On the class of admissable nonlinearities for Lur'e's Problem

by

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Let us consider the system N:

$$X' = AX + \mu S ,$$

$$\mu = \phi(\sigma) ,$$

$$\sigma = \langle C, X \rangle ,$$

where $X = (x_j(t))$ is a variable n vector, $S = (s_j)$ and $C = (c_j)$ are constant n vectors, $A = (a_{ij})$ is a constant $n \times n$ matrix, σ is a scaler, $\langle C, X \rangle = \sum_{j=1}^{n} c_j x_j$, and $\phi(\sigma)$ is in general a nonlinear function of σ. We set our problem in an appropriate ℓ^p space with ℓ^p norm. Multiplication by A is then a bounded transformation with suitable norm. We assume that the norm of S is 1 in ℓ^p and that the norm of C is also 1 in the dual space. We denote all these norms by $\|\cdot\|$.

For a system such as N, stability means that X remains bounded for

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all $t > 0$, and asymptotic stability means that X approaches zero as t approaches infinity. We wish to impose conditions on $\emptyset$ so that the system will be stable or asymptotically stable.

We assume that the linear system L:

$$Y' = AY + \mu S ,$$

$$\mu = h \sigma ,$$

$$\sigma = \langle C, Y \rangle ,$$

is stable whenever $k_1 < h < k_2$. Note that this system can be written as $Y' = BY$ where the matrix $B = (b_{ij}) = (a_{ij} + hs_i c_j)$.

<u>Lemma</u>. <u>Consider the matrix solution of</u> $Y' = BY$, $Y(o) = I$ <u>as an operator on</u> ℓ^p <u>to</u> ℓ^p. <u>If all of the characteristic roots of</u> B <u>lie in the left half of the complex plane, then there exist constants</u> $a > 0$ <u>and</u> $b > 0$ <u>such that</u> $\| Y \| < ae^{-bt}$.

<u>If all of the characteristic roots of B lie in the left half of the complex plane or as simple roots on the imaginary axis, then there exists a constant</u> $a > 0$ <u>such that</u> $\| Y \| < a$.

See Bellman [2; p.36].

Let us now approximate the nonlinearity $\emptyset(\sigma)$ by $\alpha\sigma + \psi(\sigma)$, where $k_1 < \alpha < k_2$.

Note that the linear system L with $h = \alpha$ is asymptotically stable, and that by the lemma, there exist constants $a > 0$ and $b > 0$ such that $Y(t)$, the solution of L with $h = \alpha$ satisfying $Y(o) = I$, satisfies $\| Y(t) \| < ae^{-bt}$.

Theorem. If $\phi(\sigma)$ satisfies

$$(\alpha - \frac{b}{a}) < \phi(\sigma)/\sigma < (\alpha + \frac{b}{a}) \quad ,$$

then N is asymptotically stable.

If $\phi(\sigma)$ satisfies

$$(\alpha - \frac{b}{a}) \leqq \phi(\sigma)/\sigma \leqq (\alpha + \frac{b}{a}) \quad ,$$

then N is stable.

Proof. N is equivalent to

$$X' = BX + \mu S ,$$

$$\mu = \psi(\sigma) ,$$

$$\sigma = < C, X > .$$

Thus

$$X(t) = Y(t)\, X(o) + \int_o^t Y(t-\tau)\psi(< C, X(\tau) >)\, S\, d\tau .$$

By the lemma,

$$\|X(t)\| \leqq ae^{-bt}\, \|X(o)\| + \int_o^t ae^{-b(t-\tau)}\, |\psi(<C,X(\tau)>)|\, \|S\| d\tau .$$

Now if $\phi(\sigma)$ satisfies the hypothesis of the theorem, then $|\psi(\sigma)| \leqq \beta\, |\sigma|$, where $\beta \leqq \frac{b}{a}$. Thus

$$|\psi(<C, X(\tau)>)| \leqq \beta\, |<C, X(\tau)>| ,$$

$$|\psi(-C, X(\tau)>| \leqq \beta\, \|C\|\, \|X(\tau)\| .$$

Since $\|S\| = 1$, $\|C\| = 1$, multiplying by e^{bt}, we have

$$e^{bt} \| X(t) \| \leqq a \| X(o) \| + \int_0^t a\beta e^{b\tau} \| X(\tau) \| d\tau .$$

By Gronwall s inequality [2; p. 35],

$$e^{bt} \| X(t) \| \leqq a \| X(o) \| e^{a\beta t} , \text{ and}$$

$\| X(t) \| \leqq a \| X(o) \| e^{(a\beta - b)t}$. If $\beta < \frac{b}{a}$, $\| X(t) \|$ approaches zero exponentially. If $\beta = \frac{b}{a}$, $\| X(t) \|$ is bounded.

Remarks. 1. A can have characteristic values in the right half plane. That is, the system with no feedback $\emptyset$ may be unstable.

2. a and b depend upon α. As α approaches k_1 or k_2, b/a must approach zero. Further, $\alpha - k_1 \geqq b/a$ and $k_2 - \alpha \geqq b/a$. $k_2 - k_1 \geqq 2 \frac{b}{a}$.

3. Pliss [6] has shown by example that one cannot expect the interval of stability $[\alpha - b/a , \alpha + b/a]$ to entirely fill the interval $[k_1, k_2]$. (See also Aizerman and Gantmacher [1].)

4. The optimal choice of α has not yet been determined.

References

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