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ON THE VIBRATION OF BEAMS OR RODS CARRYING AN ARBITRARY NUMBER OF CONCENTRATED MASSES

by

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SUMMARY 12928

An easy solution is given to the problem of the free vibrations of simply supported beams carrying an arbitrary number of concentrated masses in an arbitrary way. Finite Fourier sine transforms and the modification of the beam to a variable density beam by the introduction of Dirac δ - functions are utilized in the solution of the problem. The simplified frequency equations corresponding to (i) a single concentrated mass in the middle of the beam and (ii) two equally spaced identical concentrated masses, are given. For the latter case, some of the roots of the frequency equation are numerically evaluated.

Author

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LIST OF SYMBOLS

M	Mass added to beam
EI	Flexural rigidity of the beam
y	Transverse deflection of the beam
ρ	Constant mass- density of the beam material
A	Area of cross-section of the beam
p	Natural frequency of the beam
ψ	Mode shape of beam

1.0 INTRODUCTION

The problem of the vibration of beams carrying a concentrated mass in the middle has evinced considerable interest. Very recently, Yu Chen, reference (1) applying the Laplace transform technique, originally developed by Thomson, reference (2) for static beam problems, solved this problem. A review of literature indicates that the problem of vibration of beam, carrying a concentrated mass in the middle has been solved by Kármán and Biot reference (3) and Hoppmann, reference (4) by different methods. In references 1, 3, and 4, attention was focused on symmetrical modes of vibration only, because of the special position of mass on the beam.

In the present note, the authors presented an alternate method of determining the eigenvalues of the free vibration equation of a simply supported beam with a finite number of discrete masses. The partial differential equation for free vibrations of a bar with a number of discrete masses, which can be expressed by using Dirac δ - functions as a variable mass density beam, is initially reduced to an ordinary differential equation by standard methods of separation of variables. The ordinary differential equation in space variables, with its boundary conditions, is solved by an application of finite Fourier sine transforms. The infinite set of eigenvalues is obtained as the roots of a transcendental equation. If all masses are taken to be zero, we get the classical solution.

2.0 FORMULATION OF THE PROBLEM

Let us consider the free vibrations of a simply supported beam carrying an arbitrary number of discrete masses M_1, $M_2, .. M_q$ on the beam. In this analysis, it is assumed that E I, the flexural rigidity of the beam, is constant, even though the beam carries a set of masses. The equation of motion, with damping neglected, can be written as follows:

$$E I \frac{\partial^4 y}{\partial x^4} + \left[\rho A + \sum_{i=1}^{q} M_i \ \delta (x - x_i) \right] \frac{\partial^2 y}{\partial x^2} = 0 \qquad (1)$$

where

E I = flexural rigidity of the beam.
y = $y(x,t)$ is the transverse deflection of the beam.
ρ = constant mass - density of the beam material.
A = area of cross-sectionof the beam.
M_i = i^{th} mass located at $x = x_i$

and where the δ - function is defined by the following relation

$$\int_{-\infty}^{+\infty} \delta (x - x_i) \, dx = 1 . \qquad (2)$$

The boundary conditions are:

$$y(0,t) = Y(a,t) = \frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^2 y}{\partial x^2}(a,t) = 0 \qquad (3)$$

where a is the span of the beam. For a bar vibrating with amplitude $\psi(x)$, we can write for a single harmonic:

$$y(x,t) = \psi(x) \ e^{ipt} \qquad (4)$$

Substituting (4) in (1), we have

$$E I \frac{d^4 \psi}{d x^4} - p^2 \left[\rho A + \sum_{i=1}^{q} M_i \ \delta \ (x - x_i) \right] \psi = 0 \tag{5}$$

and the boundary conditions

$$\psi (0) = \psi (a) = \frac{d^2 \psi}{d x^2} (0) = \frac{d^2 \psi}{d x^2} (a) = 0 \tag{6}$$

Equations (5) and (6) constitute an eigenvalue problem, the solution of which can be obtained easily by an application of finite sine transformation. Let us denote (5)

$$\overline{\psi} (n) = \int_0^a \psi (x) \ \sin \frac{n \pi x}{a} d x \ . \tag{7}$$

Taking finite sine transforms of the differential equation (1) and using the transforms of the boundary conditions in (6), we have

$$\overline{\psi} \left[\frac{n^4 \pi^4}{a^4} - k^4 \right] - \frac{p^2}{E I} \sum_{i=1}^{q} M_i \ \psi (x_i) \ \sin \frac{n \pi x_i}{a} = 0 \tag{8}$$

where

$$k^4 = \frac{\rho A p^2}{E I}$$

Therefore:

$$\overline{\psi}(n) = \frac{p^2 \sum_{i=1}^{q} M_i \ \psi(x_i) \ \sin \frac{n \pi x_i}{a}}{E I \left(\frac{n^4 \pi^4}{a^4} - k^4 \right)} \tag{9}$$

The inverse of sine transform is given by reference (5)

$$\psi(x) = \frac{2}{a} \sum_{n=1}^{\infty} \overline{\psi}(n) \ \sin \frac{n \pi x}{a} \tag{10}$$

Applying (10) to (9) one obtains

$$\psi(x) = \frac{2 p^2}{E I a} \sum_{i=1}^{q} M_i \ \psi(x_i) \sum_{n=1}^{\infty} \frac{\sin \frac{n \pi x_i}{a} \ \sin \frac{n \pi x}{a}}{\frac{n^4 \pi^4}{a^4} - k^4} \tag{11}$$

Summation of Series

Equation (11) may be written as follows:

$$\psi(x) = \frac{2 p^2}{E I a} \sum_{i=1}^{q} M_i \ \psi(x_i) \ s_i(x) \tag{12}$$

where

$$s_i(x) = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x_i}{a} \sin \frac{n\pi x}{a}}{\frac{n^4 \pi^4}{a^4} - k^4} \tag{13}$$

We can evaluate the sum of the infinite series of (13) as follows, reference (6) . We use the known series

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^2 + \alpha^2} = -\frac{1}{2\alpha^2} + \frac{\pi}{2\alpha} \frac{\cosh \alpha(\pi - z)}{\sinh \pi\alpha} \tag{a}$$

and similarly

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha} \frac{\cos \alpha(\pi - z)}{\sin \pi\alpha} \tag{b}$$

to obtain the following relation

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^4 - \alpha^4} = \frac{1}{2\alpha^4} - \frac{\pi}{4\alpha^3}\left[\frac{\cos \alpha(\pi - z)}{\sin \pi\alpha} + \frac{\cosh \alpha(\pi - z)}{\sinh \pi\alpha}\right] \tag{c}$$

which holds for $0 < z < 2\pi$ and where $\alpha = \frac{ak}{\pi}$.

$$s_i(x) = \frac{a^4}{2\pi^4} \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{a}(x - x_i) - \cos \frac{n\pi}{a}(x + x_i)}{n^4 - a^4}$$

$$= \frac{a}{4k^3}\left[\frac{\sin k\,(a - x_i)\ \sin kx}{\sin ka} - \frac{\sinh k\,(a - x_i)\ \sinh kx}{\sinh ka}\right] \text{for } x < x_i$$

and (14)

$$= \frac{a}{4k^3}\left[\frac{\sin kx_i\ \sin k\,(a - x)}{\sin ka} - \frac{\sinh kx_i\ \sinh k\,(a - x)}{\sinh ka}\right] \text{for } x > x_i$$

or we can write (14) in a compact form as

$$s_i(x) = \frac{a}{4k^3}\left\{\frac{\sinh k\,(a - x_i)\ \sin kx}{\sin ka} - \frac{\sinh k\,(a - x_i)\ \sinh kx}{\sinh ka}\right.$$

$$\left. - U(x - x_i)\left[\sin k\,(x - x_i) - \sinh k\,(x - x_i)\right]\right\} \tag{15}$$

where $U(x)$ is the unit step function.

If $x = x_i$ then (12) reduces to

$$\psi(x_j) = \frac{2p^2}{EIa} \sum_{i=1}^{q} M_i \, \psi(x_i) \, S_{ij} \tag{16}$$

where

$$\frac{2}{a} s_{ij} = \frac{1}{2k^3} \left\{ \frac{\sin k(a-x_i) \sin k x_j}{\sin ka} - \frac{\sinh k(a-x_i) \sinh k x_j}{\sinh ka} \right.$$

$$\left. - U(x_j - x_i)\left[\sin k(x_j - x_i) - \sinh k(x_j - x_i)\right] \right\} \tag{17}$$

Thus (15) can now be written as

$$\frac{2p^2}{EIa} \sum_{i=1}^{q} M_i \, \psi(x_i) \, S_{ij} - \psi(x_j) = 0 \tag{18}$$

$$j = 1, 2, \ldots q.$$

For a non-trivial solution of the set of homogeneous equations in (18) the determinant of the coefficients must vanish. Hence

$$\begin{vmatrix} \gamma M_1 s_{11} - 1 & \gamma M_2 s_{21} & - & - & \gamma M_q s_{q1} \\ \gamma M_1 s_{12} & \gamma M_2 s_{22} - 1 & - & - & \gamma M_q s_{q2} \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ \gamma M_1 s_{1q} & \gamma M_2 s_{2q} & - & - & \gamma M_q s_{qq} - 1 \end{vmatrix} = 0 \tag{19}$$

where

$$\gamma = \frac{2 p^2}{E I a}$$

2.1 Special Cases

(a) Suppose all masses are identically zero

$$M_1 = M_2 = \text{-----------} M_q = 0$$

Then, from (19)

$$S_{i i} \quad (\text{No sum on } i) \longrightarrow \infty \qquad i = 1, 2, \ldots\ldots q$$

which implies

$$\sin ka \longrightarrow \infty \tag{20}$$

The consecutive roots of (20) are

$$k a = n \pi \qquad n = 1, 2, \ldots\ldots \infty \tag{21}$$

which agrees with the classical result.

(b) One mass at the middle of the beam, i.e.

$i = 1$; $a = 2 l$; and $x_i = l$.

Then,

$$1 = \frac{- M p^2}{4 E I k^3} \left\{ \frac{\sinh k x}{\cosh k l} - \frac{\sin k x}{\cos k l} \right.$$

$$- 2U\,(x - l)\,\left[\sinh k\,(x - l) - \sin k\,(x - l)\right]\Bigg\} \tag{22}$$

Equation (22) simplifies to

$$M p^2 \,(\tan kl - \tanh kl) - 4 E I k^3 = 0 \tag{23}$$

Equation (23) agrees with equation (12) of reference 1.

(c) Solution of the frequency determinant for two identical masses equally spaced.

In the case of 2 equal masses, equally spaced on a simply supported beam, the determinantal equation (19) reduces to:

$$(\gamma M s_{11} - 1)\,(\gamma M s_{22} - 1) - \gamma^2 M^2 s_{12}\, s_{21} = 0 \tag{24}$$

where $a = 3l \qquad x_1 = l \qquad x_2 = 2l$.

Equation (24) reduces, after lengthy computations to:

$$u^2 \left[4\,(\sinh u \sin u - \sinh 2u \sin 2u) + 3\,(\cosh 2u - \cos 2u)\right]$$
$$+ 8\lambda u \left[\sinh 2u - \sin 2u + 2\,(\sinh 2u \cos 2u - \cosh 2u \sin 2u)\right]$$
$$+ 8\lambda^2 \frac{\sinh 3u \sin 3u}{\sinh u \sin u} = 0 . \tag{25}$$

where $\lambda = \dfrac{\rho A L}{M} = \dfrac{M_o}{M}$ and $u = kl$.

The first 11 roots of the above transcendental equation are obtained on an IBM 1620 digital computer. Table 1 shows the values of kl / corresponding to the first 11 modes of vibration, for different values of λ ; (the ratio of the mass of the beam to the attached mass). The values of kl corresponding to the limiting case of $\lambda \rightarrow \infty$ are obtained from the classical theory as $\frac{n\pi}{3}$ where n is an integer. The fundamental and the lower frequencies are highly dependent on the value of λ, while the effect of λ is negligible on the higher frequencies.

Every third frequency is a constant and equals $n\pi$ where n is an integer. One can easily obtain the following relations

$$f = \frac{p}{2\pi} = \alpha \sqrt{\frac{E\,I}{\rho A l^4}} = \beta \sqrt{\frac{E\,I}{M l^3}}$$

where

$$\beta = \frac{\alpha}{\sqrt{\lambda}} \qquad \alpha = \frac{\upsilon^2}{2\pi}$$

and f = frequency in cycles.

Table 2 shows the distribution of β for different values of λ. The lowest two values of β, namely, 0.174 and 0.664, corresponding to values of $\lambda = 0.001$ coincide with the frequencies of a massless beam, ($\lambda \rightarrow 0$) treated as a system with two degrees of freedom, (0.174, 0.675)

2.2 Concluding Remarks

For a beam whose boundary conditions at $x = 0$ and $x = 2l$ are $d\psi/dx = 0$ and $d^3\psi/dx^3 = 0$, finite cosine transforms can be used. It may be noted, that for such a beam, without any masses, the classical frequency equation is the same as that of a simply supported beam, namely $\sin k a = 0$. The advantage of sine and cosine transforms over Laplace transforms is that the inverse in the former is a converging infinite series, which normally can be summed. ($a = 2l$).

TABLE 1

λ \ kℓ	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
∞						
10.	1.023	2.045	"	4.098	5.119	"
5.	1.001	2.000	"	4.030	5.027	"
2.	0.946	1.889	"	3.899	4.840	"
1.	0.880	1.753	"	3.790	4.676	"
0.5	0.796	1.579	"	3.699	4.534	"
0.001	0.186	0.366	"	3.557	4.298	"

λ \ kℓ	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	3π	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
∞					
10.	7.186	8.201	"	10.282	11.289
5.	7.093	8.075	"	10.177	11.142
2.	6.948	7.858	"	10.036	10.920
1.	6.854	7.703	"	9.957	10.781
0.5	6.790	7.587	"	9.907	10.688
0.001	6.708	7.430	"	9.849	10.572

TABLE 2

λ \ β						
10.0	0.053	0.211	0.497	0.845	1.319	
5.0	0.071	0.285	0.702	1.156	1.799	
2.0	0.100	0.402	1.111	1.711	2.636	
1.0	0.123	0.489	1.571	2.286	3.480	
0.5	0.143	0.561	2.221	3.080	4.627	
0.001	0.174	0.664	4.967	6.369	9.297	
10.0	1.987	2.600	3.385	4.468	5.321	6.414
5.0	2.810	3.581	4.641	6.319	7.372	8.836
2.0	4.443	5.433	6.949	9.991	11.335	13.420
1.0	6.283	7.477	9.444	14.180	15.779	18.499
0.5	8.886	10.377	12.956	19.983	22.091	25.712
0.001	19.869	22.648	27.784	44.683	48.819	56.251

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