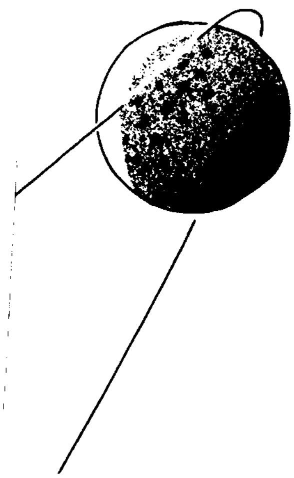

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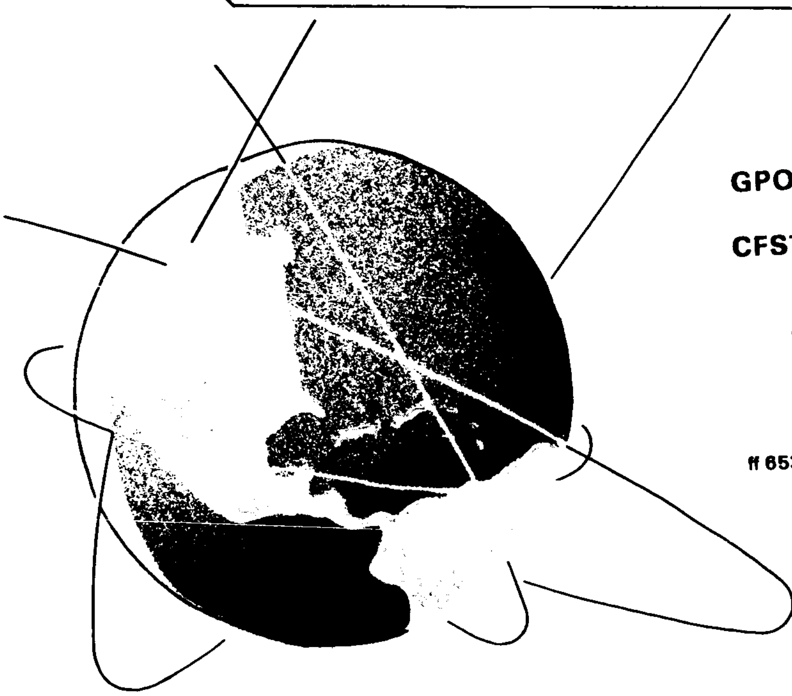

ENGINEERING MECHANICS RESEARCH LABORATORY

THE UNIVERSITY OF TEXAS · AUSTIN, TEXAS

FAILURE ANALYSIS OF THE METHOD OF ADJOINT SYSTEMS TO DETERMINE OPTIMAL TRAJECTORIES

by

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THESIS

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Assistant Professor of Engineering Mechanics

ERRATA

Page 12 In Equation (2.17), $(R_t - H)\big|_{t_f} dt_o$ should read $(R_t - H)\big|_{t_o} dt_o$

Page 12 In Equation (2.17), $R_\mu\big|_{t_f} d_\mu$ should read $R_\mu\big|_{t_o} d_\mu$

Page 55 In Equation (4.108), $\theta^T(t_o)^{-1}$ should read $\theta_\lambda^T(t_o)^{-1}$

Page 67 In Equation (4.141), $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ should read $\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Page 68 In Equation (4.144), $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ should read

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Page 77 In Equation (5.7), $dx^T(P_{xx} - w)dx\big|_{t_o}$ should read $dx^T(P_{xx} - w)dx\big|_{t_f}$

PREFACE

Due to the high degree of nonlinearity and complexity encountered in many modern trajectory optimization problems, there has been a great need for, and consequently, a great upsurge of interest in the development and application of various numerical optimization techniques in recent years. Utilization of the capabilities of high speed digital computers has resulted in solutions to problems which a few short years ago were considered too cumbersome or difficult to solve.

Although numerical techniques have been successfully applied in solving a wide class of optimization problems, occasionally someone fails to optimize a particular dynamical system by the use of a particular optimization technique. This failure can occur in several ways depending upon the technique used and the particular problem, but usually an investigation of the failure is not attempted. As a result, there is very little information presently available in the literature concerning ways in which numerical optimization techniques can fail in application to dynamical systems. The need for such information is obvious.

This thesis is a study of one particular numerical optimization technique known as the Method of Adjoint Systems. This method is developed and applied to a particular dynamical system in order to determine the effects of incorrect problem formulation. Through the properties of adjoint systems, a relationship between the effects of a conjugate point in the trajectory and a failure in the method is derived.

The author would like to express his gratitude to Dr. W. T. Fowler for bringing this area of study to the author's attention, for giving valuable direction to the study, for critical reading of the manuscript, and for serving as supervising professor. The author wishes to thank Dr. B. D. Tapley for the valuable discussions on conjugate points and for serving on the supervising committee. The author would like to thank Mr. Gary J. Lastman, fellow graduate student, for his helpful suggestions and advice in various phases of this investigation. The author is pleased to acknowledge the support of the Bureau of Engineering Research of The University of Texas. Especially, the author wishes to thank his wife, Barbara, for her understanding, encouragement, and help throughout the preparation of this thesis.

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ABSTRACT

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This thesis is concerned with the ways in which the Method of Adjoint Systems can fail to determine optimal trajectories.

The Method of Adjoint Systems is developed and applied to the solution of the two-point boundary value problem arising from the first necessary conditions for an optimal trajectory. A specific dynamical system is analyzed and various forms of incorrect problem formulation are considered in order to investigate the effects on the Method of Adjoint Systems.

The sufficiency conditions for a weak minimum are established and the Matrix Riccati Equation is shown to evolve from the requirement that the second variation be positive for a minimum. The effects of a conjugate point in the nominal trajectory on the solution of the guidance optimization problem are established through the solution to the Matrix Riccati Equation.

It is shown that the Matrix Riccati Equation can be solved by reducing it to two linear systems of differential equations, which in partitioned form are adjoint to a system of linear differential equations used in the implementation of the computational algorithm of the Method of Adjoint Systems. Through the properties of adjoint systems, a relationship is established which indicates a connection between a breakdown in the Method of Adjoint Systems due to the singularity of a matrix which must be inverted and the presence of a conjugate point in the nominal trajectory at the initial time.

Author

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LIST OF SYMBOLS

The following list tabulates all significant symbols used in the main text. Each symbol is accompanied by a brief description and the equation number where the symbol is first introduced.

Matrices:

The matrix size is indicated in the statement immediately following the symbol. The following specific indices are used.

n	-	the number of state variables
m	-	the number of control variables
q	-	the number of terminal constraint relations
r	-	the number of initial constraint relations

A	2n x 2n matrix of partial derivatives	(3.12)
I	n x n or 2n x 2n identity matrix	(3.21)
J	n x n matrix of constants	(5.69)
K	n x n matrix solution of the Matrix Riccati Equation	(5.55)
N	n x n matrix used in solving the Matrix Riccati Equation	(5.39)
T	n x n matrix used in solving the Matrix Riccati Equation	(5.43)
V	n x n matrix of feed-back gains	(5.61)
W	n x n matrix solution of the Matrix Riccati Equation	(5.5)

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(CONT'D)

α	n x n matrix equivalent of the Matrix Riccati Equation	(5.12)
β	n x n matrix used in solving the Matrix Riccati Equation	(5.13)
θ	2n x (n+1) matrix of partial derivatives resulting from n+1 backward integrations of the 2n vector of adjoint equations	(3.30)
γ	n x n matrix of partial derivatives of the Hamiltonian with respect to the control	(5.15)
Φ	n x n matrix used to require that the second variation be positive	(5.17)
ψ	n x n matrix used to require that the second variation be positive	(5.17)
ψ_1	2n x 2n fundamental matrix of the adjoint equations	(3.20)

LIST OF SYMBOLS

(CONT'D)

Vectors:

All vectors are column vectors unless otherwise noted. Vector size is indicated in the statement immediately following the symbol.

f	n vector of state variables	(2.7)
F	2n vector of state and Euler-Lagrange variable derivatives	(2.41)
g	n vector of initial constraint relations	(3.3)
h	n+1 vector of terminal constraint relations	(3.2)
dh	n+1 vector of terminal dissatisfaction change	(3.24)
L	r vector of initial state constraints	(2.5)
M	q vector of terminal state constraints	(2.6)
u	m vector of control variables	(2.4)
x	n vector of state variables	(2.7)
X	n vector of neighboring state variables	(2.27)
z	2n vector of state variables and Euler-Lagrange variables	(2.39)
ε	n arbitrary vector	(5.36)
λ	n vector of time dependent Lagrange multipliers	(2.8)
Λ	2n vector of adjoint variables	(3.15)
μ	r vector of constant Lagrange multipliers	(2.10)
ν	q vector of constant Lagrange multipliers	(2.9)

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(CONT'D)

Scalers:

a	initial value of the Lagrange multiplier $\lambda_1(t_0)$	(4.15)
b	initial value of the Lagrange multiplier $\lambda_2(t_0)$	(4.15)
c	initial value of the Lagrange multiplier $\lambda_3(t_0)$	(4.15)
C	constant value of the time rate of change of the state variable $x_3(t)$	(4.1)
E	the Weierstrass E-function	(2.27)
G	part of the performance index	(2.4)
H	the Variational Hamiltonian	(2.11)
I	auxiliary functional to be extremized	(2.8)
k	constant value of terminal constraints	(4.26)
P	auxiliary functional of terminal value variables	(2.9)
Q	part of the performance index	(2.4)
R	auxiliary functional of initial value variables	(2.11)
t	independent variable time	(2.4)

LIST OF SYMBOLS

(CONT'D)

Superscripts:

$(\dot{\ })$	differentiation with respect to time
$(\ )^T$	matrix transpose
$(\ )'$	the first variation
$(\ )''$	the second variation
$(\ )^{-1}$	matrix inverse
$(\ )^*$	evaluated along the nominal trajectory (also refers to guessed value)

Subscripts:

$(\ )_0$	initial value
$(\ )_f$	terminal value
$(\ )_{\lambda,u,x,t}$	partial derivative of the subscripted quantity with respect to λ,u,x, and t respectively with the exception of θ_λ, θ_x
$(\ )_i$	i^{th} value of subscripted quantity
$(\ )_1$	known value of subscripted quantity
$(\ )_2$	unknown value of subscripted quantity

Miscellaneous Symbols:

d	total variation or differential operator i.e., $d(\ ) = \delta(\ ) + (\dot{\ })\,dt$
δ	variational operator
$x(t_f)$	evaluation of the instantaneous variable x at time t_f

$(\ )\big|_{t_0,t_f}$ evaluation of the bracketed quantity at t_0 or t_f, respectively

Abbreviations:

PI	Performance Index
TC	Terminal Constraints

CHAPTER I

INTRODUCTION

The flight mechanics engineer today is often faced with the problem of optimizing the dynamic performance of various vehicles and systems. The optimization problem consists of two parts which are usually considered as separate problems. The first is the trajectory optimization problem; the second is the guidance optimization problem.

The trajectory optimization problem (or open-loop problem) consists of determining certain input variables (controls) required to optimize some index of performance of the system subject to (1) nonlinear differential equations of state and (2) initial and terminal state constraints. The problem may be formulated in Mayer form in which the differential equations of state and the initial and terminal state constraints are adjoined to the performance index by the use of Lagrange multipliers. This formulation yields a functional which is to be optimized. This type of problem can be readily handled by the techniques of the Calculus of Variations. The variational approach to the optimization problem yields a two-point boundary value problem whose solution is the solution to the trajectory optimization problem.

Because of model inaccuracies, disturbances of the system, etc., the control (as computed from the two-point boundary value problem) may not be optimal. Thus, a guidance optimization problem must be faced. Many of the results obtained in the trajectory optimization problem have direct counterparts in guidance optimization. The guidance optimization problem consists of correcting the open-loop control to account for small errors in the trajectory. Guidance optimization is usually achieved by the use of a linear feed-back control system which continually corrects the open-loop control program. However, before a guidance scheme can be devised, the trajectory optimization problem must be solved. Therefore, the solution of the variational two-point boundary value problem is the main topic considered in this thesis.

Numerical Trajectory Optimization

The two-point boundary value problem resulting from the variational approach to the trajectory optimization problem is characterized by the Euler-Lagrange differential equations and the differential equations of state (usually nonlinear). Analytic solutions to the boundary value problem are in most cases impossible because of the nonlinearity of the equations involved. Thus, in the course of solving many trajectory optimization problems, it is necessary to use some type of numerical technique which can be implemented on a digital computer. Several numerical techniques have been developed in recent years. These techniques, usually classified by the approach taken to the optimization problem, are divided into two groups. These groups are (1) direct methods and (2) indirect methods. Direct methods employ gradient techniques which successively improve non-optimal solutions, while indirect methods employ techniques for solving the two-point boundary value problem arising from a calculus of variations approach to the optimization problem.

The engineer, in seeking a numerical solution to a particular optimization problem, must decide which of the various numerical techniques to use. A method may be chosen which will fail to optimize a particular dynamic system. Many engineering man-hours and hours of computer time are wasted when a numerical technique is applied unsuccessfully to a particular problem. There is a clear need for criteria by which a specific optimization technique can be chosen for use in the solution of a particular problem or class of problems. At the present time such criteria do not exist. When an optimization scheme breaks down or fails to solve a particular problem, one of two procedures is usually followed: (1) another numerical technique is tried or (2) the problem is labeled "ill behaved" or "not correctly formulated" and is filed away or forgotten. Usually no attempt is made to discover the reason for the failure. Consequently, the reasons for breakdown in most numerical optimization techniques are unknown.

Recently, studies of the failure of the Steepest Ascent Method (a direct method) to optimize particular dynamical systems were undertaken by Dr. W. T. Fowler and Mr. Gary J. Lastman at The University of Texas. These studies have indicated that breakdown can occur in three distinct ways which can be related to the way in which the problem was formulated. The modes of failure in the method were found to involve (1) constrained performance indices, (2) constrained uncontrollable state variables, and (3) performance indices which were uncontrollable variables. In addition, it was found that the Steepest Ascent Method would break down for certain dynamical systems due to certain guessed control programs. This mode of failure, however, is believed to be peculiar to the Steepest Ascent Method.

Encouraged by the results of the analysis of failure of the Steepest Ascent Method, it was decided that a similar analysis should be made of other numerical optimization techniques. The technique chosen for study in this thesis is an indirect method known as the Method of Adjoint Systems. The method is an iterative technique used to solve the variational two-point boundary value problem.

The Method of Adjoint Systems

In 1956, Goodman and Lance (Ref.1) proposed a numerical technique for converting a two-point boundary value problem into an initial value problem by the use of an adjoint system of differential equations. In 1962, Jurovics and McIntyre (Ref. 2) extended this earlier work to include problems with one unknown boundary. The boundary value problem with one unknown boundary is a form of the variational boundary value problem but is quite restricted. In 1964, Jazwinski (Ref. 3) extended the scheme to include problems in which the initial and terminal boundary conditions are general functions of the problem variables and thus permitted the treatment of a general Mayer form of the trajectory optimization problem. Jazwinski's scheme will be referred to throughout this thesis as the Method of Adjoint Systems.

In addition to the method for obtaining open-loop control programs, Jazwinski proposed a feed-back control scheme to solve the guidance optimization problem in which feed-back gains could be obtained from the results of the open-loop control problem. This feed-back control scheme is simpler and requires fewer integrations of an equivalent set of equations than the method proposed by Breakwell, Speyer, and Bryson (Ref. 4).

In considering the guidance optimization problem, Jazwinski mentions the fact that his feed-back control scheme can yield infinite gains for certain systems which will cause the scheme to fail. Breakwell (Ref. 4) points out that a conjugate point in the trajectory will result in infinite gains in his guidance scheme. However, no explanation is given of how the numerical trajectory optimization scheme used to obtain the open-loop control could fail. A search of the literature on optimal control theory reveals very little information concerning modes of failure of the trajectory optimization schemes currently in use.

The Purpose of the Investigation

The purpose of this investigation is to determine the ways in which the Method of Adjoint Systems can break down, and hence fail to determine the open-loop control programs for particular dynamical systems.

In Chapter II the formulation of the trajectory optimization problem will be reviewed. The first necessary conditions for an optimal trajectory and the Legendre-Clebesch condition for a weak relative minimum will be derived. The two-point variational boundary value problem will be formed from the necessary conditions and the equations of state. In Chapter III the Method of Adjoint Systems will be reviewed and a discussion given of its application in the solution of the two-point boundary value problem for the most general case in which some of the values of the state variables and Lagrange multipliers are unknown initially and the initial and terminal time are unknown. In Chapter IV areas of breakdown and the computational algorithm due to incorrect problem formulation will be discussed with specific examples to illustrate each type of breakdown. In Chapter V the effects of a conjugate point in the trajectory will be investigated. Effects of a conjugate point on the guidance optimization problem will be looked into with the purpose of gaining some insight into the effects on the trajectory optimization problem and subsequently, the Method of Adjoint Systems. It is hoped that the investigation will provide, in some measure, usable criteria by which the Method of Adjoint Systems can be chosen or rejected as an optimization scheme for specific problems. It is also hoped that the investigation will provide guidelines by which studies of the failure of other numerical optimization schemes can follow.

CHAPTER II

THE TRAJECTORY OPTIMIZATION PROBLEM

Vector and Matrix Partial Derivative Notation

Extensive use will be made of vector and matrix forms of functions and their partial derivatives throughout this thesis without special mention. The following convention will be observed concerning partial derivatives. The n-vector of partial derivatives of a quantity $f(x,u,t)$ with respect to a vector x is denoted by either f_x or $\frac{\partial f}{\partial x}$. If f is an m-dimensional column vector, i.e. $f^T = [f_1 \ldots f_m]$ then f_x implies the following matrix.

$$f_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{\partial f_m}{\partial f_1} & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{2.1}$$

If f is a scalar function of the form $f(x_1 \ldots x_n, y_1 \ldots y_m)$ then f_x is defined as

$$f_x = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f}{\partial x_n} \end{bmatrix} \tag{2.2}$$

and f_{xy} is defined as

$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]^T = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial y_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial y_m} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial y_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial y_m} \end{bmatrix} \tag{2.3}$$

A row vector will be written as $G^T = [g_1 \; . \; . \; . \; g_n]$ while a 1 x n matrix will be written without the transpose sign, i.e., $K = [k_1 \; \ldots \; k_n]$

Statement of the Problem

The problem to be considered is that of determining the m-vector of nominal open-loop control variables $u^*(t)$ which will cause a general scalar performance index of the form

$$I = G(x_f, t_f) + \int_{t_0}^{t_f} Q(x,u,t)\, dt \tag{2.4}$$

to be an extremum in the time interval, $t_0 \leq t \leq t_f$ subject to the following constraints: (1) an r-vector of initial state constraints,

$$L(x_0, t_0) = 0 \tag{2.5}$$

(2) a q-vector of terminal state constraints,

$$M(x_f, t_f) = 0 \tag{2.6}$$

and

(3) the n first order, nonlinear, ordinary differential equations of state

$$\dot{x}(t) = f(x,u,t) \tag{2.7}$$

It is required that $r \leq n$ and $q < 2n - r + 2$

Assumptions

In considering the problem stated, the following assumptions will be made.

(1) The problem is deterministic, i.e., the state variables are known exactly at each point along the trajectory in the time interval $t_0 \leq t \leq t_f$.

(2) The n-vector function, $f(x,u,t)$, in Equation (2.7) is at least twice differentiable with respect to the state and control variables.

(3) The time rate of change of the state variables, $\dot{x}(t)$, is continuous in the interval of interest.

The Necessary Conditions for an Optimal Trajectory

The initial constraints in Equation (2.5), the terminal constraints in Equation (2.6), and the differential equations of state in Equation (2.7) may be adjoined to the performance index in Equation (2.4) as follows

$$\begin{aligned} I = {} & G(x_f,t_f) + \nu^T M(x_f,t_f) + \mu^T L(x_0,t_0) + \\ & + \int_{t_0}^{t_f} \left[\Omega(x,u,t) + \lambda^T(t) \, [f(x,u,t) - \dot{x}(t)] \right] dt \end{aligned} \tag{2.8}$$

where ν is a q-vector of constant multipliers, μ is an r-vector of constant multipliers, and $\lambda(t)$ is an n-vector of time varying Lagrange multipliers.

The functional I may be written in a more compact form by defining the following scalar quantities

$$P = G(x_f,t_f) + \nu^T M(x_f,t_f) \tag{2.9}$$

$$R = \mu^T L(x_0,t_0) \tag{2.10}$$

$$H = Q(x,u,t) + \lambda^T(t)\; f(x,u,t) \tag{2.11}$$

where H is referred to as the Variational Hamiltonian. Thus, the functional I may be written as

$$I = P(x_f,t_f,\nu) + R(x_0,t_0,\mu) + \int_{t_0}^{t_f} (H(x,\lambda,u,t) - \lambda^T\dot{x})\; dt \tag{2.12}$$

The problem written in this form is generally referred to in the literature as the Bolza form of the optimization problem.

Following Bolza (Ref. 5), the functional I may be expanded in a Taylor's series about some nominal trajectory to yield

$$dI = d'I + \frac{1}{2!}d''I + \frac{1}{3!}d'''I + \ldots \tag{2.13}$$

If deviations from the nominal trajectory are assumed to be small, the Taylor's series expansion may be truncated after the first two terms, and the total variation in I becomes

$$dI \cong d'I + \frac{1}{2} d''I \tag{2.14}$$

In order that I be an extremum in the interval of interest, it is required that $d'I$, the first variation of I, be equal to zero and that $d''I$, the second variation, be greater than zero for a minimum or less than zero for a maximum. Thus for a minimum, the following conditions are required.

$$d' I = 0 \tag{2.15}$$

$$d'' I > 0 \tag{2.16}$$

Equation (2.15) is referred to as the First Necessary Condition, and Equations (2.15) and (2.16) together are referred to as Sufficient Conditions for a minimum. It is pointed out in Ref. 6 that in most physical problems, the nature of the problem and the solution obtained leave no doubt as to the nature of the extremum. Thus, consideration of the full second variation, which is quite complex, is usually not necessary. It must be mentioned, however, that as the problems considered become more complex, it becomes difficult to rely upon the nature of the problem and the type of solution obtained by the first variation alone to insure that a maximum or a minimum performance index has been obtained. This point will be discussed further in a later chapter dealing with conjugate points.

The full expansion of the first variation of the functional I in Equation (2.12) is given in Appendix A. The resulting equation, given by Equation (A.20), is

$$d'I = (P_t+H)\big|_{t_f} dt_f + (R_t-H)\big|_{t_f} dt_0 + (P_x-\lambda^T)\big|_{t_f} dx_f +$$

$$+ (R_x+\lambda^T)\big|_{t_0} dx_0 + P_\nu\big|_{t_f} d\nu + R_\mu\big|_{t_f} d\mu + \tag{2.17}$$

$$+ \int_{t_0}^{t_f} (H_x+\dot{\lambda}^T)\delta x dt + \int_{t_0}^{t_f} (H_\lambda-\dot{x}^T)\delta\lambda dt + \int_{t_0}^{t_f} (H_u^T)\delta u dt$$

The vanishing of the first variation requires that if dx_f, dt_0, dx_0, dt_f, $d\nu$, $d\mu$, δx, $\delta\lambda$, and δu are independent variations, then each term in Equation (2.17) must vanish separately. The conditions which result from the vanishing of the terms outside the integrals in Equation (2.17) containing initial and terminal variations in the state and time are referred to as the transversality conditions. The transversality conditions yield the following conditions which must be satisfied at the end points of the trajectory.

At the initial boundary

$$(1) \quad (R_x + \lambda^T)\big|_{t_0} dx(t_0) = 0 \tag{2.18}$$

$$(2) \quad (R_t - H)\big|_{t_0} dt_0 = 0 \tag{2.19}$$

At the terminal boundary

$$(3) \quad (P_x - \lambda^T)\big|_{t_f} dx(t_f) = 0 \tag{2.20}$$

$$(4) \quad (P_t + H)\big|_{t_f} dt_f = 0 \tag{2.21}$$

The conditions which result from the vanishing of the terms containing the variations in the constant multipliers μ and ν at the initial and terminal time are referred to as the constraint conditions and are given by the following relations.

At the initial boundary

$$(5) \quad R_{\mu}\big|_{t_0} d\mu = 0 \tag{2.22}$$

At the terminal boundary

$$(6) \quad P_{\nu}\big|_{t_f} d\nu = 0 \tag{2.23}$$

In order that the integral quantities in Equation (2.17) vanish along the trajectory, the fundamental Lemma of the calculus of variations, (Ref. 7), requires that each of the coefficients of δx, $\delta\lambda$, and δu equal zero. The vanishing of these coefficients yields the following equations which must be satisfied at each point along the trajectory.

$$(7) \quad \dot{x}(t) = H_{\lambda}^{T}(x,\lambda,u,t) = f(x,u,t) \tag{2.24}$$

This equation is the n-vector of differential equations of state given in the statement of the problem.

$$(8) \quad \dot{\lambda}(t) = -H_{x}^{T}(x,\lambda,u,t) \tag{2.25}$$

This is the classical set of Euler-Lagrange equations and $\lambda(t)$ is an n-vector function of time.

$$(9) \quad H_{u}^{T}(x,\lambda,u,t) = 0 \tag{2.26}$$

This equation is known as the optimality condition and Hu is an m-vector function of time.

Equations (2.18) through (2.26) are known collectively as the first necessary conditions and must be satisfied in order that I be an extremum in the interval of interest.

The Legendre-Clebesch Condition

In addition to the first necessary conditions for I to be an extremum, it will be required in this thesis that the extremum be a weak relative minimum throughout the interval of interest.

It is proved in Ref. 5 that if I is to take on a strong minimum value in the interval of interest, the necessary condition of Weierstrass must be satisfied. This condition requires that if a trajectory $x(t)$, where $\dot{x}(t)$ is continuous in $[t_0,t_f]$, affords a minimum to I relative to all neighboring trajectories $X(t)$, where $\dot{X}(t)$ is piece-wise continuous in $[t_0,t_f]$, joining its end points, the following inequality must be satisfied.

$$E(x,\dot{x},\dot{X},t) > 0 \tag{2.27}$$

The function $E(x,\dot{x},\dot{X},t)$ is called the Weierstrass E-function, and is defined as

$$E(x,\dot{x},\dot{X},t) = F(x,\dot{X},t) - F(x,\dot{X},t) - F_{\dot{x}}(\dot{X}-\dot{x}) \tag{2.28}$$

It may be shown that for weak variations in which $|\dot{X}-\dot{x}| \leq \varepsilon$, where ε is an arbitrarily small quantity, the E-function reduces to

$$E(x,\dot{x},\dot{X},t) = - F_{\dot{x}}(\dot{X}-\dot{x}) \tag{2.29}$$

The functions $F(x,\dot{x},t)$ and $F(x,\dot{X},t)$ are defined as

$$F(x,\dot{x},t) = \lambda^T(t) \quad (f(x,u,t) - \dot{x}(t)) = 0, \tag{2.30}$$

and

$$F(x,\dot{X},t) = \lambda^T(t) \quad (f(x,U,t) \quad - \dot{X}(t)) \quad = \quad 0 \tag{2.31}$$

From the definition of the Hamiltonian, Equation (2.29) may be written as

$$E(x,\dot{x},\dot{X},t) \quad = H(x,\lambda,U,t) \quad - H(x,\lambda,u,t) \tag{2.32}$$

Letting $U = u + \delta u$, $H(x,\lambda,U,t)$ may be expanded in a Taylor's series about u and for small δu, the series may be truncated after the first two terms to yield

$$E(x,\dot{x},\dot{X},t) \cong H + H_u \ \delta u + \frac{1}{2} \ \delta u^T \ H_{uu} \ \delta u - H \tag{2.33}$$

Fulfillment of the optimality condition in Equation (2.26) reduces Equation (2.33) to

$$\frac{1}{2} \ \delta u^T \ H_{uu} \ \delta u \ > 0 \tag{2.34}$$

The quadratic form of Equation (2.34) requires that H_{uu} be a nonsingular positive definite matrix. The requirement that the relation

$$H_{uu} > \quad 0 \qquad \text{(Positive definite)} \tag{2.35}$$

holds true in the interval of interest is referred to as the Legendre-Clebesch condition. This condition is necessary in order that I be a weak relative minimum in the interval of interest.

<u>The Two-Point Boundary Value Problem</u>

The differential equations of state (Equation (2.24)) and the Euler-Lagrange equations (Equation (2.25)) constitute a set of first order, nonlinear, ordinary differential equations in terms of the variables x, λ,u, and t.

The optimality condition (Equation 2.26) may be used to solve for the m optimal control variables in terms of the Lagrange multipliers and the independent variable time yielding an m-vector equation of the form

$$u(t) = u(\lambda,t) \tag{2.36}$$

Equation (2.36) may be used to eliminate the control variables from Equations (2.24) and (2.25) yielding

$$\dot{x}(t) = H_{\lambda}^{T}(x,\lambda,t) \tag{2.37}$$

$$\dot{\lambda}(t) = -H_{x}^{T}(x,\lambda,t) \tag{2.38}$$

Equation (2.37) and (2.38) may be combined into an ordinary, first order, nonlinear vector differential equation containing 2n elements by defining a new variable, z(t) such that $x^T(t) = [z_1 \ldots z_n]$ and $\lambda^T(t) = [z_{n+1} \ldots z_{2n}]$. The variable may be written as

$$z^T = [x^T \;\vdots\; \lambda^T] \tag{2.39}$$

In view of Equation (2.39), Equations (2.37) and (2.38) written in partitioned form i.e.,

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\lambda} \end{bmatrix} \doteq \begin{bmatrix} H_{\lambda}^{T}(x,\lambda,t) \\ \hline -H_{x}^{T}(x,\lambda,t) \end{bmatrix} \tag{2.40}$$

become

$$\dot{z}(t) = F(z(t),t) \tag{2.41}$$

where $F(z(t),t)$ is defined as the right-hand side of Equation (2.40). The boundary conditions imposed on Equation (2.41) are as follows:

1) r conditions arising at t_0 from Equation (2.22), i.e., $R_\mu\big|_{t_0} d\mu = 0$. From the definition of R in Equation (2.10), this relation requires that $L(x_0,t_0)\, d\mu = 0$ but $L(x_0,t_0) = 0$ from Equation (2.5), thus yielding r conditions on the state variables at t_0. Thus, $d\mu$ does not necessarily vanish because the initial constraints must be satisfied.

2) $n - r$ conditions arising at t_0 from Equation (2.18), i.e., $(R_x + \lambda^T)\big|_{t_0} dx(t_0) = 0$. There are r conditions specified initially on the state variables, thus r of the $dx(t_0)$ will equal zero leaving $n - r$ unspecified state variations which yield $n - r$ initial conditions on the Lagrange multipliers.

3) 1 condition arising at t_0 from Equation (2.19), i.e., $(R_t - H)\big|_{t_0} dt_0 = 0$. If the initial time is specified, the equation is identically satisfied. If the initial time is unspecified, then the condition implies that the scalar function $(R_t - H)\big|_{t_0}$ is equal to zero resulting in one initial condition on the Lagrange multipliers.

4) q conditions arising at t_f from Equation (2.23), i.e., $P_\nu\big|_{t_f} d\nu = 0$. From the definition of P, this condition requires that $M(x_f,t_f)\, d\nu = 0$. But $M(x_f,t_f) = 0$ from Equation (2.6) resulting in q terminal conditions on the state variables. Thus $d\nu$ does not necessarily vanish because the terminal constraints must be satisfied.

5) $n - q$ conditions arising at t_f from Equation (2.20), i.e., $(P_x - \lambda^T)\big|_{t_f} dx(t_f) = 0$. The q terminal conditions specified on the state variables require that q of the $dx(t_f)$ equal zero thus leaving $n - q$ unspecified state variations yielding $n - q$ terminal conditions on the Lagrange multipliers.

6) 1 condition arising at t_f from Equation (2.21), i.e., $(P_t + H)\big|_{t_f} dt_f = 0$. If the terminal time is known the equation is identically satisfied. If the terminal time is unspecified, then the scalar function $(P_t + H)\big|_{t_f}$ must equal zero resulting in one terminal condition on the Lagrange multipliers.

Thus, there are $n - r + 1$ initial and $n - q + 1$ terminal conditions resulting in $2n - q - r + 2$ conditions imposed upon the Lagrange multipliers. There are r initial and q terminal conditions resulting in $r + q$ conditions imposed on the state variables. A total of $2n + 2$ conditions are imposed upon Equation (2.41). In the most general variational two-point boundary value problem, there are 2n unknown variables to be determined and the initial time and final time are unspecified, thus requiring $2n + 2$ boundary conditions to obtain a solution.

In the problems considered in this thesis, it will be assumed that the initial time is known. The optimization problem, with the initial time given, may thus be reduced to a two-point variational boundary value problem with n boundary conditions at t_0 and $n + 1$ boundary conditions at t_f. The solution of this boundary value problem by the Method of Adjoint Systems will be considered in Chapter III.

CHAPTER III

THE METHOD OF ADJOINT SYSTEMS

The solution of the two-point boundary value problem consisting of 2n first order ordinary nonlinear differential equations of the form

$$\dot{z}(t) = F(z(t),t) \tag{3.1}$$

subject to n initial boundary conditions at some unknown initial time, t_0, and n+1 boundary conditions at some unknown terminal time, t_f, is often far from simple. Except for the simplest cases, an analytic solution of the problem can not be obtained due to the nonlinearity of the differential equations involved. Numerical integration also presents a problem. In order to numerically integrate the 2n differential equations in Equation (3.1), 2n values of the variables z(t) are required at some time t in the interval $[t_0,t_f]$. However, in the variational two-point boundary value problem, there are only n values of z(t) known at the initial time. Thus, in order to obtain a solution to Equation (3.1), an iterative technique must be employed.

The Method of Adjoint Systems, proposed by Jazwinski (Ref. 3) is such an iterative technique. The procedure used in the method is as follows. The n initial values of z(t), i.e. $z(t_0)$, which are unknown are guessed. These guessed values are used with the n specified values of $z(t_0)$ to integrate Equation (3.1). A reference solution or nominal trajectory is thus obtained. Linear perturbations about this nominal trajectory are used in conjunction with an adjoint system of equations to obtain a better estimate of the initial values of the n unknown variables. The method is developed in the following sections.

The Nominal Trajectory

The n+1 terminal boundary conditions arising from the transversality conditions (Equations (2.20) and (2.21)) and the terminal constraints (Equation (2.23)), may be written as

$$h(z_f, t_f) = 0 \tag{3.2}$$

The n initial boundary conditions arising from the transversality conditions (Equation (2.18)) and the initial constraints (Equation (2.22)) may be written as

$$g(z_0, t_0) = 0 \tag{3.3}$$

The n initial values of z(t) are guessed and used with Equation (3.3) to numerically integrate Equation (3.1) from t_0 to some guessed terminal time t_f^* . The nominal variables, $z^*(t)$, may be stored at each integration step.

The nominal trajectory thus obtained will approximate the true trajectory if the n unknown values of $z^*(t_0)$ were guessed sufficiently close to the true values of $z(t_0)$. The true trajectory is defined as the trajectory which satisfied Equations (3.2) and (3.3) at the end points and Equation (3.1) in the interval $[t_0, t_f]$.

Linear Perturbations about the Nominal Trajectory

In general the nominal trajectory will not satisfy the terminal boundary conditions in Equation (3.2), i.e. $h(z^*(t_f), t_f^*) \neq 0$. However, a sufficiently close guess of $z(t_0)$ will yield nominal variables, $z^*(t)$, which differ little from the true variables, z(t), along the trajectory.

The difference between the true and nominal trajectories is defined as

$$\delta z(t) = z(t) - z^*(t) \tag{3.4}$$

The solution to the original differential equation, given by

$$\dot{z}(t) = F(z(t),t), \tag{3.5}$$

will be assumed to have been obtained when $\delta z(t)$ is made as small as desired.

Equation (3.4) may be differentiated to obtain

$$\delta\dot{z}(t) = \dot{z}(t) - \dot{z}^*(t) \tag{3.6}$$

In view of Equation (3.5), Equation (3.6) becomes

$$\delta\dot{z}(t) = F(z(t),t) - \dot{z}^*(t) \tag{3.7}$$

Equations (3.4) and (3.5) may be combined to yield

$$\dot{z}(t) = F(z(t),t) = F(z^*(t) + \delta z(t),t) \tag{3.8}$$

The function $F(z^*(t) + \delta z(t),t)$ may be expanded in a Taylor's series about the nominal trajectory, $z^*(t)$, at each point in time, and assuming that $\delta z(t)$ is small, the series may be truncated after the linear terms yielding

$$F(z^*(t) + \delta z(t),t) \cong F(z^*(t),t) + \left[\frac{\partial F}{\partial z}\right]^* \delta z(t) \tag{3.9}$$

In view of Equation (3.5), the differential equation of the nominal trajectory is

$$\dot{z}^*(t) = F(z^*(t),t) \tag{3.10}$$

Thus, from Equations (3.7), (3.8), (3.9), and (3.10), the linear perturbation equation becomes

$$\delta\dot{z}(t) = \left[\frac{\partial F}{\partial z} \right]^* \delta z(t) \tag{3.11}$$

where

$$\left[\frac{\partial F}{\partial z} \right]^* = \begin{bmatrix} \left[\frac{\partial F_1}{\partial z_1}\right]^* & \cdots & \left[\frac{\partial F_1}{\partial z_{2n}}\right]^* \\ \vdots & & \vdots \\ \left[\frac{\partial F_{2n}}{\partial z_1}\right]^* & \cdots & \left[\frac{\partial F_{2n}}{\partial z_{2n}}\right]^* \end{bmatrix}$$

Letting $A(t) = \left[\frac{\partial F}{\partial z} \right]^*$, Equation (3.11) may be written as

$$\delta\dot{z}(t) = A(t)\ \delta z(t) \tag{3.12}$$

where $\delta z(t_0)$ is given by

$$\delta z(t_0) = \begin{bmatrix} \delta x(t_0) \\ \hline \delta\lambda(t_0) \end{bmatrix}$$

It is shown in Ref. 3 that if H_{uu} is nonsingular, the 2n + 2n matrix, A(t) can be expressed as

$$A(t) = \left[\begin{array}{c|c} H_{\lambda x}-H_{\lambda u}H_{uu}^{-1}H_{ux} & -H_{\lambda u}\ H_{uu}^{-1}H_{u\lambda} \\ \hline -H_{xx}+H_{xu}H_{uu}^{-1}H_{ux} & -H_{x\lambda}+H_{xu}H_{uu}^{-1}H_{u\lambda} \end{array} \right] \tag{3.13}$$

Defining

$$A_{11} = H_{\lambda x} - H_{\lambda u} H_{uu}^{-1} H_{ux} ,$$

$$A_{12} = -H_{\lambda u} H_{uu}^{-1} H_{u\lambda} ,$$

$$A_{21} = -H_{xx} + H_{xu} H_{uu}^{-1} H_{ux} , \quad \text{and}$$

$$A_{22} = -A_{11}^{T} = - H_{x\lambda} + H_{xu} H_{uu}^{-1} H_{u\lambda} ,$$

the linear perturbation equation (Equation (3.12)) may be expressed as

$$\delta\dot{z}(t) = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & -A_{11}^{T} \end{array}\right] \delta z(t) \tag{3.14}$$

The requirement that H_{uu} be nonsingular is the Legendre-Clebesch Condition derived in Chapter II. It should be noted that this condition eliminates from consideration dynamical systems in which H is linear in control.

The Adjoint System

A system of differential equations adjoint to Equation (3.14) may be defined as

$$\dot{\Lambda}(t) = - A^{T}(t) \Lambda(t) \tag{3.15}$$

Premultiplying Equation (3.14) by $\Lambda^{T}(t)$ and postmultiplying the transpose of Equation (3.15) by $\delta z(t)$ and adding the resulting equations yields the following important relationship between a linear system and its adjoint system.

$$\Lambda^T(t)\ \delta\dot{z}(t) + \dot{\Lambda}^T(t)\ \delta z(t) = 0 \tag{3.16}$$

Noting that

$$\frac{d}{dt}\left(\Lambda^T(t)\ \delta z(t)\right) = \Lambda^T(t)\ \delta\dot{z}(t) + \dot{\Lambda}^T(t)\ \delta z(t), \tag{3.17}$$

the following relation exists from Equation (3.16)

$$\frac{d}{dt}\left[\Lambda^T(t)\ \delta z(t)\right] = 0 \tag{3.18}$$

From Equation (3.18), it is evident that

$$\Lambda^T(t)\ \delta z(t) = \Lambda^T(t_0)\ \delta z(t_0) = \Lambda^T(t_f)\ \delta z(t_f) \tag{3.19}$$

Equation (3.18) is one of the several important relationships between a linear system and its adjoint developed in Ref. 8.

The solution of the adjoint system, given in Equation (3.15), may be expressed in terms of the fundamental matrix of the system, $\psi_1(t,t_f)$, as

$$\Lambda(t) = \psi_1(t,t_f)\ \Lambda(t_f) \tag{3.20}$$

The 2nx2n fundamental matrix is the solution of the matrix differential equation

$$\dot{\psi}_1(t,t_f) = -A^T(t)\ \psi_1(t,t_f) \tag{3.21}$$

with the initial condition $\psi_1(t_f,t_f) = I$ where I is the 2nx2n identity matrix (Ref. 8). From Equation (3.18) and the transpose of Equation (3.20), it is seen that

$$\Lambda^T(t_f)\ [\psi_1^T(t,t_f)\ \delta z(t) - \delta z(t_f)] = 0 \tag{3.22}$$

It is noted that the vector $\Lambda^T(t)$ can not identically vanish in $[t_0,t_f]$ because if this occurs, the trivial solution is obtained in view of the linear vector equation (Equation (3.20)). Zero is a trivial solution and thus it must be required that $\Lambda^T(t) \neq 0$ for all $t_0 \leq t \leq t_f$. In view of this requirement, Equation (3.22) becomes

$$\delta z(t_f) = \psi_1^T(t,t_f) \; \delta z(t) \tag{3.23}$$

It was noted earlier that the nominal solution of Equation (3.1) in general fails to satisfy the terminal boundary conditions $h(z_f,t_f) = 0$. The dissatisfaction in the terminal boundary conditions is given by

$$[h(z(t_f),t_f) - h(z^*(t_f),t_f^*)] = - \Delta h$$

In order to obtain a solution to the boundary value problem, it is necessary to drive this dissatisfaction to zero. The dissatisfaction would not exist if the true initial values of the n unknown z(t) variables were known. Thus, driving the terminal dissatisfaction to zero amounts to improving the guessed $z^*(t_0)$ until $\delta z(t_0) = 0$. The relationship which exists between the linear perturbation equations and the adjoint system of equations provides the components necessary to construct a computational algorithm for successively improving the guessed initial values of z(t).

The Computational Algorithm

The terminal boundary conditions in Equation (3.2) may be perturbed about the nominal terminal boundary conditions to obtain

$$dh(z^*(t),t^*) = \left[\frac{\partial h}{\partial z}\right]^*_{t_f} dz^*(t_f) + \left[\frac{\partial h}{\partial t}\right]^*_{t_f} d^*t_f \tag{3.24}$$

where dh^* is an $n+1$ vector of the change in the dissatisfaction of the terminal boundary conditions. $\left[\frac{\partial h}{\partial z}\right]^*_{t_f}$ is an $(n+1)x2n$ matrix of partial derivatives, and $\left[\frac{\partial h}{\partial t}\right]^*_{t_f}$ is an $n+1$ vector of partial derivatives.

In the problem being considered, the terminal time is unknown and allowance must be made for possible variations in $z(t)$ due to terminal time variations. A first order approximation of the variation in $z(t)$ due to variations of the terminal time, (Ref. 9), is given as

$$\delta z^*(t_f) = dz^*(t_f) + \dot{z}(t_f^*)\, dt_f^* \tag{3.25}$$

Substitution of Equation (3.25) into Equation (3.24) yields

$$dh^* = \left[\frac{\partial h}{\partial z}\right]^*_{t_f} \delta z^*(t_f) + \left[\frac{\partial h}{\partial z}\right]^*_{t_f} \left[\frac{dz}{dt}\right]^*_{t_f} dt_f^* + \left[\frac{\partial h}{\partial t}\right]^*_{t_f} dt_f^* \tag{3.26}$$

However, the time rate of change in the terminal boundary conditions is

$$\dot{h} = \left[\frac{\partial h}{\partial z}\right] \left[\frac{dz}{dt}\right] + \left[\frac{\partial h}{\partial t}\right] \left[\frac{dt}{dt}\right]$$

and Equation (3.26) becomes

$$dh^* = \left[\frac{\partial h}{\partial z}\right]^*_{t_f} \delta z^*(t_f) + (\dot{h})^*_{t_f}\, dt_f^* \tag{3.27}$$

Substitution of Equation (3.23) into Equation (3.27) yields

$$dh^* = \left[\frac{\partial h}{\partial z}\right]^* \psi_1^T(t,t_f^*)\, \delta z(t) + (\dot{h})^*_{t_f}\, dt_f^* \tag{3.28}$$

Now the fundamental matrix differential equation (Equation (3.21)) may be postmultiplied by $\left[\frac{\partial h}{\partial z}\right]^{T*}_{t_f}$ to yield

$$\dot{\psi}_1(t,t_f^*) \left[\frac{\partial h}{\partial z}\right]_{t_f}^{T^*} = - A^T(t)\psi_1(t,t_f^*) \left[\frac{\partial h}{\partial z}\right]_{t_f}^{T^*} , \tag{3.29}$$

$$\psi_1(t_f^*, t_f^*)\left[\frac{\partial h}{\partial z}\right]_{t_f}^{T^*} = \left[\frac{\partial h}{\partial z}\right]_{t_f}^{T^*}$$

If $\theta(t)$ is defined as

$$\theta(t) = \psi_1(t,t_f^*) \left[\frac{\partial h}{\partial z}\right]_{t_f}^{T^*} \tag{3.30}$$

where $\theta(t)$ is a 2nx(n+1) matrix, Equation (3.29) may be replaced by

$$\begin{aligned} \dot{\theta}(t) &= - A^T(t)\ \theta(t), \\ \theta(t_f^*) &= \left[\frac{\partial h}{\partial z}\right]_{t_f}^{T^*} \end{aligned} \tag{3.31}$$

Equation (3.31) is integrated from t_f^* to t_0 to obtain $\theta(t_0)$. Substitution of Equation (3.30) into the perturbed equations for the change in the dissatisfaction in h i.e., Equation (3.28), yields

$$dh^* = \theta^T(t)\ \delta z(t) + (\dot{h})_{t_f}^*\ dt_f^* \tag{3.32}$$

At the initial time, Equation (3.32) becomes

$$dh^* = \theta^T(t_0)\ \delta z(t_0) + (\dot{h})_{t_f}\ \ dt_f^* \tag{3.33}$$

Letting $(\ )_1$ denote the known variables and $(\ )_2$ the unknown variables, the matrix $\theta(t_0)$ may be partitioned as

$$\theta^T(t_0) = [\theta_1^T\ (t_0) : \theta_2^T\ (t_0)] \tag{3.34}$$

The 2n $\delta z(t)$ vector may be partitioned as

$$\delta z(t_0) = \begin{bmatrix} \delta z_1(t_0) \\ \hdashline \delta z_2(t_0) \end{bmatrix} \tag{3.35}$$

Thus, Equation (3.33) may be written as

$$dh^* = \theta_1^T(t_0)\ \delta z_1(t_0) + \theta_2^T(t_0)\ \delta z_2(t_0) + (\dot{h})^*_{t_f}\ dt_f^* \tag{3.36}$$

Rearranging, Equation (3.36) becomes

$$[\theta_2^T(t_0) \ \vdots\ (\dot{h})^*_{t_f}] \begin{bmatrix} \delta z_2(t_\theta) \\ \hdashline dt_f^* \end{bmatrix} = [dh^* - \theta_1^T(t_0)\ \delta z_1(t_0)] \tag{3.37}$$

where $[\theta_2^T(t_0) \ \vdots\ (\dot{h})^*_{t_f}]$ is an $(n + 1) \times (n + 1)$ matrix, $\begin{bmatrix} \delta z_2(t_0) \\ \hdashline dt_f^* \end{bmatrix}$ is an $(n + 1) \times 1$ vector, and $[dh^* - \theta_1^T(t_0)\ \delta z_1(t_0)]$ is an $(n + 1) \times 1$ vector. Provided that $[\theta_2^T(t_0) \ \vdots\ (\dot{h})^*_{t_f}]$ is nonsingular, Equation (3.37) yields

$$\begin{bmatrix} \delta z_2(t_0) \\ \hdashline dt_f^* \end{bmatrix} = [\theta_2^T(t_0) \ \vdots\ (\dot{h})^*_{t_f}]^{-1} [dh^* - \theta_1^T(t_0)\ \delta z_1(t_0)] \tag{3.38}$$

The desired changes in the terminal boundary conditions may be specified as the terminal boundary conditions multiplied by a scaling constant given as

$$dh_i(z^*(t_f),t_f^*) = -k\, h_i(z^* t_f),t_f^*) \quad \text{where}$$

$$0 < k_i \le 1 \qquad (i = 1, \ldots, n+1) \tag{3.39}$$

where the k_i are scaling constants. With the desired changes thus specified, Equation (3.38) may be solved for the n initial variations $\delta z_2(t_0)$ and the variation in the terminal time, dt_f^*. An improved nominal trajectory is obtained by improving the guessed variables $z^*(t_0)$ and guessed terminal time t_f^* as

$$t_f^{*\,\text{new}} = t_f^{*\,\text{old}} + dt_f^{*\,\text{calculated}} \tag{3.40}$$

$$z^*(t_0)^{\text{new}} = z^*(t_0)^{\text{old}} + \delta z(t_0)^{\text{calculated}} \tag{3.41}$$

Equation (3.38), resulting from the preceding development of the computational algorithm, was derived for the two-point boundary value problem in which the terminal time was unknown and r of the n state variables were specified by $L(x_0,t_0) = 0$. For $r < n$, the guessed $z^*(t_0)$ consists of guessed initial state variables and Lagrange multipliers. $\delta z_2(t_0)$ is composed of r initial state variations and 2n-r variations of the initial values of the Lagrange multipliers.

There are less general cases of the two-point boundary value problem which result in a simplified version of Equation (3.38). These are:

Case 1: n Initial State Constraints Specified

In this case, n initial state constraints are specified in Equation (2.5), i.e. $L(x_0,t_0)$ consists of n equations in n values of the elements of the $x(t_0)$ vector, and Equations (3.34) and (3.35) may be written as

$$\theta^T(t_0) = [\theta_x{}^T(t_0) \,\vdots\, \theta_\lambda^T(t_0)\,] \tag{3.42}$$

$$\delta z(t_0) = \begin{bmatrix} \delta x(t_0) \\ ----- \\ \delta\lambda(t_0) \end{bmatrix} \tag{3.43}$$

Equation (3.38) becomes

$$\begin{bmatrix} \delta\lambda(t_0) \\ ----- \\ dt_f^* \end{bmatrix} = [\theta_\lambda^T(t_0) \,\vdots\, (\dot{h})_{t_f}]^{-1} \, [dh^*] \tag{3.44}$$

The unknown initial variables which must be guessed consists entirely of the Lagrange multipliers $\lambda^*(t_0)$.

Case 2: Final Time Specified

If the final time is specified, Equation (3.27) becomes

$$dh^* = \left[\frac{\partial h}{\partial z}\right]_{t_f}^* \delta z^*(t_f) \tag{3.45}$$

where, in this case, $\left[\frac{\partial h}{\partial z}\right]_{t_f}^*$ is an (nx2n) matrix of partial derivatives. For fixed final time problems, Equation (3.38) reduces to

$$\delta z_2(t_0) = [\theta_2^T(t_0)]^{-1} \; [dh^* - \theta_1^T(t_0) \;\; \delta z_1(t_0) \;\;] \tag{3.46}$$

where $\delta z_2(t_0)$ is an nx1 vector, $\theta_2^T(t_0)$ is an nxn matrix, and $[dh^* - \theta_1^T(t_0) \;\; \delta z_1(t_0) \;\;]$ is an nx1 vector.

Case 3: Fixed Final Time and n State Constraints Specified

This case is a combination of the first two cases. In this case, Equation (3.46) becomes

$$\delta\lambda(t_0) = [\theta_\lambda^T(t_0)]^{-1} [dh^*] \tag{3.47}$$

For the general case of Equation (3.38), it is pointed out in Ref. 3 that the improved nominal trajectory obtained by the computational algorithm presented will come closer to satisfying the terminal boundary conditions in Equation (3.2) than the old solution. The improved values of the unknown variables $z^*(t_0)$ and the unknown terminal time from Equations (3.40) and (3.41) are used as new guessed values of these variables and the process is repeated until the dissatisfaction in the terminal constraints can be made as small as desired, i.e. $|\Delta h| \leq \varepsilon$.

If for any given iteration Δh increases, the scaling constants, k_i, are reduced until a new nominal trajectory is obtained which is better than the old one. It is pointed out in Ref. 3 that the convergence characteristics of the Method of Adjoint Systems are extremely good for sufficiently good guesses of the initial values of the unknown variables.

There are conditions under which the Method of Adjoint Systems will break down in application. An examination of Equation (3.38) reveals that if the matrix which must be inverted is singular the method will fail. Another trouble spot exists if the guessed values of the unknown variables are not sufficiently close to the true values. The linear perturbation assumptions made in deriving the computational algorithm would not be valid if this were to happen. These and other areas of breakdown in the Method of Adjoint Systems will be explored in the following chapters.

CHAPTER IV

BREAKDOWNS IN THE METHOD OF ADJOINT SYSTEMS DUE TO INCORRECT PROBLEM FORMULATION

The Method of Adjoint Systems, derived in Chapter III, has proven in application to be a powerful numerical technique for solving a wide class of optimization problems which were considered too difficult or cumbersome prior to the development of the high speed digital computer. However, as in any numerical technique, the method can fail when applied to certain problems. In some cases, the failure is due to the singularity of the matrix which must be inverted in Equation (3.38). This matrix should be nonsingular on each iteration in the iteration sequence. If the matrix becomes singular, the problem is usually labeled as not correctly formulated. What constitutes an ill-behaved or incorrectly formulated two-point boundary value problem and how it relates to a failure of the Method of Adjoint Systems is not clear. Clarification of what constitutes an incorrectly formulated problem will be explored in the following sections.

Conditions for a Well Posed Variational Boundary Value Problem

In any boundary value problem, the boundary conditions imposed upon the system must be attainable by the system. In some boundary value problems, the desired boundary conditions may arise by choice or by some mission specifications which are desired. However, in the case of the variational two-point boundary value problem, part of the boundary conditions are specified at the initial and final time on the state variables and the remaining ones arise from the transversality conditions.

The transversality conditions, as shown in Chapter II, yield constraints on the initial and terminal values of the Lagrange multipliers. The constraints on the Lagrange multipliers are a direct result of the performance index chosen and the initial and terminal state constraints. Thus the choice of the performance index, initial state constraints, and terminal state constraints will determine whether or not the problem is well posed or correctly formulated. State constraints which are unattainable by the system or which when considered with the performance index result in constraints on the Lagrange multipliers which are unattainable, are usually labeled illegitimate constraints. A well posed boundary value problem must have legitimate constraints which are compatable with the performance index.

Investigation Procedure

In order to investigate the effects of incorrect problem formulation on the Method of Adjoint Systems, the following procedure will be used.

(1) A particular dynamical system will be chosen and n initial state constraints and the initial and final time will be specified.

(2) The first necessary conditions will be applied to the system in order to obtain the boundary value problem, but the performance index and terminal state constraints will be left in a general unspecified form.

(3) Various legitimate and illegitimate performance indices and terminal state constraints will be imposed on the system, and the boundary value problem will be solved analytically.

Effects of the choice of the performance index and terminal constraints on the analytical solution will be noted in each case and will be summarized in tabular form.

(4) Using the results of the analytical solution for each case as the true solution or converged solution, the Method of Adjoint Systems will be applied to each case, and the results presented in tabular form.

(5) Using the analytical solution as a basis for comparison, the effects of correct and/or incorrect problem formulation on the Method of Adjoint Systems will be discussed.

<u>Specific Dynamical System to be Analyzed</u>

In order to investigate areas of breakdown in the computational algorithm of the Method of Adjoint Systems, the following system will be analyzed.

$$\begin{aligned} \dot{x}_1(t) &= u^2(t) \\ \dot{x}_2(t) &= x_1(t) + u(t) \\ \dot{x}_3(t) &= C \end{aligned} \tag{4.1}$$

The system was chosen for the following reasons:

(1) Analytic solutions for the system are attainable.

(2) The system is such that it is easy to set up optimization problems which are clearly incorrectly formulated.

(3) The third state variable is uncontrollable, i.e., it is not linked to the other state variables or the control. The presence of this variable in the system should in some cases cause the optimization technique to fail (if, for example, such a variable is constrained).

The state variables are x(t), the control variable is u(t), and C is a constant. It will be noticed that the state variables $x_1(t)$ and $x_2(t)$ are linked together in the system through the control variable. It is required to minimize the terminal value of the scalar performance index, $G(x_f,t_f)$, subject to the terminal constraints, $M(x_f,t_f)$. It will be assumed that the initial and final time are known i.e., (t_0 = 0 sec., t_f = 1 sec.). It will also be assumed that the initial values of the state variables are given by

$$\begin{aligned} x_1(t_0) &= 0 \\ x_2(t_0) &= 0 \\ x_3(t_0) &= 0 \end{aligned} \tag{4.2}$$

Following the procedure in Chapter II, the state variables and terminal constraints may be adjoined to the performance index to yield the following functional quantity.

$$I = G(x_f,t_f) + \nu^T M(x_f,t_f) + \int_0^1 [\lambda_1 \ \lambda_2 \ \lambda_3] \begin{bmatrix} u^2 - \dot{x}_1 \\ x_1 + u - \dot{x}_2 \\ C - \dot{x}_3 \end{bmatrix} dt \tag{4.3}$$

Let P and F be defined as

$$P = G(x_f,t_f) + \nu^T M(x_f,t_f) \tag{4.4}$$

$$F = \lambda_1 u^2 + \lambda_2(x_1 + u) + \lambda_3 C - (\lambda_1\dot{x}_1 + \lambda_2\dot{x}_2 + \lambda_3\dot{x}_3) \tag{4.5}$$

The variational Hamiltonian, H, for the system is given by

$$H = F + \lambda^T \dot{x} \tag{4.6}$$

thus

$$H = \lambda_1 u^2 + \lambda_2(x_1 + u) + \lambda_3 C \tag{4.7}$$

In order that I be a weak relative minimum, the following conditions must be met at each point along the trajectory.

The Necessary Conditions are:

1 The Equations of State $\dot{x} = H_\lambda^T$ (4.8)

2 The Euler-Lagrange Equations $\dot{\lambda} = -H_x^T$ (4.9)

3 The Optimality Condition $H_u = 0$ (4.10)

4 The Legendre-Clebesch Condition $H_{uu} > 0$ (positive definite) (4.11)

In this particular problem in which n state constraints are specified initially and the initial and terminal time are known, the following conditions must be satisfied at the terminal time.

The Boundary Conditions are:

5 The Terminal Constraints $M(x_f, t_f) = 0$ (4.12)

6 The Transversality Conditions $(P_x - \lambda^T)_{t_f}\, dx(t_f) = 0$ (4.13)

Equation (4.9) yields the following Euler-Lagrange equations.

$$\begin{aligned} \dot{\lambda}_1(t) &= -\lambda_2(t) \\ \dot{\lambda}_2(t) &= 0 \\ \dot{\lambda}_3(t) &= 0 \end{aligned} \tag{4.14}$$

The initial values of the Lagrange multipliers are unknown. Let their values be given by the following relations.

$$\begin{aligned} \lambda_1(t_0) &= a \\ \lambda_2(t_0) &= b \\ \lambda_3(t_0) &= c \end{aligned} \tag{4.15}$$

where a, b, and c are arbitrary constants. These constants will be determined later. With the initial values thus defined, the Euler-Lagrange equations (Equation (4.14)), may be integrated from t_0 to t to yield

$$\begin{aligned} \lambda_1(t) &= a - bt \\ \lambda_2(t) &= b \\ \lambda_3(t) &= c \end{aligned} \tag{4.16}$$

The Optimality Condition (Equation (4.10)) yields the control variable as a function of the Lagrange multipliers given as

$$u(t) = -\frac{1}{2} \frac{\lambda_2(t)}{\lambda_1(t)} \tag{4.17}$$

Substituting the values for $\lambda(t)$ from Equation (4.16) into Equation (4.17), the control variable may be written as

$$u(t) = -\frac{1}{2} \frac{b}{a - bt} \tag{4.18}$$

The Legendre-Clebesch condition (Equation (4.11)) requires that for a weak minimum

$$H_{uu} = 2\lambda_1(t) = 2(a - bt) > 0 \tag{4.19}$$

Equation (4.19) imposes the following conditions on the initial values of the Lagrange multipliers a and b.

$$a - bt > 0 \tag{4.20}$$

At the initial time, $t = 0$

$$a > 0 \tag{4.21}$$

At the final time, $t = 1$ sec.,

$$a > b \tag{4.22}$$

The optimal control may be substituted into Equation (4.1) to yield the state variables as functions of time. Assuming that b is not equal to zero, integration of the state equations yields

$$\begin{aligned} x_1(t) &= \frac{b}{4}\left[\frac{1}{a-bt} - \frac{1}{a}\right] \\ x_2(t) &= \frac{1}{4}\ \ell n\ (1 - \frac{b}{a}t) - \frac{1}{4}\frac{b}{a}t \\ x_3(t) &= Ct \end{aligned} \tag{4.23}$$

If b is equal to zero, integration of the state equations yields

$$\begin{aligned} x_1(t) &= 0 \\ x_2(t) &= 0 \\ x_3(t) &= Ct \end{aligned} \tag{4.24}$$

The boundary value problem formed by the equations of state and the Euler-Lagrange equations with boundary conditions given by the initial values of the state variables (Equation (4.2)), the terminal values of the state variables (Equation (4.12)), and the terminal values of the

Lagrange multipliers (Equation (4.13)), may be solved after specification of the performance index and terminal state constraints. In considering the following examples of correct and incorrect problem formulation which occur for various performance indices and terminal constraints, it should be pointed out that while some of the conditions imposed seem absurd in this particular problem, in a more complex problem, such conditions might be unintentionally imposed. The conditions imposed in the following cases were purposely chosen to illustrate the effects of incorrect problem formulation and should be considered in this respect.

Analytical Solutions of the Two-Point Boundary Value Problem

Case 1

$$G(x_f,t_f) = x_1(t_f) \tag{4.25}$$

$$M(x_f,t_f) = x_1(t_f) - K = 0 \qquad K = \text{constant} \tag{4.26}$$

In this case, the performance index is terminally constrained, which is equivalent to trying to minimize a constant. The scalar P may be formed as

$$P = x_1(t_f)(1 + \nu) - \nu K \tag{4.27}$$

The transversality conditions (Equation (4.13)) yields

$$[(1+\nu-\lambda_1)(0-\lambda_2)(0-\lambda_3)]_{t_f} \begin{bmatrix} 0 \\ dx_{2f} \\ dx_{3f} \end{bmatrix} = 0 \tag{4.28}$$

For arbitrary values of x_{2f} and x_{3f}, Equations (4.26) and (4.28) yield the following terminal constraints

$$\begin{aligned} x_1(t_f) &= K \\ \lambda_2(t_f) &= 0 \\ \lambda_3(t_f) &= 0 \end{aligned} \tag{4.29}$$

From Equations (4.16) and (4.29), two of the initial values of the Lagrange multipliers are found to be

$$b = 0 \tag{4.30}$$

$$c = 0 \tag{4.31}$$

Because b is equal to zero, the integrated state equations (Equation (4.24)) must be used. The equation for $x_1(t)$ evaluated at t_f, in view of Equation (4.29), reveals the requirement that the terminal constraint K be equal to zero. Thus, the terminal state constraints may not be chosen freely in this case, but must be chosen such that the equations of state at the final time are satisfied. The problem has terminal constraints which are unattainable by the system unless Equation (4.26) is $M(x_f,t_f) = x_1(t_f) - 0 = 0$. The transversality conditions yield no conditions with which a may be evaluated. For b equal to zero, the Legendre-Clebesch condition simply requires that a be greater than zero. Thus, it appears that it is possible to satisfy the necessary conditions for a minimum performance index for certain choices of $x_1(t_f)$ namely $K = 0$. But the performance index is a constant and a constant can not be minimized. The problem is actually not an optimization problem, but simply a boundary value problem. If $x_1(t_f)$ equals zero, there is no performance index in this case. It can be shown that by considering the equation of state, $\dot{x}_1 = u^2(t)$, the initial constraints, $x_1(t_0) = 0$, and the terminal constraints, $x_1(t_f) = 0$, that the problem is not an optimal control problem. Integration of the state equation from t_0 to t_f yields

$$x_1(t_f) - x_2(t_f) = 0 = \int_{t_0}^{t_f} u^2(t)\, dt$$

It follows that

$$u(t) = 0 \qquad \text{(the trivial solution)}$$

Thus only one control exists which satisfies the initial and terminal constraints and the concept of a maximum or minimum is meaningless.

Case 2

$$G(x_f, t_f) = x_1(t_f) \tag{4.32}$$

$$M(x_f, t_f) = x_2(t_f) - K = 0 \tag{4.33}$$

In this case the performance index and terminal constraints appear to be legitimate. The scalar P may be formed as

$$P = x_1(t_f) + \nu(x_2(t_f) - K) \tag{4.34}$$

The transversality condition requires that

$$[(1-\lambda_1)\ \ (\nu-\lambda_2)\ \ (0-\lambda_3)]_{t_f} \begin{bmatrix} dx_{1f} \\ 0 \\ dx_{3f} \end{bmatrix} = 0 \tag{4.35}$$

For arbitrary values of x_{1f} and x_{2f}, Equations (4.33) and (4.35) yield

$$\begin{aligned} x_2(t_f) &= K \\ \lambda_1(t_f) &= 1 \\ \lambda_3(t_f) &= 0 \end{aligned} \tag{4.36}$$

From Equations (4.16) and (4.36) the following relations are found to exist between the initial values of the Lagrange multipliers.

$$b = a - 1 \tag{4.37}$$

$$c = 0 \tag{4.38}$$

From Equation (4.33) and the integrated state equation for $x_2(t)$ (Equation (4.23)) the following relationship occurs between the initial values of the Lagrange multipliers and the terminal value of $x_2(t_f)$.

$$K = \frac{1}{4}\,\ell n\left(1 - \frac{b}{a}\right) - \frac{1}{4}\,\frac{b}{a} \tag{4.39}$$

Substitution of Equation (4.37) into Equation (4.39) yields

$$4K + 1 = \ell n\left(\frac{1}{a}\right) + \frac{1}{a} \tag{4.40}$$

The Legendre-Clebesch condition requires that

$$\lambda_1(t) = a - a(a - 1)t > 0, \quad a > 0 \quad \text{at } t_0 \tag{4.41}$$

From Equation (4.40), a may be solved for various values of K. The following is a short table of values for a corresponding to specific values of K.

K	a	b
∞	0	-1
.423	$\frac{1}{2}$	$-\frac{1}{2}$
0	1	0
-.298	2	1
-.444	3	2

(4.42)

The constraints and the performance index in this case yield an optimal control given as

$$u(t) = -\frac{1}{2} \frac{a - 1}{a-(a - 1) t} \tag{4.43}$$

The performance index and terminal constraints stipulated in this case are noted to be legitimate and attainable. It is interesting to note the dependence of a on b given by Equation (4.37). If a = 1 then b = 0 and the trivial solution is obtained.

Case 3

$$G(x_f,t_f) = x_1(t_f) \tag{4.44}$$

$$M(x_f,t_f) = x_3(t_f) - K = 0 \tag{4.45}$$

In this case the variable being terminally constrained is uncontrollable in the system and P may be formed as

$$P = x_1(t_f) + \nu(x_3(t_f) - K) \tag{4.46}$$

The transversality conditions require that

$$\left[(1-\lambda_1) \quad (0-\lambda_2) \quad (\nu-\lambda_3) \right]_{t_f} \begin{bmatrix} dx_{1f} \\ dx_{2f} \\ 0 \end{bmatrix} = 0 \tag{4.47}$$

For arbitrary values of x_{1f} and x_{2f}, Equations (4.45) and (4.47) yield

$$\begin{aligned} x_3(t_f) &= K \\ \lambda_1(t_f) &= 1 \\ \lambda_2(t_f) &= 0 \end{aligned} \tag{4.48}$$

From Equations (4.16) and (4.48), two of the initial values of Lagrange multipliers are found to be

$$b = 0 \tag{4.49}$$

$$a = 1 \tag{4.50}$$

From the integrated state equation (Equation (4.24)) for $x_3(t)$, it is seen that

$$c = K \tag{4.51}$$

The Legendre-Clebesch condition in view of Equations (4.49) and (4.50) is satisfied, i.e.,

$$\lambda_1(t) = a - bt > 0, \qquad a > 0 \tag{4.52}$$

Case 4

$$G(x_f,t_f) = x_2(t_f) \tag{4.53}$$

$$M(x_f,t_f) = x_1(t_f) - K = 0 \tag{4.54}$$

In this case the performance index and terminal constraints appear to be legitimate and P may be formed as

$$P = x_2(t_f) + \nu(x_1(t_f) - K) \tag{4.55}$$

The transversality condition requires that

$$[(\nu-\lambda_1)(1-\lambda_2)(0-\lambda_3)]_{t_f} \begin{bmatrix} 0 \\ dx_{2f} \\ dx_{3f} \end{bmatrix} = 0 \qquad (4.56)$$

For arbitrary values of x_{2f} and x_{3f}, Equations (4.54) and (4.56) yield

$$\begin{aligned} x_1(t_f) &= K \\ \lambda_2(t_f) &= 1 \\ \lambda_3(t_f) &= 0 \end{aligned} \qquad (4.57)$$

From Equations (4.16) and (4.57), two of the initial values of Lagrange multipliers are found to be

$$b = 1 \qquad (4.58)$$

$$c = 0 \qquad (4.59)$$

From the integrated state equation (Equation (4.23)) for $x_1(t)$, and from Equation (4.54), it is seen that at the final time

$$K = \frac{1}{4} \quad \frac{1}{a - 1} \quad - \frac{1}{a} \qquad (4.60)$$

Equation (4.60) yields

$$a^2 - a - \frac{1}{4K} = 0 \qquad (4.61)$$

This is a quadratic equation in a , and the value of a depends upon the constraint K. Equation (4.61) may be solved to yield

$$a = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + (\frac{1}{4K})^2} \qquad (4.62)$$

Equation (4.62) yields two solutions for a. In view of Equation (4.58), the Legendre-Clebesch condition requires that at the initial time $a > 0$ and at the final time $a > 1$. The correct solution to Equation (4.62) is the one which satisfies these conditions. It should be noted that the terminal constraint K can not be equal to zero. For Case 4, the optimal control is given by

$$u(t) = -\frac{1}{2}\,\frac{1}{a - t} \tag{4.63}$$

where a is the solution of Equation (4.62).

Case 5

$$G(x_f,t_f) = x_2(t_f) \tag{4.64}$$

$$M(x_f,t_f) = x_2(t_f) - K = 0 \tag{4.65}$$

In this case it is noted that the performance index is being terminally constrained and P may be formed as

$$P = x_2(t_f) + \nu(x_2(t_f) - K) \tag{4.66}$$

Application of the transversality equations yield

$$[(0-\lambda_1)(\nu-\lambda_2)(0-\lambda_3)]_{t_f} \begin{bmatrix} dx_{1f} \\ 0 \\ dx_{3f} \end{bmatrix} = 0 \tag{4.67}$$

For arbitrary values of x_{1f} and x_{3f}, Equations (4.65) and (4.67) require that

$$\begin{aligned} x_2(t_f) &= K \\ \lambda_1(t_f) &= 0 \\ \lambda_3(t_f) &= 0 \end{aligned} \tag{4.68}$$

From Equations (4.16) and (4.68), it is seen that

$$a - b = 0 \tag{4.69}$$

$$c = 0 \tag{4.70}$$

Thus Case 5 requires that

$$a = b \tag{4.71}$$

There are no conditions available with which a and b may be evaluated. The Legendre-Clebesch condition in view of Equation (4.71) requires that

$$a(1 - t) > 0 \tag{4.72}$$

In the problem under consideration $t_f = 1$ sec.. Thus it is seen that the Legendre-Clebesch condition is violated at the final time. Assuming that $a \neq 0$, the control for this case becomes

$$u = -\frac{1}{2}\frac{b}{a - bt} = -\frac{1}{2}\frac{1}{1 - t} \tag{4.73}$$

At the final time, division by zero occurs and the control becomes infinite. From the integrated state equations (Equation (4.23)) for $x_2(t)$, and the constraint equation (Equation (4.65)), it is seen that at the final time

$$K = \frac{1}{4}\ln\left(1 - \frac{b}{a}\right) - \frac{1}{4}\frac{b}{a} \tag{4.74}$$

If $a = b$, Equation (4.74) requires that $K = -\infty$. Even if the terminal constraint was so chosen, the violation of the necessary Legendre-Clebesch condition at the final time yields a problem which is incorrectly formulated.

Case 6

$$G(x_f, t_f) = x_2(t_f) \tag{4.75}$$

$$M(x_f, t_f) = x_3(t_f) - K = 0 \tag{4.76}$$

It is noted that the uncontrollable state variable, $x_3(t)$, is being terminally constrained and P is formed as

$$P = x_2(t_f) + \nu(x_3(t_f) - K) \tag{4.77}$$

Application of the transversality conditions requires that

$$[(0-\lambda_1)(1-\lambda_2)(\nu-\lambda_3)]_{t_f} \begin{bmatrix} dx_{1f} \\ dx_{2f} \\ 0 \end{bmatrix} = 0 \tag{4.78}$$

For arbitrary values of x_{1f} and x_{2f}, Equations (4.76) and (4.78) yield

$$\begin{aligned} x_3(t_f) &= K \\ \lambda_1(t_f) &= 0 \\ \lambda_2(t_f) &= 1 \end{aligned} \tag{4.79}$$

From Equations (4.16) and (4.79), two of the initial values of Lagrange multipliers are found to be

$$b = 1 \tag{4.80}$$

$$a = 1 \tag{4.81}$$

The integrated state equation (Equation (4.23)) for $x_3(t)$ and the terminal constraint equation (Equation (4.76)) require that

$$c = K \tag{4.82}$$

As in Case 5, the Legendre-Clebesch condition is violated at the final time, and the control given by

$$u = -\frac{1}{2}\frac{1}{1-t} \tag{4.83}$$

becomes infinite at t_f because of division by zero.

Case 7

$$G(x_f,t_f) = x_3(t_f) \tag{4.84}$$

$$M(x_f,t_f) = x_1(t_f) - K = 0 \tag{4.85}$$

In this case, the uncontrollable state variable is the performance index. P may be formed as

$$P = x_3(t_f) + \nu(x_1(t_f) - K) \tag{4.86}$$

The transversality condition requires that at the final time

$$[(\nu-\lambda_1)(0-\lambda_2)(1-\lambda_3)]_{t_f} \begin{bmatrix} 0 \\ dx_{2f} \\ dx_{3f} \end{bmatrix} = 0 \tag{4.87}$$

For arbitrary values of x_{2f} and x_{3f}, Equation (4.87) requires that

$$\begin{aligned} x_1(t_f) &= K \\ \lambda_2(t_f) &= 0 \\ \lambda_3(t_f) &= 1 \end{aligned} \tag{4.88}$$

Equations (4.16) and (4.88) yield the following initial values of the Lagrange multipliers.

$$b = 0 \tag{4.89}$$

$$c = 1 \tag{4.90}$$

From the terminal state constraint of Equation (4.85) and the integrated state equations (Equation (4.24)) for $x_1(t)$, it is seen that

$$K = 0 \tag{4.91}$$

There are no conditions available with which a can be evaluated, however, in view of Equation (4.89), the Legendre-Clebesch condition requires that

$$a > 0 \tag{4.92}$$

As in Case 1, Case 7 is not an optimal control problem because of the fact that in view of Equation (4.91), $K = 0$, which requires that $x_{1f} = 0$. The initial value of $x_1(t_0)$ given in Equation (4.2) is $x_1(t_0) = 0$. Thus, the equation of state $\dot{x}_1 = u^2$ may be integrated from t_0 to t_f to yield

$$\int_{t_0}^{t_f} u^2(t)\, dt = 0$$

It follows that

$$u(t) = 0$$

which is the trivial solution.

Case 8

$$G(x_f,t_f) = x_3(t_f) \tag{4.93}$$

$$M(x_f,t_f) = x_2(t_f) - K = 0 \tag{4.94}$$

In this case, the performance index is an uncontrollable variable. P may be formed as

$$P = x_3(t_f) + \nu(x_2(t_f) - K) \tag{4.95}$$

The transversality condition requires that at the terminal time

$$[(0-\lambda_1)(\nu-\lambda_2)(1-\lambda_3)]_{t_f} \begin{bmatrix} dx_{1f} \\ 0 \\ dx_{3f} \end{bmatrix} = 0 \tag{4.96}$$

For arbitrary values of x_{1f} and x_{3f}, Equations (4.94) and (4.96) yield

$$\begin{aligned} x_2(t_f) - K &= 0 \\ \lambda_{1f} &= 0 \\ \lambda_{3f} &= 1 \end{aligned} \tag{4.97}$$

From Equations (4.16) and (4.97) it is seen that

$$c = 1 \tag{4.98}$$

$$a = b \tag{4.99}$$

Case 8 is exactly the same as Case 5 except for the value of c which does not effect the control.

Case 9

$$G(x_f,t_f) = x_3(t_f) \tag{4.100}$$

$$M(x_f,t_f) = x_3(t_f) - K = 0 \tag{4.101}$$

In this case the performance index is the uncontrollable variable of the system and the uncontrollable variable is terminally constrained. P may be formed as

$$P = x_3(t_f) + \nu(x_3(t_f) - K) \tag{4.102}$$

The transversality conditions require that

$$[(0-\lambda_1)(0-\lambda_2)(\nu-\lambda_3)]_{t_f} \begin{bmatrix} dx_{1f} \\ dx_{2f} \\ 0 \end{bmatrix} = 0 \tag{4.103}$$

For arbitrary values of x_{1f} and x_{2f}, Equations (4.101) and (4.103) require that

$$\begin{aligned} x_3(t_f) &= K \\ \lambda_1(t_f) &= 0 \\ \lambda_2(t_f) &= 0 \end{aligned} \tag{4.104}$$

From Equations (4.16) and (4.104)

$$a = 0 \tag{4.105}$$

$$b = 0 \tag{4.106}$$

The Legendre-Clebesch condition, which requires that $a - bt > 0$, is violated at each point in the trajectory. In view of Equations (4.105) and (4.106), the control equation yields zero divided by zero which is meaningless.

The results of the preceding nine cases are summarized in tabular form in Table 1.

CASE	PI	TC	a	b	c	Legendre-Clebesch Condition	Imposed Conditions
1	x_{1f}	$x_{1f}-K$	#	0	0	Not violated if a>0.	Performance Index is terminally constrained.
2	x_{1f}	$x_{2f}-K$	*	a-1	0	Not violated. a chosen such that a>0 from *.	Legitimate Performance Index and Terminal Constraints.
3	x_{1f}	$x_{3f}-K$	1	0	K	Not violated. a=1.	Uncontrollable State Variable terminally constrained.
4	x_{2f}	$x_{1f}-K$	**	1	0	Not violated. a chosen such that a>0 from **.	Legitimate Performance Index and Terminal Constraints
5	x_{2f}	$x_{2f}-K$	b	a	0	Violated at the final time. a=b.	Performance Index is terminally constrained.
6	x_{2f}	$x_{3f}-K$	1	1	K	Violated at the final time. a=b=1.	Uncontrollable State Variables terminally constrained.
7	x_{3f}	$x_{1f}-K$	#	0	1	Not violated if a>0.	Uncontrollable Variable is the Performance Index.
8	x_{3f}	$x_{2f}-K$	b	a	1	Violated at the final time. a=b.	Uncontrollable Variable is the Performance Index.
9	x_{3f}	$x_{3f}-K$	0	0	K	Violated at every point in time. a=b=0	Uncontrollable Variable is the Performance Index and is terminally constrained.

TABLE 1

Summary of Analytic Solutions of Cases 1-9

#No conditions available to evaluate a.

$*4K + 1 = \ell n(\frac{1}{a}) + (\frac{1}{a})$

$**a = \frac{1}{2} \pm \frac{1}{2}\ 1 + (\frac{1}{4K})^2$

Solution by the Method of Adjoint Systems

The solution of the two-point boundary value problem by the Method of Adjoint Systems will be considered for the nine cases of performance indices and terminal constraints considered in the preceding section. The improved guess of the initial values of the Lagrange multipliers is found by adding a $\delta\lambda(t_0)_i$ to the guessed value of $\lambda(t_0)_i$. For the problem under consideration, the $\delta\lambda(t_0)$ are found from Equation (3.47) in which the final time is known and n initial state constraints are specified. The relation is given by

$$\delta\lambda(t_0) = \theta_\lambda^T(t_0)^{-1}\ [dh^*(t_f)] \tag{4.107}$$

Let the guessed values of the $\lambda(t_0)$ be denoted by a^*, b^*, and c^*. Thus Equation (4.107) becomes

$$\begin{bmatrix} \delta a \\ \delta b \\ \delta c \end{bmatrix} = \theta^T(t_0)^{-1}[-k\, h^*(t_f)\ ] \tag{4.108}$$

It was shown in Chapter III that $\theta(t)$ was the solution of the $2n$xn matrix equation (Equation (3.33)) which is

$$\dot{\theta}(t) = -A^T(t)\ \theta(t), \qquad \theta(t_f) = \left[\frac{\partial h}{\partial z}\right]_{t_f}^T \tag{4.109}$$

In the problem under consideration $n = 3$, thus $\theta(t)$ is a 6x3 matrix, $-A^T(t)$ is a 6x6 matrix, and h is the terminal constraints specified on the state variables and found from the transversality conditions on the Lagrange multipliers.

In the particular problem under consideration, the $-A^T(t)$ matrix is given by

$$-A^T(t) = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{1}{2}\frac{z_5^2}{z_4^3} & -\frac{1}{2}\frac{z_5}{z_4^2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\frac{z_5}{z_4^2} & \frac{1}{2}\frac{1}{z_4} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{4.110}$$

where $z_4(t) = a - bt$

$z_5(t) = b$

$z_6(t) = c$

Equation (4.109) becomes

$$\begin{bmatrix} \dot{\theta}_{11} & \dot{\theta}_{12} & \dot{\theta}_{13} \\ \vdots & \vdots & \vdots \\ \dot{\theta}_{61} & \dot{\theta}_{62} & \dot{\theta}_{63} \end{bmatrix} = -A^T(t) \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \vdots & \vdots & \vdots \\ \theta_{61} & \theta_{62} & \theta_{63} \end{bmatrix}, \tag{4.111}$$

$$\theta(t_f) = \begin{bmatrix} \frac{\partial h_1}{\partial z_1} & \frac{\partial h_2}{\partial z_1} & \frac{\partial h_3}{\partial z_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_1}{\partial z_6} & \frac{\partial h_2}{\partial z_6} & \frac{\partial h_3}{\partial z_6} \end{bmatrix}$$

Equation (4.111) yields the following eighteen first order nonlinear ordinary differential equations.

$$\dot{\theta}_{11}(t) = -\theta_{21}(t)$$
$$\dot{\theta}_{12}(t) = -\theta_{22}(t)$$
$$\dot{\theta}_{13} = -\theta_{23}(t)$$

$$\dot{\theta}_{21}(t) = 0$$
$$\dot{\theta}_{22}(t) = 0$$
$$\dot{\theta}_{23}(t) = 0$$

$$\dot{\theta}_{31}(t) = 0$$
$$\dot{\theta}_{32}(t) = 0$$
$$\dot{\theta}_{33}(t) = 0$$

$$\dot{\theta}_{41}(t) = \frac{1}{2}\frac{b^2}{(a-bt)^3}\theta_{11}(t) - \frac{1}{2}\frac{b}{(a-bt)^2}\theta_{21}(t)$$
$$\dot{\theta}_{42}(t) = \frac{1}{2}\frac{b^2}{(a-bt)^3}\theta_{12}(t) - \frac{1}{2}\frac{b}{(a-bt)^2}\theta_{22}(t)$$
$$\dot{\theta}_{43}(t) = \frac{1}{2}\frac{b^2}{(a-bt)^3}\theta_{13}(t) - \frac{1}{2}\frac{b}{(a-bt)^2}\theta_{32}(t)$$

$$\dot{\theta}_{51} = -\frac{1}{2}\frac{b}{(a-bt)^2}\theta_{11}(t) + \frac{1}{2}\frac{1}{a-bt}\theta_{21}(t) + \theta_{41}(t)$$
$$\dot{\theta}_{52} = -\frac{1}{2}\frac{b}{(a-bt)^2}\theta_{12}(t) + \frac{1}{2}\frac{1}{a-bt}\theta_{22}(t) + \theta_{42}(t)$$
$$\dot{\theta}_{53} = -\frac{1}{2}\frac{b}{(a-bt)^2}\theta_{13}(t) + \frac{1}{2}\frac{1}{a-bt}\theta_{32}(t) + \theta_{43}(t)$$
$$\dot{\theta}_{61}(t) = 0$$
$$\dot{\theta}_{62}(t) = 0$$
$$\dot{\theta}_{63}(t) = 0$$

These eighteen differential equations are numerically integrated from $t_f = 1$ sec. to $t_0 = 0$ sec. in the Method of Adjoint Systems to form the $\theta^T(t_0) = [\theta_\lambda^T(t_0) \vdots \theta_x^T(t_0)]$ required in Equation (4.107) to obtain a better guess of a^*, b^*, and c^*.

The integration will be performed analytically in this section. It will be assumed that the values of a^*, b^*, and c^* have been guessed sufficiently close to the true values and have, after a number of iterations, converged to the true values of a, b, and c given in Table 1. It will be assumed that the values of a, b, and c obtained by the analytic solution to the two-point boundary value problem are the true values.

Case 1

$$b = 0 \qquad \text{From Equation (4.30)}$$
$$c = 0 \qquad \text{From Equation (4.31)} \qquad (4.112)$$
$$a > 0 \qquad \text{From Legendre's Condition}$$

From Equation (4.29)

$$\begin{aligned} h_1 &= x_1(t_f) - K = z_{1f} - K = 0 \\ h_2 &= \lambda_2(t_f) - 0 = z_{5f} - 0 = 0 \\ h_3 &= \lambda_3(t_f) - 0 = z_{6f} - 0 = 0 \end{aligned} \qquad (4.113)$$

The partial derivatives of h may be formed to yield $\theta(t_f)$ as

$$\theta(t_f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4.114)$$

The quantity $\theta(t_f)$ is used as the boundary condition with which the eighteen differential equations (Equation (4.111)) are integrated from t_f to t_0. The values of a, b, and c from Equation (4.112) are used as converged values of the Lagrange multipliers. The integration of the differential equations yields the following value of $\theta(t_0)$.

$$\theta(t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4.115)$$

It is noted that $\theta(t_f)$ is the same as $\theta(t_0)$ due to the fact that $b = 0$. Thus, it is seen that

$$\theta_\lambda^T(t_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.116}$$

The converged value of $\theta_\lambda^T(t_0)$ is singular and can not be inverted in Equation (4.108). Thus,the Method of Adjoint Systems fails for Case 1 in which the performance index is terminally constrained.

Case 2

$$\begin{aligned} &b = a - 1 && \text{From Equation (4.37)} \\ &c = 0 && \text{From Equation (4.38)} \\ &4K + 1 = \ell n\left(\frac{1}{a}\right) + \frac{1}{a} && \text{Equation (4.40)} \end{aligned} \tag{4.117}$$

From Equation (4.36), h may be formed as

$$\begin{aligned} h_1 &= x_{2f} - K = z_{2f} - K = 0 \\ h_2 &= \lambda_{1f} - 1 = z_{4f} - 1 = 0 \\ h_3 &= \lambda_{3f} - 0 = z_{6f} - 0 = 0 \end{aligned} \tag{4.118}$$

From Equation (4.118), $\theta(t_f)$ may be shown to be

$$\theta(t_f) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4.119}$$

Integration of Equation (4.111) yields

$$\theta(t_0) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_1 & 1 & 0 \\ a_2 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4.120}$$

where a_1 and a_2 are given as

$$a_1 = \frac{1}{4a^2} + \frac{1}{2ab} - \frac{1}{4ab} - \frac{1}{2a} - \frac{1}{(a-b)^2} - \frac{1}{2b(a-b)}$$

$$+ \frac{1}{4ab} + \frac{1}{2(a-b)}$$

$$a_2 = \frac{1}{a} \quad - \frac{1}{2} + \frac{a}{2b} + \frac{1}{4b} - \frac{a}{4} \quad + \ell n(a) \quad \frac{1}{2b} - \frac{1}{2b^2} \quad +$$

$$- \frac{1}{a-b} \quad - \frac{1}{2} + \frac{1}{2b} + \frac{1}{4b} - \frac{a}{4} \quad + \ell n(a-b) \quad \frac{1}{2b} - \frac{1}{2b^2} \quad + \ldots$$

$\theta_\lambda^T(t_0)$ may be formed as

$$\theta_\lambda^T(t_0) = \begin{bmatrix} a_1 & a_2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.121}$$

$\theta_\lambda^T(t_0)$ is nonsingular and may be inverted, however, if $a = 1$ so that $b = 0$, then $\theta_\lambda^T(t_0)$ becomes singular and the Method of Adjoint Systems fails.

Case 3

$$
\begin{aligned}
b &= 0 && \text{From Equation (4.49)} \\
a &= 1 && \text{From Equation (4.50)} \\
c &= K && \text{From Equation (4.51)}
\end{aligned}
\tag{4.122}
$$

From Equation (4.48) h becomes

$$
\begin{aligned}
h_1 &= x_{3f} - K = z_{3f} - K = 0 \\
h_2 &= \lambda_{1f} - 1 = z_{4f} - 1 = 0 \\
h_3 &= \lambda_{2f} - 0 = z_{5f} - 0 = 0
\end{aligned}
\tag{4.123}
$$

From Equation (4.123),

$$
\theta(t_f) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \tag{4.124}
$$

and integration of the Equation (4.111) yields

$$
\theta(t_0) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \tag{4.125}
$$

$\theta_\lambda^T(t_0)$ is seen to be

$$\theta_\lambda^T(t_0) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.126}$$

Due to the fact that $b = 0$, $\theta_\lambda^T(t_0)$ is singular and the Method of Adjoint Systems fails for Case 3 in which the uncontrollable variable is constrained.

Case 4

$$\begin{aligned} b &= 1 && \text{From Equation (4.58)} \\ c &= 0 && \text{From Equation (4.59)} \\ a &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1+\left(\frac{1}{4K}\right)^2} && \text{From Equation (4.62)} \end{aligned} \tag{4.127}$$

h , given by Equation (4.57), is

$$\begin{aligned} h_1 &= x_{1f} - K = z_{1f} - K = 0 \\ h_2 &= \lambda_{2f} - 1 = z_{5f} - 1 = 0 \\ h_3 &= \lambda_{3f} - 0 = z_{6f} - 0 = 0 \end{aligned} \tag{4.128}$$

The matrix of partial derivatives, $\theta(t_f)$, is formed as

$$\theta(t_f) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4.129}$$

and integration of Equation (4.111) yields that the initial time

$$\theta(t_0) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline \frac{b}{4}\left[\frac{1}{a^2} - \frac{1}{(a-b)^2}\right] & 0 & 0 \\ -\frac{1}{4a} + \frac{1}{4(a-b)} + \frac{b}{4(a-b)^2} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4.130}$$

From Equation (4.130) $\theta_\lambda^T(t_0)$ may be formed as

$$\theta_\lambda^T(t_0) = \begin{bmatrix} \frac{b}{4}\left(\frac{1}{a^2} - \frac{1}{(a-b)^2}\right) & \left(-\frac{1}{4a} + \frac{1}{4(a-b)} + \frac{b}{4(a-b)^2}\right) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.131}$$

$\theta_\lambda^T(t_0)$ is not singular due to the fact that $b = 1$ and the a, given by Equation (4.62), satisfies Legendre's condition. Thus, $\theta_\lambda^T(t_0)$ may be inverted and the Method of Adjoint Systems does not fail for Case 4.

Case 5

$$a = b \qquad \text{From Equation (4.69)}$$
$$c = 0 \qquad \text{From Equation (4.70)} \tag{4.132}$$

h may be formed as

$$\begin{aligned} h_1 &= x_{2f} - K = z_{2f} - K = 0 \\ h_2 &= \lambda_{1f} - 0 = z_{4f} - 0 = 0 \\ h_3 &= \lambda_{3f} - 0 = z_{6f} - 0 = 0 \end{aligned} \tag{4.133}$$

and $\theta(t_f)$ is given by

$$\theta(t_f) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.134}$$

The integration of Equation (4.111) yields a $\theta(t_0)$ matrix which because of the fact that $a = b$, will have elements a_1 and a_2 which approach infinity at the converged values of a and b. The matrix is found to be

$$\theta(t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_1 & 1 & 0 \\ a_2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.135}$$

From Equation (4.135) $\theta_\lambda^T(t_0)$ may be formed as

$$\theta_\lambda^T(t_0) = \begin{bmatrix} a_1 & a_2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.136}$$

Because of the fact that elements a_1 and a_2 approach infinity, the $\theta_\lambda^T(t_0)$ matrix is ill-behaved and the Method of Adjoint Systems will fail at the converged values of a and b.

Case 6

$$\begin{aligned} b &= 1 && \text{From Equation (4.80)} \\ a &= 1 && \text{From Equation (4.81)} \\ c &= K && \text{From Equation (4.82)} \end{aligned} \tag{4.137}$$

h is given by

$$\begin{aligned} h_1 &= x_{3f} - K = z_{3f} - K = 0 \\ h_2 &= \lambda_{1f} - 0 = z_{4f} - 0 = 0 \\ h_3 &= \lambda_{2f} - 1 = z_{5f} - 1 = 0 \end{aligned} \tag{4.138}$$

The $\theta(t_f)$ matrix may be formed as

$$\theta(t_f) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hdashline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \tag{4.139}$$

Integration of Equation (4.111) yields a $\theta(t_0)$ matrix given as

$$\theta(t_0) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \tag{4.140}$$

The $\theta_\lambda^T(t_0)$ matrix is seen to be

$$\theta_\lambda^T(t_0) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.141}$$

The matrix is singular and the Method of Adjoint Systems fails.

<u>Case 7</u>

$$\begin{array}{ll} b = 0 & \text{From Equation (4.89)} \\ c = 1 & \text{From Equation (4.90)} \\ a > 0 & \text{From Equation (4.92)} \end{array} \tag{4.142}$$

From Equation (4.88),

$$\begin{aligned} h_1 &= x_{1f} - K = z_{1f} - K = 0 \\ h_2 &= \lambda_{2f} - 0 = z_{5f} - 0 = 0 \\ h_3 &= \lambda_{3f} - 1 = z_{6f} - 1 = 0 \end{aligned} \tag{4.143}$$

The $\theta(t_f)$ matrix may be formed as

$$\theta(t_f) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{4.144}$$

Due to the fact that b = 0, $\theta(t_0) = \theta(t_f)$. Thus, $\theta_\lambda^T(t_0)$ is given by

$$\theta_\lambda^T(t_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.145}$$

The Method of Adjoint Systems fails in this case because $\theta_\lambda^T(t_0)$ is a singular matrix.

Case 8

$$\begin{aligned} a &= b && \text{From Equation (4.98)} \\ c &= 1 && \text{From Equation (4.99)} \end{aligned} \tag{4.146}$$

From Equation (4.97) it is seen that h is given by

$$\begin{aligned} h_1 &= x_{2f} - K = z_{2f} - K = 0 \\ h_2 &= \lambda_{1f} - 0 = z_{4f} - 0 = 0 \\ h_3 &= \lambda_{3f} - 1 = z_{6f} - 1 = 0 \end{aligned} \tag{4.147}$$

The $\theta(t_f)$ matrix may be formed as

$$\theta(t_f) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4.148}$$

Integration of Equation (4.111) yields

$$\theta(t_0) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_1 & 1 & 0 \\ a_2 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4.149}$$

This case is exactly the same as Case 5 and the $\theta_\lambda^T(t_0)$ matrix will have elements a_1 and a_2 which approach infinity at the converged values of a and b. Thus the Method of Adjoint Systems fails because the matrix contains unbounded elements.

<u>Case 9</u>

$$\begin{array}{ll} a = 0 & \text{From Equation (4.105)} \\ b = 0 & \text{From Equation (4.106)} \end{array} \tag{4.150}$$

In this case Equation (4.111) can not be integrated because the $-A^T(t)$ matrix, given by Equation (4.110), contains elements which involve zero divided by zero.

The results of the preceding nine cases are summarized in tabular form in Table 2.

CASE	a	b	c	$\theta_\lambda^T(t_0)$ Matrix	Method of Adjoint Systems
1	#	0	0	Singular because b=0.	Method fails because $\theta_\lambda^T(t_0)$ matrix cannot be inverted.
2	*	a-1	0	Nonsingular if a>1.	Method will not fail due to singularity of $\theta_\lambda^T(t_0)$ matrix.
3	1	0	K	Singular because b=0.	Method fails because $\theta_\lambda^T(t_0)$ matrix cannot be inverted.
4	**	1	0	Nonsingular.	Method will not fail due to singularity of $\theta_\lambda^T(t_0)$ matrix.
5	b	a	0	Contains infinite elements.	Method fails due to infinite elements in $\theta_\lambda^T(t_0)$ matrix.
6	1	1	K	Singular.	Method fails because $\theta_\lambda^T(t_0)$ matrix cannot be inverted.
7	#	0	1	Singular because b=0.	Method fails because $\theta_\lambda^T(t_0)$ matrix cannot be inverted.
8	b	a	1	Contains infinite elements.	Method fails due to infinite elements in $\theta_\lambda^T(t_0)$ matrix.
9	0	0	K	Cannot be formed because a=b.	Method cannot be applied because $\theta_\lambda^T(t_0)$ matrix contains elements which involve zero divided by zero.

TABLE II

Summary of Solutions by Method of Adjoint Systems

#,*,**, (See Table 1)

Discussion of Results

In considering the particular dynamical system and the nine cases of performance indices and terminal constraints, two modes of breakdown in the Method of Adjoint Systems have become apparent. The first occurred when the $\theta_\lambda^T(t_0)$ matrix was singular and the second occurred when this matrix contained elements which became infinite at the converged initial values of the Lagrange multipliers. In Table II it is observed that the $\theta_\lambda^T(t_0)$ matrix became singular in each case when $b = 0$. In the cases in which $b = 0$, an examination of the analytical solutions reveals that the performance index and terminal constraints resulted in transversality conditions which led to the trivial solution of the problem. In the Method of Adjoint Systems, the trivial solution leads to a singularity of the $\theta_\lambda^T(t_0)$ matrix. Thus the Method of Adjoint Systems fails. It should be noted that the trivial solution was obtained only in those cases in which the optimization problem was incorrectly formulated.

It was observed that the $\theta_\lambda^T(t_0)$ matrix had elements which became infinite when $a = b$, with the exception of Case 6. In the cases in which $a = b$, the analytical solutions reveal that the performance index and terminal constraints allowed transversality conditions which cause the Legendre-Clebesch condition be violated at some time in the interval $[t_0, t_f]$. In this particular example, when $a = b$ the matrix of partial derivatives, $A(t)$, had elements which became infinite.

Of the nine cases of performance indices and terminal constraints considered, only Cases 2 and 4 yielded two-point boundary value problems which could be solved by the Method of Adjoint Systems. An examination of the $\theta_\lambda^T(t_0)$ matrix for these cases reveals that unless the initial values of the Lagrange multipliers were chosen carefully, the Method of Adjoint Systems could fail to solve the boundary value problem. The effects of the initial choice of the Lagrange multipliers on the Method of Adjoint Systems will be considered in more detail in the following section.

Effects of the Initial Choice of the Lagrange Multipliers on the Method of Adjoint Systems

In developing the Method of Adjoint Systems, it was assumed that the initial values of the $z^*(t_0)$ variables could be guessed sufficiently close to the true values so that the difference between the true and nominal trajectories was small. With this assumption it was possible to consider only linear perturbations about the nominal trajectory. The subsequent development of the Method of Adjoint Systems depended upon the validity of this assumption. If this assumption is not valid, then all of the development leading to the computational algorithm is in error, and the convergence properties of the method are destroyed.

A review of the literature reveals a lack of information concerning how the initial unknown $z^*(t_0)$ are to be chosen. Apparently experience in using the method and knowledge of the problem are the best tools with which to make the choice.

In Ref. 4 it is pointed out that it is often quite difficult to get a first trial solution to the problem where the person using the method has had little previous experience. It is suggested that in some cases a gradient method may have to be used in order to obtain beginning estimates of the missing boundary conditions. In Ref. 3 it is pointed out that while gradient methods often yield control programs which bear little resemblance to the true optimal control programs, they converge nicely on the end conditions. In Ref. 10 it is stated that the Steepest-Ascent Method, after only a few iterations, provides initial values of the unknown $z^*(t_0)$ variables which are well within the convergence envelope of the Method of Adjoint Systems.

While the convergence characteristics of the Method of Adjoint Systems may be destroyed if the initial values of the unknown variables can not be guessed sufficiently close to the true values, this in itself does not constitute a breakdown. The method, while possibly unable to converge, will not fail due to division by zero or some other illegitimate operation. A person, inexperienced in the use of the Method of Adjoint Systems, may mistake the inability of the method to converge for a breakdown.

While some choices of the unknown initial values of the $z^*(t_0)$ variables may destroy the convergence properties of the Method of Adjoint Systems, other choices may actually cause a breakdown in the method when applied to a legitimate problem. For example, it was found that Case 4 was a legitimate boundary value problem and could be solved by the Method of Adjoint Systems. The $\theta_\lambda^T(t_0)$ matrix of Case 4 was found to be

$$\theta_\lambda^T(t_0) = \begin{bmatrix} \frac{b}{4}\left(\frac{1}{a^2} - \frac{1}{(a-b)^2}\right) & \left(-\frac{1}{4a} + \frac{1}{4(a-b)} + \frac{b}{4(a-b)^2}\right) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$b = 1$$

$$a = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \left(\frac{1}{4K}\right)^2}$$

If the terminal constraint, K, had been chosen as $K = 1$ then $b = 1.000$, and $a = 1.015$. Had b been chosen to be zero, the $\theta_\lambda^T(t_0)$ matrix would have been singular, or had a been chosen to equal 1.000, division by zero would have occurred. In each case, the Method of Adjoint Systems would have failed for the legitimate problem due to the choices of the initial values of the Lagrange multipliers and the failure would have resembled that for an incorrectly formulated or illegitimate problem. Had the problem been more complex, an inexperienced user of the Method of Adjoint Systems would have possibly concluded that the problem was incorrectly formulated.

In this chapter the effects of incorrect problem formulation on the Method of Adjoint Systems have been explored briefly and it has been shown that in certain cases the method will fail, however, failure in the Method of Adjoint Systems may occur due to other reasons. In Chapter V the effects of a conjugate point in the trajectory will be investigated with the purpose of determining the effects on the Method of Adjoint Systems.

CHAPTER V

THE CONJUGATE POINT: EFFECTS ON THE OPTIMAL TRAJECTORY

In the previous chapters dealing with the development and application of the Method of Adjoint Systems, it was assumed that the First Necessary Conditions for an extremum and the Legendre-Clebesch Condition for a weak relative minimum could be used to determine the trajectory which caused the performance index to be a weak minimum in the interval of interest. While these conditions are necessary, they are not sufficient.

In many problems these conditions, when used with engineering judgment and knowledge of the problem, are enough to determine the nature of the extremum. This is not always the case. An example is when the trajectory contains a conjugate point.

The classical problem illustrating the conjugate point is that of determining the shortest distance (geodesics) between points on a sphere. The classical development leads to the requirement that the second variation of the functional I (See Appendix B) be positive in order to insure that the trajectory contains no conjugate point.

In the simple problem of determining the geodesics on a sphere, it is easy to pick the minimum path without requiring that the second variation be positive. However, in complex, multi-variable trajectory optimization problems, the first necessary conditions and knowledge of the problem may not be enough to determine whether or not the trajectory leads to a minimum or whether or not a conjugate point exists in the trajectory. Very little is known about the effects of a conjugate point on the solution to the optimization problem by a numerical technique. The remainder of this chapter will be devoted to consideration of various aspects of the conjugate point condition with the purpose of establishing the effects on the Method of Adjoint Systems.

The Sufficiency Conditions for a Weak Minimum

The sufficiency conditions for the functional I to take on a weak minimum value throughout the time interval $t_0 \leq t \leq t_f$ are:

$$\text{(1)} \qquad d'I = 0 \tag{5.1}$$

The vanishing of the first variation yields the Euler-Lagrange equations and transversality conditions.

$$\text{(2)} \qquad d''I > 0 \tag{5.2}$$

The requirement that the second variation is positive.

The conditions which result are to be developed.

If (1) is satisfied, the trajectory is an extremum. If (1) and (2) are satisfied, the trajectory is a weak local minimum.

The second variation of the functional I is developed in Appendix B for the case in which all of the initial values of the state variables and the initial time are known. The second variation results in the following equation.

$$2d''I = K + \int_{t_0}^{t_f} [\delta x^T H_{xx}\, \delta x + \delta x^T H_{xu}\, \delta u + \delta u^T H_{ux}\, \delta x + \delta u^T H_{uu}\, \delta u]\, dt > 0 \tag{5.3}$$

where K is a function of the terminal conditions. In order to keep the analysis simple, only fixed final time problems will be considered. For fixed final time problems, the value of K (in this case) is given by

$K = dx^T P_{xx} dx|_{t_f}$. Thus Equation (5.3) may be expressed as

$$2d''I = dx^T P_{xx} dx|_{t_f} + \int_{t_0}^{t_f} [\delta x^T H_{xx} \delta x + \delta x^T H_{xu} \delta u + \delta u^T H_{ux} \delta x + \delta u^T H_{uu} \delta u] dt > 0 \tag{5.4}$$

In order to insure that the inequality of Equation (5.4) is satisfied for all δx and δu in $[t_0, t_f]$, it would be convenient to express the quantity under the integral in a quadratic form which contains terms involving products of δx and δu only. The reason for this will be seen later. In order to achieve the desired quadratic form, it is convenient to introduce a new variable $W(t)$ where $W(t)$ is an nxn symmetric matrix. The following equation may be written for the variable $W(t)$ (Ref. 11).

$$-\delta x^T W(t) \delta x\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \frac{d}{dt} (\delta x^T W(t) \delta x) \; dt = 0 \tag{5.5}$$

For fixed final time problems, the variation of $x(t)$ outside the integral of Equation (5.5) becomes dx (See Appendix A, Equation (A.8)). Thus, for the case under consideration i.e., fixed final time, in which all of the initial values of the state variables are known, Equation (5.5) becomes

$$-dx^T W(t) dx\Big|_{t_f} + \int_{t_0}^{t_f} \frac{d}{dt} (\delta x^T W \delta x) \; dt = 0 \tag{5.6}$$

Addition of Equations (5.4) and (5.6) yields

$$2d''I = dx^T (P_{xx} - W) dx\Big|_{t_0} + \int_{t_0}^{t_f} [\frac{d}{dt} (\delta x^T W \delta x) + \delta x^T H_{xx} \delta x + \delta x^T H_{xu} \delta u + \delta u^T H_{ux} \delta x + \delta u^T H_{uu} \delta u] \; dt \tag{5.7}$$

Considering linear perturbations in the differential equations of state, (Equation (2.24)) it is seen that

$$\delta\dot{x} = H_{x\lambda}\ \delta x + H_{u\lambda}\ \delta u \tag{5.8}$$

In examining Equation (5.6), it is noted that

$$\frac{d}{dt}(\delta x^T W\ \delta x) = \delta\dot{x}^T W\ \delta x + \delta x^T \dot{W}\ \delta x + \delta x^T W\ \delta\dot{x} \tag{5.9}$$

In view of Equation (5.8), Equation (5.9) may be written as

$$\frac{d}{dt}(\delta x^T W\ \delta x) = \delta x^T (H_{x\lambda} W)\ \delta x + \delta u^T\ (H_{u\lambda} W)\ \delta x +$$
$$+ \delta x^T (\dot{W})\ \delta x + \delta x^T (W\ H_{\lambda x})\ \delta x + \delta x^T (W\ H_{\lambda u})\ \delta u$$

or

$$\begin{aligned} \frac{d}{dt}(\delta x^T W\ \delta x) = {} & \delta x^T (\dot{W} + H_{x\lambda} W + W\ H_{\lambda x})\ \delta x + \\ & + \delta x^T (W\ H_{\lambda u})\ \delta u + \delta u^T (H_{u\lambda} W)\ \delta x \end{aligned} \tag{5.10}$$

Substitution of Equation (5.10) into Equation (5.7) yields

$$\begin{aligned} 2d''I = {} & dx^T (P_{xx} - W) dx \Big|_{t_f} + \int_{t_0}^{t_f} [\delta x^T (\dot{W} + H_{x\lambda} W + \\ & + W\ H_{\lambda x} + H_{xx})\ \delta x + \delta x^T\ (W\ H_{\lambda u} + H_{xu})\ \delta u + \\ & + \delta u^T\ (H_{ux} + H_{u\lambda} W)\ \delta x + \delta u^T\ (H_{uu})\ \delta u]\ dt \end{aligned} \tag{5.11}$$

Defining the following quantities as

$$\alpha = \dot{W} + H_{x\lambda} W + W H_{\lambda x} + H_{xx} \tag{5.12}$$

$$\beta = W H_{\lambda u} + H_{xu} \tag{5.13}$$

$$\beta^T = H_{u\lambda} W + H_{ux} \tag{5.14}$$

$$\gamma = H_{uu} \tag{5.15}$$

Equation (5.11) becomes

$$2d''I = dx^T(P_{xx}-W)\, dx\big|_{t_f} + \int_{t_0}^{t_f} [\, \delta x^T \alpha\, \delta x + \delta x^T \beta \delta u + \delta u^T \beta^T \delta x + \delta u^T \gamma\, \delta u]\, dt \tag{5.16}$$

In order to assure that the quantity under the integral of Equation (5.16) is always greater than zero, it would be desirable to express it in the form of a dot product which separates δx and δu, i.e., $(\Phi\delta x + \psi\delta u)^T\,(\Phi\delta x + \psi\delta u)$ where Φ is an mxn matrix, and ψ is an nxn matrix. All that remains is to determine the conditions which must be satisfied in order that this can be done. These conditions may be determined by requiring that

$$(\Phi\delta x + \psi\delta u)^T\,(\Phi\delta x + \psi\delta u) = \delta x^T \alpha\, \delta x + \delta x^T \beta\, \delta u + \delta u^T \beta^T \delta x + \delta u^T \gamma\, \delta u \tag{5.17}$$

Noting that

$$(\Phi\delta x + \psi\delta u)^T\,(\Phi\delta x + \psi\delta u) = \delta x^T \Phi^T \Phi\, \delta x + \delta x^T \Phi^T \psi\, \delta u + \delta u^T \psi^T \Phi\, \delta x + \delta u^T \psi^T \psi\, \delta u \tag{5.18}$$

and equating terms with like coefficients in Equations (5.17) and (5.18), the following equations are obtained.

$$\alpha = \phi^T \phi \tag{5.19}$$

$$\beta = \phi^T \psi \tag{5.20}$$

$$\beta^T = \psi^T \phi \tag{5.21}$$

$$\gamma = \psi^T \psi \tag{5.22}$$

From Equations (5.15) and (5.22) it is noted that ψ is of full rank because H_{uu} is of full rank. Thus Equation (5.22) may be inverted to obtain

$$\gamma^{-1} = [\psi^T \psi]^{-1} = \psi^{-1} \psi^{T-1} \tag{5.23}$$

Postmultiplying Equation (5.20) by γ^{-1} yields

$$\beta\gamma^{-1} = \phi^T \psi\psi^{-1} \psi^{T-1} = \phi^T \psi^{T-1} \tag{5.24}$$

Postmultiplying Equation (5.24) by β^T yields

$$\beta\gamma^{-1} \beta^T = \phi^T \psi^{T-1} \psi^T \phi = \phi^T \phi \tag{5.25}$$

From Equations (5.19) and (5.25) the following equation may be formed.

$$\alpha = \beta\gamma^{-1} \beta^T \tag{5.26}$$

Equation (5.26) represents the conditions which must be satisfied in order to express the terms under the integral of Equation (5.16) in the form of a positive scalar quantity. From Equation (5.26) and the definitions of α, β, and γ in Equations (5.12) through (5.15), the following equation results

$$\dot{W}(t) = (-H_{x\lambda} + H_{xu} H_{uu}^{-1} H_{u\lambda}) W(t) +$$

$$+ W(t) (-H_{\lambda x} + H_{\lambda u} H_{uu}^{-1} H_{ux}) + W(H_{\lambda u} H_{uu}^{-1} H_{u\lambda}) W +$$

$$+ (-H_{xx} + H_{xu} H_{uu}^{-1} H_{ux}) \tag{5.27}$$

In view of the definitions following Equation (3.13), Equation (5.27) may be written as

$$\dot{W} = -W A_{11} + A_{22}W - W A_{12}W + A_{21} \tag{5.28}$$

Equation (5.28) is referred to in the literature as the Matrix Riccati Equation. If Equation (5.28) holds true in the interval of interest, it is possible to express Equation (5.16) in the following form

$$2d''I = dx^T (P_{xx}-W) dx\Big|_{t_f} + \int_{t_0}^{t_f} (\delta x^T\Phi^T + \delta u^T\psi^T) I (\Phi\delta x + \psi\delta u)] dt \tag{5.29}$$

Consider the terms under the integral. Noting that $\psi\psi^{-1} = I$, the terms under the integral become

$$(\delta x^T\Phi^T + \delta u^T\psi^T)\psi^{T-1} \psi^T\psi\psi^{-1} (\Phi\delta x + \psi\delta u) \tag{5.30}$$

or

$$(\delta x^T\Phi^T\psi^{T-1} + \delta u^T)\psi^T\psi (\psi^{-1}\Phi \delta x + \delta u) \tag{5.31}$$

But from Equations (5.22) and (5.24),

$$\psi^T \psi = \gamma = H_{uu}$$

$$\phi^T \psi^{T-1} = \beta\gamma^{-1}$$

$$\psi^{-1} \phi = \gamma^{-1} \beta^T$$

Thus, Equation (5.31) becomes

$$(\delta x^T \beta\gamma^{-1} + \delta u^T) H_{uu} (\gamma^{-1} \beta^T \delta x + \delta u) \tag{5.32}$$

or

$$(\gamma^{-1} \beta^T \delta x + \delta u)^T H_{uu} (\gamma^{-1} \beta^T \delta x + \delta u) \tag{5.33}$$

Now Equation (5.33) is of the form $\varepsilon^T H_{uu} \varepsilon$, thus Equation (5.29) becomes

$$2''dI = dx^T (P_{xx} - W) dx \Big|_{t_f} + \int_{t_0}^{t_f} \varepsilon^T H_{uu} \varepsilon \, dt \tag{5.34}$$

By requiring that

$$d^T x (P_{xx} - W) dx \Big|_{t_f} = 0 \tag{5.35}$$

it is seen that in order that the second variation of Equation (5.34) be greater than zero, it is necessary that for any arbitrary vector ε,

$$\varepsilon^T H_{uu} \varepsilon > 0 \tag{5.36}$$

The requirement that H_{uu} be positive definite is a necessary condition in order that $2d''I > 0$. The conditions under which $2d''I > 0$, are that Equation (5.36) hold true in the interval of interest which is equivalent to requiring that the solution of the Matrix Riccati Equation (Equation (5.28)) be finite in the interval of interest.

The Matrix Riccati Equation and the Conjugate Point

If at any point in the interval $[t_0, t_f]$, $W(t)$, the solution to the Riccati Equation should become infinite, the matrix identity $\alpha = \beta\gamma^{-1}\beta^T$, i.e., (Equation (5.26)) can not be maintained and the integral expression of the second variation of I can not be expressed in the form $\varepsilon^T H_{uu}\varepsilon$. This is equivalent to saying that if $W(t)$ becomes infinite, $2d''I > 0$ can not be maintained. The point at which $W(t)$ becomes infinite is referred to as a conjugate point (Ref. 12).

The Solution of the Matrix Riccati Equation

The Matrix Riccati Equation may be solved by reducing it to two linear matrix equations. The boundary conditions on the Riccati Equation are given by Equation (5.35) as

$$W(t_f) = (P_{xx})_{t_f} \tag{5.37}$$

The Riccati Equation (Equation (5.28)) may be written as

$$\dot{W} = -W(A_{11} + A_{12}W) + A_{22}W + A_{21} \tag{5.38}$$

A system of equations adjoint to Equation (5.38) may be defined as

$$\dot{N} = N(A_{11} + A_{12}W)^T \tag{5.39}$$

The transpose of Equation (5.39) is

$$\dot{N}^T = (A_{11} + A_{12}W)N^T \tag{5.40}$$

Postmultiplying Equation (5.38) by N^T, premultiplying Equation (5.40) by W and adding the resulting equations yields

$$\dot{W}N^T + W\dot{N}^T = A_{22} W N^T + A_{21}N^T \tag{5.41}$$

Noting that $\frac{d}{dt}(WN^T) = \dot{W}N^T + W\dot{N}^T$, Equation (5.41) becomes

$$\frac{d}{dt}(WN^T) = A_{22}W N^T + A_{21} N^T \tag{5.42}$$

A new variable T^T may be defined as

$$T^T = W N^T \tag{5.43}$$

so that Equation (5.42) may be written as

$$\dot{T}^T = A_{21} N^T + A_{22} T^T \tag{5.44}$$

Equations (5.40) and (5.44) may be combined to form

$$\left[\begin{array}{c} \dot{N}^T \\ \hline \dot{T}^T \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} N^T \\ \hline T^T \end{array}\right] = A(t) \left[\begin{array}{c} N^T \\ \hline T^T \end{array}\right] \tag{5.45}$$

where $A(t)$ is a $2n\mathrm{x}2n$ matrix, and $\left[\begin{array}{c} N^T \\ \hline T^T \end{array}\right]$ is a $2n\mathrm{x}n$ matrix.

In view of Equations (5.37) and (5.43), the following relations may be defined at the terminal time

$$T^T(t_f) = (P_{xx})_{t_f} \tag{5.46}$$

$$N^T(t_f) = I \tag{5.47}$$

where I is the nxn identity matrix. Equations (5.46) and (5.47) may be partitioned as

$$\begin{bmatrix} N^T(t_f) \\ \text{- - - - -} \\ T^T(t_f) \end{bmatrix} = \begin{bmatrix} I \\ \text{- - - - -} \\ (P_{xx})_{t_f} \end{bmatrix} \tag{5.48}$$

With Equation (5.48) as boundary conditions, Equation (5.45) may be integrated from t_f to t to obtain $T^T(t)$ and $N^T(t)$.

From Equation (5.43) if $N^T(t)$ is a nonsingular matrix, W(t) is given by

$$W(t) = T^T(t)\, N^{T-1}(t) \tag{5.49}$$

Thus, the requirement that W(t) be finite in the interval of interest may be replaced by the requirement that $N^T(t)$ be nonsingular and $T^T(t)$ be finite in $[t_0,t_f]$. If $N^T(t)$ should become singular or $T^T(t)$ become infinite, a conjugate point has been reached.

Effects of a Conjugate Point on the Guidance Optimization Problem

In much of the control literature, the conjugate point and its effects upon optimal control are discussed in connection with the solution to the guidance optimization problem. In Chapter I it was mentioned that many of the results derived in connection with the trajectory optimization problem are directly applicable to the guidance optimization problem.

In the guidance optimization problem, it is required to construct some type of correction procedure by which the nominal control program is automatically corrected to produce an optimal control for small perturbations in the initial and/or terminal state. This is usually accomplished by a feed-back control system. In many guidance optimization schemes such as the ones presented in References (4) and (13), it is assumed that the changes in the boundary conditions are small and thus only small changes are required in the nominal control to preserve optimality. With this assumption the small deviations in the state are given by

$$\delta x = x(t) - x^*(t) \tag{5.50}$$

where $x(t)$ is the true state and $x^*(t)$ is the nominal state. The small deviations in the control resulting from the small deviations in the nominal control program are given as

$$\delta u(t) = u(t) - u^*(t) \tag{5.51}$$

Linear perturbations in the state variables and Lagrange multipliers result in the following linear differential equations developed in Chapter III, i.e., Equation (3.14).

$$\delta \dot{x} = A_{11}\, \delta x + A_{21}\, \delta\lambda \tag{5.52}$$

$$\delta \dot{\lambda} = A_{21}\, \delta x - A_{11}^T\, \delta\lambda \tag{5.53}$$

If linear perturbations are considered in the optimality condition, i.e., Equation (2.26), the results are

$$\delta(H_u) = H_{uu}\,\delta u + H_{ux}\,\delta x + H_{u\lambda}\,\delta\lambda = 0$$

or

$$\delta u(t) = -H_{uu}^{-1}\,[\,H_{ux}\,\delta x + H_{u\lambda}\,\delta\lambda\,] \tag{5.54}$$

In the guidance optimization problem, it is required to determine $\delta u(t)$ as a function of $\delta x(t)$. This is done in References (4) and (13) by considering the second variation of I. The procedure is quite lengthy and will not be repeated here. However, the same results may be obtained by a simpler approach.

It can be shown (Ref. 14) that $\delta x(t)$ and $\delta\lambda(t)$ are related by the expression

$$\delta\lambda(t) = K(t)\,\delta x(t) \tag{5.55}$$

The conditions for which Equation (5.55) is valid are found to be

$$\dot{K} = -KA_{11} - A_{11}^T K - KA_{12} K + A_{21} \tag{5.56}$$

Noting that $A_{22} = -A_{11}^T$, it is seen that Equation (5.56) is the Matrix Riccati Equation. A full second variation approach to the problem as presented in Ref. 13 would have revealed that for fixed final time problems, the terminal value of $K(t_f)$ is $(P_{xx})_{t_f}$. Thus, it is possible to relate small perturbations in the control to small perturbations in the state by substituting Equation (5.55) into Equation (5.54) i.e.,

$$\delta u(t) = -H_{uu}^{-1}\,[\,H_{ux} + H_{u\lambda}\,K(t)\,]\,\delta x(t) \tag{5.57}$$

It was shown in the previous section that the solution to the Riccati Equation was given by

$$K(t) = W(t) = T^T(t)\, N^T(t)^{-1} \qquad (5.58)$$

where $T^T(t)$ and $N^T(t)$ are the solutions to Equation (5.45). From Equations (5.57) and (5.58)

$$\delta u(t) = - H_{uu}^{-1} [\, H_{ux} + H_{u\lambda}\, T^T(t)\, N^T(t)^{-1} \,]\, \delta x(t) \qquad (5.59)$$

The solution to the guidance optimization problem in view of Equations (5.50), (5.51), and (5.59) is given as

$$u(t) = u^*(t) - H_{uu}^{-1} [\, H_{ux} + H_{u\lambda}\, T^T(t)\, N^T(t)^{-1} \,]\, (x(t) - x^*(t)\,) \qquad (5.60)$$

It is stated in References (4), (7), (12), and (13) that if the nominal trajectory $x^*(t)$ is not optimal but contains a conjugate point at $t = t_c$, then the solution to the Riccati Equation, W(t), becomes unbounded at $t - t_c$. The feed-back aspects of the linear guidance optimization scheme are seen by defining

$$V(t) = - H_{uu}^{-1} [\, H_{ux} + H_{u\lambda}\, T^T(t)\, N^T(t)^{-1} \,] \qquad (5.61)$$

where V(t) is the feed-back gain. If a conjugate point occurs, V(t) becomes infinite and is not physically attainable.

Thus, if the trajectory contains a conjugate point, it is impossible to construct a linear guidance scheme relating δu and δx and the guidance optimization problem can not be solved.

Through the second variation and the subsequent development of the solution to the guidance optimization problem, the relationships between the conjugate point, the Riccati variable, and the linear guidance scheme have been established. The guidance optimization problem may be solved only after the trajectory optimization problem has been solved and the existence of a conjugate point in the trajectory can be detected if the solution to the Matrix Riccati Equation becomes infinite in the interval of interest. However, it would be desirable to detect a conjugate point, if it existed, before the guidance scheme is constructed. It would be quite costly in terms of engineering man-hours and computer time to solve the trajectory optimization problem and construct a linear guidance scheme only to discover that the trajectory contains a conjugate point.

By considering the sufficiency conditions for a weak minimum and the subsequent development of the relationship between the Matrix Riccati Equation and the conjugate point, mathematical relationships have been established which indicate a possible connection between the existence of a conjugate point in the trajectory and a breakdown in the Method of Adjoint Systems. This will be investigated in the following section.

<u>Effects of a Conjugate Point on the Method of Adjoint Systems</u>

In the Method of Adjoint Systems, it is required to integrate Equation (3.31), i.e.,

$$\begin{bmatrix} \dot{\theta}_1(t) \\ ---- \\ \dot{\theta}_2(t) \end{bmatrix} = -A^T(t) \begin{bmatrix} \theta_1(t) \\ ---- \\ \theta_2(t) \end{bmatrix} \tag{5.62}$$

$$\begin{bmatrix} \theta_1(t_f) \\ ---- \\ \theta_2(t_f) \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial z} \end{bmatrix}^T_{t_f}$$

in order to improve the initial guessed values of the unknown $z(t_0)$ variables. The dissatisfaction in the terminal boundary conditions is given by Equation (3.32), which for fixed final time problems becomes

$$dh(t_f) = \theta^T(t)\ \delta z(t) \tag{5.63}$$

or

$$dh(t_f) = \theta_1^T(t)\ \delta z_1(t) + \theta_2^T(t)\ \delta z_2(t) \tag{5.64}$$

If $\theta_2^T(t)$ is nonsingular, Equation (5.64) may be solved for the unknown variation $\delta z_2(t)$ as

$$\delta z_2(t) = \theta_2^{T^{-1}}(t)\ [\ dh(t_f) - \theta_1^T(t)\ \delta z_1(t)\ ] \tag{5.65}$$

Equation (5.64) is evaluated at the initial time in the Method of Adjoint Systems because it is required to determine $\delta z_2(t_0)$, but it holds true at any time in the interval $[t_0, t_f]$.

It was found that the Matrix Riccati Equation (Equation (5.28)) could be solved by reducing it to two linear systems of differential equations (Equation (5.45)) with boundary conditions for fixed final time problems given by Equation (5.43). These equations are

$$\begin{bmatrix} \dot{N}^T(t) \\ ---- \\ \dot{T}^T(t) \end{bmatrix} = A(t) \begin{bmatrix} N^T(t) \\ ---- \\ T^T(t) \end{bmatrix} \tag{5.66}$$

$$\begin{bmatrix} N^T(t_f) \\ ---- \\ T^T(t_f) \end{bmatrix} = \begin{bmatrix} I \\ --- \\ P_{xx} \end{bmatrix}_{t_f}$$

It is noted that Equation (5.62) is adjoint to Equation (5.66). From the properties of the adjoint systems developed in Chapter III, it is seen that

$$\frac{d}{dt} \left\{ [\ \theta_1^T(t) \ \vdots \ \theta_2^T(t)\ ] \begin{bmatrix} N^T(t) \\ --- \\ T^T(t) \end{bmatrix} \right\} = 0 \tag{5.67}$$

Equation (5.67) may be integrated from t_f to t to yield

$$[\ \theta_1^T(t) \ \vdots\ \theta_2^T(t)\ ] \begin{bmatrix} N^T(t) \\ ---- \\ T^T(t) \end{bmatrix} = [\ \theta_1^T(t_f) \ \vdots\ \theta_2^T(t_f)\ ] \begin{bmatrix} N^T(t_f) \\ ---- \\ T^T(t_f) \end{bmatrix} \tag{5.68}$$

or

$$\theta_1^T(t)\ N^T(t) + \theta_2^T(t)\ T^T(t) = \left[\frac{\partial h}{\partial z}\right]_{t_f} \begin{bmatrix} I \\ ---- \\ P_{xx} \end{bmatrix}_{t_f} = J(t_f) \tag{5.69}$$

where $J(t_f)$ is an nxn constant matrix. Assuming that $N^T(t)$ is nonsingular, Equation (5.69) may be postmultiplied by $N^T(t)^{-1}$ to yield

$$\theta_2^T(t)\ T^T(t)\ N^T(t)^{-1} = J(t_f)\ N^T(t)^{-1} - \theta_1^T(t) \tag{5.70}$$

Assuming that $\theta_2^T(t)$ is nonsingular, Equation (5.70) may be solved for $T^T(t)\ N^T(t)^{-1}$ to yield

$$T^T(t)\ N^{T^{-1}}(t) = \theta_2^T(t)^{-1}\ [\ J(t_f)\ N^{T^{-1}}(t) - \theta_1^T(t)\ ] \tag{5.71}$$

If it is assumed that $[\ J(t_f)\ N^{T^{-1}}(t) - \theta_1^T(t)\ ]$ is nonsingular, Equation (5.71) may be solved for $\theta_2^{T^{-1}}(t)$ to yield

$$\theta_2^{T^{-1}}(t) = T^T(t)\ N^{T^{-1}}(t)\ [\ J(t_f)\ N^{T^{-1}}(t) - \theta_1^T(t)\ ]^{-1} \tag{5.72}$$

Now the Method of Adjoint Systems requires that $\theta_2^T(t)$ be inverted at t_0 in Equation (5.65). At t_0, Equation (5.72) becomes

$$\theta_2^{T^{-1}}(t_0) = T^T(t_0)\, N^{T^{-1}}(t_0)\, [\, J(t_f)\, N^{T^{-1}}(t_0) - \theta_1^T(t_0)\,]^{-1} \tag{5.73}$$

In examining Equation (5.73), it is seen that if $T^T(t_0)$ is infinite or $N^T(t_0)$ is singular at t_0, a conjugate point will exist at the initial time and $\theta_2^T(t_0)$ will be singular causing the Method of Adjoint Systems to fail. If a conjugate point does not exist at the initial time, but the problem is incorrectly formulated, it was found in Chapter IV that in some cases $\theta_2^T(t_0)$ was singular. From Equation (5.73) it is noted that the only way in which $\theta_2^T(t_0)$ can be singular if no conjugate points exists is for $[\, J(t_f)\, N^{T^{-1}}(t_0) - \theta_1^T(t_0)\,]$ to be singular. Thus, it is highly probable that a singularity in $[\, J(t_f)\, N^{T^{-1}}(t_0) - \theta_1^T(t_0)\,]$ can be attributed to to incorrect problem formulation.

The Method of Adjoint Systems is an iterative process and will yield the nominal trajectory for correctly formulated problems when the dissatisfaction in the terminal boundary conditions has been driven to zero. If $\theta_2^T(t_0)$ should become singular before the optimal trajectory has been obtained, it is not known at this time what this would indicate. If this condition occurs, however, the Method of Adjoint Systems fails. It is clear that more investigation is needed in order to fully understand all possible breakdowns in the Method of Adjoint Systems.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Summary

The Method of Adjoint Systems was developed as a means of solving the two-point boundary value problem arising from the first necessary conditions for an optimal trajectory. A simple dynamical system was used to investigate the ways in which illegitimate operations involving the performance index and terminal constraints of the system could result in incorrect problem formulation. The effects of incorrect problem formulation on the Method of Adjoint Systems were considered to illustrate how failure of the method could occur. Through the properties of adjoint systems, it was shown that a relationship exists between a conjugate point in the trajectory and a possible failure of the Method of Adjoint Systems.

Conclusions

The following conclusions may be drawn from this investigation:

(1) If the terminal constraints and the performance index chosen result in transversality conditions which lead to the trivial solution of the optimization problem, the Method of Adjoint Systems will fail due to the singularity of the $\theta_2^T(t_0)$ matrix. The cases in which this occurred were incorrectly formulated. It should be noted, however, that the trivial solution does not necessarily imply incorrect problem formulation. The $\theta_2^T(t_0)$ matrix may be singular due to poor choices of the initial values of the $z^*(t_0)$ variables.

(2) If the terminal constraints or performance index chosen result in transversality conditions which cause the $\theta_2^T(t_0)$ matrix to contain infinite elements at the converged values of the Lagrange multipliers, the Method of Adjoint Systems will fail. In all cases investigated, failure of this type could be traced to incorrect problem formulation. It should be noted that infinite elements in the $\theta_2^T(t_0)$ matrix does not insure incorrect problem formulation.

(3) Unless the initial values of the unknown variables $z^*(t_0)$, (in the cases considered these variables were all Lagrange multipliers) can be guessed sufficiently close to the true values, it is possible to destroy the convergence characteristics of the Method of Adjoint Systems. It is also possible to guess these initial values sufficiently close to the true values and to cause the Method of Adjoint Systems to fail due to the fact that certain choices may (1) result in the trivial solution to the problem or (2) may cause division by zero. Clearly, the choice of the unknown initial values of the $z^*(t_0)$ variables is the weakest part of the Method of Adjoint Systems.

(4) The analysis of the causes of a singularity in the $\theta_2^T(t_0)$ matrix of Equation (5.73) leads to the conclusion that the Method of Adjoint Systems can fail for problems which (1) contain a conjugate point in the nominal trajectory at the initial time, and (2) are incorrectly formulated.

The modes of breakdown in the Method of Adjoint Systems suggest that the following procedures be used when considering the use of the method on a specific problem.

(1) Always carry out the analytical work as far as possible. Relations concerning system behavior and functional dependence of the optimal control and the initial values of the $z^*(t_0)$ variables can be of great importance in foreseeing trouble spots which could cause failure of the Method of Adjoint Systems.

(2) The terminal constraints and performance index of the system should be inspected closely prior to application of the numerical technique. In more complex systems, it is sometimes difficult to determine if conditions have been imposed which constitute incorrect problem formulation. The system may be such that a subtle functional dependence could exist between the performance index and terminal constraints which would escape detection unless careful inspection of these parameters was undertaken. The performance index and terminal constraints should be independent in order to insure that the Method of Adjoint Systems will work.

(3) A system to be optimized should be checked carefully for uncontrollable state variables. In more complex systems it may be difficult to determine these variables. Uncontrollable state variables in the system may not cause trouble in the Method of Adjoint Systems unless an attempt is made to constrain these variables or incorporate them into the performance index on terminal constraints. Any relation which exists between the uncontrollable state variables and the performance index and/or terminal constraints should be noted and regarded as a possible trouble spot.

(4) If the problem is determined to be correctly formulated and yet the Method of Adjoint Systems fails, the following must be considered:

(1) The guessed $z^*(t_0)$ variables may have been such that they caused a trivial solution to the problem to be obtained or caused division by zero. Another set of these initial values should be guessed or determined by a gradient method to determine if this has occurred. (2) The possibility exists of a conjugate point at the initial time.

Recommendations for Further Study

(1) A more effective method for choosing the initial values of the unknown $z^*(t_0)$ variables needs to be developed. The use of gradient methods to obtain initial guesses of these variables is probably the best solution to this problem at present.

(2) Analysis similar to the one presented here should be undertaken on other optimization techniques in order to develop more usable criteria by which a method could be chosen or rejected for a particular problem.

(3) An investigation of the controllability of the linearized system of equations (Equation (3.14)) should be carried out. Although some effects of an uncontrollable variable in the system were studied here, no attempt was made to relate controllability, as defined by Kalman (Ref. 15), to the problem formulation.

(4) The analysis developed relating a conjugate point in the trajectory at the initial time to a singularity in the $\theta_2^T(t_0)$ matrix should be pursued further. Also the effects of conjugate points should be determined in other numerical optimization methods.

APPENDICES

APPENDIX A

THE FIRST VARIATION

It is required to determine the first variation of a functional quantity I of the form

$$I = G(x_f,t_f) + \nu^T M(x_f,t_f) + \mu^T L(x_0,t_0) + \int_{t_0}^{t_f} [\ Q(x,u,t) + \lambda^T(t)\ (f(x,u,t) - \dot{x}(t)\ )]\ dt \tag{A.1}$$

where the elements of I are defined as follows.

$G + \int Q\, dt$ - a scalar function called the Performance Index.

ν - a q-vector of constant multipliers.

M - a q-vector of terminal constraints.

μ - an r-vector of constant multipliers.

L - an r-vector of initial constraints.

$\lambda(t)$ - an n-vector of time varying multipliers.

$f(x,u,t)$ - an n-vector function.

$\dot{x}(t)$ - an n-vector of the time rate of change of the state variables.

Equation (A.1) may be written in a simpler form by defining the following scalar functions.

$$P = G + \nu^T M = P(x_f,t_f,\nu) \tag{A.2}$$

$$R = \mu^T L = R(x_0, t_0, \mu) \tag{A.3}$$

$$H = \lambda^T (f(x,u,t)) + Q(x,u,t) = H(x,\lambda,u,t) \tag{A.4}$$

In view of Equations (A.2), (A.3), and (A.4), Equation (A.1) may be written as

$$I = R + P + \int_{t_0}^{t_f} (H - \lambda^T \dot{x})\, dt \tag{A.5}$$

Denoting the total variation in I as $d'I$, an expansion of I in a Taylor's series about a nominal trajectory yields

$$dI = d'I + \frac{1}{2!} d''I + \frac{1}{3!} d'''I + \ldots \tag{A.6}$$

The first term of the series is designated the first variation, the second term is designated the second variation, etc. The first variation of I from Equation (A.5) is given as

$$d'I = dP + dR + d\int_{t_0}^{t_f} (H - \lambda^T \dot{x})\, dt \tag{A.7}$$

To the first order, the variation of a function G is given as

$$dG = \delta G + \dot{G}\, dt \tag{A.8}$$

If instead of being a function, G is a functional quantity, i.e.,

$$G = \int_{t_0}^{t_f} F\ dt \tag{A.9}$$

then Leibnitz's Rule for differentiation under the integral sign applied to Equation (A.8) for variable t_0 and t_f yields

$$d \int_{t_0}^{t_f} F\ dt = F\ dt \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \delta F\ dt \tag{A.10}$$

Thus, Leibnitz's Rule, applied to Equation (A.7) yields

$$d'I = dP\Big|_{t_f^+} dR\Big|_{t_0^+} (H - \lambda^T\dot{x})\ dt\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \delta(H - \lambda^T\dot{x})\ dt \tag{A.11}$$

The variation of (H - $^T\dot{x}$) in Equation (A.11) is given by

$$\delta(H - \lambda^T x) = H_x\ \delta x + H_u\ \delta u + H_\lambda \delta\lambda + H_t\ \delta t - \dot{x}^T\ \delta\lambda - \lambda^T\ \delta\dot{x} \tag{A.12}$$

It is noted that $\delta t = 0$ because there is no variation of time along the trajectory under the integral.

Integration of $-\lambda^T\ \delta\dot{x}$ by parts yields

$$\int_{t_0}^{t_f} (-\lambda^T\delta\dot{x})\ dt = -\lambda^T\ \delta x\ \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \dot{\lambda}^T\ \delta x\ dt \tag{A.13}$$

From Equations (A.12) and (A.13), Equation (A.11) becomes

$$d'I = [dP + (H - \lambda^T \dot{x})\, dt - \lambda^T \delta x\,]^{t_f} +$$

$$[dR - (H - \lambda^T \dot{x})\, dt + \lambda^T \delta x\,]^{t_0} + \tag{A.14}$$

$$\int_{t_0}^{t_f} [\,(H_x + \dot{\lambda}^T)\, \delta x + (H_u)\, \delta u + (H_\lambda - \dot{x}^T)\, \delta\lambda\,]\, dt$$

For the variable end point problem, variations in the state variables are, from Equation (A.8), given as

$$\delta x_i = dx_i - \dot{x}_i\, dt_i \quad , \qquad i = t_0, t_f \tag{A.15}$$

Thus, $d'I$ becomes

$$d'I = [\, dP + (H - \lambda^T \dot{x})\, dt - \lambda^T dx + \lambda^T \dot{x}\, dt\,]_{t_f}$$

$$+ [\, dR - (H - \lambda^T \dot{x})\, dt + \lambda^T dx - \lambda^T \dot{x}\, dt\,]_{t_0} \tag{A.16}$$

$$+ \int_{t_0}^{t_f} [\,(H_x + \dot{\lambda}^T)\, \delta x + (H_u)\, \delta u + (H_\lambda - \dot{x}^T)\, \delta\lambda\,]\, dt$$

The total differentials, dP and dR, in view of Equations (A.2) and (A.3), are

$$dP = \frac{\partial P}{\partial x_f}\, dx_f + \frac{\partial P}{\partial t_f}\, dt_f + \frac{\partial P}{\partial \nu}\, d\nu \tag{A.17}$$

$$dR = \frac{\partial R}{\partial x_0}\, dx_0 + \frac{\partial R}{\partial t_0}\, dt_0 + \frac{\partial R}{\partial \mu}\, d\mu \tag{A.18}$$

In view of Equations (A.17) and (A.18), Equations (A.16) becomes

$$
\begin{aligned}
d'I \; = \; & [\, P_x \, dx + P_t \, dt + P_\nu \, d\nu + (H - \lambda^T \dot{x}) \, dt + \\
& - \lambda^T \, dx + \lambda^T \dot{x} \, dt \,]_{t_f} + [\, R_x \, dx + R_t \, dt + R_\mu \, d\mu \\
& - (H - \lambda^T \dot{x}) \, dt + \lambda^T \, dx - \lambda^T \dot{x} \, dt \,]_{t_0} + \\
& + \int_{t_0}^{t_f} [\, (H_x + \dot{\lambda}^T) \, \delta x + (H_u) \, \delta u + (H_\lambda - \dot{x}^T) \, \delta\lambda \,] \, dt
\end{aligned}
\tag{A.19}
$$

Canceling the $\lambda^T \dot{x}$ dt terms and regrouping, the final form of the first variation becomes

$$
\begin{aligned}
d'I \; = \; & [\, (\, P_x - \lambda^T) \, dx + (P_t + H) \, dt + P_\nu \, d\nu \,]_{t_f} \\
& + [\, (\, R_x + \lambda^T) \, dx + (R_t - H) \, dt + R_\mu \, d\mu \,]_{t_0} \\
& + \int_{t_0}^{t_f} (\, H_x + \dot{\lambda}^T) \, \delta x \, dt + \int_{t_0}^{t_f} (H_u) \, \delta u \, dt + \int_{t_0}^{t_f} (H_\lambda - \dot{x}^T) \, \delta\lambda \, dt \, .
\end{aligned}
\tag{A.20}
$$

APPENDIX B

THE SECOND VARIATION

From Appendix A, the first variation of the functional

$$I = P + R + \int_{t_0}^{t_f} (H - \lambda^T \dot{x}) \, dt$$

is given in Equation (A.20) as

$$\begin{aligned} d'I = {} & [(P_x - \lambda^T) dx + (P_t + H) dt + P_\nu d\nu]_{t_f} + \\ & + [(R_x + \lambda^T) dx + (R_t - H) dt + R_\mu d\mu]_{t_0} + \\ & + \int_{t_0}^{t_f} [(H_x + \dot{\lambda}^T) \delta x + (H_u) \delta u + (H_\lambda - \dot{x}^T) \delta\lambda] \, dt \end{aligned} \tag{B.1}$$

If it is assumed that n of the initial state variables, $x(t_0)$, and the initial time t_0 are known, then $dx \big|_{t_0} = 0$, $dt \big|_{t_0} = 0$, $R_\mu = E(x_0,t_0) = 0$, and Equation (B.1) becomes

$$\begin{aligned} d'I = {} & [(P_x - \lambda^T) dx + (P_t + H) dt + P_\nu d\nu]_{t_f} + \\ & + \int_{t_0}^{t_f} [(H_x + \dot{\lambda}^T) \delta x + (H_u) \delta u + (H \; - \dot{x}^T) \delta\lambda] \, dt \end{aligned} \tag{B.2}$$

The second variation of I requires a variation of Equation (B.2) which yields

$$2d''I = [\ d(P_x\ dx) - d(\lambda^T\ dx) + d(P_t\ dt) +$$

$$+ d(H\ dt) + d(P_\nu\ d\nu)\ ]_{t_f} + \tag{B.3}$$

$$+ d \int_{t_0}^{t_f} [\ (H_x + \dot{\lambda}^T)\ \delta x + (H_u)\ \delta u + (H_\lambda - \dot{x}^T)\ \delta\lambda\ ]\ dt$$

Let the terms outside the integral of Equation (B.3) be denoted by G_f.

By Leibnitz's Rule for differentiation under the integral for fixed initial time, (See Appendix A, Equation (A.10)), Equation (B.3) becomes

$$2d''I = G_f + [\ (H_x + \dot{\lambda}^T)\ \delta x\ dt + (H_u)\ \delta u\ dt +$$

$$+ (H_\lambda - \dot{x}^T)\ \delta\lambda\ dt\ ]_{t_f} + \tag{B.4}$$

$$\int_{t_0}^{t_f} \delta\ [\ (H_x + \dot{\lambda}^T)\ \delta x + (H_u)\ \delta u + (H_\lambda - \dot{x}^T)\ \delta\lambda\ ]\ dt$$

or

$$2d''I = G_f + [\ (H_x + \dot{\lambda}^T)\ \delta x\ dt + (H_u)\ \delta u\ dt +$$

$$+ (H_\lambda - \dot{x}^T)\ \delta\lambda\ dt\ ]_{t_f} +$$

$$+ \int_{t_0}^{t_f} [\ \delta x^T\ (H_{xx}\ \delta x + H_{xu}\ \delta u + H_{x\lambda}\ \delta\lambda + \delta\dot{\lambda}\ ) \tag{B.5}$$

$$+ \delta u^T\ (H_{ux}\ \delta x + H_{u\lambda}\ \delta\lambda + H_{uu}\ \delta u\ ) +$$

$$+ \delta\lambda^T\ (H_{\lambda x}\ \delta x + H_{\lambda u}\ \delta u + H_{\lambda\lambda}\ \delta\lambda - \delta\dot{x}\ )\ ]\ dt$$

It is noted that $H_{\lambda\lambda}$ is zero because H is linear in λ . Integrating $\delta x^T \delta\dot{\lambda}$ by parts yields

$$\int_{t_0}^{t_f} \delta x^T \delta\dot{\lambda}\, dt = \delta x^T \delta\lambda \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \delta\dot{x}^T \delta\lambda\, dt \tag{B.6}$$

But $\delta x(t_0) = dx(t_0) - (\dot{x})t_0\, dt_0 = 0$, thus Equation (B.5) becomes

$$\begin{aligned} 2d''I = G_f &+ [\, (H_x + \dot{\lambda}^T)\, \delta x\, dt + (H_u)\, \delta u\, dt + \\ &+ (H_\lambda - \dot{x}^T)\, \delta\lambda\, dt + \delta x^T \delta\lambda\,]_{t_f} + \\ &+ \int_{t_0}^{t_f} [\, \delta x^T (H_{xx}\, \delta x + H_{xu}\, \delta u + H_{x\lambda}\, \delta\lambda) - \delta\dot{x}\, \delta\lambda + \\ &+ \delta u^T (H_{ux}\, \delta x + H_{u\lambda}\, \delta\lambda + H_{uu}\, \delta u) + \\ &+ \delta\lambda^T (H_{\lambda x}\, \delta x + H_{\lambda u}\, \delta u - \delta\dot{x})\,]\, dt \end{aligned} \tag{B.7}$$

Let the terms outside the integral of Equations (B.7) be denoted by G_{ff}. Regrouping terms, Equation (B.7) becomes

$$\begin{aligned} 2d''I = G_{ff} &+ \int_{t_0}^{t_f} [\, \delta u^T H_{xx}\, \delta x + \delta x^T H_{xu}\, \delta u + \\ &+ \delta u^T H_{ux}\, \delta x + \delta u^T H_{uu}\, \delta u + \\ &+ (\delta x^T H_{x\lambda} + \delta u^T H_{u\lambda} - \delta\dot{x}^T)\, \delta\lambda + \\ &+ \delta\lambda^T (H_{\lambda x}\, \delta x + H_{\lambda u}\, \delta u - \delta\dot{x})\,]\, dt \end{aligned} \tag{B.8}$$

From Chapter II, Equation (2.24), it is required that $\dot{x} = H_\lambda^T$. Considering linear perturbations in the equation of state, it is seen that

$$\delta\dot{x} - H_{\lambda x}\ \delta x + H_{\lambda u}\ \delta u = 0 \tag{B.9}$$

In view of Equation (B.9), Equation (B.8) becomes

$$\begin{aligned} 2d''I = G_{ff} + \int_{t_0}^{t_f} [\ & \delta x^T H_{xx}\ \delta x + \delta x^T H_{xu}\ \delta u + \\ & + \delta u^T H_{ux}\ \delta x + \delta u^T H_{uu}\ \delta u\ ]\ dt \end{aligned} \tag{B.10}$$

Now the G_{ff} terms are found to be

$$\begin{aligned} G_{ff} = [\ & dx^T P_{xx}\ dx + d\nu^T P_{x\nu}\ dx + P_{xt}\ dt + P_x\ dx^2 + \\ & + - d\lambda^T\ dx - \lambda^T\ dx^2 + dx^T P_{tx}\ dt + d\nu^T P_{t\nu}\ dt \\ & + P_{tt}\ dt\ dt + P_t\ d^2t + H_x\ dx\ dt + H_u\ du\ dt + \\ & + H_t dt\ dt + H\ d^2t + dx^T P_{\nu x}\ d\nu + d\nu^T P_{\nu\nu}\ d\nu + \\ & + P_{\nu t}\ d\nu\ dt + P_\nu\ d^2\nu + H_x\ \delta x\ dt + \dot{\lambda}^T\ \delta x\ dt + \\ & + H_u\ \delta u\ dt + H_\lambda\ \delta\lambda\ dt - \dot{x}^T\ \delta\lambda\ dt + \delta x^T\ \delta\lambda\ ]_{t_f} \end{aligned} \tag{B.11}$$

From Appendix A, Equation (A.8), it is noted that

$$\delta x(t_f) = dx(t_f) - (\dot{x})t_f \, dt_f$$

$$\delta u(t_f) = du(t_f) - (\dot{u})t_f \, dt_f = du_f, \quad \text{i.e., } \dot{u}_f = 0$$

$$\delta \lambda(t_f) = d\lambda(t_f) - (\dot{\lambda})t_f \, dt_f$$

Thus, upon regrouping and canceling of appropriate terms, G_{ff} becomes

$$\begin{aligned} G_{ff} = & [\, dx^T P_{xx} \, dx + 2dx^T P_{\nu\nu} \, d\nu + 2P_{xt} \, dx \, dt \\ & + 2 P_{\nu t} \, d\nu \, dt + P_{tt} \, dt \, dt + 2 H_x \, dx \, dt \\ & + H_t \, dt \, dt \,]_{t_f} \end{aligned} \tag{B.12}$$

From Equations (B.10) and (B.12), the final form of the second variation becomes

$$\begin{aligned} d''I = & \frac{1}{2} [\, dx^T P_{xx} \, dx + 2dx^T P_{\nu x} \, d\nu + 2P_{xt} \, dx \, dt \\ & + 2P_{\nu t} \, d\nu \, dt + P_{tt} \, dt \, dt + 2H_x \, dx \, dt \\ & + H_t \, dt \, dt \,]_{t_f} + \frac{1}{2} \int_{t_0}^{t_f} [\, \delta x^T H_{xx} \, \delta x + \\ & + \delta x^T H_{xu} \, \delta u + \delta u^T H_{ux} \, \delta x + \delta u^T H_{uu} \, \delta u \,] \, dt \end{aligned} \tag{B.13}$$

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VITA

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