AN EQUILIBRIUM STATE FOR THE INTERSTELLAR GAS AND MAGNETIC FIELD*

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ABSTRACT

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A simple periodic system is set up to demonstrate that the interstellar gas-magnetic field system, which has been shown by Parker (1966) to be unstable to a Rayleigh-Taylor type perturbation, can form an equilibrium state in which the interstellar gas is clumped discretely. It is demonstrated that in at least one instance the total energy of the clumped system is less than the total energy in the original atmosphere. It is believed that the results should be true for a much wider class of situation than the simple system for which we have been able to perform the mathematics.

Author

1. INTRODUCTION

It has been demonstrated by Parker (1966) that the galactic, or interstellar, magnetic field must be confined to the Galaxy by the weight of the interstellar gas which is threaded by the field and is distributed throughout the galaxy. It is assumed that the intergalactic medium (if any) exerts forces on the galaxy which are considerably less than about 10^{-12} dynes/cm^2. In such a case Parker shows that the interstellar gas-magnetic field system is unstable to a Rayleigh-Taylor perturbation.

The instability is of such a nature that the interstellar gas drains down the field lines into the lowest regions of fields and tends to accumulate in 'pockets' separated by a distance of the order of the wavelength of the perturbation. The gas in these pockets bears down on the field in the pocket leaving the field between pockets free to expand upward until the Maxwell stress tensor balances the weight of the gas. It is then of interest to consider the resulting configuration in order to decide whether it represents the final state of the interstellar gas or is merely a transition phase of the gas-field system.

In this paper we propose to consider the equilibrium structure of such a configuration but do not intend to examine the stability of the system. This will receive careful consideration in a later paper.

We suppose that the Rayleigh-Taylor instability is generated with the same wavelength throughout the galaxy. In such a case, since the gas is tied to the magnetic field, an initially uniform state will degenerate into sheets of material separated by some scale distance large compared to the thickness of each sheet (which is determined by the kinetic gas pressure). The sheets of material will then drop under galactic gravity until the magnetic field is deformed sufficiently, when the field stress tensor will hold the weight of the gas.

We do not believe that the particular model configuration described in this paper represents the actual behavior of the interstellar gas-magnetic field system. However we do believe that the essential physics of the real situation is captured by the model and thus any results obtained for the model will mirror in a semi-quantitative manner the true behavior of the gas-field system.

2. THE MODEL

As depicted in Figure 1 we consider an infinite array of sheets at positions $x = 2na$, ($n = 0, \pm 1, \pm 2, \ldots$) extending between $-\infty \leq y \leq \infty$. The galactic plane is chosen to be the plane $y = 0$ and the direction of the galactic gravitational acceleration reverses across $y = 0$. Further we will concern ourselves only with the case of a cold interstellar gas in order to keep the mathematics tractable.* Also since the galactic magnetic field is tied down by the weight of the discretely clumped gas it follows that there is a current flowing in each sheet, and further the magnetic field is not force-free (Parker, 1966). We shall consider a symmetric model in which the current is $j(y) = -j(-y)$ on each sheet. Further, we will discuss a rigorously two-dimensional model where the field components are in the $x-y$ plane only, so that we have uniform currents in the z -direction on each sheet which are dependent solely upon y.

We further suppose that pervading all of space there exists a tenous, conducting plasma so that the hydromagnetic equations correctly describe the system.

* We believe that the physical results derived are of a more general nature than the limited case for which we can do the algebra.

In order to keep the mathematics at a reasonable level we suppose that pressure of the tenuous plasma is negligible compared to the magnetic field stresses and the weight of the sheets of interstellar gas. Then $\nabla \times \underset{\sim}{B} = 0$ everywhere except on the current sheets. As a consequence we can compute the magnetic field due to the current sheets as though the surrounding plasma medium were vacuum.

Under the assumption of a steady-state configuration, we see that the current density, $\underset{\sim}{J}$, can be written

$$\underset{\sim}{J} = \hat{z}\, j(y) \sum_{n=-\infty}^{\infty} \delta(x - 2na) \quad . \tag{1}$$

The appropriate Maxwell equations then become

$$\nabla \cdot \underset{\sim}{H} = 0 \quad , \tag{2}$$

and

$$\nabla \times \underset{\sim}{H} = \hat{z}\, 4\pi c^{-1} j(y) \sum_{n=-\infty}^{\infty} \delta(x - 2na) \quad . \tag{3}$$

In view of the fact that the current sheet system points in the z -direction and is independent of z it follows that the magnetic field has components only in the $x-y$ plane and

$$H_x = -\frac{2}{c}\int_{-\infty}^{\infty} dy'\, j(y')\,(y-y') \sum_{n=-\infty}^{\infty} \frac{1}{[(x-2na)^2+(y-y')^2]}, \tag{4}$$

$$H_y = \frac{2}{c}\int_{-\infty}^{\infty} dy'\, j(y') \sum_{n=-\infty}^{\infty} \frac{(x-2na)}{[(x-2na)^2+(y-y')^2]}. \tag{5}$$

Now it is well known that

$$\sum_{n=-\infty}^{\infty} \frac{1}{[(x-2na)^2+(y-y')^2]} = \frac{\pi \sinh \pi(y-y')/a}{2a(y-y')[\cosh \pi(y-y')/a - \cos \pi x/a]}, \tag{6}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(x-2na)}{[(x-2na)^2+(y-y')^2]} = \frac{\pi \sin \pi x/a}{2a[\cosh \pi(y-y')/a - \cos \pi x/a]} \tag{7}$$

Use of Equations (6) and (7) in Eqs. (4) and (5) together with the fact that

$j(-y) = -j(y)$ yields

$$H_x = -\frac{2}{c}\int_0^{\infty} \frac{\sinh \xi\,(\cos X \cosh Y - \cosh \xi)\, j(\xi)\, d\xi}{(\cosh^2 \xi + \cosh^2 Y - 2\cos X \cosh Y \cosh \xi - \sin^2 X)}, \tag{8}$$

$$H_y = \frac{2\sin X \sinh Y}{c} \int_0^{\infty} \frac{\sinh \xi \; j(\xi)\, d\xi}{(\cosh^2\xi + \cosh^2 Y - 2\cos X \cosh Y \cosh \xi - \sin^2 X)} \tag{9}$$

where $\xi = \pi y'/a$, $Y = \pi y/a$, $X = \pi x/a$.

Before we can proceed with the evaluation of the integrals which occur in Eqs. (8) and (9) it is obvious that the current $j(\xi)$ must be specified. However we are not at liberty to specify $j(\xi)$ in an arbitrary manner. We must ensure that the $\underset{\sim}{J} \times \underset{\sim}{B}$ force on each sheet balances ρg at every point, y , on the sheet; ρ is the mass density of the material. Thus on each and every sheet we must choose $j(Y)$ so that the Lorentz force $2j(Y)H_x c^{-1}$ supports the weight of the gas ρg on each sheet. Thus $2j(Y)H_x c^{-1} = +\sigma g$. Use of (8) enables us to write

$$\frac{2j(Y)}{c} \, P\!\int_0^{\infty} \frac{\sinh\xi \;\; j(\xi)\, d\xi}{(\cosh Y - \cosh \xi)} = -\sigma(Y)\, g(Y) , \tag{10}$$

where

$$\rho(Y, X) = \sigma(Y) \sum_{n=-\infty}^{\infty} \delta(x - 2na)$$

Setting $\cosh Y - 1 = f$ and $\cosh \xi - 1 = u$ we see that Eq. (10) can be written

$$\frac{2j(f)}{c} \, P\!\int_0^{\infty} \frac{j(u)\, du}{(f-u)} = -\sigma(f)\, g(f) . \tag{11}$$

It is at this stage that some difficulty arises. We can either specify $\sigma(f)g(f)$ (which is positive definite) and try to find $j(f)$ or we can guess a functional form for $j(f)$ and hope that the resulting $\sigma(f)g(f)$ determined from Eq. (11) turns out to be positive definite. It is clear that, in general, we have an exceedingly difficult non-linear integral equation to solve on the one hand and that only a lucky guess or some flash of inspiration will help us on the other.

However, there exists a particular class of solution to Eq. (11) which has the advantage of being mathematically tractable and at the same time it yields a reasonable dependence for $\sigma(f)$ and $j(f)$. It is clear that if we can find a $j(f)$ such that

$$P\int_0^\infty \frac{j(u)\,du}{(f-u)} = -\alpha\, j(f) , \tag{12}$$

where $\alpha > 0$ then $\sigma(f)g(f)$ is always positive definite. It merely remains to show that the $j(f)$ and $\sigma(f)$ so derived both tend to zero as f, and thus Y, tends to infinity.

One can solve Eq. (12) formally in terms of Mellin transforms, however, a simpler procedure is to set $j(f) = j_0 f^{-\gamma}$ where j_0 is constant. It follows that this is indeed a solution of Eq. (12) with

$$\alpha = P\int_0^\infty \frac{du}{u^\gamma(u-1)} \equiv \pi \cot \pi\gamma > 0 \tag{13}$$

provided $0 < \gamma < \frac{1}{2}$.

Thus a particular class of solution is

$$j(Y) = \frac{j_0 \, Y}{|Y| \, (\cosh Y - 1)^{\gamma}} \quad , 0 < \gamma < 1/2 ; \tag{14}$$

and

$$\sigma(Y) g(Y) = \frac{\pi \cot \pi \gamma \, j_0^2}{2^{2\gamma - 1} c^2 \sinh^{4\gamma}(Y/2)} \quad , 0 < \gamma < 1/2 . \tag{15}$$

We will see later that γ has to be restricted even further from energy considerations but at the moment we have that any γ in $0 < \gamma < 1/2$ yields a reasonable variation for both $j(Y)$ and $\sigma(Y) g(Y)$.

It is clear by inspection that for a fixed γ both $j(Y)$ and σg are monotonically decreasing functions of Y and vanish as Y tends to infinity.

With $j(Y)$ given by Eq. (14) we see that the general expressions for $H_x(X, Y)$ and $H_y(X, Y)$ become

$$H_x = -\frac{2 j_0}{c} \int_0^{\infty} \frac{(\cos X \cosh Y - 1 - \mu) \, d\mu}{\mu^{\gamma} [\mu^2 + 2\mu(1 - \cos X \cosh Y) + (\cosh Y - \cos X)^2]} , \tag{16}$$

$$H_y = \frac{2 j_0 \sin X \sinh Y}{c} \int_0^{\infty} \frac{d\mu}{\mu^{\gamma} [\mu^2 + 2\mu(1 - \cos X \cosh Y) + (\cosh Y - \cos X)^2]} . \tag{17}$$

The integrals over μ can readily be evaluated* with $0 < \gamma < 1/2$ and upon so doing we find that

$$H_x = \frac{2\pi j_0}{c \sin \pi\gamma (\cosh Y - \cos X)^{\gamma}} \cos \left[\gamma \left\{ \pi - \tan^{-1} \left[\frac{|\sin X \sinh Y|}{(\cos X \cosh Y - 1)} \right] \right\} \right] \quad (18)$$

and

$$H_y = \frac{2\pi j_0 \sin X \sinh Y \sin \left[\gamma \left\{ \pi - \tan^{-1} \left[\frac{|\sin X \sinh Y|}{(\cos X \cosh Y - 1)} \right] \right\} \right]}{c \sin \pi\gamma (\cosh Y - \cos X)^{\gamma} |\sin X \sinh Y|} \quad (19)$$

where

$$\pi \geqslant \tan^{-1} \left[\frac{|\sin X \sinh Y|}{(\cos X \cosh Y - 1)} \right] \geqslant 0 \quad . \quad (20)$$

By definition the current sheet system is periodic in X with period 2π and by inspection both H_x and H_y are periodic in X with the same period. Thus it is sufficient to consider boundary conditions on $\underset{\sim}{H}$ in $0 \leqslant X \leqslant 2\pi$ and $-\infty \leqslant Y \leqslant \infty$. The conditions that the field must meet are:

i) $H_y (X = \pi) = 0$ (symmetry),

ii) $H_x, H_y \to 0$ as $Y \to \pm\infty$ for all X (energy conservation),

iii) $\dfrac{\partial H_x}{\partial X} + \dfrac{\partial H_y}{\partial Y} = 0$ for all X and Y (Maxwell's equation),

iv) $H_y (X = +0) - H_y (X = -0) = \dfrac{4\pi j_0}{c(\cosh Y - 1)^{\gamma}}$ (Maxwell's equation),

v) $H_y (X = 2\pi + 0) - H_y (X = 2\pi - 0) = \dfrac{4\pi j_0}{c(\cosh Y - 1)^{\gamma}}$ (Maxwell's equation),

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*We treat the integral as a contour integral in the complex μ plane. Note that $\mu = |\mu| \arg(\mu)$ so that on $\mu = |\mu| e^{2\pi i}$, $\mu^{\gamma} = |\mu|^{\gamma} e^{2\pi\gamma i}$ For $\gamma < 1/2$ the poles occur at $\mu = \cos X \cosh Y - 1 \pm i |\sin X \sinh Y|$.

vi) H_x continuous across $X=0$ for all Y (Flux conservation),

vii) $H_x(-Y) = H_x(Y)$ for all X (symmetry),

viii) $H_y(-Y) = -H_y(Y)$ for all X (asymmetry through $g(-Y)=-g(Y)$)

ix) $H_y(X=+0)\,H_x(X=+0) = 2\pi\sigma(Y)\,g(Y)$ (equilibrium state).

It is clear by inspection that conditions i, ii, iv, v, vi, vii, and viii are satisfied. Further, condition ix is automatically satisfied since we chose the current system so that it would be. In fact substitution of Eq. (18) and (19) into the left hand side of ix) and use of Eq. (15) shows directly that ix is satisfied. By direct calculation with Eqs. (18) and (19), and after some tedious algebra, it can be shown that the magnetic field also satisfies iii.

3. THE ORIGINAL ATMOSPHERE

Having set up a periodic equilibrium system of current sheets we now have the following sequence of events. Under Parker's Rayleigh-Taylor type instability we expect that a cold atmosphere with an X -pointing magnetic field and a density, both varying with Y but not with X, will condense into current sheets in a manner similar to that sketched above. We can then ask: Is it possible to find a method of transforming the current sheet equilibrium system back into a uniform atmosphere with an X -pointing magnetic field and a density which are functions of Y only?

It is clear that there is a transformation from the uniform atmosphere to the current sheet system. To show this we start with a field which is in the X -direction and is solely a function of Y. If we now clamp the field lines we are at liberty to move the material along the field lines into sheets. Having done so we then clamp the sheets and release the field which clearly expands upwards between sheets. If we now release the sheets they will drop under gravity until the magnetic field becomes sufficiently bent when the $\underset{\sim}{J} \times \underset{\sim}{B}$ force will balance ρg on each sheet. It follows that we can reverse the process.

We denote by primed coordinates the original atmosphere. Then conservation of matter demands that

$$2a\,\rho(y')\,dy' = \sigma(y)\,dy \quad , \tag{21}$$

where $\rho(y')$ is the mass density of the original atmosphere.

Since the magnetic field threads through the material it follows that

$$H_x(y, x=0)\,dy = H(y')\,dy' \quad , \tag{22}$$

where $H(y')$ is the X -pointing magnetic field in the original atmosphere.

Further since the atmosphere is an equilibrium situation (admittedly unstable) it follows that we require

$$\frac{d}{dy'}\left(\frac{H^2(y')}{8\pi}\right) = -\rho(y')\,g(y') \quad , \tag{23}$$

for cold gas.

We can write Eq. (23) in the form

$$\frac{d}{dy}\left(\frac{H^2(y')}{8\pi}\right) = -\rho(y')\,g(y')\frac{dy'}{dy} = \frac{-\sigma(y)\,g(y')}{2a} . \tag{24}$$

Thus we can integrate to find $H^2(y')$ expressed as a function of y .

$$\frac{H^2(y')}{8\pi} = \frac{H^2(0)}{8\pi} - \frac{1}{2a}\int_0^y \sigma(y)\,g(y')\,dy . \tag{25}$$

Under the reasonable assumption that the point $y = \infty$ maps into the point $y' = \infty$ where $H^2(y' = \infty) = 0$ it follows that

$$\frac{H^2(y')}{8\pi} = \frac{1}{2a}\int_y^\infty \sigma(y)\,g(y')\,dy . \tag{26}$$

In general it is extremely difficult to evaluate the integral transformation expressed by Eqs. (26) and (22) since $g(y')$ rather than $g(y)$ occurs in the integral in Eq. (26). However, if g is independent of Y (and hence of Y' since the gravity is imposed on the interstellar gas by the galaxy rather than being a self gravity)*, it is possible to gain some insight into the behavior of the original atmosphere in a relatively simple manner. For the remainder of this paper we will concern ourselves solely with the case of constant gravitational acceleration.

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* There is a discontinuity in the gravitational field on $Y = 0$ or $Y' = 0$ in this case.

We then have

$$\frac{H^2(y')}{8\pi} = \frac{j_0^2 \cot\pi\gamma}{2^{2\gamma} c^2} \int_Y^\infty \frac{dY}{\sinh^{4\gamma}(Y/2)} \qquad (27)$$

Use of Eq. (27) in Eq. (22) shows that the differential scaling is

$$\frac{dY'}{dY} = \sqrt{\left(\frac{\pi\cot\pi\gamma}{2}\right)} \frac{\left[\int_Y^\infty \frac{dp}{\sinh^{4\gamma}(p/2)}\right]^{-1/2}}{\sinh^{2\gamma}(Y/2)}, \qquad (28)$$

where $Y' = \pi y'/a$.

Further we have

$$\rho(Y') = \frac{(\pi\cot\pi\gamma)^{3/2}\, j_0^2 \left[\int_Y^\infty \frac{dp}{\sinh^{4\gamma}(p/2)}\right]^{-1/2}}{2^{2\gamma+1/2}\, g a c^2 \sinh^{6\gamma}(Y/2)}, \qquad (29)$$

Before we can find explicit expressions for $H^2(Y')$, $\rho(Y')$ and Y' as functions of Y it is necessary that we assign a particular value to γ . However before this is done we need to consider the energy states of the original atmosphere and the periodic current sheet system.

4. ENERGY CONSIDERATIONS

It is clear that a necessary, but by no means sufficient, condition that the uniform atmosphere degenerate into current sheets is that the energy in the original atmosphere be greater than the energy in the current sheet system.

From symmetry it is sufficient to consider the energy in the range $0 \leq X \leq 2\pi$. In this range we see that the original atmosphere has a total energy, say E_0 , of

$$\mathcal{E}_0 = 2a \int_0^\infty \rho(y') g(y') y' dy' + \frac{2a}{8\pi} \int_0^\infty H^2(y') dy' \quad . \tag{30}$$

In view of Eq. (23) we see that $\mathcal{E}_0$ becomes

$$\mathcal{E}_0 = \frac{a}{2\pi} \int_0^\infty H^2(y') dy' + \frac{a}{4\pi} \Big[y' H^2(y') \Big]_{y'=0}^{y'=\infty} \quad . \tag{31}$$

If the point $y' = \infty$ is not to produce a contribution to the energy we require that $H^2(y')$ tend to zero faster than $(y')^{-1}$ for large y'. Further we also require that $H^2(y')$ vary less rapidly than $(y')^{-1}$ for small y' or the point $y' = 0$ produces a large (or infinite) contribution to the energy. It can easily be shown that for y' large, i.e. y large, $y' H^2(y') = O(y e^{-2\gamma y})$* and thus vanishes as y tends to large values.

It can also be shown that for small y', and thus small y, that $y' H^2(y') = O(Y^{3/2 - 4\gamma})$.** Thus the point $y' = 0$ produces zero contribution to the energy provided $\gamma < 3/8$. Thus in order to obtain a physically reasonable energy in the original atmosphere we must now restrict γ to the range $0 < \gamma < 3/8$.

In such a case we have

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* One considers the asymptotic functional form of (27) and (28).

** One considers the limiting values of (27) and (28) as $y \to +0$.

$$\mathcal{E}_0 = \frac{a}{2\pi}\int_0^\infty H^2(y')\,\frac{dy'}{dy}\,dy \qquad , \tag{32}$$

and upon making use of Eqs. (27) and (28) in Eq. (32) we see that

$$\mathcal{E}_0 = \frac{a^2 j_0^2 (2\cot\pi\gamma)^{3/2}}{2^{2\gamma} c^2 \sqrt{\pi}} \int_0^\infty \frac{\left[\int_Y^\infty \frac{dp}{\sinh^{4\gamma}(p/2)}\right]^{1/2} dY}{\sinh^{2\gamma}(Y/2)} \tag{32a}$$

Also in the range $0 \leq X \leq 2\pi$ we see that the current sheet system has an energy, say $\mathcal{E}_1$, given by

$$\mathcal{E}_1 = \int_{0-}^{2a-0} dx \int_0^\infty dy\; y g(y)\sigma(y)\delta(x) + \frac{1}{8\pi}\int_0^\infty dy \int_0^{2\pi} dx\left[H_x^2(x,y) + H_y^2(x,y)\right]. \tag{33}$$

Making use of Eqs. (15), (18) and (19) in Eq. (33) we see that

$$\mathcal{E}_1 = \frac{a^2 j_0^2 \cot\pi\gamma}{2^{2\gamma-1}\pi c^2}\int_0^\infty \frac{Y\,dY}{\sinh^{4\gamma}(Y/2)} + \frac{a^2 j_0^2}{2\pi c^2 \sin^2\pi\gamma}\int_0^\infty dY \int_0^{2\pi}\frac{dX}{[\cosh Y - \cos X]^{2\gamma}}\,. \tag{34}$$

Thus if the current sheet system is to be energetically preferred to the original atmosphere we require $\mathcal{E}_0 > \mathcal{E}_1$ which can be written

$$\cos\pi\gamma\sqrt{(\pi\sin 2\pi\gamma)}\int_0^\infty \frac{\left[\int_Y^\infty \frac{dp}{\sinh^{4\gamma}(p/2)}\right]^{1/2} dY}{\sinh^{2\gamma}(Y/2)} >$$

$$\frac{1}{2}\sin 2\pi\gamma \int_0^\infty \frac{Y\,dY}{\sinh^{4\gamma}(Y/2)} + 4^{\gamma-1}\int_0^\infty dY \int_0^{2\pi}\frac{dX}{(\cosh Y - \cos X)^{2\gamma}}\,. \tag{35}$$

It can be shown (Appendix) that

$$\int_0^\infty dY \int_0^{2\pi} \frac{dX}{(\cosh Y - \cos X)^{2\gamma}} = \frac{4^{\gamma-1} \tan\pi\gamma \, [\Gamma(\gamma)]^4}{[\Gamma(2\gamma)]^2} \quad . \tag{36}$$

Thus Eq. (35) becomes

$$\cos\pi\gamma \sqrt{(2\pi \sin 2\pi\gamma)} \int_0^\infty \frac{\left[\int_z^\infty \frac{d\varphi}{\sinh^{4\gamma}(\varphi)} \right]^{1/2} dz}{\sinh^{2\gamma}(z)} >$$

$$\sin 2\pi\gamma \int_0^\infty \frac{z\, dz}{\sinh^{4\gamma}(z)} + \frac{2^{4\gamma-5} \tan\pi\gamma \, [\Gamma(\gamma)]^4}{[\Gamma(2\gamma)]^2} \quad . \tag{37}$$

Clearly if the current sheet system is energetically more favorable than the uniform atmosphere then there exists at least one value γ in the range $0 < \gamma < 3/8$ such that (37) is obeyed.

For arbitrary γ we have been unable to demonstrate analytically that Eq. (37) is satisfied. However in the particular case of $\gamma = 1/4$ we find that Eq. (37) becomes

$$8\sqrt{\pi} \int_0^\infty \frac{y^2 e^{-y^2} dy}{\sqrt{(1 - e^{-4y^2})}} > \frac{\pi^2}{4} + \frac{[2\Gamma(5/4)]^4}{\pi} \tag{38}$$

Upon evaluating the integral in Eq. (38) numerically we obtain

$$6 \cdot 30 - 2 \cdot 47 - 3 \cdot 44 \geq 0 \tag{39}$$

for less energy in the current sheet system.

Thus for the particular value of $\gamma = 1/4$ we obtain a 6 per cent reduction in total energy if the original atmosphere degenerates into sheets. The decrease in gravitational energy is some 21 per cent while the magnetic field energy undergoes a 9 percent *increase*. It may at first sight seem a little surprising that the magnetic field energy shows an increase. However we can readily demonstrate that this is indeed the case. Consider the uniform atmosphere where we first clamp the field and move the material into sheets and then release the field, but clamp the material. It is then clear that the field expands upwards thereby *decreasing* its energy. If we now unclamp the material it drops under gravity thereby *decreasing* its gravitational energy but dragging the field with it. Thus the magnetic field is stressed by the dropping and hence this tends to try and *increase* the magnetic field energy. Thus, whether the field energy increases or decreases depends upon how the material is distributed on the sheets and in the original atmosphere. In the case of $\gamma = 1/4$ it happens that we obtain an increase.

We do not contend that $\gamma = 1/4$ is the case which leads to the largest positive value for $E_0 - E_1$ but it does illustrate the non-linear behavior of the system. For other distributions of matter the linear calculations (Parker, 1966) indicate the trend of the instability and the general method of approach used in this paper can be employed. However the calculations would probably have to be performed numerically. With $\gamma = 1/4$ it can be easily shown that

$$H^2(Y') = 4\sqrt{2}\,\pi j_0^2 c^{-2} \ln[\coth(Y/2)], \tag{40}$$

$$\rho(Y') = \frac{\pi^{3/2}\, j_0^2}{2^{3/2} g a c^2 (\sinh(Y/2))^{3/2} [\ln(\coth(Y/2))]^{1/2}}, \tag{41}$$

as functions of Y .

Further we have

$$Y' \simeq 4\sqrt{\pi} \left\{ \frac{-\tanh(Y/2)}{\ln[\tanh(Y/2)]} \right\}^{1/2}, \qquad Y \ll 1 \;; \tag{42a}$$

$$Y' \simeq \sqrt{\pi}\, Y \quad , \qquad Y \gg 1 \tag{42b}$$

From Eqs. (40) and (42a) it can be seen that $Y' H^2(Y')$ tends to zero with Y' tending to zero or infinity as is required by Eq. (31).

5. DISCUSSION

In the above somewhat idealized calculation we have attempted to illustrate the behavior of the interstellar gas after it deviates from a uniform atmosphere due to a Rayleigh-Taylor instability. Despite the fact that there are several restrictions built into the model, for example no kinetic gas pressure, no cosmic rays, infinitely thin current sheets; we nevertheless believe that the physics described in this paper does apply to the real interstellar gas and that the results derived give a reasonable indication of the behavior of the galactic interstellar gas.

On the basis of the model chosen it appears that it is energetically more favorable for the uniformly spread interstellar gas to form into sheets of material separated by roughly the wavelength of the most unstable Rayleigh-Taylor perturbation. However a word of caution needs to be inserted here. Although we have calculated the field and gas sheet system as though it were an equilibrium state no proof has been given that it is a stable equilibrium. In fact we will see in a later paper that it is unstable and thus the current sheet system can at best represent a reasonably long-lived transition phase of the interstellar gas.

We have done this calculation in order to try and demonstrate that the interstellar gas-magnetic field system, which starts off as a uniform atmosphere, can lower its total energy content by breaking up into a discrete periodic structure; and further to illustrate that in so doing the gas tends to come to a lower level in the galactic gravitational field.

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APPENDIX

Let

$$I = \int_0^\infty dY \int_0^{2\pi} \frac{dX}{(\cosh Y - \cos X\ )^{2\gamma}} \quad . \tag{A1}$$

With $\cos X = \beta$ we have

$$I = 2\int_0^\infty dY \int_{-1}^{+1} \frac{d\beta}{\sqrt{(1-\beta^2)}\ (\cosh Y - \beta\ )^{2\gamma}} \quad . \tag{A2}$$

Now it can be shown (Gradshteyn and Ryzhik, 1965) that

$$\int_{-1}^{+1} \frac{(1-t^2)^\nu dt}{(z-t)^{\nu+1-\mu}} = \frac{\sqrt{\pi}\ \Gamma(\nu+1)(z^2-1)^\mu}{\Gamma(\nu+3/2)\ \ z^{\mu+\nu+1}}\ {}_2F_1\left(\frac{\mu+\nu+1}{2},\ 1+\frac{\mu+\nu}{2};\ \nu+\frac{3}{2};\ z^{-2}\right), \tag{A3}$$

provided $Re(\mu+\nu) > -1$, $Re(\mu) > -1$; $|\arg(z \pm 1)| < \pi$.

Use of Eq. (A3) in Eq. (A2) yields

$$I = 2\pi \int_0^\infty [\sinh(Y)]^{1-4\gamma} [\cosh(Y)]^{2\gamma-1}\ {}_2F_1\left(\tfrac{1}{2}-\gamma,\ 1-\gamma;\ 1; \operatorname{sech}^2 Y\right) dY, \tag{A4}$$

provided $0 < \gamma < 1/2$.

With $p = \mathrm{sech}^2 y$ we have

$$I = \pi \int_0^1 \frac{p^{\gamma-1}}{(1-p)^{2\gamma}}\, {}_2F_1\left(\tfrac{1}{2}-\gamma, 1-\gamma; 1; p\right) dp \quad . \tag{A5}$$

It can be shown (Erdelyi et. al., 1954) that

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\, {}_2F_1(a, b; c; x)\, dx = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\, {}_3F_2(a, b, \rho; c, \rho+\sigma; 1) , \tag{A6}$$

provided $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(c+\sigma \; a-b) > 0$.

Use of Eq. (A6) in Eq. (A5) yields

$$I = \frac{\pi\, \Gamma(\gamma)\Gamma(1-2\gamma)}{\Gamma(1-\gamma)}\, {}_3F_2\left(\tfrac{1}{2}-\gamma, 1-\gamma, \gamma; 1, 1-\gamma; 1\right) , \tag{A7}$$

and

$$ {}_3F_2(a, b, c; d, c; 1) = {}_2F_1(a, b; d; 1) . \tag{A8}$$

Thus

$$I = \frac{\pi\, \Gamma(\gamma)\Gamma(1-2\gamma)}{\Gamma(1-\gamma)}\, {}_2F_1\left(\tfrac{1}{2}-\gamma, \gamma; 1; 1\right) . \tag{A9}$$

It is well known that

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} . $$

Thus

$$ I = \frac{\pi^{3/2}\,\Gamma(\gamma)\,\Gamma(1-2\gamma)}{\Gamma(\frac{1}{2}+\gamma)\,[\Gamma(1-\gamma)]^2} . \qquad \text{(A10)} $$

Use of $\sin \pi z\,\Gamma(z)\,\Gamma(1-z) = \pi$ together with the duplication formula

$$ \Gamma(\tfrac{1}{2}+z)\,\Gamma(z) = \sqrt{\pi}\; 2^{1-2z}\,\Gamma(2z) , $$

gives

$$ I = \frac{4^{\gamma-1}\tan\pi\gamma\,[\Gamma(\gamma)]^4}{[\Gamma(2\gamma)]^2} , \qquad \text{(A11)} $$

which is the required result.

REFERENCES

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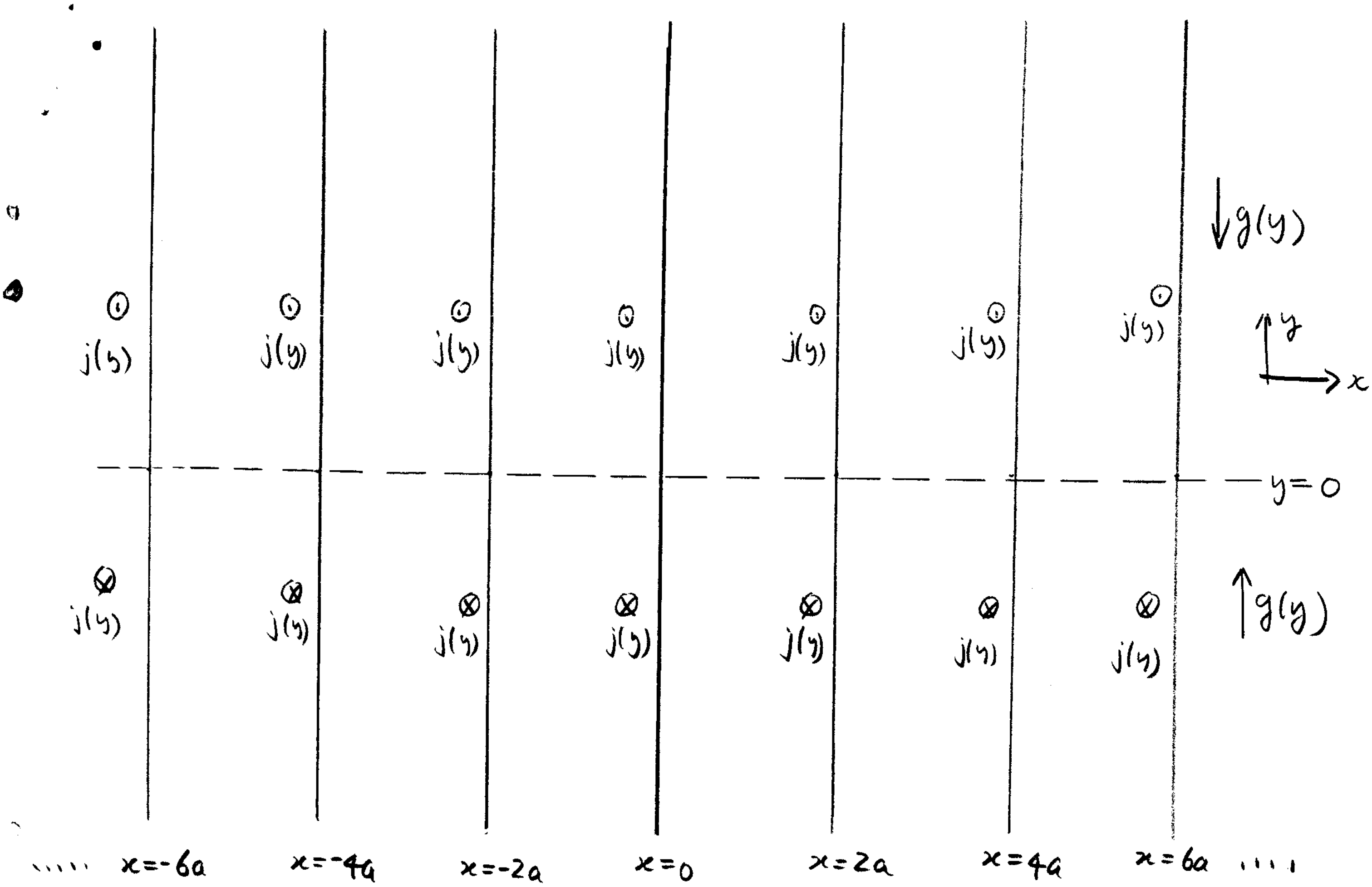

The array of current sheets which are referred to in Section II.