N67 12156 (ACCESSION NUMBER) — 31 (PAGES) — CR-80096 (NASA CR OR TMX OR AD NUMBER)
(THRU) — 1 (CODE) — 19 (CATEGORY)
FACILITY FORM 602

WYLE LABORATORIES RESEARCH STAFF
Report WR 66-24

SOME INTEGRATION FORMULAE WHICH SIMPLIFY THE EVALUATION OF CERTAIN INTEGRALS IN COMMON USE BY ENGINEERS

GPO PRICE $ ______

CFSTI PRICE(S) $ ______

Hard copy (HC) 2.00

Microfiche (MF) .50

ff 653 July 65

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This Report reflects work conducted under
Contract No. NAS8 - 5384

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COPY NO. 37

N67-12156

SUMMARY

A table of fifty-six commonly used integrals is presented in this report. Many of these integrals are not included in standard tables of integrals and it is left to the user of such tables to reduce the complex integrals considered in this report to the standard elementary forms found in such tables. These integrals are of the mixed polynominal, exponential, and trigonometric type. Some of these integrals are already available in the standard tables of integrals, although they are usually given without phase constants included in the arguments. Use of this table of integrals reduces the amount of algebraic manipulation needed to evaluate many common integrals. Four integration formulae are also derived which show that the fifty-six integrals given here are rather special; these formulae show how the integrals presented may be derived from the simpler integral tables, in which phase angles are neglected, by the process of induction. The integrals contained herein have proved extremely valuable in reducing analysis time in many vibration, acoustic, and aerodymanic problems

Author

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LIST OF SYMBOLS

a	=	coefficient of exponent of e
e	=	2.718, base of natural numbers
f	=	function of x
F_2	=	value of second intermediate integral
F_4	=	value of fourth intermediate integral
g	=	function of x
I_n	=	value of integral of type n $(n = 1,2,3,4)$
m	=	coefficient of x
n	=	coefficient of x
p	=	constant
q	=	constant
x	=	variable of integration
x_1	=	lower limit of integration
x_2	=	upper limit of integration

Greek Alphabet

ϵ_1	=	± 1	the ambiguous sign depends upon the function under consideration
ϵ_2	=	± 1	

Superscript

$'$	prime indicates differentiation with respect to x
$''$	double prime indicates differentiation twice with respect to x

1.0 INTRODUCTION

When faced with the task of evaluating an integral of the type:

$$I = \int_{x_1}^{x_2} x\,e^{ax} \sin(mx + p) \cos(nx + q)\; dx \qquad \text{(i)}$$

the average engineer will either expand the sine and cosine terms and then at this stage integrate the resulting four terms by parts or with the aid of one of the commonly available books of tabulated integrals (References 1 and 2). The problem in evaluating an integral of the above type is that it is not possible to use the method of substitution of another variable for the arguments of sine or cosine, since the coefficients of x and the constants in each argument are different.

In Section 2 of this report, a table of fifty-six indefinite integrals is listed. Some of these were evaluated from first principles and some were evaluated by induction using the integration formulae developed in Section 3 of this report and given by Equations (7), (18), (25), and (36). Some of these integrals are available in References 1 and 2 and more are available in such specialized works as Reference 3. However, in general, if these integrals are given they are usually given with $p = q = 0$ and often with $m = n = 1$ [see Equation (i) above] .

An engineer often needs to evaluate an integral where constants are included in the arguments of the functions of x [as in Equation (i) above] and the authors believe the fifty-six integrals presented in this report will prove to be very useful. In particular, one author found these integrals to be invaluable in reducing the amount of algebraic manipulation needed in the complex integration carried out in References 4 and 5.

In Section 4 of this report, it is shown that the fifty-six integrals presented are rather special cases in that if the integral with no constants in the arguments of the functions of x is known, then the integral with constants present in the argument is also known. The reason for this is demonstrated by examining the integration formulae derived in Section 3.

2.0 TABLE OF INDEFINITE INTEGRALS

2.1 Integrals of the First Type.

1). $$\int_{x_1}^{x_2} \sin(mx+p)\,dx = -\frac{1}{m}\Big\{\cos(mx+p)\Big\}_{x_1}^{x_2}$$

2). $$\int_{x_1}^{x_2} \cos(mx+p)\,dx = \frac{1}{m}\Big\{\sin(mx+p)\Big\}_{x_1}^{x_2}$$

3). $$\int_{x_1}^{x_2} \sin(mx+p)\cos(nx+q)\,dx = -\frac{1}{2}\left\{\frac{\cos[(m-n)x+p-q]}{m-n} + \frac{\cos[(m+n)x+p+q]}{m+n}\right\}_{x_1}^{x_2}$$

4). $$\int_{x_1}^{x_2} \sin(mx+p)\sin(nx+q)\,dx = \frac{1}{2}\left\{\frac{\sin[(m-n)x+p-q]}{m-n} - \frac{\sin[(m+n)x+p+q]}{m+n}\right\}_{x_1}^{x_2}$$

5). $$\int_{x_1}^{x_2} \cos(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{\frac{\sin[(m-n)x+p-q]}{m-n} + \frac{\sin[(m+n)x+p+q]}{m+n}\right\}_{x_1}^{x_2}$$

6). $$\int_{x_1}^{x_2} \sinh(mx+p)\,dx = \frac{1}{m}\Big\{\cosh(mx+p)\Big\}_{x_1}^{x_2}$$

7). $$\int_{x_1}^{x_2} \cosh(mx+p)\,dx = \frac{1}{m}\Big\{\sinh(mx+p)\Big\}_{x_1}^{x_2}$$

8). $$\int_{x_1}^{x_2} \sinh(mx+p)\cosh(nx+q)\,dx = \frac{1}{2}\left\{\frac{\cosh[(m+n)x+p+q]}{m+n} + \frac{\cosh[(m-n)x+p-q]}{m-n}\right\}_{x_1}^{x_2}$$

9) $$\int_{x_1}^{x_2} \sinh(mx+p)\sinh(nx+q)\,dx = \frac{1}{2}\left\{\frac{\sinh[(m+n)x+p+q]}{m+n} - \frac{\sinh[(m-n)x+p-q]}{m-n}\right\}_{x_1}^{x_2}$$

10). $$\int_{x_1}^{x_2} \cosh(mx+p)\cosh(nx+q)\,dx = \frac{1}{2}\left\{\frac{\sinh[(m+n)x+p+q]}{m+n} + \frac{\sinh[(m-n)x+p-q]}{m-n}\right\}_{x_1}^{x_2}$$

11). $$\int_{x_1}^{x_2} \sinh(mx+p)\sin(nx+q)\,dx = \left\{\frac{m\cosh(mx+p)\sin(nx+q) - n\sinh(mx+p)\cos(nx+q)}{m^2+n^2}\right\}_{x_1}^{x_2}$$

12). $$\int_{x_1}^{x_2} \sinh(mx+p)\cos(nx+q)\,dx = \left\{\frac{m\cosh(mx+p)\cos(nx+q) + n\sinh(mx+p)\sin(nx+q)}{m^2+n^2}\right\}_{x_1}^{x_2}$$

13). $$\int_{x_1}^{x_2} \cosh(mx+p)\sin(nx+q)\,dx = \left\{\frac{m\sinh(mx+p)\sin(nx+q) - n\cosh(mx+p)\cos(nx+q)}{m^2+n^2}\right\}_{x_1}^{x_2}$$

14). $$\int_{x_1}^{x_2} \cosh(mx+p)\cos(nx+q)\,dx = \left\{\frac{m\sinh(mx+p)\cos(nx+q) + n\cosh(mx+p)\sin(nx+q)}{m^2+n^2}\right\}_{x_1}^{x_2}$$

2.2 Integrals of the Second Type

1). $$\int_{x_1}^{x_2} e^{ax}\sin(mx+p)\,dx = \left\{\frac{e^{ax}}{a^2+m^2}\left[a\sin(mx+p) - m\cos(mx+p)\right]\right\}_{x_1}^{x_2}$$

2) $$\int_{x_1}^{x_2} e^{ax}\cos(mx+p)\,dx = \left\{\frac{e^{ax}}{a^2+m^2}\left[a\cos(mx+p) + m\sin(mx+p)\right]\right\}_{x_1}^{x_2}$$

3). $$\int_{x_1}^{x_2} e^{ax}\sin(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{e^{ax}\left[\frac{a\sin[(m-n)x+p-q] - (m-n)\cos[(m-n)x+p-q]}{a^2+(m-n)^2}\right] + e^{ax}\left[\frac{a\sin[(m+n)x+p+q] - (m+n)\cos[(m+n)x+p+q]}{a^2+(m+n)^2}\right]\right\}_{x_1}^{x_2}$$

4). $$\int_{x_1}^{x_2} e^{ax} \sin(mx+p)\sin(nx+q)\,dx = \frac{1}{2}\left\{ e^{ax}\left[\frac{(m-n)\sin[(m-n)x+p-q] + a\cos[(m-n)x+p-q]}{a^2+(m-n)^2}\right] - e^{ax}\left[\frac{(m+n)\sin[(m+n)x+p+q] + a\cos[(m+n)x+p+q]}{a^2+(m+n)^2}\right]\right\}_{x_1}^{x_2}$$

5). $$\int_{x_1}^{x_2} e^{ax} \cos(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{ e^{ax}\left[\frac{(m-n)\sin[(m-n)x+p-q] + a\cos[(m-n)x+p-q]}{a^2+(m-n)^2}\right] + e^{ax}\left[\frac{(m+n)\sin[(m+n)x+p+q] + a\cos[(m+n)x+p+q]}{a^2+(m+n)^2}\right]\right\}_{x_1}^{x_2}$$

6). $$\int_{x_1}^{x_2} e^{ax} \sinh(mx+p)\,dx = \left\{\frac{e^{ax}}{a^2-m^2}\left[a\sinh(mx+p) - m\cosh(mx+p)\right]\right\}_{x_1}^{x_2}$$

7). $$\int_{x_1}^{x_2} e^{ax} \cosh(mx+p)\,dx = \left\{\frac{e^{ax}}{a^2-m^2}\left[a\cosh(mx+p) - m\sinh(mx+p)\right]\right\}_{x_1}^{x_2}$$

8). $$\int_{x_1}^{x_2} e^{ax} \sinh(mx+p)\cosh(nx+q)\,dx = \left\{ e^{ax}\left[\frac{a\sinh[(m+n)x+p+q] - (m+n)\cosh[(m+n)x+p+q]}{a^2-(m+n)^2}\right] + e^{ax}\left[\frac{a\sinh[(m-n)x+p-q] - (m-n)\cosh[(m-n)x+p-q]}{a^2-(m-n)^2}\right]\right\}_{x_1}^{x_2}$$

9). $$\int_{x_1}^{x_2} e^{ax} \sinh(mx+p)\sinh(nx+q)\,dx = \left\{ e^{ax}\left[\frac{a\cosh[(m+n)x+p+q] - (m+n)\sinh[(m+n)x+p+q]}{a^2-(m+n)^2}\right] - e^{ax}\left[\frac{a\cosh[(m-n)x+p-q] - (m-n)\sinh[(m-n)x+p-q]}{a^2-(m-n)^2}\right]\right\}_{x_1}^{x_2}$$

10). $$\int_{x_1}^{x_2} e^{ax} \cosh(mx+p)\cosh(nx+q)\,dx = \left\{ e^{ax}\left[\frac{a\cosh[(m+n)x+p+q] - (m+n)\sinh[(m+n)x+p+q]}{a^2-(m+n)^2}\right] - e^{ax}\left[\frac{a\cosh[(m-n)x+p-q] - (m-n)\sinh[(m-n)x+p-q]}{a^2-(m-n)^2}\right]\right\}_{x_1}^{x_2}$$

11). $$\int_{x_1}^{x_2} e^{ax} \sinh(mx+p)\sin(nx+q)\,dx = \frac{1}{2}\left\{ e^{(a+m)x+p}\left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2}\right] - e^{(a-m)x-p}\left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2}\right]\right\}_{x_1}^{x_2}$$

12). $$\int_{x_1}^{x_2} e^{ax} \sinh(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{ e^{(a+m)x+p}\left[\frac{(a+m)\cos(nx+q) + n\sin(nx+q)}{(a+m)^2+n^2}\right] - e^{(a-m)x-p}\left[\frac{(a-m)\cos(nx+q) + n\sin(nx+q)}{(a-m)^2+n^2}\right]\right\}_{x_1}^{x_2}$$

13). $$\int_{x_1}^{x_2} e^{ax} \cosh(mx+p)\sin(nx+q)\,dx = \frac{1}{2}\left\{ e^{(a+m)x+p}\left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2}\right] + e^{(a+m)x-p}\left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2}\right]\right\}_{x_1}^{x_2}$$

14). $$\int_{x_1}^{x_2} e^{ax} \cosh(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{ e^{(a+m)x+p}\left[\frac{(a+m)\cos(nx+q) + n\sin(nx+q)}{(a+m)^2+n^2}\right] + e^{(a+m)x-p}\left[\frac{(a-m)\cos(nx+q) + n\sin(nx+q)}{(a-m)^2+n^2}\right]\right\}_{x_1}^{x_2}$$

2.3 Integrals of the Third Type

1). $$\int_{x_1}^{x_2} x\sin(mx+p)\,dx = \frac{1}{m}\left\{\frac{1}{m}\sin(mx+p) - x\cos(mx+p)\right\}_{x_1}^{x_2}$$

2). $$\int_{x_1}^{x_2} x\cos(mx+p)\,dx = \frac{1}{m}\left\{\frac{1}{m}\cos(mx+p) + \sin(mx+p)\right\}_{x_1}^{x_2}$$

3). $$\int_{x_1}^{x_2} x \sin(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{\frac{\sin[(m+n)x+p+q]}{(m+n)^2} - \frac{x\cos[(m+n)x+p+q]}{m+n} + \frac{\sin[(m-n)x+p-q]}{(m-n)^2} - \frac{x\cos[(m-n)x+p-q]}{m-n}\right\}_{x_1}^{x_2}$$

4). $$\int_{x_1}^{x_2} x \sin(mx+p)\sin(nx+q)\,dx = \frac{1}{2}\left\{\frac{\cos[(m-n)x+p-q]}{(m-n)^2} + \frac{x\sin[(m-n)x+p-q]}{m-n} - \frac{\cos[(m+n)x+p+q]}{(m+n)^2} - \frac{x\sin[(m+n)x+p+q]}{m+n}\right\}_{x_1}^{x_2}$$

5). $$\int_{x_1}^{x_2} x \cos(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{\frac{\cos[(m-n)x+p-q]}{(m-n)^2} + \frac{x\sin[(m-n)x+p-q]}{m-n} + \frac{\cos[(m+n)x+p+q]}{(m+n)^2} - \frac{x\sin[(m+n)x+p+q]}{m+n}\right\}_{x_1}^{x_2}$$

6). $$\int_{x_1}^{x_2} x \sinh(mx+p)\,dx = \frac{1}{m}\left\{x\cosh(mx+p) - \frac{1}{m}\sinh(mx+p)\right\}_{x_1}^{x_2}$$

7). $$\int_{x_1}^{x_2} x \cosh(mx+p)\,dx = \frac{1}{m}\left\{x\sinh(mx+p) - \frac{1}{m}\cosh(mx+p)\right\}_{x_1}^{x_2}$$

8). $$\int_{x_1}^{x_2} x \sinh(mx+p)\cosh(nx+q)\,dx = \frac{1}{2}\left\{\frac{x\cosh[(m+n)x+p+q]}{m+n} - \frac{\sinh[(m+n)x+p+q]}{(m+n)^2} + \frac{x\cosh[(m-n)x+p-q]}{m-n} - \frac{\sinh[(m-n)x+p-q]}{(m-n)^2}\right\}_{x_1}^{x_2}$$

9). $$\int_{x_1}^{x_2} x \sinh(mx+p)\sinh(nx+q)\,dx = \frac{1}{2}\left\{ \frac{x\sinh[(m+n)x+p+q]}{m+n} - \frac{\cosh[(m+n)x+p+q]}{(m+n)^2} - \frac{x\sinh[(m-n)x+p-q]}{m-n} + \frac{\cosh[(m-n)x+p-q]}{(m-n)^2} \right\}_{x_1}^{x_2}$$

10). $$\int_{x_1}^{x_2} x \cosh(mx+p)\cosh(nx+q)\,dx = \frac{1}{2}\left\{ \frac{x\sinh[(m+n)x+p+q]}{m+n} - \frac{\cosh[(m+n)x+p+q]}{(m+n)^2} + \frac{x\sinh[(m-n)x+p-q]}{m-n} - \frac{\cosh[(m-n)x+p-q]}{(m-n)^2} \right\}_{x_1}^{x_2}$$

11). $$\int_{x_1}^{x_2} x \sinh(mx+p)\sin(nx+q)\,dx = \left\{ x\left[\frac{m\cosh(mx+p)\sin(nx+q) - n\sinh(mx+p)\cos(nx+q)}{m^2+n^2}\right] - \frac{(m^2-n^2)\sinh(mx+p)\sin(nx+q) - 2mn\cosh(mx+p)\cos(nx+q)}{(m^2+n^2)^2} \right\}_{x_1}^{x_2}$$

12). $$\int_{x_1}^{x_2} x \sinh(mx+p)\cos(nx+q)\,dx = \left\{ x\left[\frac{m\cosh(mx+p)\cos(nx+q) + n\sinh(mx+p)\sin(nx+q)}{m^2+n^2}\right] - \frac{(m^2-n^2)\sinh(mx+p)\sin(nx+q) + 2mn\cosh(mx+p)\sin nx+q}{(m^2+n^2)^2} \right\}_{x_1}^{x_2}$$

13). $$\int_{x_1}^{x_2} x \cosh(mx+p)\sin(nx+q)\,dx = \left\{ x\left[\frac{m\sinh(mx+p)\sin(nx+q) - n\cosh(mx+p)\cos(nx+q)}{m^2+n^2}\right] - \frac{(m^2-n^2)\cosh(mx+p)\sin(nx+q) - 2mn\sinh(mx+p)\cos(nx+q)}{(m^2+n^2)^2} \right\}_{x_1}^{x_2}$$

$$14).\ \int_{x_1}^{x_2} x \cosh(mx+p)\cos(nx+q)\,dx = \left\{ x \left[\frac{m \sinh(mx+p)\cos(nx+q) + n\cosh(mx+p)\sin(nx+q)}{m^2+n^2} \right] - \right.$$

$$\left. - \frac{(m^2-n^2)\cosh(mx+p)\cos(nx+q) + 2mn\sinh(mx+p)\sin(nx+q)}{(m^2+n^2)^2} \right\}_{x_1}^{x_2}$$

2.4 Integrals of the Fourth Type

$$1).\ \int_{x_1}^{x_2} x e^{ax}\sin(mx+p)\,dx = \left\{ x e^{ax}\left[\frac{a\sin(mx+p) - m\cos(mx+p)}{a^2+m^2} \right] - e^{ax}\left[\frac{(a^2-m^2)\sin(mx+p) - 2am\cos(mx+p)}{(a^2+m^2)^2} \right] \right\}_{x_1}^{x_2}$$

$$2).\ \int_{x_1}^{x_2} x e^{ax}\cos(mx+p)\,dx = \left\{ x e^{ax}\left[\frac{a\cos(mx+p) + m\sin(mx+p)}{a^2+m^2} \right] - e^{ax}\left[\frac{(a^2-m^2)\cos(mx+p) + 2am\sin(mx+p)}{(a^2+m^2)^2} \right] \right\}_{x_1}^{x_2}$$

$$3).\ \int_{x_1}^{x_2} x e^{ax}\sin(mx+p)\cos(nx+q)\,dx$$

$$= \left\{ x e^{ax}\left[\frac{a\sin[(m+n)x+p+q] - (m+n)\cos[(m+n)x+p+q]}{a^2+(m+n)^2} \right] - \right.$$

$$- e^{ax}\left[\frac{[a^2-(m+n)^2]\sin[(m+n)x+p+q] - 2a(m+n)\cos[(m+n)x+p+q]}{[a^2+(m+n)^2]^2} \right] +$$

$$+ x e^{ax}\left[\frac{a\sin[(m-n)x+p-q] - (m-n)\cos[(m-n)x+p-q]}{a^2+(m-n)^2} \right] -$$

$$\left. - e^{ax}\left[\frac{[a^2-(m-n)^2]\sin[(m-n)x+p-q] - 2a(m-n)\cos[(m-n)x+p-q]}{[a^2+(m-n)^2]^2} \right] \right\}_{x_1}^{x_2}$$

4). $\displaystyle\int_{x_1}^{x_2} x\, e^{ax} \sin(mx+p) \cos(nx+q)\, dx$

$$= \left\{ x\, e^{ax} \left[\frac{a \cos[(m-n)x + p - q] + (m-n) \sin[(m-n)x + p - q]}{a^2 + (m-n)^2} \right] - \right.$$

$$- e^{ax} \left[\frac{[a^2 - (m-n)^2] \cos[(m-n)x + p - q] + 2a(m-n) \sin[(m-n)x + p - q]}{[a^2 + (m-n)^2]^2} \right] -$$

$$- x\, e^{ax} \left[\frac{a \cos[(m+n)x + p + q] + (m+n) \sin[(m+n)x + p + q]}{a^2 + (m+n)^2} \right] +$$

$$\left. + e^{ax} \left[\frac{[a^2 - (m+n)^2] \cos[(m+n)x + p + q] + 2a(m+n) \sin[(m+n)x + p + q]}{[a^2 + (m+n)^2]^2} \right] \right\}_{x_1}^{x_2}$$

5). $\displaystyle\int_{x_1}^{x_2} x\, e^{ax} \cos(mx+p) \cos(nx+q)\, dx$

$$= \left\{ x\, e^{ax} \left[\frac{a \cos[(m-n)x + p - q] + (m-n) \sin[(m-n)x + p - q]}{a^2 + (m-n)^2} \right] - \right.$$

$$- e^{ax} \left[\frac{[a^2 - (m-n)^2] \cos[(m-n)x + p - q] + 2a(m-n) \sin[(m-n)x + p - q]}{[a^2 + (m-n)^2]^2} \right] +$$

$$+ x\, e^{ax} \left[\frac{a \cos[(m+n)x + p + q] + (m+n) \sin[(m+n)x + p + q]}{a^2 + (m+n)^2} \right] -$$

$$\left. - e^{ax} \left[\frac{[a^2 - (m+n)^2] \cos[(m+n)x + p + q] + 2a(m+n) \sin[(m+n)x + p + q]}{[a^2 + (m+n)^2]^2} \right] \right\}_{x_1}^{x_2}$$

6). $$\int_{x_1}^{x_2} x\, e^{ax} \sinh(mx+p)\, dx = \left\{ \frac{e^{ax}}{a^2 - m^2} \left[\left[ax - \frac{a^2 + m^2}{a^2 - m^2} \right] \sinh(mx+p) - \left[mx - \frac{2am}{a^2 - m^2} \right] \cosh(mx+p) \right] \right\}_{x_1}^{x_2}$$

7). $$\int_{x_1}^{x_2} x\,e^{ax}\cosh(mx+p)\,dx = \left\{ \frac{e^{ax}}{a^2-m^2}\left[\left[ax - \frac{a^2+m^2}{a^2-m^2}\right]\cosh(mx+p) - \left[mx - \frac{2am}{a^2-m^2}\right]\sinh(mx+p)\right]\right\}_{x_1}^{x_2}$$

8). $$\int_{x_1}^{x_2} x\,e^{ax}\sinh(mx+p)\cosh(nx+q)\,dx$$

$$= \left\{ \frac{e^{ax}}{a^2-(m+n)^2}\left[\left[ax - \frac{a^2+(m+n)^2}{a^2-(m-n)^2}\right]\sinh[(m+n)x+p+q] - \left[mx - \frac{2a(m+n)}{a^2-(m+n)^2}\right]\cosh[(m+n)x+p+q]\right] + \right.$$

$$\left. + \frac{e^{ax}}{a^2-(m-n)^2}\left[\left[ax - \frac{a^2+(m-n)^2}{a^2-(m-n)^2}\right]\sinh[(m-n)x+p-q] - \left[mx - \frac{2a(m-n)}{a^2-(m-n)^2}\right]\cosh[(m-n)x+p-q]\right]\right\}_{x_1}^{x_2}$$

9). $$\int_{x_1}^{x_2} x\,e^{ax}\sinh(mx+p)\sinh(nx+q)\,dx$$

$$= \left\{ \frac{e^{ax}}{a^2-(m+n)^2}\left[\left[ax - \frac{a^2+(m+n)^2}{a^2-(m+n)^2}\right]\cosh[(m+n)x+p+q] - \left[mx - \frac{2a(m+n)}{a^2-(m+n)^2}\right]\sinh[(m+n)x+p+q]\right] - \right.$$

$$\left. - \frac{e^{ax}}{a^2-(m-n)^2}\left[\left[ax - \frac{a^2+(m-n)^2}{a^2-(m-n)^2}\right]\cosh[(m-n)x+p-q] - \left[mx - \frac{2a(m-n)}{a^2-(m+n)^2}\right]\sinh[(m-n)x+p-q]\right]\right\}_{x_1}^{x_2}$$

10). $$\int_{x_1}^{x_2} x\,e^{ax}\cosh(mx+p)\cosh(nx+q)\,dx$$

$$= \left\{ \frac{e^{ax}}{a^2-(m+n)^2}\left[\left[ax - \frac{a^2+(m+n)^2}{a^2-(m+n)^2}\right]\cosh[(m+n)x+p+q] - \left[mx - \frac{2a(m+n)}{a^2-(m+n)^2}\right]\sinh[(m+n)x+p+q]\right] + \right.$$

$$\left. + \frac{e^{ax}}{a^2-(m-n)^2}\left[\left[ax - \frac{a^2+(m-n)^2}{a^2-(m-n)^2}\right]\cosh[(m-n)x+p-q] - \left[mx - \frac{2a(m-n)}{a^2-(m-n)^2}\right]\sinh[(m-n)x+p-q]\right]\right\}_{x_1}^{x_2}$$

11). $\int_{x_1}^{x_2} x\, e^{ax} \sinh(mx+p) \sin(nx+q)\, dx$

$$= \frac{1}{2}\left\{ xe^{(a+m)x+p}\left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2}\right] - e^{(a+m)x+p}\left[\frac{[(a+m)^2-n^2]\sin(nx+q) - 2(a+m)n\cos(nx+q)}{((a+m)^2+n^2)^2}\right] - \right.$$

$$\left. - xe^{(a-m)x-p}\left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2}\right] + e^{(a+m)x+p}\left[\frac{[(a-m)^2-n^2]\sin(nx+q) - 2(a-m)n\cos(nx+q)}{((a-m)^2+n^2)^2}\right]\right\}_{x_1}^{x_2}$$

12). $\int_{x_1}^{x_2} x\, e^{ax} \sinh(mx+p) \cos(nx+q)\, dx$

$$= \frac{1}{2}\left\{ xe^{(a+m)x+p}\left[\frac{(a+m)\cos(nx+q) + n\sin(nx+q)}{(a+m)^2+n^2}\right] - e^{(a+m)x+p}\left[\frac{[(a+m)^2-n^2]\cos(nx+q) + 2(a+m)n\sin(nx+q)}{((a+m)^2+n^2)^2}\right] - \right.$$

$$\left. - xe^{(a-m)x-p}\left[\frac{(a-m)\cos(nx+q) + n\sin(nx+q)}{(a-m)^2+n^2}\right] + e^{(a-m)x-p}\left[\frac{[(a-m)^2-n^2]\cos(nx+q) + 2(a-m)n\sin(nx+q)}{((a-m)^2+n^2)^2}\right]\right\}_{x_1}^{x_2}$$

13). $\int_{x_1}^{x_2} x\, e^{ax} \cosh(mx+p) \sin(nx+q)\, dx$

$$= \frac{1}{2}\left\{ xe^{(a+m)x+p}\left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2}\right] - e^{(a+m)x+p}\left[\frac{[(a+m)^2-n^2]\sin(nx+q) - 2(a-m)n\cos(nx+q)}{((a+m)^2+n^2)^2}\right] + \right.$$

$$\left. + xe^{(a-m)x-p}\left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2}\right] - e^{(a-m)x-p}\left[\frac{[(a-m)^2-n^2]\sin(nx+q) - 2(a-m)n\cos(nx+q)}{((a-m)^2+n^2)^2}\right]\right\}_{x_1}^{x_2}$$

14). $\int_{x_1}^{x_2} x\, e^{ax} \cosh(mx+q) \cos(nx+q)\, dx$

$$= \frac{1}{2}\left\{ xe^{(a+m)x+p}\left[\frac{(a+m)\cos(nx+q) + n\sin(nx+q)}{(a+m)^2+n^2}\right] - e^{(a+m)x+p}\left[\frac{[(a+m)^2-n^2]\cos(nx+q) + 2(a+m)n\sin(nx+q)}{((a+m)^2+n^2)^2}\right] + \right.$$

$$\left. + xe^{(a-m)x-p}\left[\frac{(a-m)\cos(nx+q) + n\sin(nx+q)}{(a-m)^2+n^2}\right] - e^{(a-m)x-p}\left[\frac{[(a-m)^2-n^2]\cos(nx+q) + 2(a-m)n\sin(nx+q)}{((a-m)^2+n^2)^2}\right]\right\}_{x_1}^{x_2}$$

3.0 INTEGRATION FORMULAE

In the analysis which follows in this Section of the report, for the sake of brevity:

f (mx + p) is abbreviated to f
g (nx + q) is abbreviated to g

$\left\{ \quad \right\}$ represents evaluation of the limits: $\left\{ \quad \right\}_{x_1}^{x_2}$

$\int$ is used as an abbreviation of the integral $\int_{x_1}^{x_2}$

the prime ' is used to refer to differentiation with respect to x, the double prime " is used to refer to differentiation twice with respect to x.

3.1 A Formula for Integrals Involving the Product of an Exponential with a Trigonometric or Hyperbolic Function

Consider an integral of the type:

$$I_1 = \int e^{ax} f \, dx \tag{1}$$

Let:

$$\frac{d^2 f}{dx^2} = f'' = \epsilon_1 m^2 f \tag{2}$$

where $\epsilon_1 = \pm 1$. The ambiguous sign depends upon the function f under consideration. Thus the function f is restricted to sine, cosine, exponential, and hyperbolic sine or cosine functions.

Integrating Equation (1) by parts gives:

$$I_1 = \left\{ \frac{e^{ax}}{a} \cdot f \right\} - \frac{1}{a} \int e^{ax} f' \, dx \tag{3}$$

Integrating the second term of Equation (3) by parts gives:

$$I_1 = \left\{ \frac{e^{ax}}{a} \cdot f \right\} - \left\{ \frac{e^{ax}}{a^2} \cdot f' \right\} + \frac{1}{a^2} \int e^{ax} f'' \, dx \tag{4}$$

and making use of Equation (2):

$$I_1 = \left\{ \frac{e^{ax}}{a} \cdot f - \frac{e^{ax}}{a^2} \cdot f' \right\} + \epsilon_1 \left(\frac{m}{a} \right)^2 \int e^{ax} f \, dx \tag{5}$$

and thus substituting Equation (1) into the third term of Equation (5) gives:

$$I_1 = \left\{ \frac{e^{ax}}{a} \cdot f - \frac{e^{ax}}{a^2} \cdot f' \right\} + \epsilon_1 \left(\frac{m}{a} \right)^2 I_1 \tag{6}$$

which gives on rearranging and rewriting in full:

$$I_1 = \int_{x_1}^{x_2} e^{ax} f(mx+p)\,dx = \left\{ e^{ax} \left[\frac{af(mx+p) - f'(mx+p)}{a^2 - \epsilon_1 m^2} \right] \right\}_{x_1}^{x_2} \tag{7}$$

provided $a^2 - \epsilon_1 m^2 \neq 0$

3.2 A Formula for Integrals Involving the Product of an Exponential with Two Trigonometric or Hyperbolic Functions

Consider an integral of the type:

$$I_2 = \int e^{ax} f\, g\, dx \tag{8}$$

Let:

$$\frac{d^2 f}{dx^2} = f'' = \epsilon_1 m^2 f \tag{9}$$

$$\frac{d^2 g}{dx^2} = g'' = \epsilon_2 n^2 g \tag{10}$$

where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. The ambiguous signs depending upon the functions f and g under consideration. Again this restricts the functions f and g to sine, cosine, exponential, and hyperbolic sine or cosine functions.

Integrating Equation (8) by parts gives:

$$I_2 = \left\{ \frac{e^{ax}}{a} f\, g \right\} - \frac{1}{a}\int e^{ax} f' g\, dx - \frac{1}{a}\int e^{ax} f g'\, dx \tag{11}$$

Integrating the last two terms of Equation (11) by parts gives:

$$I_2 = \left\{ \frac{e^{ax}}{a} fg \right\} - \left\{ \frac{e^{ax}}{a^2} f'g \right\} + \frac{1}{a}\int \frac{e^{ax}}{a} f'' g\, dx + \frac{1}{a}\int \frac{e^{ax}}{a} f'g'\, dx - \left\{ \frac{e^{ax}}{a^2} fg' \right\} + \frac{1}{a}\int \frac{e^{ax}}{a} f'g' dx + \frac{1}{a}\int \frac{e^{ax}}{a} fg'' dx \tag{12}$$

Regrouping terms and using Equations (8), (9), and (10), Equation(12) may be rewritten:

$$I_2 = \left\{ \frac{e^{ax}}{a^2} \left[afg - (fg)' \right] \right\} + \frac{1}{a^2}\left[\epsilon_1 m^2 + \epsilon_2 n^2 \right] I_2 + \frac{2}{a^2}\int e^{ax} f' g'\, dx \tag{13}$$

Thus:

$$(a^2 - \epsilon_1 m^2 - \epsilon_2 n^2)\, I_2 = \left\{ e^{ax} \left[afg - (fg)' \right] \right\} + 2F_2 \tag{14}$$

where:

$$F_2 = \int e^{ax} f' g'\, dx \tag{15}$$

Comparing Equation (15) with Equation (8), the result of Equation (13) may be utilitzed if f and g are replaced by f' and g' respectively, thus:

$$F_2 = \left\{\frac{e^{ax}}{a^2}\left[\ af'g' - (f'g')'\right]\right\} + \frac{1}{a^2}\left[\epsilon_1 m^2 + \epsilon_2 n^2\ \right]F_2 + \frac{2}{a^2}\int e^{ax} f''g''\ dx \tag{16}$$

Regrouping terms and using Equations (9) and (10), Equation (16) may be rewritten:

$$(a^2 - \epsilon_1 m^2 - \epsilon_2 n^2)\ F_2 = \left\{e^{ax}\left[af'g' - (f'g')'\ \right]\right\} \qquad + 2\epsilon_1\epsilon_2 m^2 n^2 I_2 \tag{17}$$

Thus provided $a^2 - \epsilon_1 m^2 - \epsilon_2 n^2 \neq 0$, F_2, as given by Equation (17), may be substituted into Equation (14), which gives after rearrangement and writing I_2 in full:

$$I_2 = \int_{x_1}^{x_2} e^{ax} f(mx+p)\ g\ (nx+q)\ dx = \left\{e^{ax}\left[\frac{(a^2-\epsilon_1 m^2-\epsilon_2 n^2)\left[\ afg-(fg)'\ \right] +2\left[\ af'g' -(f'g')'\right]}{(a^2-\epsilon_1 m^2-\epsilon_2 n^2)^2 -4\epsilon_1\epsilon_2 m^2 n^2}\right]\right\} \tag{18}$$

provided $(a^2-\epsilon_1 m^2-\epsilon_2 n^2)^2 -4\epsilon_1\epsilon_2 m^2 n^2 \neq 0$.

3.3 A Formula for Integrals Involving the Product of an Exponential with a First Order Polynomial and a Trigonometric or Hyperbolic Function

Consider an integral of the type:

$$I_3 = \int x e^{ax} f\ dx \tag{19}$$

Let:

$$\frac{d^2 f}{dx^2} = f'' = \epsilon_1 m^2 f \tag{20}$$

where $\epsilon_1 = \pm 1$. The ambiguous sign depends upon the function f under consideration. Again this restricts the function f to sine , cosine, exponential, and hyperbolic sine or cosine functions.

Integrating Equation (19) by parts gives:

$$I_3 = \left\{\frac{e^{ax}}{a^2}\ (ax-1)f\right\} - \frac{1}{a^2}\int e^{ax}(ax-1)f'\ dx \tag{21}$$

$$I_3 = \left\{\frac{e^{ax}}{a^2}\ (ax-1)f\ \right\} + \frac{1}{a^2}\int e^{ax} f'\ dx - \frac{1}{a}\int x e^{ax} f'\ dx \tag{22}$$

The second term on the right hand side of Equation (22) may be evaluated by replacing f by f' in Equation (7).

Thus:

$$I_3 = \left\{\frac{e^{ax}}{a^2}(ax-1)f\right\} + \left\{\frac{e^{ax}[af'-f'']}{a^2(a^2-\epsilon_1 m^2)}\right\} - \frac{1}{a}\int xe^{ax}f'\,dx \tag{23}$$

The third term on the right hand side of Equation (22) may be evaluated by replacing f by f' in Equation (23), thus Equation (23) becomes [using Equation (20)] :

$$I_3 = \left\{\frac{e^{ax}}{a^2}(ax-1)f\right\} + \left\{\frac{e^{ax}[af'-\epsilon_1 m^2 f]}{a^2(a^2-\epsilon_1 m2)}\right\} - \frac{1}{a^3}\left\{e^{ax}(ax-1)f'+e^{ax}\epsilon_1 m^2\left[\frac{af-f'}{a^2-\epsilon_1 m^2}\right]\right\} + \epsilon_1\frac{m^2}{a^2}\int xe^{ax}f\,dx \tag{24}$$

On rearranging, noting that the last term of Equation (24) is $\epsilon_1 m^2 I_3/a^2$, and writing I_3 in full gives:

$$I_3 = \int_{x1}^{x2} xe^{ax}f(mx+p)\,dx = \left\{\frac{e^{ax}}{a^2-\epsilon_1 m^2}\left[(ax-1)(f-f'/a) + \frac{(a+\epsilon_1 m^2/a)f'-2\epsilon_1 m^2 f}{a^2-\epsilon_1 m^2}\right]\right\} \tag{25}$$

provided $a^2 - \epsilon_1 m^2 \neq 0$

3.4 A Formula for Integrals Involving the Product of an Exponential with a First Order Polynomial and Two Trigonometric or Hyperbolic Functions

Consider an integral of the type:

$$I_4 = \int xe^{ax}\,f\,g\,dx \tag{26}$$

Let:

$$\frac{d^2f}{dx^2} = f'' = \epsilon_1 m^2 f \tag{27}$$

$$\frac{d^2g}{dx^2} = g'' = \epsilon_2 n^2 g \tag{28}$$

where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. The ambiguous sign again depends upon the function f and g under consideration. Again this restricts the functions f and g to sine, cosine, exponential, and hyperbolic sine or cosine functions.

Integrating Equation (26) by parts gives:

$$I_4 = \left\{\frac{e^{ax}}{a^2}(ax-1)fg\right\} - \frac{1}{a^2}\int(ax-1)e^{ax}fg'\,dx - \frac{1}{a^2}\int(ax-1)e^{ax}f'g\,dx \tag{29}$$

$$I_4 = \left\{\frac{e^{ax}}{a^2}(ax-1)\,fg\right\} + \frac{1}{a^2}\int e^{ax}fg'\,dx + \frac{1}{a^2}\int e^{ax}f'g\,dx - \frac{1}{a}\int xe^{ax}fg'\,dx - \frac{1}{a}\int xe^{ax}f'g\,dx \tag{30}$$

The second through fifth terms on the right hand side of Equation (30) may be integrated by parts to give:

$$I_4 = \left\{\frac{e^{ax}}{a^2}(ax-1)fg\right\} + \left\{\frac{e^{ax}}{a^3} fg'\right\} - \frac{1}{a^3}\int e^{ax} fg'' \, dx - \frac{1}{a^3}\int e^{ax} f'g' \, dx + \left\{\frac{e^{ax}}{a^3} f'g\right\} -$$

$$- \frac{1}{a^3}\int e^{ax} f''g \, dx - \frac{1}{a^3}\int e^{ax} fg'' \, dx - \left\{\frac{e^{ax}}{a^3}(ax-1)fg'\right\} + \frac{1}{a^3}\int (ax-1)e^{ax} fg'' \, dx +$$

$$+ \frac{1}{a^3}\int (ax-1)e^{ax} f'g' \, dx - \left\{\frac{e^{ax}}{a^3}(ax-1)f'g\right\} + \frac{1}{a^3}\int (ax-1)e^{ax} f''g \, dx + \frac{1}{a^3}\int (ax-1)e^{ax} f'g' \, dx \qquad (31)$$

Substituting Equations (27) and (28) into Equation (31), gives (after rearranging) collecting like terms and splitting the ninth, tenth, twelth and thirteenth terms in the above equation:

$$I_4 = \left\{\frac{e^{ax}}{a^3}(ax-1)fg\right\} + \left\{\frac{e^{ax}}{a^3}(fg)'\right\} - \frac{\epsilon_2 n^2}{a^3}\int e^{ax} fg \, dx - \frac{\epsilon_1 m^2}{a^3}\int e^{ax} fg \, dx - \frac{2}{a^3}\int e^{ax} f'g' \, dx -$$

$$- \left\{\frac{e^{ax}}{a^3}(ax-1)(fg)'\right\} + \frac{\epsilon_2 n^2}{a^2}\int xe^{ax} fg \, dx - \frac{\epsilon_2 n^2}{a^3}\int e^{ax} fg \, dx + \frac{\epsilon_1 m^2}{a^2}\int xe^{ax} fg \, dx -$$

$$- \frac{\epsilon_1 m^2}{a^3}\int e^{ax} fg \, dx + \frac{2}{a^2}\int xe^{ax} f'g' \, dx - \frac{2}{a^3}\int e^{ax} f'g' \, dx \qquad (32)$$

However, noting that the seventh and ninth terms of the right hand side of Equation (32) are equal to I_4 multiplied by constants, the equation may be rearranged and written:

$$I_4(a^2-\epsilon_1 m^2-\epsilon_2 n^2) = \left\{e^{ax}\left[(ax-1)fg - \frac{ax-2}{a}(fg)'\right]\right\} - \frac{2}{a}\left[\epsilon_1 m^2 + \epsilon_2 n^2\right]\int e^{ax} fg \, dx -$$

$$- \frac{4}{a}\int e^{ax} f'g' \, dx + \frac{2}{a}\int xe^{ax} f'g' \, dx \qquad (33)$$

The second term on the right hand side of Equation (33) is:

$$- \frac{2}{a}\left[\epsilon_1 m^2 + \epsilon_2 n^2\right] I_2$$

where I_2 is given by Equation (18). Also the third term on the right hand side of Equation (33) may be easily evaluated from Equations (13) and (14), and thus:

$$- \frac{4}{a}\int xe^{ax} f'g' \, dx = -\frac{2}{a}\left[a^2-\epsilon_1 m^2-\epsilon_2 n^2\right] I_2 + 2\left\{\frac{e^{ax}}{a}\left[afg-(fg)'\right]\right\} \qquad (14)$$

Substituting for these terms in Equation (33) gives on simplifying:

$$\left[a^2-\epsilon_1 m^2-\epsilon_2 n^2\right] I_4 = \left\{e^{ax}\left[(ax+1)fg - x(fg)'\right]\right\} - 2_a I_2 + 2\int xe^{ax} f'g' \, dx \qquad (34)$$

but:

$$- 2aI_2 = -2a\left\{e^{ax}\left[\frac{(a^2-\epsilon_1 m^2-\epsilon_2 n^2)\left[afg-(fg)'\right] + 2\left[af'g' - (f'g')'\right]}{(a^2 - \epsilon_1 m^2 - \epsilon_2 n^2)^2 - 4\epsilon_1\epsilon_2 m^2 n^2}\right]\right\} \tag{18}$$

It is seen that the last term of Equation (34) is equivalent to $2I_4$ (with f and g replaced by f' and g' respectively in the integral I_4). This term (which will be called $2F_4$) may be evaluated from Equation (34) when the substitution of Equation (18) has been made, by replacing f and g by f' and g' respectively. Thus:

$$2F_4 = \frac{2}{a^2-\epsilon_1 m^2-\epsilon_2 n^2}\left\{e^{ax}\left[(ax+1)\ f'g' - x(f'g')'\right] - \right.$$

$$\left. - 2ae^{ax}\left[\frac{(a^2-\epsilon_1 m^2-\epsilon_2 n^2)\left[af'g'-(f'g')'\right] +2\left[af''g''-(f''g'')'\right]}{(a^2-\epsilon_1 m^2-\epsilon_2 n^2)^2 - 4\epsilon_1\epsilon_2 m^2 n^2}\right]\right\} + \left[\frac{2\epsilon_1\epsilon_2 m^2 n^2}{a^2-\epsilon_1 m^2-\epsilon_2 n^2}\right] I_4 \tag{35}$$

where use has been made of Equations (27) and (28) in evaluating the last term of the equation. If further use is made of these equations and Equations (18) and (35) are substituted into Equation (34), after rearrangement (34) becomes (writing I_4 in full):

$$I_4 = \int_{x_1}^{x_2} xe^{ax} f(mx+p)\, g(nx+q)\, dx$$

$$I_4 = \left\{\frac{e^{ax}}{K}\left[(a^2-\epsilon_1 m^2-\epsilon_2 n^2)\left[(ax+1)fg-x(fg)'-\frac{2a}{K}\left[(a^2-\epsilon_1 m^2-\epsilon_2 n^2)(afg-(fg)') + \right.\right.\right.\right.$$

$$\left.\left. +2(af'g-(f'g')')\right]\right] + 2\left[(ax+1)f'g'-x(f'g')'-\frac{2a}{K}\left[(a^2-\epsilon_1 m^2-\epsilon_2 n^2)(af'g-(f'g')') + \right.\right.$$

$$\left.\left.\left.\left. +2\epsilon_1\epsilon_2 m^2 n^2 (afg-(fg)')\right]\right]\right]\right\} \tag{36}$$

where $K = (a^2 - \epsilon_1 m^2 - \epsilon_2 n^2)^2 - 4\epsilon_1\epsilon_2 m^2 n^2$

and provided $K \neq 0$ and $a^2-\epsilon_1 m^2-\epsilon_2 n^2 \neq 0$.

4.0 DISCUSSION AND CONCLUSIONS

If a study is made of the fifty-six integrals presented in Section 2 of this report, it is observed that the constants p and q which appear as constants of the arguments of the functions of the integrands always remain within the arguments of the functions of the integrated results. This is demonstrated to be true also by the integration formulae derived in Section 3 and given in Equations (7), (18), (25), and (36); these formulae show clearly that the constants p and q are always contained as arguments of the functions of x and never emerge as constants multiplying the functions.

This is an important result since if the integral is known for $p = q = 0$, it can also be formulated by induction when $p \neq 0$ and $q \neq 0$. This also means that if this table of integrals is not available to the engineer he can formulate the more complex case where $p \neq 0$ and $q \neq 0$ from the simpler integrals, if these are available in a standard table of integrals. However, it is shown in Section 3 of this report that the integration formulae are only valid for the case where the second differential of the function of x equals plus or minus one times a constant squared times the original function. This restricts this result to functions of x such as sine, cosine, exponential, and hyperbolic sine or cosine functions.

It is obvious by studying the integration formulae derived that this result should hold true for such integrals as $\int x^n e^{ax} f\, g\, dx$, however, these integrals have not been investigated in this report.

The reader may care to check for himself that when $g = 1$ and $n = 1$ in Equation (36), this equation reduces to Equation (25) and if the same conditions are applied to Equation (18), it reduces to Equation (7). This gives some confidence in the correctness of the more complicated integration formulae. Integrals of the first type (Section 2.1) are also seen to be covered by Equation (18) if $a \to 0$.

The value of the integration formulae derived in Section 3 is essentially that they show the reason for the forms of the integrals given in Section 2 and the use which can be made of them by induction. In general, it usually takes longer to evaluate an integral using one of the integration formulae of Section 3, than it does from first principles.

The authors have found the fifty-six integrals to be of great use in the study of linear and linearized physical problems in the fields of vibration, acoustics, and aerodynamics.

5.0 ACKNOWLEDGEMENTS

The authors would like to thank Mr. F. V. Bracco of Wyle Laboratories for his assitance in checking the algebra in Sections 3.3 and 3.4 of this report.

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REPORT WR 66-24

Errata

SOME INTEGRATION FORMULAE WHICH SIMPLIFY THE EVALUATION OF CERTAIN INTEGRALS IN COMMON USE BY ENGINEERS

by

M. J. Crocker and R.W. White

Report Submitted Under Contract No. NAS8-5384

1. On page 4, alter formula 10). (by changing a minus to a plus sign) to read:

$$10).\ \int_{x_1}^{x_2} e^{ax}\cosh(mx+p)\cosh(nx+q)\,dx = \left\{ e^{ax}\left[\frac{a\cosh[(m+n)x+p+q] - (m+n)\sinh[(m+n)x+p+q]}{a^2-(m+n)^2}\right] + e^{ax}\left[\frac{a\cosh[(m-n)x+p-q] - (m-n)\sinh[(m-n)x+p-q]}{a^2-(m-n)^2}\right]\right\}_{x_1}^{x_2}$$

2. On page 5, alter formulae 13). and 14). (by changing a sign in the power of one of the exponential functions) so that they read:

$$13).\ \int_{x_1}^{x_2} e^{ax}\cosh(mx+p)\sin(nx+q)\,dx = \frac{1}{2}\left\{ e^{(a+m)x+p}\left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2}\right] + e^{(a-m)x-p}\left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2}\right]\right\}_{x_1}^{x_2}$$

$$14).\ \int_{x_1}^{x_2} e^{ax}\cosh(mx+p)\cos(nx+q)\,dx = \frac{1}{2}\left\{ e^{(a+m)x+p}\left[\frac{(a+m)\cos(nx+q) + n\sin(nx+q)}{(a+m)^2+n^2}\right] + e^{(a-m)x-p}\left[\frac{(a-m)\cos(nx+q) + n\sin(nx+q)}{(a-m)^2+n^2}\right]\right\}_{x_1}^{x_2}$$

3. On page 5, alter formula 2). (by addition of x) to read:

$$2).\quad \int_{x_1}^{x_2} x \cos(mx+p)\, dx = \frac{1}{m}\left\{\frac{1}{m}\cos(mx+p) + x\sin(mx+p)\right\}_{x_1}^{x_2}$$

4. On page 6, alter formula 5). (by changing a minus to a plus sign) to read:

$$5).\quad \int_{x_1}^{x_2} x\cos(mx+p)\cos(nx+q)\, dx = \frac{1}{2}\left\{\frac{\cos[(m-n)x+p-q]}{(m-n)^2} + \frac{x\sin[(m-n)x+p-q]}{m-n} + \right.$$

$$\left. + \frac{\cos[(m+n)x+p+q]}{(m+n)^2} + \frac{x\sin[(m+n)x+p+q]}{m+n}\right\}_{x_1}^{x_2}$$

5. On page 8, alter formula 3). (by adding the factor $\frac{1}{2}$) to read:

$$3).\quad \int_{x_1}^{x_2} x\, e^{ax} \sin(mx+p)\cos(nx+q)\, dx$$

$$= \frac{1}{2}\left\{x\, e^{ax}\left[\frac{a\sin[(m+n)x+p+q] - (m+n)\cos[(m+n)x+p+q]}{a^2+(m+n)^2}\right] - \right.$$

$$- e^{ax}\left[\frac{[a^2-(m+n)^2]\sin[(m+n)x+p+q] - 2a(m+n)\cos[(m+n)x+p+q]}{[a^2+(m+n)^2]^2}\right] +$$

$$+ x\, e^{ax}\left[\frac{a\sin[(m-n)x+p-q] - (m-n)\cos[(m-n)x+p-q]}{a^2+(m-n)^2}\right] -$$

$$\left. - e^{ax}\left[\frac{[a^2-(m-n)^2]\sin[(m-n)x+p-q] - 2a(m-n)\cos[(m-n)x+p-q]}{[a^2+(m-n)^2]^2}\right]\right\}_{x_1}^{x_2}$$

6. On page 9, alter formula 4). (by changing a cos to a sin, and adding the factor $\frac{1}{2}$) to read:

4). $$\int_{x_1}^{x_2} x\, e^{ax} \sin(mx+p) \sin(nx+q)\, dx$$

$$= \frac{1}{2}\left\{ x\, e^{ax} \left[\frac{a \cos[(m-n)x+p-q] + (m-n)\sin[(m-n)x+p-q]}{a^2 + (m-n)^2} \right] - \right.$$

$$- e^{ax} \left[\frac{[a^2-(m-n)^2] \cos[(m-n)x+p-q] + 2a(m-n)\sin[(m-n)x+p-q]}{[a^2+(m-n)^2]^2} \right] -$$

$$- x\, e^{ax} \left[\frac{a\cos[(m+n)x+p+q] + (m+n)\sin[(m+n)x+p+q]}{a^2+(m+n)^2} \right] +$$

$$\left. + e^{ax} \left[\frac{[a^2-(m+n)^2] \cos[(m+n)x+p+q] + 2a(m+n)\sin[(m+n)x+p+q]}{[a^2+(m+n)^2]^2} \right] \right\}_{x_1}^{x_2}$$

7. On page 9, alter formula 4). (by adding the factor $\frac{1}{2}$) to read:

5). $$\int_{x_1}^{x_2} x\, e^{ax} \cos(mx+p) \cos(nx+q)\, dx$$

$$= \frac{1}{2}\left\{ x\, e^{ax} \left[\frac{a \cos[(m-n)x+p-q] + (m-n)\sin[(m-n)x+p-q]}{a^2 + (m-n)^2} \right] - \right.$$

$$- e^{ax} \left[\frac{[a^2-(m-n)^2] \cos[(m-n)x+p-q] + 2a(m-n)\sin[(m-n)x+p-q]}{[a^2+(m-n)^2]^2} \right] +$$

$$+ x\, e^{ax} \left[\frac{a\cos[(m+n)x+p+q] + (m+n)\sin[(m+n)x+p+q]}{a^2+(m+n)^2} \right] -$$

$$\left. - e^{ax} \left[\frac{[a^2-(m+n)^2] \cos[(m+n)x+p+q] + 2a(m+n)\sin[(m+n)x+p+q]}{[a^2+(m+n)^2]^2} \right] \right\}_{x_1}^{x_2}$$

8. On page 11, alter formula 11). (by changing the signs in the power of one of the exponential functions) to read:

$$11).\ \int_{x_1}^{x_2} x\, e^{ax} \sinh(mx+p) \sin(nx+q)\, dx$$

$$= \frac{1}{2}\Bigg\{ x e^{(a+m)x+p} \left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2} \right] - e^{(a+m)x+p} \left[\frac{[(a+m)^2-n^2]\sin(nx+q) - 2(a+m)n\cos(nx+q)}{((a+m)^2+n^2)^2} \right] -$$

$$- x e^{(a-m)x-p} \left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2} \right] + e^{(a-m)x-p} \left[\frac{[(a-m)^2-n^2]\sin(nx+q) - 2(a-m)n\cos(nx+q)}{((a-m)^2+n^2)^2} \right] \Bigg\}_{x_1}^{x_2}$$

9). On page 11, alter formula 13). (by changing a minus to a plus sign) to read:

$$13).\ \int_{x_1}^{x_2} x\, e^{ax} \cosh(mx+p) \sin(nx+q)\, dx$$

$$= \frac{1}{2}\Bigg\{ x e^{(a+m)x+p} \left[\frac{(a+m)\sin(nx+q) - n\cos(nx+q)}{(a+m)^2+n^2} \right] - e^{(a+m)x+p} \left[\frac{[(a+m)^2-n^2]\sin(nx+q) - 2(a+m)n\cos(nx+q)}{((a+m)^2+n^2)^2} \right] +$$

$$+ x e^{(a-m)x-p} \left[\frac{(a-m)\sin(nx+q) - n\cos(nx+q)}{(a-m)^2+n^2} \right] - e^{(a-m)x-p} \left[\frac{[(a-m)^2-n^2]\sin(nx+q) - 2(a-m)n\cos(nx+q)}{((a-m)^2+n^2)^2} \right] \Bigg\}_{x_1}^{x_2}$$

10). On page 15, alter Equation (24) to read:

$$I_3 = \left\{ \frac{e^{ax}}{a^2}(ax-1)f \right\} + \left\{ \frac{e^{ax}[af' - \epsilon_1 m^2 f]}{a^2\ (a^2 - \epsilon_1 m^2)} \right\} - \frac{1}{a^3} \left\{ e^{ax}(ax-1)f' + e^{ax}\epsilon_1 m^2 \left[\frac{af-f'}{a^2-\epsilon_1 m^2} \right] \right\} + \epsilon_1 \frac{m^2}{a^2} \int x e^{ax} f\, dx \quad (24)$$

11). On page 15, alter Equation (25) to read:

$$I_3 = \int_{x_1}^{x_2} x e^{ax} f(mx+p)\, dx = \left\{ \frac{e^{ax}}{a^2 - \epsilon_1 m^2} \left[(ax-1)(f - f'/a) + \frac{(a + \epsilon_1 m^2/a)f' - 2\epsilon_1 m^2 f}{a^2 - \epsilon_1 m^2} \right] \right\} \quad (25)$$

provided $a^2 - \epsilon_1 m^2 \neq 0$

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REPORT WR 66-24

ERRATA

SOME INTEGRATION FORMULAE WHICH SIMPLIFY THE EVALUATION OF CERTAIN INTEGRALS IN COMMON USE BY ENGINEERS

by

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Report Submitted Under Contract No. NAS8 5384

On page 7, alter formula 12 (by changing a sin to a cos and adding a parenthesis) to read:

$$12).\ \int_{x_1}^{x_2} x \sinh(mx+p)\cos(nx+q)\,dx = \left\{ x\left[\frac{m\cosh(mx+p)\cos(nx+q) + n\sinh(mx+p)\sin(nx+q)}{m^2+n^2}\right] - \right.$$

$$\left. - \frac{(m^2-n^2)\sinh(mx+p)\cos(nx+q) + 2mn\cosh(mx+p)\sin(nx+q)}{(m^2+n^2)^2}\right\}_{x_1}^{x_2}$$