Root Locus Asymptotes For The Sum Of Two Polynomials Of The Same Degree

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by

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Introduction. There seems to be a considerable amount of confusion concerning the asymptotes of the root-locus diagram of the sum of two polynomials having the same degree. The subject is either ignored [8], an improper impression is given because the parameter involved is restricted to non-negative values [4], or else incorrect statements are made concerning it [7; page 195, last line]. The fact is that asymptotes do exist when the parameter is real valued and are important in the construction of the root-locus diagram. The confusion arises because the root-locus does not approach the asymptotes as the parameter approaches $\pm\infty$, but instead as it approaches -1.

The reason for this is quite simple. As the parameter approaches -1, the coefficient of the highest power of the polynomial approaches zero, and the polynomial is reduced in degree

A Theorem. Let

$$g(z) = z^n + \sum_{0}^{n-1} a_{n-i} z^i ,$$

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$$h(z) = z^n + \sum_{0}^{n-1} b_{n-i} z^i ,$$

be relatively prime, and let

$$k(z) = z^m + \sum_{0}^{m-1} \left(\frac{a_{n-i} - b_{n-i}}{a_{n-m} - b_{n-m}}\right) z^i ,$$

where a_i, b_i are complex constants, and $a_{n-m} - b_{n-m}$ is the coefficient of the highest power of $g(z) - h(z)$ which is non-zero. Clearly $m < n$. We wish to consider the root-locus of

$$F(z) = g(z) - Ke^{i\theta} h(z) = 0$$

as K varies from $-\infty$ to $+\infty$. θ is a fixed, real constant. We prove the following.

Theorem. Let $\theta \neq 0$, $\theta \neq \pi$. Let

$$M = |\csc \theta| \, [\sup_i (|a_i| + |b_i|)].$$

The root locus of $F(z)$ (as K varies between $-\infty$ and $+\infty$) lies within a circle centered at the origin with radius $p = M + 1$.

Let $\theta = \pi$. ($\theta = 0$ is the same except K changes sign.) As K approaches -1 from above and below, the root locus of $F(z)$ becomes asymptotic to the $2(n-m)$ lines passing through

$$z = \left[b_1 - \left\{\frac{a_{n-m+1} - b_{n-m+1}}{a_{n-m} - b_{n-m}}\right\}\right]/(n-m) ,$$

making angles

$$[\arg(a_{n-m} - b_{n-m}) - \arg(K+1) + (2r+1)\pi]/(n-m) ,$$

where $r = 0, 1, \ldots, n-m-1$. Further the $2(n-m)$ quantities (for $K > -1$ and $K < -1$)

$$z = [b_1 - \left\{\frac{a_{n-m+1} - b_{n-m+1}}{a_{n-m} - b_{n-m}}\right\}]/(n-m) +$$

$$+ [-(a_{n-m} - b_{n-m})/(K+1)]^{\frac{1}{n-m}}$$

<u>become arbitrarily good estimates for n-m roots of $F(z) = 0$ as K approaches -1.</u>

Proof. Let $\theta \neq 0$, $\theta \neq \pi$. Consider

$$\phi(z) = F(z)/[1 - Ke^{i\theta}] \quad ,$$

$$= z^n + \sum_{0}^{n-1} [\frac{a_{n-i} - Ke^{i\theta} b_{n-i}}{1 - Ke^{i\theta}}] z^i.$$

The roots of $\phi(z) = 0$ and $F(z) = 0$ are the same. It is well known that if M is an upper bound for the coefficients of z^0, ..., z^{n-1}, then a circle centered at the origin with radius $p = M + 1$ contains all of the roots $\phi(z)$. Thus we must find an upper bound for the quantities $(a_i - Ke^{i\theta} b_i)/(1 - Ke^{i\theta})$ which is independent of K.

Let α, β be any of the pairs a_i, b_i, and consider the mapping

$$f = (\alpha - \omega\beta)/(1 - \omega).$$

As ω varies along the straight line $Ke^{i\theta}$, which passes through the origin with inclination θ, b describes a circle with center

$$\text{center} = (\alpha - e^{2i\theta}\beta)/(1 - e^{2i\theta})$$

and radius

$$\text{radius} = |\beta - \alpha|/|1 - e^{2i\theta}|.$$

Therefore a maximum for f on the circle is |center| + radius.

$$|f| \leqq [|\alpha - e^{2i\theta}\beta| + |\alpha - \beta|]/|1 - e^{2i\theta}|.$$

Using the triangle inequality, we note that

$$|f| \leqq [|\alpha| + |\beta|]/|\sin \theta|.$$

Thus if

$$M = |\csc \theta| \sup_i [|a_i| + |b_i|],$$

the result follows.

Let $\theta = \pi$. If $W = (a_{n-m} - b_{n-m})/(K + 1)$, elimination of K in $F(z) = 0$ yields

$$h(z) + Wk(z) = 0.$$

The result then follows from theorem 2 of [5].

An Example. Let us consider

$$z(z+1)(z+2) + K(z+3)(z+i\sqrt{2})(z-i\sqrt{2}) = 0.$$

It is apparent from the relative locations of the points $z = 0$, -1 and -2 and the points $z = -3$, $i\sqrt{2}$ and $-i\sqrt{2}$, that the positive root-locus $(K > 0)$ consists of two branches beginning at $z = 0$ and $z = -1$, which meet at $z = -1 + \frac{\sqrt{3}}{3}$, and then approach $z = i\sqrt{2}$ and $z = -i\sqrt{2}$ respectively. A third branch begins at $z = -2$ and approaches $z = -3$. The negative root-locus $(K < 0)$ is not at all clear.

However, if we let $W = -6/(K + 1)$, the equation is transformed into

$$(z+3)(z+i\sqrt{2})(z-i\sqrt{2}) + W = 0.$$

The root locus of this polynomial has $z = -1$ as its center of gravity (see [5]) and three asymptotic lines passing through it at angles of $60°$, $180°$ and $300°$ when $W > 0$, at angles $120°$, $240°$ and $0°$ when $W < 0$.

The positive root-locus $(W > 0)$ consists of a branch beginning at $z = i\sqrt{2}$, becoming asymptotic to the line through the center of gravity at an angle of 60°; a branch beginning at $z = -i\sqrt{2}$, becoming asymptotic to the line through the center of gravity at an angle of 300°; and a third branch beginning at $z = -3$, which approaches $z = -\infty$ along the real axis.

The negative root-locus $(W < 0)$ has two branches which begin at $z = \pm i\sqrt{2}$ and meet at $z = -1 + \frac{\sqrt{3}}{3}$. One branch then approaches $+\infty$ along the real axis. The other moves leftward along the real axis and meets the third branch, which began at $z = -3$, at $z = -1 - \frac{\sqrt{3}}{3}$. These then split to become asymptotic to the lines through the center of gravity at angles of 120° and 240°. It is now apparent what the original negative root locus consists of.

If $z = x + iy$, the root-locus satisfies the equation

$$y[3x^2 + 6x + 2 - y^2] = 0.$$

That is, it consists of the line $y = 0$ and the hyperbola $3(x + 1)^2 - y^2 = 1$. (See [6].)

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