DIFFERENTIAL OPERATORS AND THEIR ADJOINTS UNDER INTEGRAL AND MULTIPLE POINT BOUNDARY CONDITIONS

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1. Introduction. Since the turn of the century there has been significant progress in the area of linear matrix differential systems under various boundary (or auxiliary) conditions.

Bounitzky [3], Birkhoff and Langer [1], [2] considered systems where the differential operator is of the form $LY = Y' + PY$ (P is a continuous $n \times n$ matrix) and the boundary conditions are end point conditions $AY(a) + BY(b) = 0$.

Wilder [10], Langer [7] and Cole [4] extended the results to systems having boundary conditions at interior points $\sum_{j=o}^{m} A_j Y(a_j) = 0$, where $a = a_o < a_1 \dots < a_m = b$.

Whyburn [12] and Krall [6] considered integral boundary conditions

$$AY(a) + BY(b) + \int_a^b K(x)Y(x)dx = 0.$$

Cole [5] considered the more general condition

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$$\sum_{j=0}^{m} A_j Y(a_j) + \int_{a_0}^{a_m} K(x)Y(x)dx = 0.$$

In addition, Stallard [9] has considered interface conditions $Y(a_j+) - A_j Y(a_j-) = 0$. Krall [6] has generalized the differential operator by including a boundary form with the differential operator L.

All of these people have defined an "adjoint" system of one form or another, which was usually motivated by the properties of the Green's matrix of the original system. (For the exception, see Whyburn [12].) The original system and the "adjoint" are mutually compatable or incompatable. If $G_s(x,t)$ is the Green's function of the original system and $G_a(x,t)$ that of the "adjoint", then $G_s(x,t) = -\overline{G_a}(t,x)$.

A generalization of these boundary conditions would be one of the form $\int_a^b K(x)\, Y(x)dx + \sum_{i=0}^{m} A_i Y(a_i+) + \sum_{i=0}^{m} B_i Y(a_i-) = 0$, which includes all the other conditions as special cases. We shall see that if there are $m + 1$ points in question (m intervals), then m such conditions are necessary to achieve the systems previously mentioned. (This has already been done implicitly, since, when interior point conditions were employed, the matrices involved were assumed to be continuous at those interior points. That is, $Y(a_j+) - Y(a_j-) = 0$.)

There are also systems for which a Green's matrix is impossible to define. These also have "adjoint" systems. Results of this kind concerning nth order ordinary systems with end point conditions have been known for a long time [8]. A Hilbert space setting is employed in deriving them.

We believe that such a setting is indeed the proper one for matrix systems under boundary conditions previously discussed. We shall show that the adjoint system in a Hilbert space is precisely the one defined previously by using Green's matrices. In addition we shall derive adjoint systems when no Green's matrix exists. Since our boundary conditions contain all the others as special cases, we can consider all such systems at once.

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2. The Differential Operator L. Let us consider an interval $[a_o, a_m]$ which is subdivided into m subintervals by $a_1 \ldots a_{m-1}$ $(a_o < a_1 < \ldots < a_m)$.

Definition. Let $\mathcal{H}$ denote the Hilbert space of $n \times n$ matrices $Y = (y_{ij})$ with inner product

$$< Y, Z > = \sum_{ij} \left(\int_{a_o}^{a_m} Z^*Y \, dx \right),$$

where * denotes complex conjugate transpose.

Let P be an $n \times n$ matrix which is continuous in x. Let A_{ij} and B_{ij} be $n \times n$ matrices of constants. Let $K_i(x)$ be $n \times n$ matrices in $\mathcal{H}$. We will consider boundary conditions of the form

$$M_i(Y) = \int_{a_o}^{a_m} K_i Y dx + \sum_{j=o}^{m} U_{ij}(Y) , \quad i = 1, \ldots k ,$$

where $U_{ij}(Y) = A_{ij}Y(a_j +) + B_{ij}Y(a_j -)$, and $A_{im} = 0$, $B_{io} = 0$.
$Y(a_j \pm)$ indicates the limit of $Y(x)$ as x approaches a_j from above or below.

Definition. The boundary conditions $M_i(Y)$ are said to be acceptable if whenever there exist constant matrices C_i such that

$$\sum_{i=1}^{k} C_i A_{ij} = 0 , \quad \sum_{i=1}^{k} C_i B_{ij} = 0 , \quad j = 0, \ldots m,$$

then $\sum_{i=1}^{k} C_i K_i \equiv 0$ also.

Definition. Let $\mathfrak{D}_o$ denote the set of all $n \times n$ matrices Y satisfying

1. Y is in $\mathfrak{H}$.
2. Y is absolutely continuous on each of the open intervals (a_j, a_{j+1}), $j = 0, \ldots, m - 1$.
3. $Y' + PY$ is in $\mathfrak{H}$.

Definition. Let $\mathfrak{D}$ denote the set of all $n \times n$ matrices Y satisfying

1. Y is in $\mathfrak{D}_o$.
2. $M_i(Y) = 0$, $i = 1, \ldots k$.

Definition. We define L by letting $LY = Y' + PY$ for all Y in $\mathfrak{D}$.

Theorem 2.1. If the boundary conditions are acceptable, then $\mathfrak{D}$ is dense in $\mathfrak{H}$.

Proof. Let

$$\mathfrak{H}_o = \left\{ Y : Y \text{ is in } \mathfrak{D}_o, Y(a_j) = 0, \; j = 0, \ldots m, \; \int_{a_o}^{a_m} K_i Y dx = 0, \; i = 1, \ldots k \right\}.$$

$\mathfrak{H}_o$ is orthogonal to the space $\mathfrak{K}$ spanned by $K_1^*, \ldots, K_k^*$.

Clearly $\mathfrak{H}_o$ is dense in $\mathfrak{H} - \mathfrak{K}$. Therefore $\mathfrak{D}$ is dense in $\mathfrak{H} - \mathfrak{K}$. If $\mathfrak{D}$ were not dense in $\mathfrak{H}$, then there would exist an element K^* in $\mathfrak{K}$

such that $K = \sum_{i=1}^{k} C_i K_i$ and $\int_{a_o}^{a_m} KY\,dx = 0$ for all Y in $\mathfrak{A}$. But then $\sum_{i=1}^{k} C_i \sum_{j=1}^{m} U_{ij}(Y) = 0$. Since the boundary conditions are acceptable, $K^* \equiv 0$, and we have arrived at a contradiction.

Throughout the remainder of this article we will assume that the boundary conditions $M_i(Y)$ are acceptable.

3. The Adjoint Operator L^*. Various examples [5] [6] have shown that integral terms in the boundary conditions affect the form of the adjoint operator. We shall see precisely why this is so as we derive the adjoint L^*.

Lemma 3.1. If Z is in the domain of L^*, then Z is absolutely continuous in each of the intervals (a_j, a_{j+1}), $j = 0, \ldots m-1$. There exist linear functional matrices $\phi_1(Z), \ldots \phi_k(Z)$ such that

$$L^*Z = -Z' + P^*Z - \sum_{i=1}^{k} K_i^* \phi_i(Z).$$

Proof. Let Y be in $\mathfrak{H}_o$ (Theorem 2.1).

Then

$$\int_{a_o}^{a_m} (L^*Z)^* Y\,dx = \int_{a_o}^{a_m} Z^*(Y' + PY)dx .$$

Therefore

$$\int_{a_o}^{a_m} Z^*Y'\,dx = \int_{a_o}^{a_m} (L^*Z - P^*Z)^*Y\,dx .$$

Since Y vanishes at $a_o, \dots a_m$, integrating by parts,

$$\int_{a_o}^{a_m} Z^*Y' \, dx = - \int_{a_o}^{a_m} \left[\int^x (L^*Z - P^*Z) dt \right]^* Y' dx$$

Hence

$$\int_{a_o}^{a_m} \left[Z + \int^x (L^*Z - P^*Z) dt \right]^* Y' dx = 0.$$

We must therefore find all those functions J such that $\int_{a_o}^{a_m} JY' dx = 0.$

Suppose $\int_{a_o}^{a_m} JY' \, dx = 0.$ Let J_i be differentiable except at $a_o, \dots a_m$, and suppose $J_i \to J$ in $\mathfrak{H}$. Upon integration by parts,

$\int_{a_o}^{a_m} J_i'Y \, dx = - \int_{a_o}^{a_m} J_i Y' \, dx \to 0.$ Thus there exists a W such that $\lim J_i' = W$ and $\int_{a_o}^{a_m} WY \, dx = 0.$ However, the differentiation operator is closed. So if $J_i \to J$ and $J_i' \to W$, then J' exists and $J' = W.$ $J = \int^x W dt.$

Conversely, since $\int_{a_o}^{a_m} K_i Y \, dx = 0$, it is easy to show that

$$\int_{a_o}^{a_m} \left[\int^x K_i dt \right] Y' dx = 0.$$

Thus $Z + \int^x (L^*Z - P^*Z)dt$ is a linear combination of $\int^x K_1^* dt, \ldots, \int^x K_k^* dt$.
That is, there exist linear functional matrices $\phi_1(Z), \ldots \phi_k(Z)$ such that

$$Z + \int^x (L^*Z - P^*Z)dt = \sum_{i=1}^{k} \int^x K_i^* \, dt \, \phi_i(Z).$$

Z is easily seen to be absolutely continuous (except possibly at $a_0, \ldots, a_m$), and

$$L^*Z = -Z' + P^*Z = \sum_{i=1}^{k} K_i^* \, \phi_i(Z) \; .$$

We have L^* represented in terms of k parameters $\phi_1(Z) \ldots \phi_k(Z)$. We still need to find boundary conditions for Z, as well as see when these parameters can be eliminated.

Lemma 3.2. *If* Z *is in the domain of* L^*, *then* Z *satisfies the following equations.*

$$-Z^*(a_j -) + \sum_{i=1}^{k} \phi_i(Z)^* B_{ij} = 0,$$

$$Z^*(a_{j-1} +) + \sum_{i=1}^{k} \phi_i(Z)^* A_{ij-1} = 0,$$

$$j = 1, \ \ldots m.$$

Proof. We easily compute

$$0 = \int_{a_0}^{a_m} (L^*Z)^* \, Y \, dx - \int_{a_0}^{a_m} Z^*(LY)dx$$

$$= \sum_{j=1}^{m} \int_{a_{j-1}}^{a_j} \left[(L^*Z)^*Y - Z^*(LY) \right] dx$$

$$= \sum_{j=1}^{m} \left[-Z^*Y \Big|_{a_{j-1}}^{a_j} - \sum_{i=1}^{k} \phi_i(Z)^* \int_{a_{j-1}}^{a_j} \right] K_i Y \, dx$$

$$= \sum_{j=1}^{m} - Z^*Y \Big|_{a_{j-1}}^{a_j} - \sum_{i=1}^{k} \phi_i(Z)^* \int_{a_o}^{a_m} K_i Y \, dx$$

$$= \sum_{j=1}^{m} -Z^*Y \Big|_{a_{j-1}}^{a_j} + \sum_{i=1}^{k} \phi_i(Z)^* \sum_{j=o}^{m} U_{ij}(Y)$$

$$= \sum_{j=}^{m} \left[-Z^*(a_{j+1}-)Y(a_{j+1}-) + Z^*(a_j+)Y(a_j+) + \sum_{i=1}^{k} \phi_i(Z)^* U_{ij}(Y) \right]$$

Since $Y(a_j +)$ and $Y(a_j -)$ may be arbitrary,

$$-Z^*(a_j -) + \sum_{i=1}^{k} \phi_i(Z)^* B_{ij} = 0,$$

$$Z^*(a_{j-1}+) + \sum_{i=1}^{k} \phi_i(Z)^* A_{ij-1} = 0,$$

$j = 1, \ldots m.$

The converse of these calculations is trivial. Hence we are led to the following.

Theorem 3.1. The domain of L^*, $\mathfrak{D}^*$, consists of all those $n \times n$ matrices Z which satisfy

1. Z is in $\mathfrak{D}_0$.
2. There exist parametric linear matrix functionals $\phi_1(Z) \ldots \phi_k(Z)$ such that

$$-Z(a_j-) + \sum_{i=1}^{k} B_{ij}^* \phi_i(Z) = 0,$$

$$Z(a_{j-1}+) + \sum_{i=1}^{k} A_{ij-1}^* \phi_i(Z) = 0,$$

$j = 1, \ldots m.$

If Z is in $\mathfrak{D}^*$, then

$$L^*Z = -Z' + P^*Z - \sum_{i=1}^{k} K_i^* \phi_i(Z).$$

There are $2m$ equations for the k linear functionals $\phi_1, \ldots, \phi_k$. It is sometimes possible to achieve two expressions for each. Substitution into the expression for L^*, eliminates them there. Equating the different expressions for each gives boundary conditions.

4. Differential-Boundary Operators. If any of the K's in the boundary conditions determining $\mathfrak{D}$ is different from zero, L cannot be self-adjoint. In order to have selfadjointness as a possibility, the form of the operator L must be extended so that it resembles the form of the adjoints we have just found. A beginning step is bound in [6]. We extend those results to the present situation.

Definition. Let C_{ij} and D_{ij}, $i = 1, \dots \ell$. Let $\Psi_i(Y) = \sum_{j=0}^{m} [C_{ij} Y(a_j +) + {}_{ij}Y(a_j -)]$, $i = 1, \dots \ell$. Let H_i be in $\mathfrak{H}$, $i = 1, \dots \ell$. We define L_b by letting $L_b Y = Y' + PY + \sum_{i=1}^{\ell} H_i \Psi_i(Y)$ for all Y in $\mathfrak{D}$.

Lemma 4.1. If Z is in the domain of L_b^*, then Z is absolutely continuous in each of the intervals (a_j, a_{j+1}), $j = 0, \dots, m-1$. There exist linear functional matrices $\phi_1(Z), \dots \phi_k(Z)$ such that

$$L_b^* Z = -Z' + P^* Z - \sum_{i=1}^{k} K_i^* \phi_i(Z).$$

The proof is identical to that of Lemma 3.1.

Lemma 4.2. If Z is in the domain of L_b^*, then Z satisfies the following equations.

$$-Z^*(a_j -) + \sum_{i=1}^{k} \phi_i(Z)^* B_{ij} - \sum_{i=1}^{\ell} \int_{a_0}^{a_m} Z^* H_i dx \, D_{ij} = 0,$$

$$Z^*(a_{j-1}+) + \sum_{i=1}^{k} \phi_i(Z)^* A_{ij-1} - \sum_{i=1}^{\ell} \int_{a_0}^{a_m} Z^* H_i dx\, C_{ij-1} = 0,$$

$j = 1, \ldots m.$

The proof is only slightly more complicated than that of Lemma 3.2.

Again the converse of these statements is trivial, and we are led to the following.

Theorem 4.1. The domain of L_b^*, $\mathcal{D}_b^*$, consists of all those $n \times n$ matrices Z which satisfy

1. Z is in $\mathcal{D}_o$.
2. There exist parametric linear matrix functionals $\phi_1(Z), \ldots \phi_k(Z)$ such that

$$-Z(a_j -) + \sum_{i=1}^{k} B_{ij}^* \phi_i(Z) - \sum_{i=1}^{\ell} D_{ij}^* \int_{a_0}^{a_m} H_i^* Z\, dx = 0,$$

$$Z(a_{j-1}+) + \sum_{i=1}^{k} A_{ij-1}^* \phi_i(Z) - \sum_{i=1}^{\ell} C_{ij-1}^* \int_{a_0}^{a_m} H_i^* Z dx = 0,$$

$j = 1, \ldots m.$

If Z is m $\mathcal{D}_b^*$, then

$$L_b^* Z = -Z' + P^* Z - \sum_{i=1}^{k} K_i^* \phi_i(Z).$$

5. Some Examples.

1. Consider the system $LY = Y' + PY$, with endpoint boundary condition $AY(a_o) + BY(a_1) = 0$. The adjoint is then defined by $L^*Z = -Z' + P^*Z$ with boundary conditions $-Z(a_1) + B^*\phi_1(Z) = 0$, $Z(a_o) + A^*\phi_1(Z) = 0$. If A or B has an inverse, $\phi_1(Z)$ can be eliminated to give the adjoint the form usually found.

2. Consider the system $LY = Y' + PY$ with boundary conditions

$\sum_{j=1}^{m} A_j Y(a_j) = 0$, $Y(a_j -) - Y(a_j +) = 0$, $j = 1, \ldots m-1$. The adjoint operator is the same as before. $L^*Z = -Z' + P^*Z$. The boundary conditions are

$$-Z(a_j -) - \phi_{j+1}(Z) = 0 \quad j = 1, \ldots m-1,$$
$$-Z(a_m -) - A_m^* \phi_1(Z) = 0,$$

and

$$Z(a_o +) + A_o^* \phi_1(Z) = 0$$
$$Z(a_j +) + A_j^* \phi_1(Z) + \phi_{j+1}(Z) = 0, \ j = 1, \ldots m-1.$$

If we agree that $Z(a_o -) = 0$ and $Z(a_m +) = 0$, then $\phi_2(Z)$ through $\phi_m(Z)$ can be eliminated, giving

$$Z(a_j +) - Z(a_j -) = -A_j^* \phi_1(Z), \quad j = 0, \ldots m.$$

3. If the m point boundary condition in the previous example is replaced by $\sum_{j=1}^{m} A_j Y(a_j) + \int_{a_o}^{a_m} K_1 Y dx = 0$, the boundary values for Z are the same, but $L^*Z = -Z' + P^*Z - K_1^* \phi_1(Z)$.

4. Consider the system $LY = Y' + PY$ with boundary conditions
$A_o Y(a_o +) + B_m Y(a_m -) = 0$, $A_j Y(a_j +) + B_j Y(a_j -) = 0$, $j = 1, \ldots m-1$.
The adjoint operator is $L^*Z = -Z' + P^*Z$. The adjoint boundary conditions are

$$-Z(a_j -) + B_j^* \phi_{j+1}(Z) = 0, \quad j = 1, \ldots m-1,$$
$$-Z(a_m -) + B_m^* \phi_1(Z) = 0,$$

and

$$Z(a_j +) + A_j^* \phi_{j+1}(Z) = 0, \quad j = 0, \ldots m-1.$$

Again, if the various A's or B's have inverses, the parameters $\phi_j(Z)$ can be eliminated.

5. Consider the system.

$$L_b Y = Y' + PY + H_1 [CY(a_o) + D Y(a_1)],$$

$$AY(a_o) + BY(a_1) + \int_{a_o}^{a_1} K_1 Y dx = 0.$$

The adjoint system is

$$L_b^* Z = -Z' + P^*Z - K_1^* \phi_1(Z),$$

$$-Z(a_1) + B^* \phi_1(Z) - D^* \int_{a_o}^{a_1} H_1^* Z dx = 0,$$

$$Z(a_o) + A^* \phi_1(Z) - C^* \int_{a_o}^{a_1} H_1^* Z dx = 0.$$

Under the conditions given in [6], this adjoint system is equivalent to the one defined in [6]. We note that in all of these examples, there were m boundary conditions for the m intervals ($m + 1$ points) under discussion. In all of these examples Green's matrices have been derived, and, with the exception of the last one, each has an eigenfunction expansion associated with it. It seems reasonable to expect that this will always be true. Solving a nonhomogeneous equation in each of m intervals produces m arbitrary constants of integration. In order to specify them, m equations are needed. More equations overdetermine the system. Fewer underdetermine it.

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