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Ionospheric E-Region Irregularities Produced by Non-Linear Coupling of Unstable Plasma Waves

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ABSTRACT. The irregularities responsible for the strongest VHF echoes from the equatorial E-region and the aurora appear to have been adequately explained as arising from a form of two-stream plasma instability. Many of the weaker features of the echoes from the electrojet irregularities cannot be explained on this basis, however. The waves generated by the instability will grow in amplitude until they are limited by non-linear effects. These effects, furthermore, will cause coupling between waves which will generate new waves that cannot be directly excited. In the present paper we show that scattering from such secondary irregularities can account for many of the weak features of the VHF echoes. The theory is only qualitative, but the agreement between its predictions and most of the experimental observations is quite good.

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INTRODUCTION

The scattering of radio waves by the ionosphere, using frequencies well above the penetration frequency, is now well established as a technique for ionospheric investigation. When the ionospheric plasma is in thermal equilibrium, or fairly close to equilibrium, the scattering is extremely weak, and is due merely to the thermal fluctuations in electron density--the so-called incoherent scattering. There are other circumstances in which the scattering is much stronger (though still weak in the sense that the Born approximation applies). One such case occurs in the equatorial E-region, and is due to electron density fluctuations associated with the equatorial electrojet [Bowles, Cohen, Ochs, and Balsley, 1960; Bowles, Balsley and Cohen, 1963; Cohen and Bowles, 1963; Balsley, 1965].

The three most striking features of this equatorial scatter are the following:

1. The strength of the scattered signal is closely associated with the strength of the electrojet. Furthermore, the jet current must generally be greater than a certain threshold value if strong echoes are to be obtained.
2. The echoes are aspect sensitive; i.e., the direction of the radar beam must be nearly perpendicular to the magnetic field.
3. The frequency spectrum of the echoes, when the radar is not pointing vertically, usually consists mainly of a fairly sharp peak at a discrete Doppler shift. The magnitude of the shift is roughly independent of the angle between the beam and the jet current; the sign depends on whether one is looking east or west. Echoes from overhead are weaker

than those from oblique angles and have a more variable, often diffuse, frequency spectrum.

Farley [1963] (hereafter referred to as I) suggested that the scattering is due to the disturbance of electron density resulting from the two-stream instability which occurs when there is a sufficient relative drift of the electrons through the ions. By extending the theory of this instability to allow for the presence of both a magnetic field and collisions with neutral molecules, he showed that (1) and (2) are readily explained. The threshold effect is accounted for by the fact that the strength of the electrojet is of course proportional to the relative ion-electron drift velocity, and it is really this velocity which has a threshold value. Thus at night, when the electron density is very small, it is still possible to have an instability, even though no current flow is observable. The predicted critical velocity agrees with reasonable estimates of the velocities present in the electrojet. The aspect sensitivity arises from the fact that only waves propagating very nearly perpendicular to the earth's magnetic field are unstable. All other waves are heavily damped. The quantitative agreement between the predicted and observed aspect sensitivity is good.

The discrete Doppler shift (3) corresponds to the speed of the plasma waves in the medium. The echoes come only from those waves whose wave vector is parallel to the direction of the radar beam. At oblique angles the speed of these waves is simply the acoustic velocity. The main peak in the observed spectrum does indeed have a Doppler shift in agreement with this prediction. The sign of the Doppler shift is accounted for by the fact that only waves having a component of velocity in the direction of the electron flow can be excited. Simple ideas of "blobs" convected by the jet current would lead to a cosine dependence of the Doppler shift on elevation angle, contrary to the observations.

There are, however, some further features of the scattering not directly explained by the theory given in I. When the line of sight is perpendicular to both the magnetic field and the current vector, i.e. when it is vertical, the scattering should disappear, since there is no longer an appropriate unstable plasma wave. Nevertheless, some scattering is observed in those circumstances. Again, the theory predicts a definite Doppler shift, so that the frequency spectrum of the received signal should consist of just a single line, broadened only to the extent already involved in the incident signal. As already mentioned, the spectrum is usually quite strongly peaked at the appropriate frequency, but there is a substantial scattered signal over a band of frequencies.

In this paper we show that these secondary features exhibited by the scattering can be explained in some detail by an extension of the theory to include non-linear coupling between the unstable plasma waves. Such coupling produces further wave-like disturbances which can also scatter the radio waves. The possibility of such an explanation was actually mentioned in I.

EXPERIMENTAL RESULTS

Some details of the technique for observing the power spectrum of the scattered signal, together with early specimens of the results obtained, are given by Bowles, Balsley, and Cohen [1963]. A more complete presentation and discussion of the experimental results relevant to the theory developed here are given by Cohen and Bowles [1966]. Figure 1, which also shows the geometry of the experiment, is taken from the latter paper. We let α denote the angle between the radar and the earth's magnetic field (as in I), and β the angle between the beam and the vertical. In fact, α must be close to 90°, as already mentioned. It should be recalled that the spectra are shown normalized, and the total scattered signal also varies with β; it is seriously reduced when β is small.

Schematically, we may represent all the spectra as in Figure 2, drawn for the case when the radar looks east (β positive). The spectrum may be divided into three regions, numbered as shown. Region I is the part due to direct scattering by the unstable plasma waves themselves, as explained in paper I. The Doppler shift corresponds to the speed of the plasma waves along the line of sight, in this case towards the observer. Region II consists of Doppler shifts with the same sense as region I, but smaller in magnitude, while region III consists of Doppler shifts of the opposite sense.

When the electrojet is strong and β is large, region I is well-developed, region II weak, and region III practically non-existent (Figure 1a and 1b). At less oblique angles, regions II and III become relatively more important and region I less so. Finally, when $\beta = 0$ (vertical beam), region I disappears, while regions II and III become of equal importance, i.e., the spectrum becomes roughly symmetric. For negative values of β, Figure 2 should of course be reflected about its vertical axis.

The form of the spectrum may be somewhat different if the electrojet is quite weak, although still strong enough to produce some irregularities. In such cases region I may partially or completely disappear, even when β is large, leaving only a weak echo that is mainly in region II. For small β, the spectrum can become very narrow and centered about zero Doppler shift. Various intermediate results are also possible; for example, at moderately oblique angles when the current is fairly weak, it is possible to have a peak in the spectrum near zero Doppler shift as well as a peak corresponding to region I (cf. Figures 6 and 7, Cohen and Bowles, [1966]).

THEORETICAL INTERPRETATION

We recall the standard result for scattering of radio waves in the Born approximation (see, for example, Dougherty and Farley [1960]). We have plane waves of angular frequency ω_o incident on a medium of mean electron density N_o but subject to small fluctuations ΔN. The scattering cross-section for waves Doppler shifted by an amount between ω and $\omega + d\omega$ is proportional to

$$\left\langle \left| \Delta N(\underline{k},\omega) \right|^2 \right\rangle d\omega \quad . \tag{1}$$

Here $\Delta N(\underline{k},\omega)$ denotes the Fourier transform in both space and time, the angular brackets denote the ensemble average of the statistical fluctuations, and ω_o is assumed to be well above the critical frequency for the plasma. The appropriate wave-vector $\underline{k}$ depends on the geometry of the transmitter and receiver, but in the simple case of back scattering, $\underline{k}$ is along the line of sight towards the observer, and has magnitude $4\pi/\lambda$, where λ is the radio wavelength, $2\pi c/\omega_o$.

In the case of a stable plasma, this scattering cross-section is extremely small, and represents spontaneous thermal fluctuations, giving rise to the so-called incoherent scattering. If the plasma is able to sustain an undamped (or very slightly damped) longitudinal wave, such as a plasma oscillation with wave-vector $\underline{k}$, the cross-section (1) has a sharp maximum at the frequency ω of the longitudinal wave. As a result the scattering has a definite Doppler shift, in this case the plasma frequency. However, in ionospheric cases most of the cross-section for incoherent scatter, as given by (1), lies under a quite smooth curve not associated with free oscillations [Dougherty and Farley, 1960].

For a plasma subject to microinstabilities, such as the two-stream instability, the situation is much more similar to that of turbulence. The total cross-section is far larger (by a factor of up to 10^8 in the observations summarized above), and the spectrum cannot be calculated by means of statistical mechanics.

The idea developed in I was this. We start with a stable (say Maxwellian) plasma and gradually increase the relative velocity of the ions and electrons, as actually happens in the electrojet. The spectral distribution (1) changes gradually from the incoherent to the turbulent form, and we anticipate that in the early stages the main feature of the change is the appearance of sharp peaks at frequencies corresponding to waves which begin to grow when the threshold for instability is reached. Indeed this behaviour has been demonstrated by Rosenbluth and Rostoker [1962] in another case, namely a two-stream instability in the absence of magnetic field or collisions. Accordingly, paper I was devoted to the study of the dispersion relation for longitudinal waves, say

$$F(\underline{k},\omega) = 0 \,, \qquad (2)$$

taking account of electrons, ions, the external magnetic field, and the collisions with neutral molecules. With $\underline{k}$ fixed and real, solutions were sought for which ω/k is roughly the ion thermal speed. These waves, normally highly damped, change to growing waves when the speed of counterstreaming reaches a critical value, which is approximately the ion thermal speed if $\underline{k}$ is almost perpendicular to the magnetic field. This affords a natural explanation of region I of Figure 2.

It is extremely difficult to answer quantitatively the question of what happens when the threshold of instability is crossed (see Kadomtsev [1965] for an account of such progress as has been made). If there are waves with a substantial growth rate, strong disturbances are set up which are governed by non-linear equations, so that the linearized treatment of the Fourier components, leading to (2), is no longer relevant. In other words the waves grow to an extent limited only by non-linear effects, and also interact strongly with each other, producing additional waves. However, in practice it often happens that the threshold is exceeded only slightly, so that the coupling is fairly weak. Then the waves given by (2) still preserve their identity and behave to some extent as in the linear theory. They still, of course, reach an amplitude far higher than the thermal level. We believe that regions II and III of the spectrum, and the existence of scattering at vertical incidence, can be attributed to the non-linear coupling of the unstable plasma waves.

Several properties of this coupling can be deduced quite easily in the weakly unstable case. Consider two plasma waves in which the perturbations are proportional to

$$\exp\left[i(\omega_j t - \underline{k}_j \cdot \underline{r})\right] \qquad (j=1,2) \qquad (3)$$

where the $\underline{k}_j$ are real, and the ω_j are almost real and satisfy (2). The small imaginary part of the frequency is not important in the arguments that follow, and so for simplicity we will ignore it. The non-linear interaction involving products of terms such as (3) gives rise to further Fourier components of the same form, with $j = 3$, and

$$\underline{k}_3 = s_1\underline{k}_1 + s_2\underline{k}_2 \,, \quad \omega_3 = s_1\omega_1 + s_2\omega_2 \qquad (4)$$

Here s_1 and s_2 are each ± 1. The possibility of negative values for s_1 and s_2 is due to the fact that the factor (3) arises in the Fourier transform of real quantities (so terms involving the complex conjugate of (3) must also exist), and it is the product of the real quantities which is actually required.

The procedure is, now, to enumerate all the marginally unstable waves $(\underline{k}, \omega)$ and so construct all the available new Fourier components $(\underline{k}_3, \omega_3)$. If the latter themselves satisfy (2), we simply get a further contribution to another wave which could already exist; in such a case wave 3 is in strong interaction with waves 1 and 2. This does approximately happen when $\underline{k}_1$ and $\underline{k}_2$ are parallel, but no new feature of the spectrum can be explained on this account. When $(\underline{k}_3, \omega_3)$ does not correspond to a freely propagating wave, it represents something new. Now as the radar experiment picks out a definite $\underline{k}$ and gives us the spectrum function (1) as a function of ω, we set $\underline{k}_3$ equal to this fixed value and try in turn all values of $\underline{k}_1$ and $\underline{k}_2$ perpendicular to the magnetic field and satisfying (4). With ω_1 and ω_2 following from (2), we obtain a contribution to the scattered signal at a Doppler shift ω_3. It follows of course that $\underline{k}_3$ must itself be perpendicular to the field.

As we already have alternative signs in (4), we need not consider as distinct the waves $(\underline{k}, \omega)$ and $(-\underline{k}, -\omega)$, and so we adopt the convention that ω_1 and ω_2 are positive. It was shown in I that for marginally unstable waves (2) reduces to

$$\omega = V_a \, |\underline{k}| \, , \tag{5}$$

where V_a is roughly independent of $\underline{k}$ over a wide range, and is approximately the ion thermal speed. The wave vector $\underline{k}$, besides being orthogonal or nearly orthogonal to the field, must have a positive component along the direction of the electron stream (we regard the ions as at rest), to make ω positive. The electrons cannot excite waves traveling in a direction opposite to themselves.

Let us consider the case of the radar looking east. Figures 3 and 4 illustrate how the vectors $\underline{k}_1$ and $\underline{k}_2$ can combine as a difference or sum, respectively, to form $\underline{k}_3$. Since both $\underline{k}_1$ and $\underline{k}_2$ must have a component in the direction of the electron velocity $\underline{V}_2$, at least one of s_1 and s_2 must be +1. Without loss of generality we can consider s_1 to be +1 in all cases. In Figure 3, s_2 is then -1, and we have

$$\omega_3 = V_a \, (|\underline{k}_1| - |\underline{k}_2|)$$

and

$$\frac{\omega_3}{|\underline{k}_3|} = \frac{V_a \, (|\underline{k}_1| - |\underline{k}_2|)}{|\underline{k}_1 - \underline{k}_2|} \leq V_a \, . \tag{6}$$

If the origin of $\underline{k}_1$ and $\underline{k}_2$ is at some point such as P, ω_3 will be positive and will contribute to region II of the spectrum in Figure 2; if the origin is at P' in region III of Figure 3, ω_3 will be negative and contribute to region III of the spectrum.

It can be seen from Figure 3 that region III of the spectrum will be very weak if β is large, since $\underline{k}_1'$ and $\underline{k}_2'$ will then make large angles with the mean electron velocity $\underline{V}_e$. Similarly, the strongest waves corresponding to region II will have velocities that are only slightly less than V_a. When β is small, however, contributions to region III will be appreciable, and in the limit of vertical propagation ($\beta = 0$), regions II and III become identical, leading to a symmetrical spectrum. When β is large, it will usually be possible to excite directly the wave represented by $\underline{k}_3$, producing region I of the spectrum. This will generally be the strongest echo, since no non-linear coupling is involved (although non-linear effects of course eventually limit the growth of the wave). As β becomes smaller, region I will grow weaker and eventually disappear. Even if the electrojet is very strong, the component of $\underline{V}_e$ in the direction of the wave vector will be less than the critical value for direct excitation of the instability when β is sufficiently small.

On the other hand, if the electrojet is weak and $\underline{V}_e$ is only slightly above the minimum threshold for excitation, only waves traveling very nearly parallel to $\underline{V}_e$ will be excited. In such cases region I may disappear, even if β is quite large, and only signals in region II (and III if β is sufficiently small) will be observed. Since $\underline{k}_1$ and $\underline{k}_2$ must be nearly parallel to each other, the spectrum will consist mainly of a peak near zero Doppler shift when β is small. As β is increased, the spectrum will move towards larger shifts. If $\underline{k}_1$ and $\underline{k}_2$ are nearly parallel and horizontal, it follows from (6) and Figure 3 that the velocity of wave 3 should be just slightly greater than $V_a \sin\beta$. When the instability is weakly excited, then, we should expect to find a peak in the spectrum near Doppler shifts corresponding to the latter velocity. The results of Cohen and Bowles [1966] are in agreement with this prediction when β is small (cf. their Figure 4). When β is moderately large, however, their observations when the electrojet is weak show a Doppler shift of the correct sense, but considerably smaller than $V_a \sin\beta$ (cf. their Figures 6-9). At present we have no explanation for the discrepancy.

If the electrojet is weak and β is small, the central peak in the spectrum should be narrow. When the electron velocity is well above the minimum threshold, however, the peak should be broader, since $\underline{k}_1$ and $\underline{k}_2$ can then diverge at an appreciable angle. This behaviour is observed.

When $\underline{k}_1$ and $\underline{k}_2$ are nearly parallel, at least one of them must be greater in magnitude than $\underline{k}_3$. Therefore one or both of the interacting waves will have a shorter wavelength than wave 3. Reducing the wavelength makes the wave somewhat harder to excite [Farley, 1963], but this dependence is very weak for wavelengths in the vicinity of 3 meters (the wavelength of interest in most of the VHF experiments referred to), and is less important than the $\csc \beta$ dependence of the critical value of $\underline{V}_e$ for excitation. Thus, in spite of the wavelength dependence, it should be possible in some cases to produce wave 3 by a non-linear interaction when it cannot be produced by direct excitation. The limited vertical extent (which has been ignored in all calculations) of the electrojet region may also favor the excitation of the horizontally traveling waves to some extent. This effect should be small, however, since the thickness of the region is generally of the order of 10^3 times as great as the wavelengths of interest.

For intermediate values of β and electrojet strength, one might expect to observe a fairly broad peak in the spectrum at small Doppler shifts, as well as a comparatively weak maximum corresponding to region I.

Let us now consider Figure 4, in which $\underline{k}_3$ is the sum of $\underline{k}_1$ and $\underline{k}_2$ (i.e., $s_2 = +1$). In this case $\omega_3/|\underline{k}_3| \geq V_a$. However, only pairs of wave vectors such as $\underline{k}_1'$ and $\underline{k}_2'$ will lead to waves whose velocity is substantially greater than V_a, and such waves will be very weak due to the large angle between the primed vectors and $\underline{V}_e$. A wave resulting from a combination of the unprimed waves might be quite strong if β were large, but its velocity would be only slightly greater than $\underline{V}_a$, and so it would be indistinguishable from a directly excited wave.

It is a trivial matter to make the necessary changes in the above arguments to treat the case in which the radar points westward. We see, therefore, that the existence of strong radar echoes at vertical incidence, and most of the characteristics of the spectrum at all angles of incidence, can be accounted for by simple arguments involving the properties of non-linear interactions.

Non-linear combinations of more than two plasma waves could be considered also, but they do not introduce any new qualitative features and are in any case weaker.

CONCLUSION

This qualitative discussion suggests that most of the secondary features observed in the scattering of radar waves by electrojet irregularities may be explained in terms of non-linear coupling of the plasma waves which have already been shown in I to be the source of the main effect.

As in turbulence waves of all scale sizes, or eddy sizes, interact with each other in varying degrees. In turbulence the flow of energy is predominantly from the large eddies to the small. For the electrojet irregularities, the mean energy flow is presumably in the same direction, for waves having parallel $\underline{k}$ vectors, since the larger scale sizes can be excited somewhat more easily by the two-stream instability. Such coupling, however, is not of much interest for the radar experiments, since it leads to echoes that generally can be explained equally well as being due to direct excitation. The new effects of main concern in this paper arise from coupling in which at least one of the interacting waves has a smaller wavelength than that of the resulting wave, and is traveling more nearly in the direction of the electron stream.

A few features of the echoes still await a complete explanation. The spectral results at oblique angles when the electrojet is quite weak seem to be in partial disagreement with the theory given here. Another unexplained result is the slight east-west asymmetry reported by Cohen and Bowles [1966]. The echoes from the east, which are due to waves traveling downward, are slightly weaker than those from the west. This asymmetry may be related to the fact that the properties of the electrojet region vary rapidly with altitude. Perhaps the upgoing waves are more easily generated. The theory developed here and in I has been based on the assumption of a uniform, infinite medium, and is therefore necessarily symmetric.

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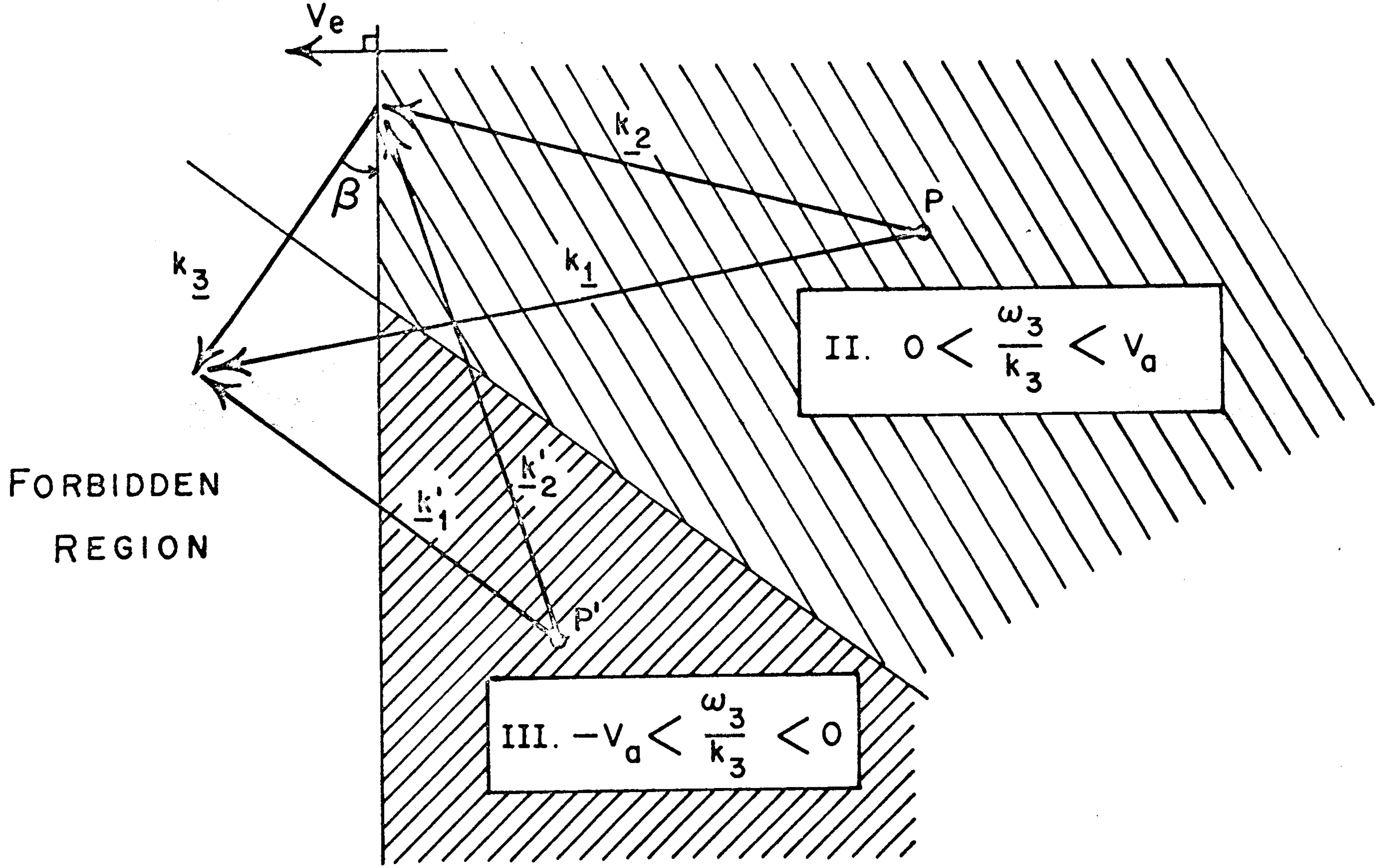

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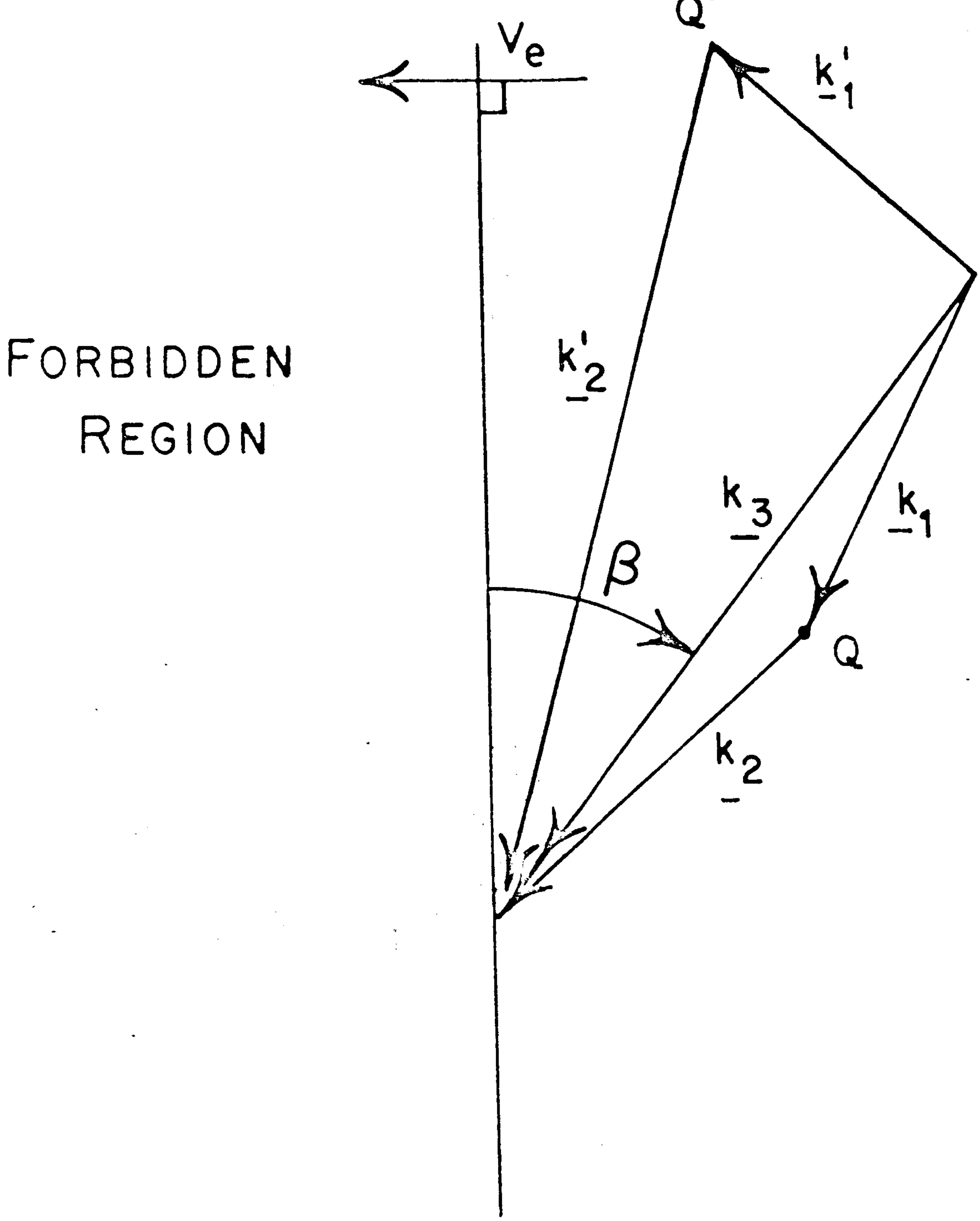
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