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REPRESENTATION OF SPINORS IN THE n — DIMENSIONAL SPACE BY A SYSTEM OF TENSORS

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1. BASIC DETERMINATIONS.

Let us consider at the outset a four-dimensional complex Euclidean space $R_n{}^+$, $n = 2\nu$, referred to the orthonormalized base e_i. Let $\gamma_1, \gamma_2, \ldots, \gamma_{2\nu}$ be the dimensionality matrices 2^ν, satisfying by definition the equation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}I, \qquad (1)$$

where δ_{ij} is the Kronecker symbol, I is the unitary dimensionality matrix 2^ν. We shall introduce the denotation $\gamma_{i_1, i_2 \ldots i_k} = i^{k(k-1)/2}\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_k}$, $i_1 < i_2 < \ldots < i_k$. The matrices $I, \gamma_i, \gamma_{i_1}\gamma_{i_2}, \ldots, \gamma_{i_1}\gamma_{i_2}\ldots i_n$ are linearly independent and form a group of elements. As is well known, any two solutions γ_i, γ_i of Eq. (1) are linked by the equality $\overline{\gamma}_i = T\gamma_iT^{-1}$, $\det T \neq 0$, whereupon matrices γ_i may be chosen Hermitian, and in such a fashion that $\gamma_1, \gamma_2, \ldots, \gamma_\nu$ be symmetrical, and $\gamma_{\nu+1}, \gamma_{\nu+2}, \ldots, \gamma_{2\nu}$ antisymmetrical [1].

Let $L = \| l_q{}^p \|$ be the orthogonal transformation of the space $R_{2\nu}{}^+$. The multiplicity of unimodular matrices S, determined from the equation

$$\gamma_p = l_p{}^q S\gamma_q S^{-1}, \qquad (2)$$

form a group materializing the representation of the group L, called spinor representation.

The object $\psi = \{\psi^i\}$ with components ψ^i determined with a precision to the sign transforming according to the representation of S, is called the first rank spinor in the space $R_{2\nu}{}^+$.

If γ_i is the solution of (1), it is obvious that $\gamma_i{}^T$ ($\gamma_i{}^T$ being the transposed γ_i) is also a solution of (1); this is why there exists a matrix C such that $\gamma_i{}^T = C\gamma_iC^{-1}, \det C = 1$.

If ν is odd, it is easy to see that $C = \gamma_\nu\gamma_{\nu-1}\ldots\gamma_1$; if ν is even, then $C = \gamma_{2\nu}\ldots\gamma_{\nu+1}$ in the case when $\gamma_1, \gamma_2, \ldots, \gamma_\nu$ are symmetrical, but $\gamma_{\nu+1}, \gamma_{\nu+2}, \ldots, \gamma_{2\nu}$ are asymmetrical. Hence it follows that $C^T = (-1)^{\nu(\nu-1)/2}C$. If ψ_i are the covariant components of the spinor ψ, then we have by definition $\psi_i = e_{ij}\psi^j$, where $E = \| e_{ij} \| = i^{n(n-1)/2}\gamma_n\gamma_{n-1}\ldots\gamma_1C$.

We shall outline in the space R_n^+ the pseudo-Euclidean space $R_n(s)$ index s, upon fixing the base $ie_1 \dots ie_s e_{s+1} \dots e_n$. We shall introduce the Hermitian matrix Π determined from the equations

$$\Pi\dot{\gamma}_i\Pi^{-1} = \pm\gamma_i, \quad \Pi\dot{\Pi} = (-1)^{(\nu-s)(\nu-s+1)/2} I; \tag{3}$$

where the sign minus is for $i = 1, \dots, s$ and the sign plus is for $i = s + 1, \dots, 2\nu$. The dot above the letter denotes the complex mating. The spinor $\psi = \Pi\dot{\psi}$ is called conjugate relative to ψ.

Let us consider now the odd-dimensional spaces R_n^+, $n = 2\nu + 1$. We shall denote $\gamma_{2\nu+1} = i^\nu \gamma_{1,2,\dots,2\nu}$. The matrices with an even number of indices I, $\gamma_{i_1 i}, \dots, \gamma_{i_1 i_2 \dots i_m}$ $(i_p = 1, 2, \dots, 2\nu + 1)$ are linearly independent. The spinor representation of the intrinsic orthogonal group L of space $R^{(s)}_{2\nu+1}$, transformation is given by the group of matrices S determined from Eq. (2), in which the indices p, q acquire values from 1 to $2\nu + 1$. The covariant components of spinor ψ_1 are determined by the matrix E:

$$(-1)^\nu \gamma_i^T = E\gamma_i E^{-1}.$$

The conjugate spinor ψ in space $R^{(s)}_{2\nu+1}$ is determined by the matrix Π

$$\pm\gamma_i = (-1)^{\nu-s}\Pi\dot{\gamma}_i\Pi^{-1};$$

where the sign minus is for $i = 1, 2, \dots, s$.

2. REPRESENTATION OF SPINORS BY COMPLEX TENSOR SYSTEM

Let us consider a complex matrix $\Psi = \{\psi^{ij}\}$ of dimensionality r. If $\psi^{ij} = \psi^i\psi^j$, ψ^{ij} must satisfy the equalities

$$\psi^{ij}\psi^{kl} = \psi^{ik}\psi^{jl} = \psi^{il}\psi^{kj}, \quad \psi^{ij} = \psi^{ji}, \tag{4}$$

among which $r(r + 1)/2 - r$ equations $\psi^{\nu\nu}\psi^{ij} = \psi^{\nu i}\psi^{\nu j}$ $(i, j \neq \nu, \psi^{\nu\nu} \neq 0)$ and $\frac{1}{2}r(r-1)$ equations $\psi^{ij} = \psi^i\psi^j$, are independent. There are in all $(r^2 - r)$ independent equations. Reciprocally, it follows from (4) that there exists a system of r components ψ^k, determined with a precision to signs such that $\psi^{ij} = \psi^i\psi^j$. In reality, if $\psi^{\nu\nu} = 0$ for all ν, it follows from (4) that $\psi^{ij} = 0$ for all values of indices i, j. If $\psi^{\nu\nu} \neq 0$, we postulate

$$\psi^k = \psi^{\nu k} / \pm\sqrt{\psi^{\nu\nu}}. \tag{5}$$

By the strength of (4) such a definition of ψ^k does not depend on the value of the index ν. Let us now assume that ψ^{ij} are components of the object Ψ, transforming according to the representation S x S, where S is any representation of a certain group, and let Eqs. (4) be invariant relative to the group S x S. Then the components ψ^k, determined according to (5) will be transformed according to the representation S. Indeed, it follows from (5) that the transformation of ψ^k is determined by the transformation of ψ^{ij} in a single fashion. Evidently, the identity (4) invariantness is maintained if ψ^k is transformed according to the representation S, consequently, by the strength of the uniqueness, ψ^k may be transformed only by the representation S. Thus the object ψ^{ij}, satisfying the identities (4) is equivalent to the object ψ^k.

Let S be the spinor representation of a 2ν-dimensional orthogonal group. It is well known that

$$S \times S \sim \sum_{k=0}^{2\nu} D^k,$$

where D^k is a representation by which a tensor of rank k, asymmetrical by all indices, is transformed. This means that in this case the object ψ^{ij} is equivalent to the tensor aggregate $\Lambda = \{c_0, c_i, c_{i_1 i_2}, \ldots, c_{i_1 i_2 \ldots i_{2\nu}}\}$. If S is the spinor representation of a $(2\nu + 1)$-dimensional group

$$S \times S \sim \sum_{k=0}^{\nu} D^{2k},$$

and consequently, the object ψ^{ij} is equivalent to the tensor aggregate consisting of even rank tensors. The components of the antisymmetrical tensors $C_{i_1 i_2 \ldots i_k}$ may be determined as follows:

$$c_{i_1 i_2 \ldots i_k} = (A_{i_1 i_2 \ldots i_k})_{\alpha\beta} \psi^{\alpha\beta}, \quad A_{i_1 i_2 \ldots i_k} = E\gamma_{i_1}\gamma_{i_2} \ldots \gamma_{i_k}. \qquad (6)$$

Inasmuch as $\det E \neq 0$ and $\gamma_{i_1 i_2 \ldots i_k}$ are linearly independent, $A_{i_1 i_2 \ldots i_k}$ are also linearly independent and consequently, the aggregates Λ are really equivalent to objects ψ^{ij}.

Taking advantage of symmetry properties of C, it may be shown that matrices $A_{i_1 i_2 \ldots i_k}$ have the following symmetry properties:

$$(A_{i_1 i_2 \ldots i_k})^T = (-1)^{(\nu(\nu+1)+k(k+1))/2} A_{i_1 i_2 \ldots i_k}.$$

This is why, so long as ψ^{ij} satisfies the identities (4), part of tensors $c_{i_1 i_2 \ldots i_k}$ $([\nu(\nu+1) + k(k+1)]/2$ being odd) become zero. If ψ^{ij} satisfy the identities (4), tensors $c_{i_1 i_2 \ldots i_k}$ satisfy $1/2\, 2^\nu(2^\nu - 1)$ independent bilinear identities, all of which are included in the generalization of the Pauli identity for the case of a n-dimensional space, (See [2])

$$2^\nu (\psi^+\theta\psi)(\psi^+\theta'\psi) = \sum_{k=1}^{2\nu} \sum_{i_1 < i_2 < \ldots < i_k}^{2\nu} (\psi^+\gamma_{i_1 i_2 \ldots i_k}\psi)(\psi^+\theta'\gamma_{i_1 i_2 \ldots i_k}\theta\psi) + $$
$$+ (\psi^+\psi)(\psi^+\theta'\theta\psi), \quad n = 2\nu. \qquad (7)$$
$$2^\nu (\psi^+\theta\psi)(\psi^+\theta'\psi) = \sum_{k=1}^{2\nu} \sum_{i_1 < i_2 < \ldots < i_k}^{2\nu+1} (\psi^+\gamma_{i_1 i_2 \ldots i_{2k}}\psi)(\psi^+\theta'\gamma_{i_1 i_2 \ldots i_{2k}}\theta\psi) + $$
$$+ (\psi^+\psi)(\psi^+\theta'\theta\psi), \quad n = 2\nu + 1,$$

where θ', θ are arbitrary matrices of dimensionality 2^ν; ψ^+, ψ are covariant and contravariant component of the spinor.

Therefore, spinor ψ^k in the space $R_n{}^+$, $n = 2\nu, 2\nu + 1$ is equivalent to the tensor aggregate Λ, consisting of complex antisymmetrical tensors satisfying $1/2\, 2^\nu(2^\nu - 1)$ bilinear identities (7). By virtue of this, any spinor equation may be written in an equivalent manner as an equation in tensor components $c_{i_1 i_2 \ldots i_k}$.

Note that formula (3) determines the components ψ^k in any system of coordinates, but the transformation of components ψ^k to curvilinear coordinates is found to be nonlinear relative to ψ^k.

3. REPRESENTATION OF SPINORS BY A SYSTEM OF REAL TENSORS

We shill consider an r-dimensional complex matrix $\psi^{p\dot{q}}$. Assume that $\psi^{p\dot{q}} = \psi^p \psi^{\dot{q}}$.

Then $\psi^{\dot{p}q}$ will satisfy the identities

$$\psi^{\dot{p}q} = (\psi^{\dot{q}p})^{\cdot}, \quad \psi^{\dot{m}n}\psi^{\dot{p}r} = \psi^{\dot{m}r}\psi^{\dot{p}n} \tag{8}$$

among which $(r - 1)$ real equations $\psi^{\dot{\nu}\nu}\psi^{\dot{p}q} = \psi^{\dot{\nu}q}\psi^{\dot{p}\nu}$ $(p, q \neq \nu, \psi^{\dot{\nu}\nu} \neq 0)$ and r^2 real equations $\psi^{\dot{p}q} = (\psi^{\dot{q}p})^{\cdot}$ are independent. Obviously, if the components ψ^k determine the matrix $\psi^{\dot{p}q}$ the components $\psi^k e^{i\varphi}$, and only they, determine the same matrix $\psi^{\dot{p}q}$.

Reciprocally, it follows from (8) that there exists a system of component determined with a precision to the phase $e^{i\varphi}$, such that $\psi^{\dot{p}q} = \psi^{\dot{p}}\psi^q$. In reality if $\psi^{\dot{\nu}\nu} = 0$ for all ν, it will follow from (8) that $\psi^{\dot{p}q} = 0$ for all p, q. In this case we postulate $\psi^k = 0$. If $\psi^{\dot{\nu}\nu} \neq 0$, we postulate

$$\psi^k = \frac{\psi^{\dot{\nu}k}}{\pm\sqrt{\psi^{\dot{\nu}\nu}}} e^{i\varphi}, \tag{9}$$

where φ is an arbitrary real number.

By virtue of (8) such a determination of ψ^k multiplicity does not depend on the value of ν. It may be shown that only the components ψ^k, determined according to (9) satisfy the equation $\psi^{\dot{p}q} = \psi^{\dot{p}}\psi^q$.

Assume that $\psi^{\dot{p}q}$ are components of an object transforming according to the representation S x S, where S is any representation of a certain group, and let the equalities (8) be invariant relative to the group S x S. Then, we may evidently point to such a law of transformation of φ that the components ψ^k transform according to the representation S.

In this way the assignment of the object $\psi^{\dot{p}q}$ satisfying the identities (8) and of the argument φ of one of the components ψ^k fully determine the object Let S be the spinor representation of a 2ν-dimensional orthogonal group of transrmations of the space $R_{2\nu}^{(s)}$. It is well known that

$$\dot{S} \times S \sim \sum_{k=0}^{2\nu} D^k.$$

This means that the object $\psi^{\dot{p}q}$ is equivalent to the tensor aggregate $\Omega = \{\Omega_0 \Omega_i \ldots \ldots \Omega_{i_1 i_2 \ldots i_{2\nu}}\}$, consisting of antisymmetrical tensors.

If S is a spinor representation in the space $R_{2\nu+1}^{(s)}$, $\dot{S} \times S \sim \sum_{k=0}^{\nu} D^{2k}$, and in this case the object $\psi^{\dot{p}q}$, is equivalent to the aggregate $\Omega = \{\Omega_0 \Omega_{i_1 i_2} \ldots \Omega_{i_1 i_2 \ldots i_{2\nu}}\}$, consisting of antisymmetrical tensor of even rank.

The components of these tensors my be determined as follows :

$$\Omega_{i_1 i_2 \ldots i_k} = (D_{i_1 i_2 \ldots i_k})_{\dot{\alpha}\beta}\psi^{\dot{\alpha}\beta}, \quad D_{i_1 i_2 \ldots i_k} = E\Pi\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_k} i^{k(k+1)\,2}.$$

Utilizing (3), we may show that matrices $D_{i_1 i_2 \ldots i_k}$ are Hermitian.

This is why, the components $\Omega_{i_1 i_2 \ldots i_k}$ are real if $\psi^{\dot{p}q}$ satisfy the identity $\psi^{\dot{p}q} = (\psi^{\dot{q}p})^{\cdot}$ If $\psi^{\dot{p}q}$ satisfy the identities (8), the components of tensors $\Omega_{i_1 i_2 \ldots i_k}$ satisfy $(2^{\nu} - 1)^2$ bilinear identities, of which everyone is contained in the identity (7).

Therefore, the assignment of the aggregate Ω and of argument φ of one of the components fully determines the spinor. This means that the spinor equations may be written in an equivalent manner in components of the aggregate Ω and φ. Then, eliminating from such equations the argument φ, it is possible to obtain a closed system of equations in components of aggregate Ω.

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