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DETERMINATION OF INTEGRATION CONSTANTS IN THE GENERALIZED PROBLEM OF TWO FIXED CENTERS

by

E. I. Timoshkova

(USSR)

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A method is proposed for the determination of six integration constants in the problem of two fixed centers. The expressions are also derived for the spheroidal coordinates of the moving point in terms of power series by two small parameters.

Author

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The solution of the generalized problem of two fixed centers dependent on six arbitrary constants have been obtained in the works by E. P. Aksenova, E. A. Grevenikova and V. G. Demina. These arbitrary constants must be found from some initial data. We shall formulate the problem of determination of arbitrary constants of integration when rectangular coordinates are given of a material point of which the motion is investigated for two moments of time.

Let us consider the motion of a material point in a gravitational field, induced by two fixed masses. The force function characterizing this field has the form

$$U=\frac{fM}{r}\left[1+\sum_{k=2}^{\infty}\frac{\gamma_k}{r^k}P_k\left(\frac{z}{r}\right)\right],$$

$$\gamma_k=\frac{c^k}{2}\left[(1+i\sigma)(\sigma+i)^k+(1-i\sigma)(\sigma-i)^k\right], \tag{1}$$

where f is the gravitational constant; M is the mass, equal to the sum of masses of attracting centers; r is the distance of the moving point from the center of masses; $P_k\left(\frac{z}{r}\right)$ are the Legendre polynomials; i is the imaginary unity; c and σ are certain real numbers.

As is shown in the work [1], the above expression for the potential U is a good approximation of the real Earth's gravitational potential, provided c and σ are chosen from the conditions

$$c=R\sqrt{-J_2-\left(\frac{J_3}{2J_2}\right)^2},\qquad \sigma=\frac{J_3}{2J_2}\frac{1}{\sqrt{-J_2-\left(\frac{J_3}{2J_2}\right)^2}}, \tag{2}$$

where R is the equatorial radius of the Earth, J_2, J_3 are the constant factors in the expansion of the real Earth's potential by Legendre polynomials.

We shall introduce into the consideration a fixed rectangular system of coordinates x, y, z with origin at the center of masses. The axiz z is directed along the line linking the fixed attracting centers. The differential equations of motion ofthe material point in this system of coordinates will be written in the form

$$\frac{d^2x}{dt^2}=\frac{\partial U}{\partial x}, \quad \frac{d^2y}{dt^2}=\frac{\partial U}{\partial y}, \quad \frac{d^2z}{dt^2}=\frac{\partial U}{\partial z}. \tag{3}$$

We shall pass in Eqs (3) to new variables by the formulas

$$\begin{aligned} x&=\sqrt{(\xi^2+c^2)(1-\eta^2)}\cos w,\\ y&=\sqrt{(\xi^2+c^2)(1-\eta^2)}\sin w,\\ z&=c\sigma+\xi\eta \end{aligned} \tag{4}$$

and to a new independent variable

$$dt=(\xi^2+c^2\eta^2)\,d\tau. \tag{5}$$

Then, as is shown in the work [1], Eqs. (3), transformed at first in an appropriate fashion, become integrable in quadratures. These quadratures may be written as follows:

$$\begin{aligned} \int\frac{d\eta}{\sqrt{F(\eta)}}&=\sqrt{-2hc^2}\,(\tau+c_3),\\ \int\frac{d\xi}{\sqrt{\Phi(\xi)}}&=\sqrt{-2h}\,(\tau+c_4), \end{aligned} \tag{6}$$

$$\begin{aligned} w&=c_1\int\frac{\xi^2+c^2\eta^2}{(\xi^2+c^2)(1-\eta^2)}\,d\tau+c_5,\\ t&=\int(\xi^2+c^2\eta^2)\,d\tau, \end{aligned} \tag{7}$$

where

$$\begin{aligned} F(\eta)&=\eta^4-f\frac{M\sigma}{hc}\eta^3-\left(\frac{c_2}{c^2h}+1\right)\eta^2+f\frac{M\sigma}{hc}\eta+\left(\frac{c_2}{c^2h}+\frac{1}{2}\frac{c_1^2}{c^2h}\right),\\ \Phi(\xi)&=-\xi^4-f\frac{M}{h}\xi^3-\left(\frac{c_2}{h}+c^2\right)\xi^2-f\frac{M}{h}c^2\xi-c^2\left(\frac{c_2}{h}+\frac{1}{2}\frac{c_1^2}{h}\right). \end{aligned} \tag{8}$$

The quantities $h, c_1, c_2, c_3, c_4, c_5$ in the expressions (6) – (8) are arbitrary integration constants, whereupon h is the constant of the energy integral, and c_1 is the constant of area integral.

In the furthest we shall limit ourselves to the case of elliptic-type motions for which the inequalities

$$h<0; \quad 2c_2+c_1^2<0.$$

are fulfilled [2]. Instead of constants $h, c_1, c_2, c_3, c_4, c_5$ we shall introduce into the consideration new, often more convenient integration constants, which we shall denote by analogy with the Kepler elements by $a, e, s, \Omega, \omega, M_0$.

The expressions linking the old integration constants with the new ones may be obtained analogously to what was done in the work [3]. This is why here we shall bring forth these expressions in their final form.

The constants h, c_1 and c_2 may be expressed by means of new constants a, e and s with the aid of the following relations :

$$f\frac{M}{h} = -2a\{1+\varepsilon^2(1-e^2)(1-s^2)+2\varepsilon^3\sigma s(s^2-1)(1-e^2)+ \\ +\varepsilon^4[(1-e^2)^2-s^2(1-e^2)(5-e^2)+4s^4(1-e^2)]+\dots\},$$

$$\frac{c_2}{h} = a^2(1-e^2)\{1+\varepsilon^2(1-s^2)(3+e^2)+2\varepsilon^3\sigma s(s^2-1)(3+e^2)+ \\ +\varepsilon^4[4(1-e^2)-16s^2+4(3+e^2)s^4]+\dots\}, \tag{9}$$

$$c_1 = \pm\sqrt{fMa(1-e^2)(1-s^2)}\Big\{1-\varepsilon\sigma s+\varepsilon^2\Big[(1+e^2)-\frac{1}{2}s^2(3+e^2)\Big]- \\ -\frac{1}{2}\varepsilon^2\sigma^2 s^2+\varepsilon^3\sigma s\Big[(1+e^2)-\frac{1}{2}s^2(3+e^2)\Big]- \\ -\varepsilon^4\Big[2e^2+\frac{1}{2}(1+e^2)(3+e^2)s^2-\frac{1}{8}s^4(11+34e^2+3e^4)\Big]+\dots\Big\},$$

where $\varepsilon = \dfrac{c}{a(1-e^2)}$ is a small parameter. The quantity σ may also be considered as a small parameter. It is not difficult to see that parameters ε and σ have the same order of smallness.

The conversion of elliptical integrals (6) allows to obtain the expressions for the coordinates ξ, η, w in the form of power series by small parameters ε and σ :

$$\eta = \frac{\alpha+\beta\sin\varphi}{1+\gamma\sin\varphi},$$

$$\xi = \frac{\bar{p}(1+\varkappa\cos v)}{1+\bar{e}\cos v},$$

$$w-\Omega = \pm A\,\mathrm{arctg}\left(\sqrt{1-s^2}\,\mathrm{tg}\,\varphi\right)\pm\mu u\pm a_{1,0}\cos u\pm a_{2,0}\cos 2u\pm \\ \pm a_{3,0}\cos 3u\pm a_{4,0}\cos 4u\pm a_{2,2}\cos 2(u+v)\pm a_{2,-2}\cos 2(u-v)\pm \\ \pm b_{1,0}\sin u\pm b_{2,0}\sin 2u\pm b_{3,0}\sin 3u\pm b_{4,0}\sin 4u\pm \\ \pm b_{2,2}\sin 2(u+v)\pm b_{2,-2}\sin 2(u-v)\pm b_{0,1}\sin v\pm b_{0,2}\sin 2v\pm \\ \pm b_{0,3}\sin 3v\pm b_{0,4}\sin 4v\pm d\sin u\sin 2v\pm g\sin u\sin 2u, \tag{10}$$

$$\varphi = \mathrm{am}\,[\sigma_1(\tau+c_3),\ k_1],$$

$$v = \mathrm{am}\,[\sigma_2(\tau+c_4),\ k_2],$$

$$\alpha = -\varepsilon\sigma(1-2s^2)+4\varepsilon^2\sigma^2 s(1-s^2)+\varepsilon^3\sigma\{e^2(1-2s^4)+ \\ +(3-12s^2+10s^4)\}+\dots, \tag{11}$$

$$\beta = s+\varepsilon\sigma(1-s^2)-3\varepsilon^2\sigma^2 s(1-s^2)+\varepsilon^3\sigma(1-s^2)(-9+8s^2+e^2)+\dots,$$

$$\gamma = \varepsilon\sigma s+\varepsilon^2\sigma^2(1-s^2)-\varepsilon^3\sigma s[4(1-s^2)+(1-e^2)(1-2s^2)]+\dots,$$

$$\sigma_1 = \sqrt{fMa(1-e^2)}\Big\{1+\varepsilon^2\Big[\frac{1}{2}(3+e^2)(1-s^2)\Big]+\varepsilon^2\sigma^2\Big(3-\frac{7}{2}s^2\Big)- \\ -\varepsilon^3\sigma s(1-s^2)(3+e^2)-\varepsilon^4\Big[\frac{1}{8}(9+6e^2+e^4)+\frac{1}{4}(1+14e^2+e^4)s^2- \\ -\frac{1}{8}(11+34e^2+3e^4)s^4\Big]+\dots\Big\},$$

$$k_1^2 = (1-e^2)\{\varepsilon^2 s^2+2\varepsilon^3\sigma s(1-s^2)-4\varepsilon^4 s^2(1-s^2)\}+\varepsilon^2\sigma^2 s^2+\dots, \\ \bar{p} = a(1-e\bar{e}), \tag{12}$$

../..

$$\bar{e} = e\{1 + \varepsilon^2(1-e^2)(1-2s^2) - 4\varepsilon^3\sigma s(1-e^2)(1-s^2) + \\ + \varepsilon^4(1-e^2)[3 - 16s^2 + 14s^4 - 2e^2(1-s^2)^2] + \ldots\},$$

$$\varkappa = \varepsilon^2 e\{(1-2s^2) - 4\varepsilon\sigma s(1-s^2) + \varepsilon^2[3 - 16s^2 + 14s^4 - e^2(1-2s^4)]\} + \ldots,$$

$$k_2^2 = \varepsilon^2 e^2 s^2 + 4\varepsilon^3\sigma s e^2(1-s^2) - \varepsilon^4 e^2(1 - 10s^2 + 11s^4 + e^2 s^4) + \ldots,$$

$$\sigma_2 = \sqrt{fMa(1-e^2)}\left\{1 - \varepsilon^2\left[\frac{1}{2}(3-e^2) - 2s^2\right] - 8\varepsilon^3\sigma s e^2(1-s^2) - \\ - \frac{1}{8}\varepsilon^4[(9 + 2e^2 + e^4) - (72 + 40e^2)s^2 + (63 + 48e^2)s^4] + \ldots\right\},$$

$$A = 1 + \frac{1}{2}\varepsilon^3\sigma s[(-13 + 9e^2) + 2s^2(1-e^2)] + \ldots,$$

$$\mu = \sqrt{1-s^2}\left\{-\frac{3}{2}\varepsilon^2 - \frac{3}{2}\varepsilon^2\sigma^2 + \frac{3}{2}\varepsilon^3\sigma s + \\ + \frac{1}{16}\varepsilon^4[(54 - 39s^2) + 72e^2 s^2] + \ldots\right\},$$

$$a_{1,0} = \sqrt{1-s^2}\,[2\varepsilon\sigma s + \varepsilon^2\sigma^2(2 + 5s^2 - 8s^4) - \varepsilon^3\sigma s(9 - 10s^2 + 2e^2 s^2 - e^2) + \ldots],$$

$$a_{2,0} = \sqrt{1-s^2}\,[\varepsilon\sigma s + \varepsilon^2\sigma^2(1 - 6s^2 + 4s^4) - \varepsilon^3\sigma s(4 - 5s^2 + e^2 s^2) + \ldots],$$

$$a_{3,0} = -\sqrt{1-s^2}\,\varepsilon^2\sigma^2 s^2 + \ldots,$$

$$a_{4,0} = \sqrt{1-s^2}\left[-\frac{1}{2}\varepsilon^2\sigma^2 s^2 + \frac{1}{8}\varepsilon^3\sigma s^3(1-e^2) + \ldots\right],$$

$$a_{2,2} = -\frac{1}{8}\sqrt{1-s^2}\,\varepsilon^3\sigma s^3 e^2 + \ldots,$$

$$a_{2,-2} = \frac{1}{8}\sqrt{1-s^2}\,\varepsilon^3\sigma s^3 e^2 + \ldots,$$

$$b_{1,0} = 2\sqrt{1-s^2}\,\varepsilon^2\sigma^2 s^2(7 - 8s^2) + \ldots, \qquad (12)$$

$$b_{2,0} = \sqrt{1-s^2}\left[-\frac{1}{2}\varepsilon\sigma s + \frac{1}{4}\varepsilon^2\sigma^2 s^2(15 + 24s^2) + \frac{1}{2}\varepsilon^3\sigma s(9 - 8s^2 - e^2) + \\ + \frac{1}{32}\varepsilon^4 s^2(1-e^2)^2 + \ldots\right],$$

$$b_{3,0} = -2\sqrt{1-s^2}\,\varepsilon^2\sigma^2 s^2 + \ldots$$

$$b_{4,0} = -\sqrt{1-s^2}\left[\frac{3}{8}\varepsilon^2\sigma^2 s^2 + \frac{1}{16}\varepsilon^3\sigma s^3(1-e^2) + \ldots\right],$$

$$b_{2,2} = \frac{1}{16}\sqrt{1-s^2}\,\varepsilon^3\sigma s^3 e^2 + \ldots,$$

$$b_{2,-2} = -\frac{1}{16}\sqrt{1-s^2}\,\varepsilon^3\sigma s^3 e^2 + \ldots,$$

$$b_{0,1} = -\sqrt{1-s^2}\left[2\varepsilon^2 e - 2\varepsilon^3\sigma s e + \varepsilon^4 e\left(1 - \frac{3}{2}e^2 - 7s^2 - \frac{7}{4}e^2 s^2\right) + \ldots\right],$$

$$b_{0,2} = -\sqrt{1-s^2}\,e^2\left[\frac{1}{4}\varepsilon^2 - \frac{1}{4}\varepsilon^3\sigma s - \frac{1}{16}\varepsilon^4(22 + s^2 + 2e^2 + e^2 s^2) + \ldots\right],$$

$$b_{0,3} = -\sqrt{1-s^2}\,\varepsilon^4 e^3\left(-\frac{1}{2} + \frac{1}{4}s^2\right) + \ldots,$$

$$b_{0,4} = \frac{1}{32}\sqrt{1-s^2}\,\varepsilon^4 e^4\left(1 + \frac{1}{2}s^2\right) + \ldots,$$

$$d = \frac{1}{4}\sqrt{1-s^2}\,\varepsilon^3\sigma s^3 e^2 + \ldots,$$

$$g = -\frac{1}{4}\sqrt{1-s^2}\,\varepsilon^3\sigma s^3(1-e^2) + \ldots$$

The relations (11) may be utilized for the elimination of the indeoendent variable τ. As a result of such an elimination we shall obtain

$$\varphi = u + A_{0,2}\sin 2v + A_{2,0}\sin 2u + A_{2,-2}\sin 2(u-v) + A_{2,2}\sin 2(u+v) + \dots, \qquad (13)$$

where

$$u = (1+\nu)\,v + \omega,$$

$$\nu = \varepsilon^2\left[\frac{1}{4}(12-15s^2)\right] + \varepsilon^2\sigma^2\left[\frac{1}{4}(12-15s^2)\right] - \varepsilon^3\sigma s(1-s^2)\left(\frac{7}{2}-\frac{17}{2}e^2\right) +$$
$$+ \varepsilon^4\left[\frac{1}{64}(288-1296s^2+1035s^4) - \frac{e^2}{64}(144+288s^2-510s^4)\right] + \dots$$
$$A_{0,2} = -\frac{1}{8}\varepsilon^2 e^2 s^2 - \frac{1}{2}\varepsilon^3\sigma s e^2(1-s^2) +$$
$$+ \frac{1}{8}\varepsilon^4 e^2\left(1-13s^2+\frac{59}{4}s^4+\frac{1}{2}e^2 s^4\right) + \dots, \qquad (14)$$
$$A_{2,0} = \frac{1}{8}\varepsilon^2 s^2(1-e^2) + \frac{1}{8}\varepsilon^2\sigma^2 s^2 + \frac{1}{4}\varepsilon^3\sigma s(1-s^2) -$$
$$- \frac{1}{16}\varepsilon^4 s^2\left[8-9s^2-e^2(8-10s^2)-e^4 s^2\right] + \dots,$$
$$A_{2,-2} = \frac{1}{64}\varepsilon^4 e^2 s^4(1-e^2),$$
$$A_{2,2} = -\frac{1}{64}\varepsilon^4 e^2 s^4(1-e^2),$$

ω is a certain constant function of constants a, e, s, c_3 and c_4. In the furthest we shall consider ω as a new integration constant, alongside with a, e, s .

In order to compute ξ, η, w for any moment of time, it is necessary to obtain the expression linking the variable v with the time t.

If we compute integral (7) taking into account (10) and (12), after a few transformations we shall have :

$$\bar{n}(t-t_0) + M_0 = 2\,\mathrm{arctg}\left(\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\,\mathrm{tg}\frac{v}{2}\right) - e^*\sqrt{1-\bar{e}^2}\frac{\sin v}{1+\bar{e}\cos v} +$$
$$+ \lambda v + \lambda_{0,1}\sin v + \lambda_{0,2}\sin 2v + \lambda_{2,0}\sin 2u + \lambda_{4,0}\sin 4u +$$
$$+ \lambda_{2,-2}\sin 2(u-v) + \lambda_{2,2}\sin 2(u+v) + \beta_{1,0}\cos u + \beta_{3,0}\cos 3u + \dots, \qquad (15)$$

where

$$\bar{n} = \sqrt{\frac{fM}{a^3}}\left\{1 - \frac{3}{2}\varepsilon^2(1-e^2)(1-s^2) - 2\varepsilon^3\sigma s(1-s^2)(1+5e^2) +\right.$$
$$\left. + \frac{3}{8}\varepsilon^4(1-e^2)(1-s^2)\left[(1+11s^2)-e^2(1-5s^2)\right] + \dots\right\},$$
$$e^* = e\{1-\varepsilon^2(1-e^2)(1-s^2) + 4\varepsilon^3\sigma s e^2(1-s^2) +$$
$$+ \varepsilon^4 s^2(1-e^2)(1-s^2)(3+e^2) + \dots\},$$
$$\lambda = -(1-e^2)^{3/2}\left\{2\varepsilon^3\sigma s(1-s^2) + \frac{\varepsilon^4}{16}(24-96s^2+75s^4) + \dots\right\},$$
$$\lambda_{0,1} = -\frac{1}{4}\varepsilon^4 e s^2(1-e^2)^{3/2}(4-5s^2) + \dots,$$
$$\lambda_{0,2} = \frac{1}{32}\varepsilon^4 e^2 s^4(1-e^2)^{3/2} + \dots, \qquad (16)$$

... cont...

$$\lambda_{2,0} = -\frac{1}{4}\varepsilon^2 s^2 (1-e^2)^{3/2} - \frac{1}{2}\varepsilon^3 \sigma s (1-s^2)(1-e^2)^{3/2} + \tag{16}$$
$$+\frac{1}{16}\varepsilon^4 s^2 (1-e^2)^{3/2} [(12-13s^2) - e^2(4-5s^2)] + \dots,$$
$$\lambda_{4,0} = -\frac{1}{64}\varepsilon^4 s^4 (1-e^2)^{5/2} + \dots,$$
$$\lambda_{2,-2} = -\frac{1}{32}\varepsilon^4 e^2 s^4 (1-e^2)^{3/2} + \dots,$$
$$\lambda_{2,2} = \frac{1}{32}\varepsilon^4 e^2 s^4 (1-e^2)^{3/2} + \dots,$$
$$\beta_{1,0} = \frac{1}{2}\varepsilon^3 \sigma s (4-5s^2)(1-e^2)^{3/2} + \dots,$$
$$\beta_{3,0} = -\frac{1}{6}\varepsilon^3 \sigma s^3 (1-e^2)^{3/2} + \dots$$

Here M_0 is a new arbitrary constant of integration.

If we assume in formulas (10 – (16) the small parameter σ to be zero, we shall obtain the formulas brought out in the work [3]. At $\varepsilon=0$ and $\sigma=0$ simultaneously, formulas (10) – (16) pass into Kepler motion formulas, and the constants a, e, $s=\sin i$, Ω, ω, M_0 become the standard Kepler elements.

We shall consider the problem of the determination of six constants of integration , if the rectangular coordinates are known of the point for two moments of time, also known.

Let us consider that for the moments of time t_1 and t_2 the rectangular coordinates of respectively the points x_1, y_1, z_1 and x_2, y_2, z_2 are known. Eqs. (4) determine in a unique fashion the spheroidal coordinates ξ, η and w. Indeed the first two Eqs. (4) give the longitude

$$w = \operatorname{arctg}\frac{y}{x}. \tag{17}$$

From the last Eq. (4) we find

$$\eta = \frac{z - c\sigma}{\xi}. \tag{18}$$

Eliminating from the first two Eqs. (4) the longitude w and substituting according to formula (9), we obtain the equation for the determination of ξ in the following form:

$$\xi^4 + p\xi^2 + q = 0, \tag{19}$$

where

$$p = c^2(1-\sigma^2) + 2c\sigma z - (x^2+y^2+z^2),$$
$$q = -c^2(z-c\sigma)^2.$$

It may be shown that for real motions of the elliptic type the inequalities

$$-1 \leqslant \eta \leqslant 1; \quad \xi > 0 \quad \& \quad \xi^2 + c^2 > R^2_{pol} \tag{20}$$

where R_{pol} is the Earth's polar radius, are valid.

Then, obviously, Eqs. (17) – (10) determine unilaterally the spheroidal coordinates ξ, η and w.

Consequently, for the moments of time t_1 and t_2, the spheroidal coordinates $\xi_1, \eta_1, w_1, \xi_2, \eta_2, w_2$ of the point may be considered as known.

Let us now examine how the constants h, c_1 and c_2 may be found, the latter, determining, according to (9), the shape and the dimensions of the orbit. Let us now turn to Eqs. (6), (7), linking the constants h, c_1, c_2 with the coordinates ξ, η, w. It is not difficult to see that these equations may be written in the form

$$\begin{aligned} \frac{d\xi}{dt} &= F_1(\xi, \eta, h, c_1, c_2), \\ \frac{d\eta}{dt} &= F_2(\xi, \eta, h, c_1, c_2), \\ \frac{dw}{dt} &= c_1 F_3(\xi, \eta), \end{aligned} \tag{21}$$

where

$$\begin{aligned} F_1(\xi, \eta, h, c_1, c_2) &= \frac{\sqrt{-2h\Phi(\xi)}}{\xi^2 + c^2\eta^2}, \\ F_2(\xi, \eta, h, c_1, c_2) &= \frac{\sqrt{-2hc^2F(\eta)}}{\xi^2 + c^2\eta^2}, \\ F_3(\xi, \eta) &= \frac{1}{(\xi^2 + c^2)(1-\eta^2)}, \end{aligned} \tag{22}$$

$\Phi(\xi)$ and $F(\eta)$ being determined by formulas (8).

If ξ_2, η_2, w_2 are known, the system of differential equations (21) with initial data

$$\xi(t_1) = \xi_1, \quad \eta(t_1) = \eta_1, \quad w(t_1) = w_1 \tag{23}$$

do generally determine the constants h, c_1, c_2. For the computation of the latter the following method may be proposed.

Assume that certain values of constants $h^{(m)}, c_1^{(m)}, c_2^{(m)}$ are known to us. We shall consider the system of differential equations

$$\begin{aligned} \frac{d\bar{\xi}}{dt} &= F_1(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)}), \\ \frac{d\bar{\eta}}{dt} &= F_2(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)}), \\ \frac{d\bar{w}}{dt} &= c_1^{(m)} F_3(\bar{\xi}, \bar{\eta}) \end{aligned} \tag{24}$$

with the initial data

$$\bar{\xi}(t_1) = \xi_1, \quad \bar{\eta}(t_1) = \eta_1, \quad \bar{w}(t_1) = w_1. \tag{25}$$

Besides, we shall consider that the following correlations

$$\begin{aligned} \xi &= \bar{\xi} + \alpha_1 \Delta h + \alpha_2 \Delta c_2 + \alpha_3 \Delta c_1 = \bar{\xi} + \Delta\xi, \\ \eta &= \bar{\eta} + \beta_1 \Delta h + \beta_2 \Delta c_2 + \beta_3 \Delta c_1 = \bar{\eta} + \Delta\eta, \\ w &= \bar{w} + \gamma_1 \Delta h + \gamma_2 \Delta c_2 + \gamma_3 \Delta c_1 = \bar{w} + \Delta w, \end{aligned} \tag{26}$$

are valid; here $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ are certain still unkmown functions of the time t, and

$$\Delta h = h - h^{(m)},$$
$$\Delta c_2 = c_2 - c_2^{(m)},$$
$$\Delta c_1 = c_1 - c_1^{(m)}.$$

Assume that at $t = t_1$ functions $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ become identically zero, i. e.

$$\alpha_1(t_1) \equiv \alpha_2(t_1) \equiv \ldots \equiv \gamma_3(t_1) \equiv 0. \tag{27}$$

We shall obtain the differential equations determining the functions $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$. To that effect we shall subtract the system (24) from the system (21), as a result of which we shall have

$$\begin{aligned} \frac{d}{dt}(\Delta\xi) &= F_1(\xi, \eta, h, c_1, c_2) - F_1(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)}), \\ \frac{d}{dt}(\Delta\eta) &= F_2(\xi, \eta, h, c_1, c_2) - F_2(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)}), \\ \frac{d}{dt}(\Delta w) &= c_1 F_3(\xi, \eta) - c_1^{(m)} F_3(\bar{\xi}, \bar{\eta}). \end{aligned} \tag{28}$$

We shall expand the functions $F_1(\xi, \eta, h, c_1, c_2)$, $F_2(\xi, \eta, h, c_1, c_2)$ in exponential Taylor series by variables ξ, η, h, c_1, c_2 in the vicinity of the point $(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)})$, limiting ourselves at the same time only to terms linear relative to $\Delta\xi, \Delta\eta, \Delta h, \Delta c_2, \Delta c_1$.

$$\begin{aligned} F_1(\xi, \eta, h, c_1, c_2) &= F_1(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)}) + \Delta\xi \frac{\partial F_1}{\partial \xi} + \\ &+ \Delta\eta \frac{\partial F_1}{\partial \eta} + \Delta h \frac{\partial F_1}{\partial h} + \Delta c_1 \frac{\partial F_1}{\partial c_1} + \Delta c_2 \frac{\partial F_1}{\partial c_2}, \\ F_2(\xi, \eta, h, c_1, c_2) &= F_2(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)}) + \Delta\xi \frac{\partial F_2}{\partial \xi} + \\ &+ \Delta\eta \frac{\partial F_2}{\partial \eta} + \Delta h \frac{\partial F_2}{\partial h} + \Delta c_1 \frac{\partial F_2}{\partial c_1} + \Delta c_2 \frac{\partial F_2}{\partial c_2}. \end{aligned} \tag{29}$$

The expansion in Taylor series for the function $F_3(\xi, \eta)$ in the vicinity of the point $(\bar{\xi}, \bar{\eta})$ provided we limit ourselves to linear terms, has the form

$$F_3(\xi, \eta) = F_3(\bar{\xi}, \bar{\eta}) + \Delta\xi \frac{\partial F_3}{\partial \xi} + \Delta\eta \frac{\partial F_3}{\partial \eta}. \tag{30}$$

We shall now substitute expressions (26), (29), (30) into Eqs. (28) and equate the coefficients at identical powers $\Delta h, \Delta c_2, \Delta c_1$ Then, we shall obtain with a precision to linear terms, the following differential equations for the determinations of functions $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$:

$$\begin{aligned} \frac{d\alpha_1}{dt} &= \alpha_1 \frac{\partial F_1}{\partial \xi} + \frac{\partial F_1}{\partial h} + \beta_1 \frac{\partial F_1}{\partial \eta}, \\ \frac{d\alpha_2}{dt} &= \alpha_2 \frac{\partial F_1}{\partial \xi} + \frac{\partial F_1}{\partial c_2} + \beta_2 \frac{\partial F_1}{\partial \eta}, \\ \frac{d\alpha_3}{dt} &= \alpha_3 \frac{\partial F_1}{\partial \xi} + \frac{\partial F_1}{\partial c_1} + \beta_3 \frac{\partial F_1}{\partial \eta}, \end{aligned} \tag{31}$$

../..

.. continuation ...

$$
\begin{aligned}
\frac{d\beta_1}{dt} &= \beta_1 \frac{\partial F_2}{\partial \eta} + \frac{\partial F_2}{\partial h} + \alpha_1 \frac{\partial F_2}{\partial \xi}, \\
\frac{d\beta_2}{dt} &= \beta_2 \frac{\partial F_2}{\partial \eta} + \frac{\partial F_2}{\partial c_2} + \alpha_2 \frac{\partial F_2}{\partial \xi}, \\
\frac{d\beta_3}{dt} &= \beta_3 \frac{\partial F_2}{\partial \eta} + \frac{\partial F_2}{\partial c_1} + \alpha_3 \frac{\partial F_2}{\partial \xi}, \\
\frac{d\gamma_1}{dt} &= c_1^{(m)} \left[\alpha_1 \frac{\partial F_3}{\partial \xi} + \beta_1 \frac{\partial F_3}{\partial \eta} \right], \\
\frac{d\gamma_2}{dt} &= c_1^{(m)} \left[\alpha_2 \frac{\partial F_3}{\partial \xi} + \beta_2 \frac{\partial F_3}{\partial \eta} \right], \\
\frac{d\gamma_3}{dt} &= F_3(\bar{\xi}, \bar{\eta}) + c_1^{(m)} \left[\alpha_3 \frac{\partial F_3}{\partial \xi} + \beta_3 \frac{\partial F_3}{\partial \eta} \right]
\end{aligned} \tag{31}
$$

with the initial data (27).

The partial derivatives $\frac{\partial F_i}{\partial \xi}, \frac{\partial F_i}{\partial \eta}, \frac{\partial F_i}{\partial h}, \frac{\partial F_i}{\partial c_2}, \frac{\partial F_i}{\partial c_1}$ $(i=1, 2, 3)$, standing in the right-hand parts of Eqs. (31), must be computed at the point $(\bar{\xi}, \bar{\eta}, h^{(m)}, c_1^{(m)}, c_2^{(m)})$.

Let us bring forth the final expressions for the partial derivatives

$$
\begin{aligned}
\frac{\partial F_1}{\partial \xi} &= -\frac{2\sqrt{2}}{(\xi^2 + c^2\eta^2)^2} \xi f_1^{1/2} + \frac{\sqrt{2}}{2(\xi^2 + c^2\eta^2)} [2h\xi(\xi^2 + c^2) + 2h\xi^3 + \\
&\quad + fM(\xi^2 + c^2) + 2fM\xi^2 + 2c_2\xi] f_1^{-1/2}, \\
\frac{\partial F_1}{\partial \eta} &= -\frac{2\sqrt{2}}{(\xi^2 + c^2\eta^2)^2} c^2 \eta f_1^{1/2}, \\
\frac{\partial F_1}{\partial h} &= \frac{\sqrt{2}\xi^2(\xi^2 + c^2)}{2(\xi^2 + c^2\eta^2)} f_1^{-1/2}, \\
\frac{\partial F_1}{\partial c_2} &= \frac{\sqrt{2}(\xi^2 + c^2)}{2(\xi^2 + c^2\eta^2)} f_1^{-1/2}, \\
\frac{\partial F_1}{\partial c_1} &= \frac{\sqrt{2}c_1}{2(\xi^2 + c^2\eta^2)} f_1^{-1/2}, \\
\frac{\partial F_2}{\partial \xi} &= -\frac{2\sqrt{2}}{(\xi^2 + c^2\eta^2)^2} \xi f_2^{1/2}, \\
\frac{\partial F_2}{\partial \eta} &= -\frac{2\sqrt{2}c^2\eta}{(\xi^2 + c^2\eta^2)^2} f_2^{1/2} + \frac{\sqrt{2}}{2(\xi^2 + c^2\eta^2)} [2hc^2\eta(1 - \eta^2) - \\
&\quad - 2hc^2\eta^3 - fM\sigma c(1 - \eta^2) + 2fM\sigma c\eta^2 + 2c_2\eta] f_2^{-1/2}, \\
\frac{\partial F_2}{\partial h} &= \frac{\sqrt{2}c^2\eta^2(1 - \eta^2)}{2(\xi^2 + c^2\eta^2)} f_2^{-1/2}, \\
\frac{\partial F_2}{\partial c_2} &= -\frac{\sqrt{2}(1 - \eta^2)}{2(\xi^2 + c^2\eta^2)} f_2^{-1/2}, \\
\frac{\partial F_2}{\partial c_1} &= -\frac{\sqrt{2}c_1}{2(\xi^2 + c^2\eta^2)} f_2^{-1/2}, \\
\frac{\partial F_3}{\partial \xi} &= -\frac{2\xi}{(1 - \eta^2)(\xi^2 + c^2)^2}, \\
\frac{\partial F_3}{\partial \eta} &= \frac{2\eta}{(\xi^2 + c^2)(1 - \eta^2)^2}, \\
\frac{\partial F_3}{\partial h} &= \frac{\partial F_3}{\partial c_2} = \frac{\partial F_3}{\partial c_1} = 0.
\end{aligned} \tag{32}
$$

where

$$f_1 = h\xi^4 + fM\xi^3 + (c_2 + c^2h)\xi^2 + fMc^2\xi + c^2\left(c_2 + \frac{1}{2}c_1^2\right),$$

$$f_2 = -hc^2\eta^4 + fMc\sigma\eta^3 + (c_2 + c^2h)\eta^2 - fMc\sigma\eta - \left(c_2 + \frac{1}{2}c_1^2\right).$$

The solution of the linear system of differential equations (24), (31) with the initial data (27), (27), may be performed by any numerical method, for example by that of Runge-Kutta . Consequently, the values of the functions

$$\overline{\xi}(t),\ \overline{\eta}(t),\ \overline{w}(t),\ \alpha_1(t),\ \alpha_2(t),\ \dots,\ \gamma_3(t)$$

for the moment of time $t = t_2$ may be computed. Since the values of functions $\xi(t)$, $\eta(t)$ and $w(t)$ for the moment of time $t = t_2$ are known, by resolving for $t = t_2$ the system of algebraic equations (26) we shall find the corrections Δh, Δc_2, Δc_1, and consequently also certain new values of integration constants h, c_1, c_2.

$$\begin{aligned} h^{(m+1)} &= h^{(m)} + \Delta h, \\ c_2^{(m+1)} &= c_2^{(m)} + \Delta c_2, \\ c_1^{(m+1)} &= c_1^{(m)} + \Delta c_1. \end{aligned} \tag{33}$$

Such a calculation process of consecutive approximations may be repeated more than once to obtain the required precision.

The question of convergence of the approximations (33) has not heretofore been investigated theoretically. On the basis of certain preliminary numerical computations it is possible to assert, however, that the approximation process is converging only if the initial approximations of constants $h^{(0)}$, $c_1^{(0)}$, $c_2^{(0)}$ are taken within a sufficiently small neighborhood of the true values of h, c_1, c_2

The initial approximations of constants $h^{(0)}$, $c_1^{(0)}$, $c_2^{(0)}$ for small time intervals may also be found. We shall indicate the way this can be done.

It is easy to see from Eqs. (6) and (7), that the equality

$$w_2 - w_1 = c_1 \int_{t_1}^{t_2} \frac{dt}{(1-\eta^2)(\xi^2 + c^2)}. \tag{34}$$

is valid for the moments of time t and t_2. In order to find the zero approximation $c_1^{(0)}$ we shall compute a specific integral, standing in the right-hand part of (34), according to the trapeze formula

$$\int_{t_1}^{t_2} \frac{dt}{(1-\eta^2)(\xi^2 + c^2)} = \frac{t_2 - t_1}{2}\left[\frac{1}{(1-\eta_1^2)(\xi_1^2 + c^2)} + \frac{1}{(1-\eta_2^2)(\xi_2^2 + c^2)}\right].$$

Then $c_1^{(0)}$ is found according to formula:

$$c_1^{(0)} = \frac{2(w_2 - w_1)}{t_2 - t_1}\left[\frac{1}{(1-\eta_1^2)(\xi_1^2 + c^2)} + \frac{1}{(1-\eta_2^2)(\xi_2^2 + c^2)}\right]^{-1}. \tag{35}$$

For the determination of the zero approximation of constants $c_2^{(0)}$ and $h^{(0)}$ we shall consider that

$$\xi_2 - \xi_1 \doteq \frac{d\xi}{dt}(t_1)\,[t_2 - t_1],$$

$$\eta_2 - \eta_1 \doteq \frac{d\eta}{dt}(t_1)\,[t_2 - t_1],$$

and the first two Eqs. (21) will be rewritten in the form

$$\frac{\xi_2-\xi_1}{t_2-t_1} = \frac{\sqrt{2}}{\xi_1^2+c^2\eta_1^2} \times \sqrt{h^{(0)}\xi_1^4 + fM\xi_1^3 + (c_2^{(0)}+c^2h^{(0)})\,\xi_1^2 + fMc^2\xi_1 + c^2\left(c_2^{(0)} + \frac{1}{2}c_1^{(0)^2}\right)}, \qquad (36)$$

$$\frac{\eta_2-\eta_1}{t_2-t_1} = \frac{\sqrt{2}}{\xi_1^2+c^2\eta_1^2} \times \sqrt{-h^{(0)}c^2\eta_1^4 + fM\sigma c\eta_1^3 + (c_2^{(0)}+c^2h^{(0)})\,\eta_1^2 - fM\sigma c\eta_1 - \left(c_2^{(0)} + \frac{1}{2}c_1^{(0)^2}\right)}.$$

Eqs (36) allow the finding of $h^{(0)}$, $c_2^{(0)}$, provided only $c_1^{(0)}$ was already found:

$$h^{(0)} = \frac{\xi_1^2+c^2\eta_1^2}{2}\left[\frac{(\xi_2-\xi_1)^2}{(t_2-t_1)^2(\xi_1^2+c^2)} + \frac{(\eta_2-\eta_1)^2}{(t_2-t_1)^2(1-\eta_1^2)}\right] - \frac{fM(\xi_1-c\sigma\eta_1)}{\xi_1^2+c^2\eta_1^2} + \frac{1}{2}\,\frac{c_1^{(0)^2}}{(\xi_1^2+c^2)(1-\eta_1^2)}, \qquad)37)$$

$$c_2^{(0)} = \frac{(\xi_2-\xi_1)^2(\xi_1^2+c^2\eta_1^2)^2}{2(t_2-t_1)^2(\xi_1^2+c^2)} - h^{(0)}\xi_1^2 - fM\xi_1 - \frac{1}{2}\,\frac{c^2c_1^{(0)^2}}{(\xi_1^2+c^2)}.$$

After the constants h, o_1, o_2 have been found, it is possible to compute the new constants a, e, s by utilizing Eqs. (9). The latter will have to be resolved by some approximate method. However, for the determination of constant quantities a, e, s it is more practical to apply another method. Indeed, when obtaining Eqs. (9) we considered [3], that the multiple term $\Phi(\xi)$ has in the general case two real roots, which we shall denote by a (1 + e) and a)1 —e). Neither is it difficult to show that the multiple term $F(\eta)$ has in the region (20) of variable η variation two real roots, one of which is denoted by s. This is why, having determined the real roots of the multiple terms $\Phi(\xi)$ and $F(\eta)$, and the corresponding regions (20) of variables ξ and η variation, we shall be able to determine a, e, s. However, it still remains unclear which of the two real roots corresponding to region (20) ought to be taken for s. This duality of the solution of the constant s will be eliminated in the furthest, when the remaining constants Ω, ω and M_0 are found. The calculation of the latter offers no particular difficulties.

Let us take for s either of the real roots of the multiple term $F(\eta)$. Now, knowing a, e, s, we shall compute all the coefficients determinable by formulas (12). Since the quantities η_1, η_2, ξ_1, ξ_2 are well known, we may determine with the aid of the first two Eqs. (10) the values φ_1, φ_2, v_1 and v_2. Then, resolving Eqs(13), for example, by the approximate Newton method [4], for the moments of time t_1 and t_2, we shall find the quantities u_1 and u_2. Then Eq. (14) will give us the value of the constant ω. If s was chosen correctly, it is evident that the constant ω, found from Eq, (14) for the moments of time t_1 and t_2, will be one and the same within the precision of calculations.

In the opposite case, one should take for the constant s the other root of the term $F(\eta)$ from region (20) and recompute the quantities $\varphi_1, \varphi_2, v_1, v_2, u_1, u_2$ and the constant ω. The constant Ω, corresponding to the longitude of the node in the Keplerian motion, will be found without difficulty from the last equality (10).

Finally, Eq. (15), linking the independent variable t with the variable v, allows the finding of the last constant M_0.

Thus, all the six constants of integration a, e, s Ω, ω, M_0 are determined, and, moreover, they are determined in a unique fashion.

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****** T H E E N D ******

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