.969	1 254.016	0.063	1 258.509
	3	7 211.323	6 793.473	7 192.587	0.260	7 215.055
	4	23 732.84	22 636.78	23 378.85	1.503	23 736.58
	5	59 001.70	56 723.86	57 173.23	3.148	59 005.45
21.00	1	7.271955	----	7.220387	0.712	14.5044
	2	1 222.460	1 106.989	1 221.621	0.069	1 226.927
	3	7 139.458	6 673.771	7 119.119	0.285	7 143.994
	4	23 605.69	22 387.01	23 215.54	1.667	23 610.25
	5	58 803.48	56 273.92	56 782.65	3.497	58 808.06
22.00	1	2.538900	----	2.489753	1.955	10.5565
	2	1 206.270	1 084.293	1 205.413	0.071	1 211.136
	3	7 103.499	6 613.559	7 082.395	0.298	7 108.563
	4	23 542.09	22 261.51	23 133.93	1.749	23 547.08
	5	58 704.35	56 047.99	56 587.34	3.672	58 709.36

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.10$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Consent End Load
22.25	1	1.347959	----	1.298447	3.741	9.56961
	2	1 202.220	1 078.596	1 201.356	0.072	1 207.187
	3	7 094.506	6 598.468	7 073.216	0.301	7 099.580
	4	23 526.18	22 230.07	23 113.49	1.770	23 531.29
	5	58 679.56	55 991.40	56 538.51	3.717	58 684.69
22.50	1	0.1538999	----	0.1040043	38.682	8.58265
	2	1 198.17	1 072.890	1 197.298	0.073	1 203.329
	3	7 085.512	6 583.360	7 064.037	0.304	7 090.697
	4	23 510.28	22 198.61	23 093.09	1.790	23 515.50
	5	58 654.77	55 934.77	56 489.67	3.761	58 660.02

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
1.00	1	94.92801	94.59147	94.92394	0.004	94.9417
	2	1 548.664	1 543.772	1 548.610	0.003	1 548.675
	3	7 867.917	7 847.274	7 866.778	0.014	7 867.929
	4	24 897.23	24 842.28	24 878.15	0.077	24 897.24
	5	60 818.97	60 703.77	60 722.68	0.158	60 818.99
5.00	1	84.72365	82.85905	84.70111	0.027	85.0721
	2	1 508.919	1 483.754	1 508.651	0.018	1 509.197
	3	7 778.833	7 673.832	7 773.258	0.072	7 779.102
	4	24 739.06	24 460.84	24 644.12	0.385	24 739.33
	5	60 571.98	59 990.19	60 090.62	0.798	60 572.25
10.00	1	71.30376	67.34356	71.27029	0.047	72.7351
	2	1 458.752	1 409.939	1 458.265	0.033	1 459.849
	3	7 667.004	7 459.246	7 656.303	0.140	7 668.069
	4	24 540.88	23 979.70	24 352.25	0.772	24 541.94
	5	60 262.77	59 089.72	59 300.19	1.610	60 263.83
15.00	1	57.08734	50.16623	57.03932	0.083	60.3981
	2	1 408.065	1 334.337	1 407.394	0.048	1 410.501
	3	7 554.657	7 241.396	7 539.280	0.204	7 557.036
	4	24 342 18	23 489.36	24 061.14	1.161	24 344.55
	5	59 953.04	58 174.63	58 509.38	2.606	59 955.40

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
20.00	1	42.00398	30.90309	41.95275	0.122	48.0611
	2	1 356.890	1 257.749	1 356.070	0.060	1 361.153
	3	7 441.805	7 021.661	7 422.214	0.264	7 446.003
	4	24 142.97	22 990.42	23 770.82	1.553	24 147.15
	5	59 642.79	57 245.57	57 718.18	3.280	59 646.98
25.00	1	25.97378	8.89325	25.91837	0.214	35.7241
	2	1 305.264	1 179.673	1 304.336	0.071	1 311.805
	3	7 328.459	6 802.433	7 305.086	0.319	7 334.970
	4	23 943.26	22 482.28	23 481.30	1.948	23 949.76
	5	59 332.04	56 301.99	56 926.59	4.138	59 338.55
30.00	1	8.906012	----	8.848833	0.644	23.3871
	2	1 253.232	1 100.010	1 252.223	0.081	1 262.457
	3	7 214.636	6 579.224	7 187.091	0.371	7 223.937
	4	23 743.05	21 964.30	23 192.62	2.345	23 752.37
	5	59 020.79	55 388.08	56 134.63	5.013	59 030.12
32.00	1	1.765746	----	1.710738	3.165	18.4522
	2	1 232.318	1 067.875	1 231.279	0.084	1 242.718
	3	7 168.977	6 489.106	7 141.022	0.391	7 179.524
	4	23 662.83	21 755.39	23 077.39	2.505	23 673.41
	5	58 896.14	54 954.51	55 817.73	5.367	58 906.75

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average	Lumped Constant End Load
	1	0.8600384	----	0.8096142	6.038	17.8354
	2	1 229.700	1 063.699	1 228.664	0.084	1 240.250
32.25	3	7 163.264	6 478.028	7 135.161	0.393	7 173.972
	4	23 652.79	21 725.42	23 062.99	2.525	23 663.54
	5	58 880.55	54 906.15	55 778.11	5.412	58 891.33

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
1.00	1	97.39548	97.05219	97.38885	0.007	97.4091
	2	1 558.533	1 553.621	1 558.481	0.003	1 558.545
	3	7 890.123	7 869.443	7 888.977	0.015	7 890.135
	4	24 936.71	24 881.71	24 917.63	0.077	24 936.72
	5	60 880.66	60 765.39	60 784.37	0.158	60 880.68
5.00	1	97.06945	95.02756	97.05359	0.016	97.4091
	2	1 558.264	1 532.551	1 558.000	0.017	1 558.545
	3	7 889.865	7 783.922	7 884.291	0.071	7 890.135
	4	24 936.45	24 656.89	24 841.50	0.381	24 936.73
	5	60 880.40	60 296.88	60 399.04	0.794	60 880.68
10.00	1	96.05006	91.43544	96.01405	0.037	97.4091
	2	1 557.425	1 506.462	1 556.936	0,031	1 558.545
	3	7 889.057	7 677.593	7 878.347	0.136	7 890.134
	4	24 935.66	24 368.81	24 747.01	0.759	24 936.73
	5	60 879.61	59 699.54	59 917.04	1.594	60 880.68
20.00	1	91.96428	78.64063	91.91397	0.055	97.4091
	2	1 554.071	1 446.067	1 553.239	0.054	1 558.545
	3	7 885.830	7 451.941	7 866.169	0.250	7 890.135
	4	24 932.48	23 756.87	24 560.22	1.504	24 936.73
	5	60 876.46	58 451.08	58 951.90	3.212	60 880.68

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
30.00	1	85.12538	56.42424	85.06259	0.074	97.4091
	2	1 548.503	1 375.513	1 547.459	0.067	1 558.545
	3	7 880.455	7 210.208	7 853.494	0.343	7 890.135
	4	24 927.18	23 095.63	24 376.41	2.234	24 936.73
	5	60 871.19	57 129.12	57 985.23	4.856	60 880.68
40.00	1	75.48948	21.68015	75.42380	0.087	97.4091
	2	1 540.754	1 292.824	1 539.589	0.076	1 558.545
	3	7 872.940	6 950.046	7 840.212	0.417	7 890.135
	4	24 919.78	22 387.13	24 195.62	2.949	24 936.73
	5	60 863.84	55 727.85	57 017.05	6.527	60 880.68
50.00	1	62.99520	----	62.93328	0.098	97.4091
	2	1 530.869	1 195.846	1 529.658	0.079	1 558.545
	3	7 863.296	6 669.137	7 826.211	0.473	7 890.135
	4	24 910.26	21 611.65	24 017.91	3.648	24 936.73
	5	60 854.38	54 239.88	56 047.35	8.224	60 880.68
60.00	1	47.56379	----	47.49907	0.136	97.4091
	2	1 518.906	1 082.880	1 517.698	0.080	1 558.545
	3	7 851.536	6 364.444	7 811.384	0.513	7 890.135
	4	24 898.63	20 767.33	23 843.33	4.330	24 936.73
	5	60 842.82	52 657.41	55 076.16	9.949	60 880.68

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
70.00	1	29.09930	----	29.03900	0.207	97.4091
	2	1 504.937	951.7511	1 503.762	0.078	1 558.545
	3	7 837.675	6 034.095	7 795.619	0.538	7 890.135
	4	24 884.90	19 863.03	23 671.88	4.996	24 936.73
	5	60 829.17	50 971.82	54 103.51	11.704	60 880.68
80.00	1	7.488954	----	7.427070	0.830	97.4091
	2	1 489.045	801.0849	1 487.923	0.075	1 558.545
	3	7 821.731	5 689.257	7 778.816	0.550	7 890.135
	4	24 869.08	18 858.86	23 503.62	5.646	24 936.73
	5	60 813.43	49 175.77	53 129.41	13.487	60 880.68
81.00	1	5.150067	----	5.085385	1.264	97.4091
	2	1 487.354	785.6676	1 486.242	0.075	1 558.545
	3	7 820.022	5 652.000	7 777.073	0.551	7 890.135
	4	24 8 7.38	18 753.30	23 486.96	5.710	24 936.73
	5	60 811.74	48 988.74	53 031.92	13.668	60 880.68
82.00	1	2.778307	----	2.713758	2.351	97.4091
	2	1 485.644	768.9310	1 484.539	0.074	1 558.545
	3	7 818.294	5 615.842	7 775.319	0.551	7 890.135
	4	24 86 .67	18 646.77	23 470.35	5.773	24 936.73
	5	60 810.03	48 801.56	52 934.43	13.848	60 880.68

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	99.86295	99.51297	99.86157	0.001	99.8765
	2	1 568.403	1 563.469	1 568.352	0.003	1 568.415
1.00	3	7 912.330	7 891.613	7 911.188	0.014	7 912.342
	4	24 976.19	24 921.13	24 957.11	0.076	24 976.20
	5	60 942.34	60 827.01	60 846.06	0.158	60 942.36
	1	109.4148	107.2052	109.3978	0.016	109.7461
	2	1 607.610	1 581.359	1 607.345	0.016	1 607.893
5.00	3	8 000.896	7 894.022	7 995.323	0.070	8 001.168
	4	25 133.84	24 852.95	25 038.90	0.378	25 134.11
	5	61 1881.83	60 603.59	60 707.47	0.790	61 189.10
	1	120.7894	115.5869	120.7548	0.029	122.0831
	2	1 656.102	1 603.075	1 655.616	0.029	1 657.241
10.00	3	8 111.112	7 896.020	8 100.397	0.132	8 112.202
	4	25 330.43	24 758.00	25 141.78	0.748	25 331.51
	5	61 496.46	60 309.52	60 533.90	1.578	61 497.53
	1	141.8121	126.6892	141.7554	0.040	146.7571
	2	1 751.334	1 635.145	1 750.491	0.048	1 755.937
20.00	3	8 329.371	7 880.263	8 310.140	0.237	8 334.268
	4	25 721.99	24 523.88	25 349.63	1.458	25 726.29
	5	62 110.12	59 656.94	60 185.61	3.147	62 114.38

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
30.00	1	160.7671	129.6790	160.7000	0.042	171.4311
	2	1 844.200	1 653.497	1 843.138	0.058	1 854.633
	3	8 546.365	7 840.957	8 519.187	0.319	8 556.334
	4	26 111.35	24 234.53	25 560.22	2.133	26 121.079
	5	62 721.62	58 916.90	59 835.84	4.709	62 731.23
40.00	1	177.8850	123.8125	177.8083	0.043	196.1051
	2	1 934.700	1 656.769	1 933.479	0.063	1 953.329
	3	8 760.552	7 776.235	8 727.331	0.380	8 778.400
	4	26 498.48	23 875.30	25 773.49	2.774	26 515.86
	5	63 330.94	58 083.83	59 484.58	6.264	63 348.08
50.00	1	193.3510	108.4000	193.2635	0.045	220.7791
	2	2 022.857	1 643.877	2 021.548	0.065	2 052.0256
	3	8 972.409	7 684.302	8 934.409	0.424	9 000.466
	4	26 883.35	23 454.02	25 989.35	3.382	26 910.64
	5	63 938.06	57 149.33	59 131.86	7.811	63 964.93
60.00	1	207.3155	82.13972	207.2298	0.041	245.4531
	2	2 108.712	1 611.067	2 107.367	0.064	2 150.726
	3	9 181.915	7 572.918	9 140.270	0.455	9 222.532
	4	27 265.95	22 945.47	26 207.73	3.958	27 305.43
	5	64 542.95	56 109.45	58 777.69	9.350	64 581.78

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
70.00	1	219.9011	45.50295	219.8101	0.041	270.1271
	2	2 192.316	1 565.173	2 190.959	0.062	2 249.417
	3	9 389.065	7 423.49	9 344.764	0.473	9 444.598
	4	27 646.24	22 435.10	26 428.54	4.504	27 700.22
	5	65 145.60	54 952.82	58 422.11	10.882	65 198.63
80.00	1	231.2094	----	231.1131	0.042	294.8011
	2	2 273.728	1 497.529	2 272.369	0.060	2 348.114
	3	9 593.853	7 241.964	9 547.784	0.481	9 666.664
	4	28 024.21	21 680.87	26 651.69	5.021	28 094.99
	5	65 745.98	53 671.96	58 065.14	12.407	65 815.48
90.00	1	241.3245	----	241.2256	0.041	319.4752
	2	2 353.010	1 410.343	2 351.666	0.057	2 446.809
	3	9 796.286	7 026.768	9 749.202	0.482	9 888.731
	4	28 399.86	20 992.29	26 877.07	5.510	28 489.78
	5	66 344.09	52 257.63	57 706.82	13.925	66 432.33
100.00	1	250.3169	----	250.2154	0.041	344.1492
	2	2 430.225	1 303.072	2 428.900	0.055	2 545.505
	3	9 996.371	6 795.214	9 948.945	0.476	10 110.79
	4	28 773.16	20 018.08	27 104.57	5.972	28 884.57
	5	66 939.92	50 702.86	57 347.18	15.436	67 047.18

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -0.75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	280.2315	----	280.0769	0.057	467.5192
	2	2 787.495	439.3405	2 786.273	0.044	3 038.986
150.00	3	10 962 20	5 034.360	10 919.79	0.406	11 221.12
	4	30 604.40	13 548.06	28 269.28	7.933	30 858.49
	5	69 884.39	40 397.25	55 531.11	22.889	70 133.43

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
1.00	1	102.3304	101.9738	102.3264	0.004	102.3439
	2	1 578.273	1 573.317	1 578.219	0.003	1 578.284
	3	7 934.537	7 913.782	7 933.391	0.014	7 934.549
	4	25 015.66	24 960.05	24 996.58	0.076	25 015.68
	5	61 004.03	60 888.62	60 907.74	0.158	61 004.05
5.00	1	121.7597	119.3913	121.7338	0.021	122.0831
	2	1 656.956	1 630.178	1 656.690	0.016	1 657.241
	3	8 111.928	8 004.133	8 106.352	0.069	8 112.202
	4	25 331.23	25 049.03	25 236.29	0.375	25 331.51
	5	61 497.25	60 910.29	61 015.89	0.786	61 497.52
10.00	1	145.6227	139.7893	145.4817	0.028	145.7571
	2	1 754.785	1 699.563	1 754.292	0.028	1 755.937
	3	8 333.166	8 114.520	8 322.439	0.087	8 334.268
	4	25 724.21	25 148.07	25 536.55	0.736	25 726.29
	5	62 113.30	60 914.02	61 150.74	1.562	62 114.38
20.00	1	191.5759	174.8713	191.5207	0.029	196.1051
	2	1 948.654	1 825.068	1 947.809	0.043	1 953.329
	3	8 773.930	8 311.577	8 754.135	0.226	8 778.400
	4	26 511.51	25 294.93	26 139.05	1.415	26 515.86
	5	63 343.78	60 863.53	61 419.33	3.085	63 348.08

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
30.00	1	236.0319	203.0425	235.9576	0.031	245.4531
	2	2 140.146	1 933.952	2 139.055	0.051	2 150.721
	3	9 212.350	8 476.840	9 184.954	0.298	9 222.532
	4	27 295.54	25 372.56	26 744.07	2.041	27 304.43
	5	64 572.05	60 706.93	61 686.44	4.571	64 581.78
40.00	1	279.2148	224.5836	279.1330	0.029	294.8012
	2	2 329.327	2 026.117	2 328.056	0.055	2 348.114
	3	9 648.386	8 615.351	9 614.699	0.350	9 666.664
	4	28 077.27	25 380.02	27 351.47	2.619	28 094.99
	5	65 798.06	60 444.15	61 952.11	6.021	65 815.48
50.00	1	321.3589	239.9198	321.2640	0.030	344.1492
	2	2 516.301	2 099.304	2 514.910	0.055	2 545.506
	3	10 082.02	8 716.340	10 043.17	0.386	10 110.79
	4	28 856.64	25 305.54	27 961.06	3.152	28 884.57
	5	67 021.81	60 068.90	62 215.35	7.437	67 049.18
75.00	1	423.2159	252.5182	423.102	0.027	467.5192
	2	2 974.865	2 216.332	2 973.316	0.052	3 038.986
	3	11 155.69	8 839.134	11 108.01	0.428	11 221.13
	4	30 794.66	24 732.32	29 493.62	4.316	30 858.49
	5	70 070.98	58 596.06	62 870.95	10.832	70 133.43

TABLE I - BOUNDS FOR THE EIGENVALUES OF THE SIMPLY SUPPORTED BEAM $\beta = -1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
100.00	1	521.3549	230.0603	521.2197	0.026	590.8893
	2	3 422.233	2 221.951	3 420.612	0.047	3 532.466
	3	12 214.89	8 756.443	12 162.74	0.428	12 331.46
	4	32 717.68	23 562.65	31 036.50	5.274	32 832.41
	5	73 105.38	56 270.24	63 517.38	14.036	73 217.68
125.00	1	616.7877	179.7252	616.6295	0.026	714.2593
	2	3 860.049	2 144.900	3 858.391	0.043	4 025.946
	3	13 260.35	8 451.943	13 206.70	0.405	13 441.79
	4	34 625.77	21 666.47	32 587.38	6.065	34 806.33
	5	76 124.82	52 967.30	64 156.24	17.064	76 301.94
150.00	1	710.1394	99.52777	709.9560	0.026	837.6294
	2	4 289.673	1 958.556	4 287.981	0.039	4 519.426
	3	14 292.91	7 961.67	14 239.73	0.373	14 552.12
	4	36 519.20	18 874.88	34 143.94	6.723	36 780.25
	5	79 129.26	48 545.04	64 788.19	19.930	79 386.18

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 0.0$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
0.0	1	500.564	500.564	500.564	0.000	500.564
	2	3 803.54	3 803.54	3 803.54	0.000	3 803.54
	3	14 617.6	14 617.6	14 617.6	0.000	14 617.6
	4	39 943.8	39 943.8	39 943.8	0.000	39 943.8
10.0	1	438.485	436.964	438.281	0.047	438.857
	2	3 572.38	3 570.81	3 571.05	0.037	3 573.03
	3	14 122.2	14 119.9	14 115.8	0.045	14 123.0
	4	39 084.9	39 081.3	39 069.3	0.040	39 085.8
20.0	1	375.192	366.109	374.825	0.098	376.742
	2	3 339.36	3 320.29	3 336.87	0.074	3 342.01
	3	13 624.9	13 589.2	13 612.6	0.091	13 628.2
	4	38 224.2	38 189.7	38 196.0	0.074	38 227.7
30.0	1	310.543	289.161	310.010	0.171	314.184
	2	3 104.42	3 062.73	3 100.83	0.115	3 110.44
	3	13 125.9	13 047.8	13 107.8	0.138	13 133.2
	4	37 361.7	37 285.5	37 318.9	0.114	37 369.5
40.0	1	244.366	200.691	243.766	0.245	251.143
	2	2 867.50	2 786.55	2 863.13	0.152	2 878.30
	3	12 625.1	12 472.4	12 601.6	0.186	12 638.1
	4	36 497.3	36 364.3	36 440.4	0.156	36 511.2

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 0.0$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
50.0	1	176.455	110.389	175.815	0.363	187.572
	2	2 628.54	2 505.85	2 623.39	0.195	2 645.57
	3	12 122.6	11 891.2	12 094.1	0.234	12 142.9
	4	35 631.2	35 422.2	35 562.0	0.194	35 652.8
60.0	1	106.557	5.130	105.877	0.639	123.421
	2	2 387.52	2 164.11	2 381.38	0.257	2 412.22
	3	11 618.4	11 249.9	11 584.8	0.288	11 647.6
	4	34 763.3	34 453.4	34 679.2	0.242	34 794.3
70.0	1	34.354	----	33.672	2.004	58.631
	2	2 144.43	1 801.43	2 137.92	0.304	2 178.22
	3	11 112.6	10 592.4	11 074.6	0.342	11 152.3
	4	33 893.8	33 459.1	33 799.5	0.278	33 935.9
72.0	1	19.603	----	18.915	3.567	45.590
	2	2 095.57	1 735.77	2 088.83	0.321	2 131.34
	3	11 011.2	10 464.7	10 971.8	0.358	11 053.2
	4	33 719.7	33 261.5	33 621.8	0.290	33 764.2
73.0	1	12.185	----	11.517	5.632	39.059
	2	2 071.11	1 702.82	2 064.43	0.322	2 107.88
	3	10 960.5	10 400.7	10 920.7	0.363	11 003.6
	4	33 632.5	33 097.8	33 535.9	0.287	33 678.3

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 0.0$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
74.0	1	4.740	----	4.101	14.457	32.521
	2	2 046.62	1 669.78	2 039.78	0.334	2 084.43
	3	10 909.8	10 336.5	10 870.1	0.364	10 954.1
	4	33 545.5	32 997.1	33 445.2	0.299	33 592.5

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = .25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
5.0	1	454.229	454.011	454.138	0.020	454.321
	2	3 630.54	3 629.90	3 629.87	0.018	3 630.71
	3	14 246.5	14 245.5	14 243.2	0.022	14 246.7
	4	39 300.1	39 298.6	39 291.9	0.021	39 300.3
10.0	1	407.474	403.672	407.270	0.050	407.853
	2	3 456.93	3 449.07	3 455.61	0.038	3 457.59
	3	13 874.8	13 859.6	13 868.4	0.046	13 875.6
	4	38 655.9	38 641.5	38 641.2	0.038	38 656.8
15.0	1	360.266	352.036	359.978	0.080	361.146
	2	3 282.68	3 265.48	3 280.76	0.058	3 284.16
	3	13 502.7	13 469.4	13 493.3	0.069	13 504.5
	4	38 011.2	37 979.6	37 989.0	0.058	38 013.2
20.0	1	312.568	297.357	312.215	0.113	314.184
	2	3 107.77	3 078.04	3 105.32	0.079	3 110.44
	3	13 130.0	13 072.4	13 117.7	0.094	13 133.2
	4	37 366.1	37 310.7	37 335.5	0.082	37 369.5
25.0	1	264.339	241.495	263.913	0.161	266.951
	2	2 932.19	2 881.39	2 929.19	0.102	2 936.39
	3	12 756.1	12 669.1	12 741.7	0.119	12 761.9
	4	36 720.5	36 635.3	36 684.8	0.097	36 725.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = .25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
30.0	1	215.531	181.248	215.058	0.220	219.427
	2	2 755.92	2 684.91	2 752.53	0.123	2 762.01
	3	12 383.3	12 259.8	12 365.3	0.145	12 390.5
	4	36 074.4	35 947.8	36 030.6	0.121	36 082.0
35.0	1	166.090	117.249	165.609	0.289	171.591
	2	2 578.95	2 485.09	2 575.09	0.149	2 587.29
	3	12 009.3	11 824.7	11 988.7	0.171	12 019.1
	4	35 427.8	35 258.3	35 376.8	0.144	35 438.2
40.0	1	115.952	45.459	115.458	0.426	123.422
	2	2 401.27	2 266.53	2 397.11	0.173	2 412.22
	3	11 634.8	11 399.8	11 611.7	0.198	11 647.6
	4	34 780.9	34 546.6	34 724.5	0.162	34 794.4
45.0	1	65.043	----	64.531	0.789	74.892
	2	2 222.86	2 033.24	2 218.17	0.211	2 236.78
	3	11 259.9	10 969.9	11 234.2	0.228	11 276.1
	4	34 133.4	33 841.8	34 070.1	0.185	34 150.5
49.0	1	23.705	----	23.192	2.186	35.791
	2	2 079.62	1 823.99	2 074.83	0.230	2 096.16
	3	10 959.7	10 601.7	10 932.1	0.251	10 978.9
	4	33 615.2	33 227.0	33 546.8	0.203	33 635.4

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = .25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	13.279	----	12.767	3.927	25.975
	2	2 043.73	1 779.62	2 038.75	0.243	2 060.96
50.0	3	10 884.7	10 513.8	10 856.5	0.258	10 904.6
	4	33 485.6	33 082.9	33 415.6	0.209	33 506.6

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = .50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
5.0	1	438.765	437.247	438.674	0.021	438.857
	2	3 572.87	3 569.78	3 572.20	0.018	3 573.03
	3	14 122.8	14 116.7	14 119.6	0.023	14 123.0
	4	39 085.1	39 079.9	39 077.9	0.019	39 085.8
10.0	1	376.346	370.642	376.159	0.052	376.743
	2	3 341.35	3 329.51	3 340.05	0.039	3 342.01
	3	13 627.4	13 604.2	13 621.1	0.046	13 628.2
	4	38 226.8	38 204.9	38 211.4	0.040	38 227.7
15.0	1	313.277	300.721	312.998	0.089	314.183
	2	3 108.94	3 080.15	3 107.06	0.060	3 110.437
	3	13 131.5	13 081.0	13 122.1	0.071	13 133.2
	4	37 367.6	47 313.1	37 343.8	0.063	37 369.5
20.0	1	249.456	226.799	249.107	0.140	251.143
	2	2 875.62	2 826.68	2 873.20	0.084	2 878.30
	3	12 635.0	12 548.4	12 622.7	0.097	12 638.1
	4	36 507.9	36 413.4	36 478.8	0.079	36 511.2
25.0	1	184.806	148.818	184.415	0.212	187.573
	2	2 641.34	2 568.21	2 638.51	0.107	2 645.57
	3	12 138.0	12 007.4	12 122.9	0.124	12 142.9
	4	35 647.7	35 503.7	35 610.2	0.105	35 652.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = .50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
30.0	1	119.229	66.432	118.810	0.351	123.422
	2	2 406.08	2 292.07	2 402.80	0.136	2 412.22
	3	11 640.6	11 458.7	11 622.7	0.153	11 647.6
	4	34 787.0	34 555.9	34 745.1	0.120	34 794.4
35.0	1	52.603	----	52.188	0.791	58.631
	2	2 169.82	2 000.18	2 166.18	0.167	2 178.22
	3	11 142.7	10 874.9	11 122.3	0.182	11 152.3
	4	33 925.9	33 570.4	33 875.6	0.148	33 935.9
37.9	1	13.422	----	13.003	3.165	20.733
	2	2 032.32	1 806.94	2 028.44	0.190	2 042.19
	3	10 853.7	10 546.2	10 831.8	0.201	10 864.9
	4	33 426.2	33 016.6	33 372.7	0.160	33 437.9

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM β = .75

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
5.0	1	423.275	421.095	423.173	0.024	423.368
	2	3 515.16	3 510.73	3 514.51	0.018	3 515.33
	3	13 999.1	13 990.4	13 995.9	0.023	13 999.3
	4	38 871.1	38 862.9	38 864.1	0.018	38 871.3
10.0	1	345.128	336.790	344.945	0.053	345.522
	2	3 225.64	3 208.86	3 224.36	0.040	3 226.29
	3	13 379.9	13 346.6	13 373.6	0.047	13 380.7
	4	37 797.8	37 766.3	37 783.4	0.038	37 798.6
15.0	1	266.015	247.405	265.745	0.101	266.951
	2	2 934.90	2 894.78	2 933.03	0.063	2 936.39
	3	12 760.2	12 688.5	12 750.8	0.073	12 761.9
	4	36 724.0	36 655.3	36 700.9	0.063	36 725.8
20.0	1	185.806	153.039	185.486	0.172	187.573
	2	2 642.88	2 575.69	2 640.56	0.087	2 645.57
	3	12 139.9	12 018.2	12 127.6	0.100	12 142.9
	4	35 649.6	35 516.0	35 620.7	0.081	35 652.8
25.0	1	104.342	51.289	103.995	0.333	107.287
	2	2 349.54	2 237.66	2 346.74	0.119	2 353.78
	3	11 519.0	11 337.2	11 504.04	0.129	11 523.8
	4	34 574.8	34 344.0	34 537.7	0.107	34 579.7

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = .75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
30.0	1	21.426	----	21.064	1.703	25.9749
	2	2 054.82	1 852.78	2 051.66	0.153	2 060.96
	3	10 897.7	10 617.7	10 880.1	0.161	10 904.6
	4	33 499.6	33 181.0	33 456.0	0.130	33 506.6
30.5	1	13.045	----	12.685	2.799	17.782
	2	2 025.27	1 818.03	2 022.09	0.156	2 031.62
	3	10 835.6	10 547.5	10 817.7	0.164	10 842.7
	4	33 392.0	33 061.3	33 348.2	0.131	33 399.2

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
5.0	1	407.759	404.794	407.663	0.023	407.853
	2	3 457.42	3 451.38	3 456.75	0.019	3 457.59
	3	13 875.4	13 863.4	13 872.2	0.023	13 875.6
	4	38 656.6	38 645.4	38 648.7	0.020	38 656.8
10.0	1	313.783	302.702	313.598	0.059	314.184
	2	3 109.78	3 084.36	3 108.52	0.040	3 110.44
	3	13 132.5	13 087.2	13 126.2	0.048	13 133.2
	4	37 368.7	37 325.8	37 354.4	0.038	37 369.5
15.0	1	218.460	192.281	218.202	0.117	219.427
	2	2 760.52	2 700.07	2 758.67	0.066	2 762.01
	3	12 388.9	12 292.5	12 379.5	0.075	12 390.5
	4	36 080.3	35 975.2	36 058.3	0.061	36 082.0
20.0	1	121.567	74.571	121.262	0.250	123.422
	2	2 409.53	2 295.25	2 407.26	0.094	2 412.22
	3	11 644.6	11 482.5	11 632.4	0.104	11 647.6
	4	34 791.4	34 611.8	34 761.8	0.085	34 794.4
22.0	1	82.309	22.824	82.001	0.374	84.628
	2	2 268.63	2 135.06	2 266.17	0.108	2 271.90
	3	11 346.8	11 154.9	11 333.5	0.116	11 350.4
	4	34 275.6	34 051.7	34 243.6	0.093	34 279.3

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM β = 1.00

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
25.0	1	22.822	----	22.515	1.354	25.975
	2	2 056.73	1 865.36	2 054.05	0.130	2 060.96
	3	10 899.9	10 574.8	10 885.1	0.135	10 904.6
	4	33 501.9	33 194.9	33 465.6	0.108	33 506.6
25.6	1	10.832	----	10.502	3.086	14.173
	2	2 014.26	1 814.86	2 011.50	0.136	2 081.71
	3	10 810.6	10 472.3	10 795.1	0.143	10 815.4
	4	33 347.2	33 026.9	33 310.1	0.111	33 352.1

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 1.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
5.0	1	376.647	371.774	376.549	0.026	376.743
	2	3 341.85	3 331.81	3 341.18	0.020	3 342.01
	3	13 628.0	13 607.9	13 624.8	0.023	13 628.2
	4	38 227.5	38 208.8	38 220.1	0.019	38 227.7
10.0	1	250.726	231.953	250.551	0.069	251.143
	2	2 877.65	2 836.71	2 876.39	0.044	2 878.30
	3	12 637.4	12 563.1	12 631.1	0.050	12 638.1
	4	36 510.5	36 430.0	36 494.9	0.043	36 511.2
15.0	1	122.384	80.317	122.166	0.174	123.422
	2	2 410.73	2 316.69	2 408.97	0.071	2 412.22
	3	11 646.1	11 490.9	11 636.8	0.079	11 647.6
	4	34 792.9	34 621.3	34 770.9	0.063	34 794.4
17.0	1	70.251	15.309	70.006	0.349	71.644
	2	2 223.15	2 090.54	2 221.21	0.087	2 225.07
	3	11 249.4	11 032.7	11 238.9	0.092	11 251.33
	4	34 105.7	33 859.8	34 081.1	0.072	34 107.6
18.0	1	43.993	----	43.747	0.561	45.590
	2	2 129.18	1 984.01	2 127.15	0.095	2 131.34
	3	11 051.0	10 812.0	11 039.9	0.099	11 053.2
	4	33 752.1	33 489.6	33 736.1	0.076	33 764.2

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM β = 1.50

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
19.0	1	17.599	----	17.348	1.432	19.422
	2	2 035.09	1 850.68	2 032.97	0.104	2 037.49
	3	10 852.6	10 574.2	10 841.0	0.106	10 885.1
	4	33 418.4	33 070.8	33 390.8	0.082	33 420.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 2.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
5.0	1	345.425	343.790	345.341	0.024	345.521
	2	3 226.13	3 221.23	3 225.48	0.020	3 226.29
	3	13 380.6	13 372.4	13 377.3	0.024	13 380.7
	4	37 798.5	37 785.5	37 790.6	0.021	37 798.6
10.0	1	187.138	158.651	186.972	0.088	187.573
	2	2 644.93	2 577.60	2 643.70	0.046	2 645.57
	3	12 142.3	12 032.7	12 136.0	0.052	12 142.9
	4	35 652.3	35 532.4	35 637.5	0.041	35 652.8
12.0	1	122.762	80.012	122.576	0.151	123.422
	2	2 411.29	2 305.74	2 409.85	0.059	2 412.22
	3	11 646.7	11 494.8	11 639.2	0.064	11 674.6
	4	34 793.6	34 625.7	34 775.8	0.051	34 794.4
14.0	1	57.682	----	57.474	0.360	58.6312
	2	2 176.94	2 018.56	2 175.32	0.075	2 178.22
	3	11 151.1	10 928.2	11 142.3	0.078	11 152.3
	4	33 934.9	33 681.5	33 914.1	0.061	33 935.9
15.0	1	24.852	----	24.652	0.806	25.975
	2	2 059.50	1 883.64	2 057.77	0.083	2 060.96
	3	10 903.3	10 652.4	10 893.9	0.085	10 904.6
	4	33 595.4	33 218.8	33 483.6	0.065	33 506.6

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = 2.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
	1	14.954	----	14.757	1.388	16.142
	2	2 024.23	1 843.09	2 022.46	0.087	2 025.75
15.3	3	10 828.9	10 569.5	10 819.4	0.087	10 830.3
	4	33 376.6	33 079.7	33 353.8	0.068	33 377.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
10.0	1	469.394	467.968	469.193	0.043	469.760
	2	3 687.69	3 684.62	3 686.35	0.036	3 688.35
	3	14 369.5	14 364.0	14 363.1	0.044	14 370.3
	4	39 513.9	39 508.5	39 498.7	0.038	39 514.8
20.0	1	437.367	431.682	436.975	0.090	438.857
	2	3 570.41	3 558.22	3 567.84	0.072	3 573.03
	3	14 119.7	14 097.8	14 107.3	0.088	14 123.0
	4	39 082.3	39 060.8	39 051.9	0.078	39 085.8
40.0	1	370.529	347.856	369.834	0.188	376.743
	2	3 331.41	3 283.41	3 326.61	0.144	3 342.01
	3	13 615.1	13 529.4	13 591.1	0.176	13 628.2
	4	38 213.6	38 129.0	38 156.6	0.149	38 227.7
60.0	1	299.567	244.237	298.689	0.293	314.184
	2	3 086.33	2 949.91	3 079.92	0.207	2 110.44
	3	13 103.8	12 914.7	13 069.2	0.263	13 133.2
	4	37 337.9	37 149.8	37 257.5	0.215	37 369.5
80.0	1	223.878	116.025	222.877	0.448	251.143
	2	2 835.04	2 625.53	2 826.75	0.292	2 878.30
	3	12 585.8	12 256.2	12 541.8	0.349	12 638.1
	4	36 455.1	36 081.7	36 344.4	0.303	36 511.2

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -.25$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
100.0	1	142.697	----		----	187.573
	2	2 577.45	2 255.24		13.33	2 645.57
	3	12 061.2	11 555.8		4.25	12 142.9
	4	35 565.4	34 990.1		1.63	35 652.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
10.0	1	500.205	499.105	499.992	0.042	500.564
	2	3 802.89	3 800.44	3 801.55	0.035	3 803.54
	3	14 616.8	14 613.5	14 610.4	0.044	14 617.6
	4	39 942.9	39 938.6	39 926.8	0.040	39 943.8
20.0	1	499.129	494.724	498.723	0.081	500.564
	2	3 800.94	3 791.15	3 798.33	0.068	3 803.54
	3	14 614.4	14 601.4	14 601.8	0.086	14 617.6
	4	39 940.2	39 923.2	39 909.6	0.077	39 943.8
40.0	1	494.821	477.099	494.061	0.153	500.564
	2	3 793.17	3 763.95	3 788.27	0.129	3 803.54
	3	14 604.6	14 552.6	14 580.3	0.166	14 617.6
	4	39 929.6	39 861.4	39 874.9	0.137	39 943.8
60.0	1	487.621	447.372	486.533	0.223	500.564
	2	3 780.22	3 591.84	3 773.26	0.184	3 803.54
	3	14 588.3	14 471.2	14 552.9	0.243	14 617.6
	4	39 912.0	39 758.5	39 829.7	0.206	39 943.8
80.0	1	477.550	405.005	476.125	0.288	500.564
	2	3 762.09	3 604.65	3 753.56	0.227	3 803.54
	3	14 565.6	14 357.2	14 519.8	0.314	14 617.6
	4	39 887.3	39 14.4	39 779.1	0.271	39 943.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -.50$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
100.0	1	464.419	349.230	462.905	0.326	500.564
	2	3 738.80	3 492.14	3 728.65	0.271	3 803.54
	3	14 536.3	14 210.1	14 481.2	0.379	14 617.6
	4	39 855.6	39 429.1	39 726.5	0.324	39 943.8
125.0	1	443.820	259.076	442.025	0.405	500.564
	2	3 702.43	3 315.45	3 689.41	0.352	3 893.54
	3	14 490.7	13 981.0	14 423.9	0.461	14 617.6
	4	39 806.0	39 139.5	39 640.2	0.417	39 943.8

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -.75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
10.0	1	530.922	529.723	530.701	0.042	531.274
	2	3 917.96	3 915.17	3 916.53	0.036	3 918.60
	3	14 864.1	14 859.8	14 857.6	0.043	14 864.9
	4	40 371.8	40 366.3	40 355.9	0.039	40 372.7
20.0	1	560.511	555.692	560.062	0.080	561.893
	2	4 030.98	4 020.35	4 028.28	0.067	4 033.54
	3	15 108.8	15 091.5	15 096.1	0.084	15 112.1
	4	40 798.1	40 768.6	40 768.9	0.072	40 801.7
40.0	1	617.532	599.287	616.673	0.139	622.874
	2	4 252.96	4 212.01	4 247.82	0.121	4 263.06
	3	15 593.4	15 523.6	15 568.6	0.158	15 606.3
	4	41 645.3	41 566.3	41 585.6	0.143	41 659.4
60.0	1	671.896	634.041	670.667	0.183	683.530
	2	4 469.68	4 380.84	4 462.22	0.167	4 492.12
	3	16 071.4	15 819.7	16 035.1	0.225	16 100.2
	4	42 485.3	42 309.6	42 399.3	0.203	42 517.0
80.0	1	723.829	655.591	722.256	0.217	743.883
	2	4 681.35	4 521.11	4 671.69	0.206	4 720.74
	3	16 542.9	16 294.4	16 495.6	0.286	16 494.0
	4	43 318.2	43 016.2	43 203.9	0.264	43 374.4

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -.75$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
100.0	1	773.522	671.565	770.979	0.329	803.955
	2	4 888.16	4 646.52	4 864.31	0.489	4 948.93
	3	17 008.0	16 648.5	16 936.9	0.419	17 087.5
	4	44 144.0	43 687.5	43 861.5	0.642	44 231.7
125.0	1	832.739	683.535	829.534	0.386	878.676
	2	5 140.13	4 790.86	5 110.31	0.582	5 233.58
	3	17 580.6	17 009.6	17 493.0	0.499	17 704.0
	4	45 166.2	44 443.2	44 827.8	0.752	45 303.1
150.0	1	888.985	696.494	885.218	0.425	953.016
	2	5 385.16	4 896.41	5 350.45	0.646	5 517.62
	3	18 143.7	17 377.3	18 040.9	0.568	18 230.2
	4	46 177.5	45 167.2	45 775.5	0.874	46 374.2

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problems	Gap/Average Per Cent	Lumped Constant End Load
10.0	1	561.548	559.521	561.338	0.037	561.893
	2	4 032.90	4 028.51	4 031.55	0.033	4 033.54
	3	15 111.2	15 103.9	15 104.9	0.042	15 112.1
	4	40 800.7	40 792.7	40 784.9	0.039	40 801.7
20.0	1	621.541	613.689	621.114	0.069	622.874
	2	4 260.54	4 243.43	4 257.90	0.062	4 263.06
	3	15 603.1	15 577.1	15 590.7	0.079	15 606.3
	4	41 655.9	41 623.8	41 608.6	0.114	41 659.4
40.0	1	738.893	709.280	738.076	0.111	743.883
	2	4 710.92	4 642.44	4 705.95	0.105	4 720.74
	3	16 581.3	16 482.1	16 557.3	0.145	16 594.0
	4	43 360.5	43 233.6	43 308.4	0.120	43 374.4
60.0	1	853.195	793.191	851.688	0.177	863.763
	2	5 155.16	5 001.03	5 147.73	0.144	5 176.70
	3	17 552.5	17 339.1	17 513.6	0.222	17 580.8
	4	45 057.6	44 775.5	44 901.7	0.347	45 088.8
80.0	1	964.891	857.790	963.176	0.178	982.652
	2	5 593.76	5 332.86	5 584.65	0.163	5 631.07
	3	18 516.9	18 153.4	18 471.8	0.244	18 566.5
	4	46 747.3	46 251.4	46 584.8	0.348	46 802.6

TABLE II - BOUNDS FOR THE EIGENVALUES OF THE CLAMPED BEAM $\beta = -1.00$

α	Order	Upper Bound by Rayleigh-Ritz	Lower Bound by Kato's Method	Lower Bound by Intermediate Problem	Gap/Average Per Cent	Lumped Constant End Load
100.0	1	1 074.33	917.851	1 071.61	0.253	1 100.66
	2	6 027.13	5 637.97	6 015.50	0.193	6 083.95
	3	19 474.6	18 929.7	19 407.2	0.346	19 551.3
	4	48 429.6	47 680.0	48 255.9	0.359	48 515.6
125.0	1	1 208.37	975.666	1 204.80	0.296	1 247.09
	2	6 562.08	5 998.45	6 548.45	0.209	6 648.11
	3	20 662.6	19 837.7	20 578.1	0.411	20 780.8
	4	50 522.1	49 425.7	50 209.0	0.626	50 655.8
150.0	1	1 339.76	1 019.69	1 336.44	0.249	1 392.44
	2	7 090.13	6 269.69	7 075.31	0.209	7 210.27
	3	21 881.0	20 689.1	21 763.0	0.358	22 008.6
	4	52 603.5	50 963.6	52 352.5	0.478	52 794.9
175.0	1	1 468.84	1 051.54	1 464.90	0.269	1 536.86
	2	7 611.88	6 679.71	7 594.8	0.224	7 770.58
	3	23 010.1	21 434.2	22 919.3	0.396	23 234.9
	4	54 673.8	52 581.9	54 546.7	0.233	54 9328
200.0	1	1 595.89	1 072.52	1 592.01	0.243	1 680.44
	2	8 127.84	6 940.16	8 110.06	0.219	8 329.17
	3	24 170.5	22 240.3	24 084.8	0.355	24 459.6
	4	56 733.4	54 163.9	56 511.6	0.392	57 069.4

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