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TEMPERATURE DETERMINATION OF PLANETARY ATMOSPHERES

OPTIMUM BOUNDARY CONDITIONS FOR BOTH LOW AND HIGH SOLAR ACTIVITY

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SUMMARY

Accurate temperature-altitude profiles of planetary atmospheres for altitudes above the diffusive separation level may be determined from the simultaneously observed altitude profiles of the number density of two inert gases having markedly different molecular weights M, without any assumptions concerning reference-level temperatures. In the earth's atmosphere, such gases would preferably be helium and argon. If the altitude profile of the number density of but a single inert heavy gas is measured, the temperature-altitude profile is obtainable only for the lower portion of a sufficiently large altitude interval of observation by means of the downward application of a single-gas equation. If, on the other hand, the observed number-density data are for a single light gas such as helium, a temperature-altitude profile may be determined by an upward application of the single-gas equation, but only if the temperature is independently known at the lowest altitude of observed number-density data. If both the light and heavy gases are measured simultaneously these two sets of number desities introduced into a dual-gas equation permit the determination of the temperature over the entire altitude interval of the dual-gas observation. Previously described methods using the mass-density profile or the total number-density profile in a mixed or diffusively separating atmosphere permitted the determination of only the ratio of temperature to mean molecular weight and that over only a limited portion of the altitude interval of the observed data. In contrast, to this situation the one-gas and two-gas methods described yield values of kinetic temperature directly. In the two-gas method the temperatures is determined not only at the lower altitudes where both heavy-gas and light-gas data may be measured but also up to the highest altitudes for which the light-gas number density has been measured, but where the heavy-gas number density has decreased to values below the detection sensitivity of the sensor.

An analytical and numerical examination of the single-gas and double-gas equations for both upward and downward calculations (that is, for both high-altitude and 10w-altitude reference levels) using atmospheric models for both high and low solar activity, indicates the conditions which optimize each type of calculation. The method depends upon recently developed air-borne mass spectrometers with detection sensitivities of the order of 10^4 to 10^5 particles per cubic centimeter.

An error analysis based on the gaussian method has been applied to the various temperature equations where the perfect integral of number density over a specified altitude interval has been approximated by a numerical-integration expression developed from a logarithmic tropezoidal rule. Number-density uncertainties based upon the sensitivities of present-day mass spectrometers were used in the numerical evaluation of the error expressions. The error analysis demonstrates that two sets of single-gas data applied consecutively and iteratively for a number of cycles to two appropriate single-gas equations yield temperature uncertainties which are essentially indentical to those obtained by the single application of each of two appropriate double-gas equations. The error analysis further demonstrates: (1) That for optimum conditions,

the high-altitude reference level should be chosen as the altitude for which the uncertainty in observed number-density data is 100 percent. (2) That the absolute temperature uncertainty is not strongly influenced by variations in number-density models associated with variations in solar activity. (3) That the percentage uncertainty at high altitudes, however, is strongly influenced by such variations in number-density models.

PART I

EQUATION ANALYSIS

INTRODUCTION

Knowledge of the atmospheric temperature profile as a function of height is of great importance for the interpretation of many physical effects in any planetary atmosphere and for the understanding of the mechanisms involved. Temperature with its altitude-dependent variations is a basic atmospheric parameter since it is the defining property of many other physical properties of an atmosphere. Even small changes in the temperature profile can considerably affect the pressure and density distribution in any atmosphere. Absorption of solar radiation within some altitude region, for instance, may result in an increase of temperature over an extended altitude range, and consequently may change the density at much higher altitudes by orders of magnitude.

Any direct approach to the measurement of temperature such as with a thermistor or thermocouple is practically impossible, in a gas as rarefied as that of the earth's atmosphere above 120 km altitude, and indirect methods applied to the upper regions of any planetary atmosphere except those involving long time averaging have to date yielded results with rather limited accuracy. On the other hand, any discussion of atmospheric properties must be based on realistic model atmospheres which cannot be created without reasonable estimates of the temperature distribution with height. In the case of the earth's atmosphere, some model atmospheres [1-4]* have been developed from assumed temperature profiles which were adjusted repeatedly until the pressure and density values calculated from these temperature profiles matched the observed rocket and satellite data within limits compatible with the wide spread of data. In other cases, various authors [5-7] deducing temperatures from the integral of total number density or mass density, obtained results which, at the upper end of the density-altitude profile are heavily biased by the assumed reference temperature, and for altitudes above about 100 km yield only the ratio T/M rather than kinetic temperature.

Since rocket-borne or satellite-borne, high-sensitivity mass spectrometers have recently become available [8-10], a new method for obtaining temperature profiles can now be suggested. The previously used methods for obtaining temperatures from altitude profiles of total mass density or total pressure required the additional knowledge of the altitude profile of mean molecular weight. Since accurate molecular-weight information has not been available, absolute temperature determinations could not be made accurately, especially

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* Numbers in [ ] represent reference numbers.

at altitudes above 250 km where estimates of mean molecular weight may be off by a factor of two. The method suggested in this paper, however, relies on the knowledge of the number-density profiles of two inert gases such as helium and argon with widely differing atomic weights. The vertical distribution of these two gases is known to follow the thermodynamical condition of the atmosphere. It is believed that dissociation, ionization effects, and charge exchange processes do not affect the distribution of these two gases, at least up to altitudes of approximately 1000 km. Consequently, the equation of state and the hydrostatic equation without additional expressions govern the individual vertical distributions of these gases above the level of the onset of diffusive separation.

It will be shown that no initial knowledge of the temperature at any altitude is necessary to establish a temperature profile over the entire altitude range of observed number density of either gas as long as both are in diffusive equilibrium, and as long as number densities for both are determined simultaneously for a limited altitude region. In order to verify the relative suitability of the several approaches, it has been necessary to perform a rigorous error analysis for each of the cases considered. The error analysis is discussed in Part II of this paper.

This report represents an extension to work previously reported by Minzner, et al., [11].

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THE RELATIONSHIP OF TEMPERATURE TO THE NUMBER DENSITY
PROFILE OF A SINGLE GAS

Basic Equation

We consider an altitude region in which diffusive separation dominates the distribution of the separate atmospheric gases and in which dissociation, ionization, and chemical reactions involving the measured gases can be neglected. One may then use the hydrostatic equation and kinetic-theory considerations to obtain the following well-known equation [5-7], giving temperature $T(h)$ in terms of the number density $n(h)$ for a particular atmospheric constituent of molecular weight M;

$$T_q = \frac{n_o}{n_q} T_o - \frac{GM}{R} \frac{1}{n_q} \int_{h_o}^{h_q} n(h) \, dh \qquad (1)$$

where R is the universal gas constant, and the subscript o specifies the value of n or T for a particular reference altitude h_o consistent with the limits of the integral. The expression is written in terms of geopotential altitude "h" [1,12,13], which is related to geometric altitude by means of the defining transformation $g(z) \, dz = Gdh$, where $g(z)$ is the altitude-dependent acceleration of gravity at geometric altitude "z", and where G is a constant scale factor. The use of geopotential altitude eliminates the need to account further for changes in the acceleration of gravity with altitude, in the integration process. Reconversion of h to z permits computed results to be presented in terns of geometric altitude.

It should be noted that T_q the temperature at the running altitude h_q is a frunction of four variables; (1) the reference temperature T_o at the reference altitude h_o, (2) the number density n_o at this same reference altitude, (3) the number density n_q at altitude h_q, and (4) the integral of the number density $n(h)$ with respect to h over the entire altitude interval between h_o and h_q. Consequently, the temperature at any altitude h_q can be determined from number density data only when the number-density profile is known with reasonable accuracy over the entire interval h_q to h_o. Since, on the other hand, the independent variable h_q appears explicity only as the upper limit of the integral, the temperature T_q is defined uniquely for altitude h_q: the number density n_q at this altitude is thus involved in the determination of T_q both directly and as a parameter of the integral function.

It has long been recognized that while Equation (1) is very useful for computing temperature in some situations, it also has practical limitations which make it almost useless in other situations. A principal part of this portion of the study concerns itself therefore with the determination of the conditions of utility of Equation (1) by an examination of the behavior of the

terms of Equation (1) for various combinations of two values of each of three major variables or boundary conditions of analysis. These variables consist of (1) the type (or molecular weight) of the gas whose number densities are used, (2) the particular profile of number density versus altitude, which depends not only upon the molecular weight of the gas but also upon the related temperature altitude profile, and (3) the direction of calculation; that is, from a reference level h_0 at the greatest altitude of available data downward, when $h < h_0$, or from a reference level at the lowest altitude of available data upward, when $h > h_0$.

Argon and Helium Number Density Profiles

The two gases selected for the analysis are helium and argon, the lightest and heaviest essentially inert gases having a significant concentration in the earth's atmosphere. The number-density profiles of helium or argon which should be used in an actual temperature determination would be sets of observed values of the individual number-density profiles $n_i(h)$. These values would implicitly carry the information concerning the related temperature-altitude profile, the determination of which is the object of the measurement. While Reber and Nicolet [10] have observed number densities of helium, molecular nitrogen, andother atmospheric constituents during various portions of satellite orbits, sets of simultaneously observed values of $n(h)$ for helium and argon, extending over a considerable altitude range at any single geographic location and observed at any single value of local time, are not known to exist at present. Consequently, two theoretically determined sets of both helium and argon number densities have been used in this study. One set is based upon the temperature-altitude profile of the United States Standard Atmosphere [14], which model is representative of daytime conditions during the period of relatively high solar activity. The second set is based upon the temperature-altitude profile of the Jacchia 750 degree Model [15], which is representative of a period of considerably lower solar activity. Both sets of number densities have been computed using the assumption of diffusive equilibrium at altitudes above 120 geopotential kilometers (km') and the following arbitrarily selected values of helium and argon number densities at 120 km', that is, 2.7384×10^{12} m^{-3}, and 4.86087×10^{15} m^{-3}, respectively. The helium number-density value is about one tenth of that assumed by Jacchia, and from the point of view of this study is conservative, in that, if the actual values are in fact ten times greater at 120 km' than this paper assumes, the method discussed would have greater accuracy than indicated.

The resulting two pairs of number-density profiles are presented in Figure 1. The pair depicted by solid lines is based on the temperature of the United States Standard Atmosphere which has an exospheric temperature of about 1500°K; the pair depicted by dashed lines is based on the temperatures of that one of Jacchia's static diffusion models having an exospheric temperature of 750°K. These four sets of values of $n(h)$ comprise the basic data for the analysis of Equation (1). Some preliminary examinations of the graphs of these profiles is of interest.

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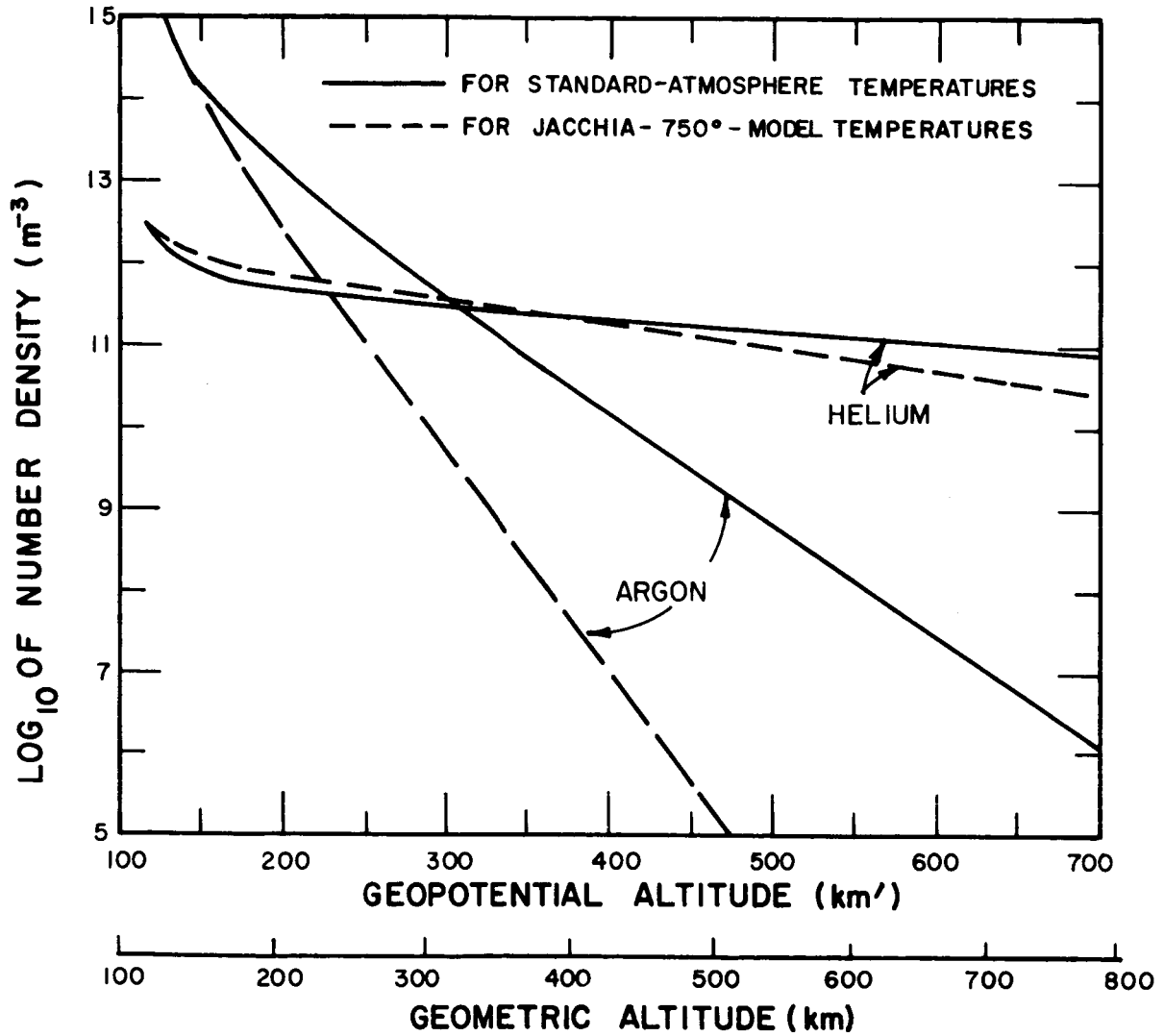
Figure 1. Calculated values of helium and argon number densities as a
function of altitude for conditions of two atmospheric models,
the United States Standard Atmosphere and for the Jacchia 750-
Degree Model Atmosphere, when diffusive separation is assumed
to begin abruptly at 120 km altitude in each model.

It is significant to note that for each profile the slopes $d \log_{10} n_i/dh$ remain constant above the thermopause (above 400 km for the solid-line profiles, and above 250 km for the dashed-line profiles). This situation is consistent with a constant exospheric temperature in each model as predicted by the following equation in which dT/dh, for the case in question, is zero

$$2.303 \frac{d \log_{10} n_i}{dh} = \frac{d \ln n_i}{dh} = -\frac{1}{T}\frac{GM_i}{R} + \frac{dT}{dh} \tag{2}$$

This equation also verifies the fact that for any single temperature-altitude profile, the negative slopes of the graphs, $\log_{10} n_i$ versus h, for different species are seen to be proportional to the molecular weight of the species. Thus, the negative slope of an argon profile is about ten times that of the corresponding helium profile.

Finally, both graph and equation indicate that for any single species of gas, (M_i = constant), the slope $d \log_{10} n_i/dh$, in the altitude region of the exosphere where dT/dh = 0, is proportional to the negative reciprocal of the exospheric temperature. Thus, the absolute value of the slope of the dashed-line argon profile for which T = 750°K is twice that of the solid-line argon profile for which T = 1500°K. An examination of the two profiles of helium number density shows a similar relationship. The slope of the solid-line profile is so small, however, that twice its value is still very small. It is apparent therefore that in any profile of measured values of helium number density, observational errors could produce variations in the point-to-point slopes comparable to wide variations in temperature. Consequently, any attempt to use the slope of the logarithm of helium number-density data directly, as a means of determining the temperature, will lead to extremely large errors.

While the slope $d \log_{10} n_i/dh$ of any of the number-density functions is not directly of concern in the further study of Equation (1), these slopes are closely related to number-density ratios which do figure prominently in Equation (1). In an isothermal atmosphere, the number-density ratio may be shown to be directly proportional to $\exp M_i/T$. Obviously, the variability of the terms of Equations (1) is strongly dependent upon both the molecular weight of the gas and upon the temperature profile.

Direction of Calculation

The direction of calculation is an extremely important consideration influencing the magnitude of the several variables and thus the utility of Equation (1). This equation consists of either the sum (for $h < h_o$) or the difference (for $h > h_o$) of two positive terms referred to in the following discussion as the "ratio term" and the "integral term." The magnitude of each of these two terms relative to the value of T_q is strongly influenced by the direction of calculation. Thus, in the examination of the behavior of Equation (1) we consider two cases, one for each of the two directions of calculations, where each of the two terms of Equation (1) are appropriately evaluated for each of the four number-density profiles.

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Case 1: $h_q < h_o$, Single-Gas Downward Calculation Process

In this situation the reference level h_o represents the greatest altitude of observed number-density data for a particular species of gas. From an application of this condition to the limits of integration in Equation (1) the temperature T_q is seen to be the sum of two positive quantities, the ratio term and the integral term. The value of the integral term is zero for $h_q = h_o$ and increases rapidly with increasing values of $(h_o - h_q)$, approaching the values of T_q when $(h_o - h_q)$ becomes sufficiently large. The value of the ratio for $h_q = h_o$ is equal to the reference-level temperature T_o, the usually unknown temperature at h_o, and decreases rapidly, approaching zero asymptotically as $(h_o - h_q)$ increases to sufficiently large values.

This case-1 type of evaluation of Equation (1) is hereinafter referred to as the single-gas downward calculation process. Figure 2 shows the variation of the values of the integral term, and of two possible ratio terms, as a function of h_q for $h_o = 450$ km, when these quantities are computed from that set of previously discussed argon number densities which are related to the temperatures of the United States Standard-Atmosphere. Sets of values of each of these terms are compared with the set of Standard-Atmosphere temperatures corresponding to the same altitude interval.

In the case of an observed-data situation, the integral term would be evaluated to a close approximation of its true unique value by some numerical integration process involving only the density-altitude data and the values of the constants G, R, and M_i, the latter for the particular gas involved. Such a process could also have been employed in the evaluation of the synthetic data used in this study. Since the synthetic data stem from either of two internally-consistent calculated models, however, it is clear from Equation (1) that the integral term may be evaluated exactly, and much more easily, from the difference $n_q T_q - n_o T_o$, using the appropriate values from the model under consideration. This is the method which was employed in evaluating the integral term for Case 1 as shown in Figure 2, and is the method employed in other integral-term evaluations in Cases 2, 3, and 4 discussed below.

Figure 2 shows two sets of values for the ratio term in accordance with two arbitrarily assumed values of T_o, 2500°K and 600°K, since, in the analysis of real observed data, the value of T_o would probably not be known independently. These assumed values approximate the probable maximum and minimum of thermospheric temperatures [15]. Thus, the vertical separation between the two curves represents approximately the uncertainty in the value of the ratio term, and consequently the uncertainty introduced into the calculated values of T_q by the uncertainty in the reference-level temperature. This source of uncertainty in T_q vanishes as the value of the ratio term approaches zero, and the value of the integral term simultaneously approaches the value of the actual temperature T_q. It is apparent that no reliable temperatures are determined in the upper 25 percent of the altitude interval (about 100 km) of the observed argon number-density data, unless smaller bounds can independently be placed upon the value of T_o. For the lower 50 percent of the altitude interval, the

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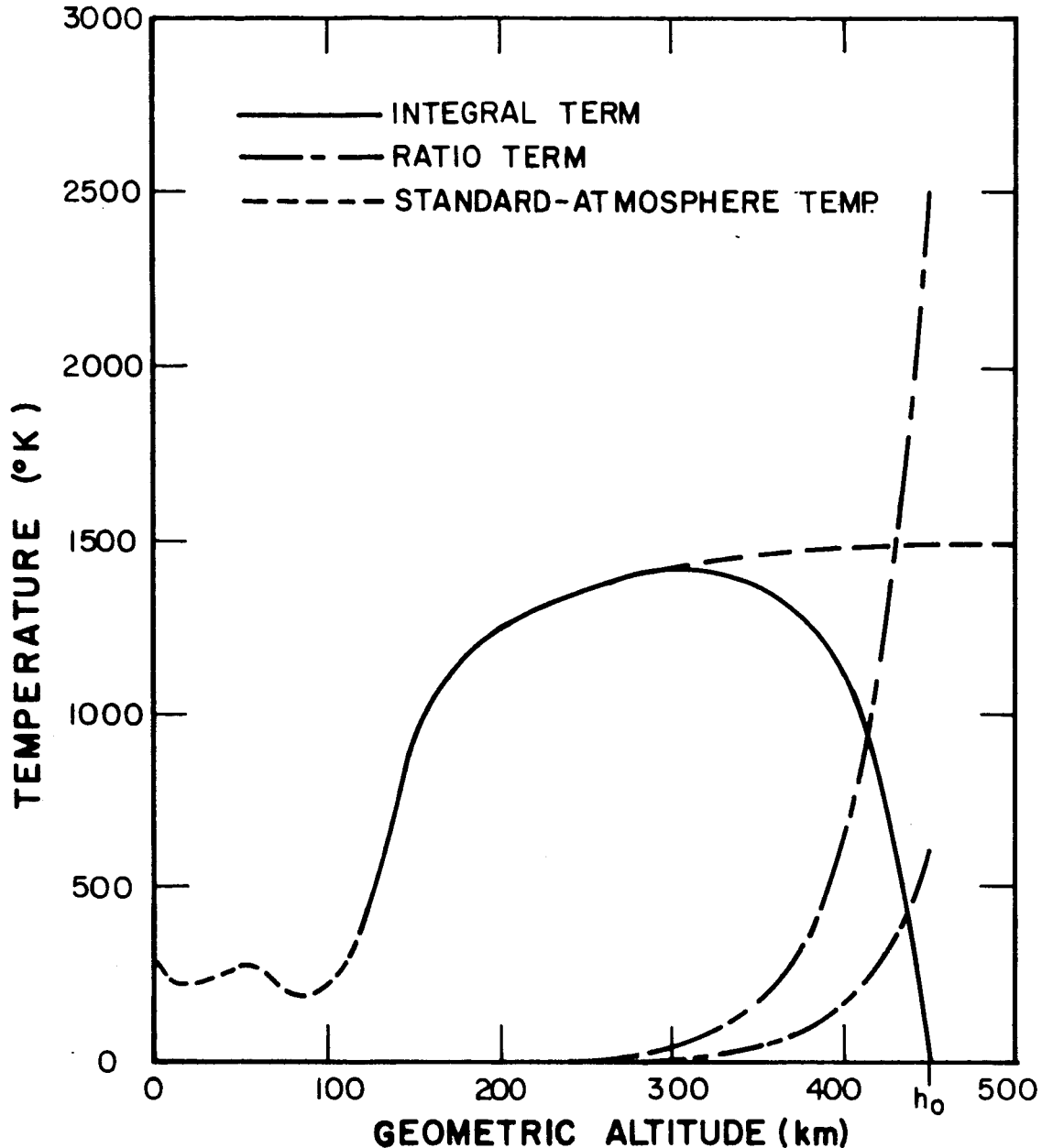
Figure 2. Two of an infinite number of possible sets of altitude-dependent
values of the ratio term of Equation (1) (determined on the basis
of the particular reference-level values of temperature of 2500°K
and 600°K respectively) and the unique set of altitude-dependent
values of the integral term of that equation, all compared with
the temperature-altitude profile of the U. S. Standard Atmosphere,
when the evaluation of the terms of Equation (1) is based upon
those argon number densities related to the Standard Atmosphere
and upon a downward calculation from a reference level of 450 km.

values of T_q are essentially uninfluenced by T_o, and in this region the accuracy of the inferred values of T_q would depend in some fashion upon the accuracy of only the number-density data.

For observed density data limited to altitudes below 120 km, the ratio term becomes negligibly small in a much smaller altitude interval $(h_o - h_q)$ of the order of 15 to 20 km, than that suggested in Figure 2. This is due to:

(1) The uncertainty in T_o below 120 km is very much smaller, (that is, of the order of $200^\circ K$) than at altitudes above 200 km.

(2) The ratio n_o/n_q approaches zero at a much greater rate for increasing values of $(n_o - h_q)$ below 120 km then at altitudes above 200 km because of the large increase in the absolute value of $d \ln n/dh$ as h decreases from the higher to the lower altitude as seen in Figure (1).

A situation identical to th shown in Figure 2 does not prevail at high altitudes for all gases and all temperature profiles. A comparison of the altitude intervals $(h_o - h_q)$ required for a given degree of relative convergence between each of four sets of integral-term values and the related set of values of T_q, one for each of the four previously discussed number-density profiles, is of interest. Such a comparison can best be made in terms of normalized values where the direct influence of T_q is eliminated by deviding each member of a set of ratio-term values by the appropriate member of the set of related values of T_q. The four sets of ratio-term data normalized as indicated above are plotted in Figure 3. Both normalized sets related to argon employ $h_o = 450$, the same as in Figure 2. For those normalized sets of data related to helium, however, the reference level is changed to $h_o = 700$ km because the helium number densities are still quite large at this altitude.

At 450-km altitude, the argon number densities related to the United States Standard-Atmosphere approach a value of 9.5×10^{-9} m^{-3}, which is close to the minimum detectable level for present-day mass spectrometers [8]. For this reason, the value of h_o was taken to be 450 km for those evaluations involving argon number density depicted in Figures 2 and 3. In the case of helium data, the value of h_o was arbitrarily taken to be 700 km on the basis of rocket performance considerations, but could have been set at a much higher altitude on the basis of the criterion of minimum detectable level.

It is apparent from Figure 3 that the altitude interval $(h_o - h_q)$ required for any specific degree of convergence of the integral-term value to T_q is greatly dependent upon the species of the gas, (that is, upon the molecular weight of the gas), as well as upon the average temperature and hence the number-density distribution of the related atmosphere for the altitude interval under consideration.

The integral term approaches (T_q) most rapidly for gases with the largest molecular weight, and for atmospheres with the smallest mean value of temperature. Thus, if T_q of the earths atmosphere is to be determined by the downward-calculation process using number-density data for a single gas, it appears from convergence conditions, at least, that the gas should be argon, the heaviest
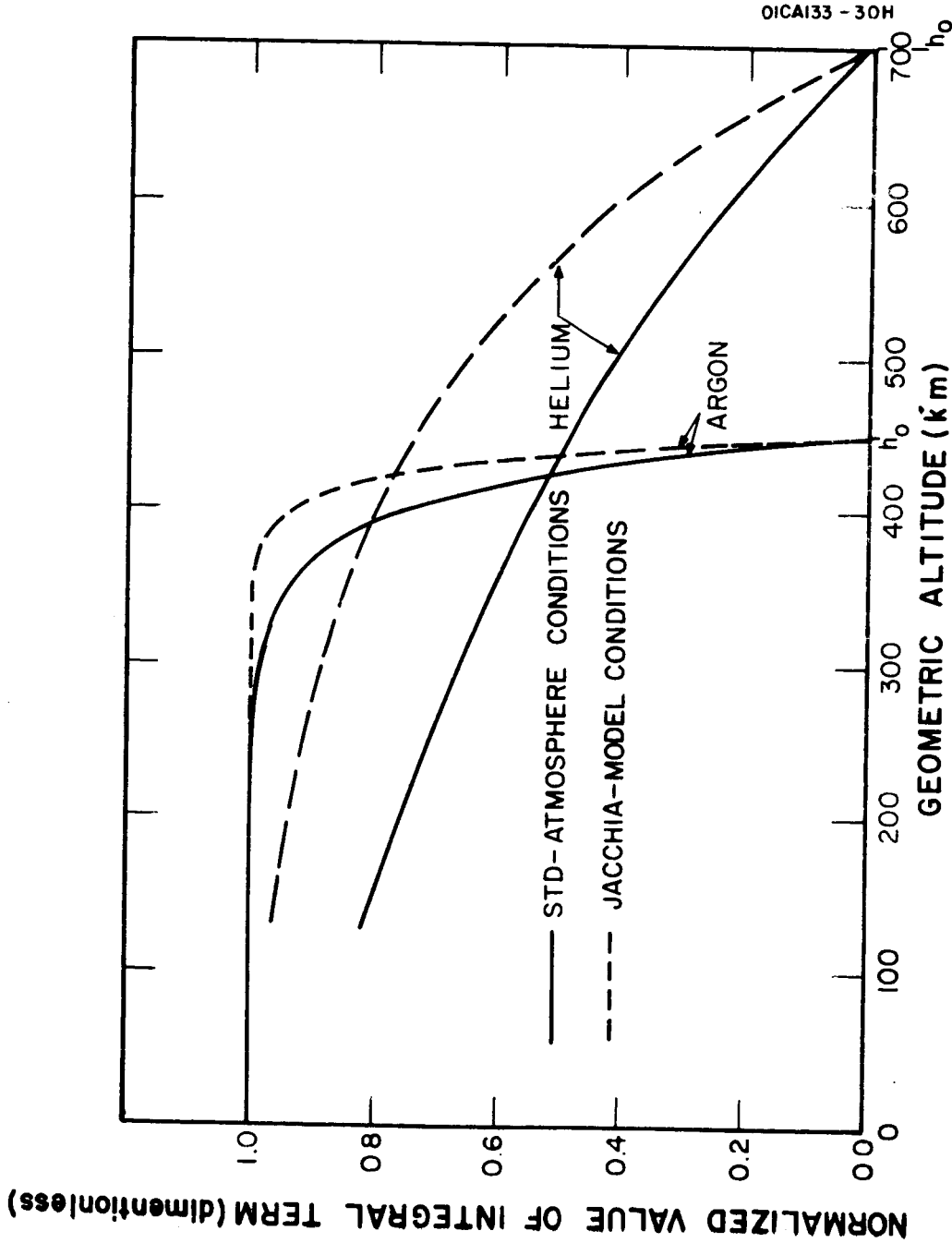
Figure 3. Four sets of normalized values of the integral term of Equation (1) as a function of altitude, each set applying respectively to one of four of the eight possible combinations of conditions resulting from simultaneously applying one member from each of three following two-member sets of conditions: (1) atmospheric gases (argon and helium), (2) atmospheric models (U.S. Standard Atmosphere and Jacchia 750-degree model), and (3) reference altitude h_o (450 and 700 km), as specified for a downward-type calculation.

inert gas having a reasonable concentration above the diffusive-separation level. Helium number-density data alone are essentially worthless for this type of calculations.

One has no control over the mean value of the temperature-altitude profile to be measured, but a knowledge of the influence of the mean temperature on the integral term is an aid in estimating the maximum altitude and sensitivity of number-density observations required for a specific maximum altitude of reliable temperature data. At times of minimum solar activity or at night, the altitude range of integration required for a specified relative development of the integral term would be minimized.

The difference between the integral term and T_q may be referred to as the temperature insufficiency of the integral term, that is, the amount by which the value of the integral term fails to equal the actual temperature. Since the values of T_q related to the number-density profiles being employed in this study are known, it has been possible to determine precisely the temperature insufficiency of the integral as well as the value of the ratio of the temperature insufficiency to T_q as a function of h_q for each of the previously discussed number-density profiles. Figure 4 depicts these ratios as a function of h_q for the same conditions used in the determination of the curves of Figure 3. From Figure 4 one can determine the altitude interval of integration required for the integral term to approach T_q within any specified percentage, for each of the four number-density profiles presented. Estimates of the relative temperature insufficiency for other gas species and temperature conditions can also be readily made. Again it is apparent that the heavier the gas and/or the lower the mean temperature, the more powerful is the downward-calculation process for evaluating Equation (1).

Case 2: $h_q > h_o$, Single-Gas Upward Calculation Process

For this situation, h_o ideally is taken to be the lowest altitude for which number-density data (for some specific gas species) are available, consistent with the condition of the existence of diffusive separation of the measured gas at that altitude. From an application of this condition to the limits of integration of Equation 1, the temperature T_q is seen to be the difference between two positive quantities, the ratio term and the integral term. In this case, neither term becomes insignificantly small or approaches any finite limit for sufficiently large values of altitude interval $(h_q - h_o)$. On the contrary, both terms grow indefinitely with $(h_q - h_o)$ to values many times as great as T_q. The integral term grows from zero at $h_q = h_o$ while the ratio term grows from T_o at the same altitude, both terms growing in such a manner that theoretically, at least, the difference is always exactly equal to T_q. This type of use of Equation (1) is hereinafter referred to as the single-gas upward-calculations process.

As in Case 1 (for the single-gas downward calculation) the integral term in Case 2 is also uniquely dependent upon only the values of number density

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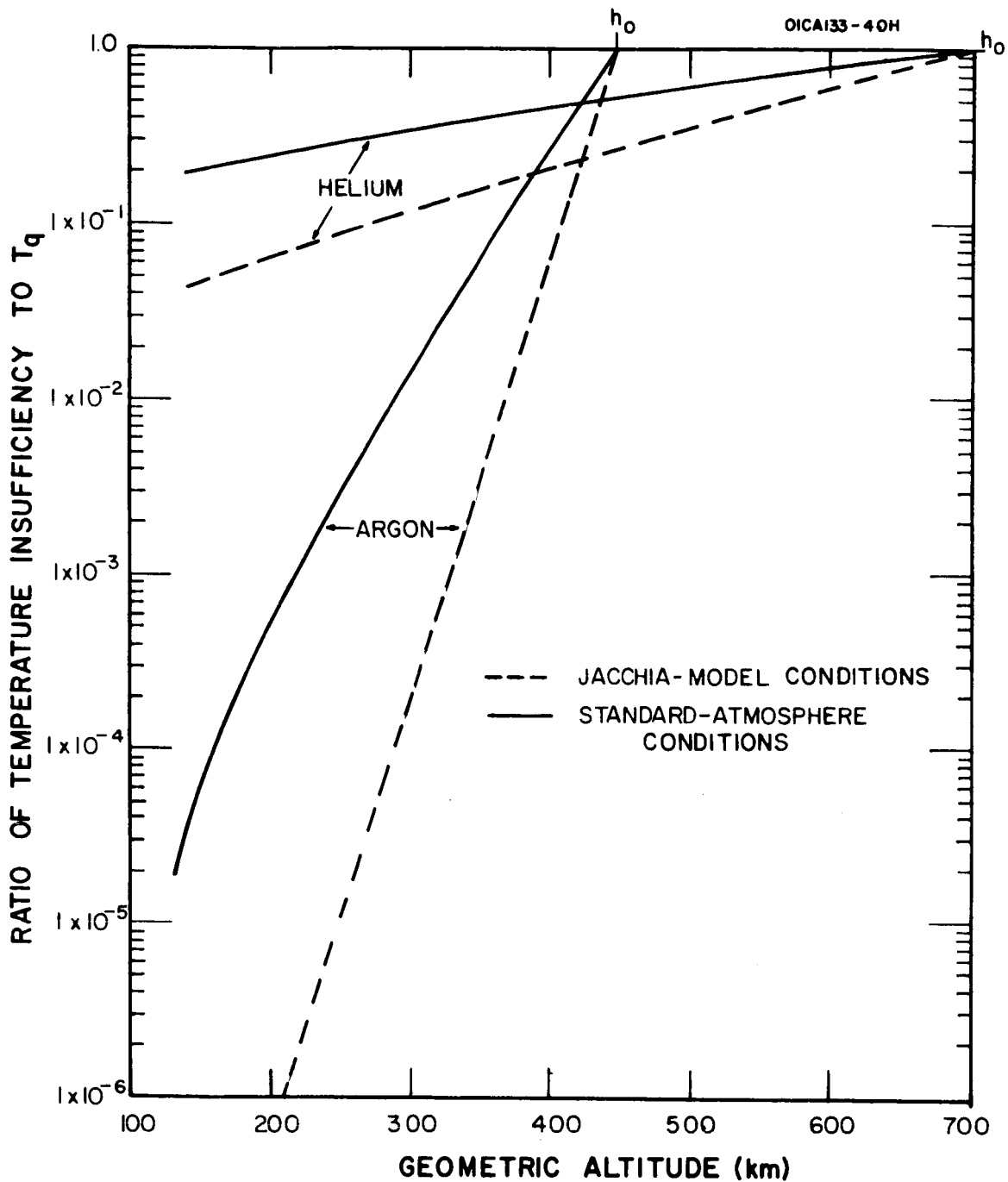
Figure 4. Four altitude-dependent set of values of the ratio of the temperature insufficiency of the integral term of Equation (1) to the actual temperature, one set for each of the four combinations of conditions employed in the graphs of Figure 3, for a downward type calculation from the indicated reference levels.

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versus altitude and upon the altitude interval $(h_q - h_o)$, while the ratio term is dependent not only upon these variables, but also upon the value of T_o which is generally not well known.

Figure 5 shows the altitude variation of the standard-atmosphere temperature as well as the altitude variation of the value of both the integral term and the ratio term, for the arbitrarily chosen condition $h_o = 150$ km, and for two sets of related number-density data, that is, for argon and helium number densities consistent with the standard-atmosphere temperatures. Because of the wide range and nearly exponential form of these values over the altitude range of interest, the ordinate is presented in the form of $\log_{10}$ of temperature. The values of the integral term for $h_q = 450$ km are seen to be 3875.7^oK for the helium and 2091.8×10^4 ^oK and for argon data. These values correspond to 2.60376 T_q or 4.3411 T_o for the helium data and $14{,}053$ T_q or $23{,}430$ T_o for the argon data. Because of the very large numerical values of both the ratio and integral terms relative to T_q in the case of argon data, very small uncertainties in any of the factors governing either of these two terms would make the results of an argon calculation useless. Calculations involving helium data, on the other hand, would not suffer this limitation even to altitudes up to 700 km or above.

In an observed real-data situation, the uncertainty in estimating T_o may be as great as 25 percent; or in the event that T_o is independently measured, the uncertainty at best may be as low as 2 percent. To expect an uncertainty as small as 0.007 percent would be completely unrealistic. The absolute and percentage uncertainty introduced into the ratio term at $h_q = 450$ km for the three above-mentioned percentage errors in T_o are given in Table 1.

TABLE 1

UNCERTAINTY INTRODUCED INTO RATIO-TERM VALUES AT $h_q = 450$ km
FOR A CASE-2 CALCULATION INVOLVING BOTH HELIUM AND ARGON
NUMBER DENSITIES AS SHOWN IN FIGURE 5, FOR VARIOUS PERCENTAGE
ERRORS IN T_o.

	Helium Evaluation		Argon Evaluation	
Error in T_o (%)	Absolute Uncertainty ^oK	Uncertainty Relative to T_q	Absolute Uncertainity ^oK	Uncertainty Relative to T_q (h)
25	9.689×10^2	6.509×10^{-1}	5.229×10^6	3.513×10^3
2.0	7.751×10^1	5.207×10^{-2}	4.183×10^5	2.811×10^2
0.007	2.713×10^{-1}	1.822×10^{-4}	1.488×10^3	1.000

The very large errors introduced into the argon term by even small uncertainties in T_o suggest a reversal of the operation; that is, adjust T_o in
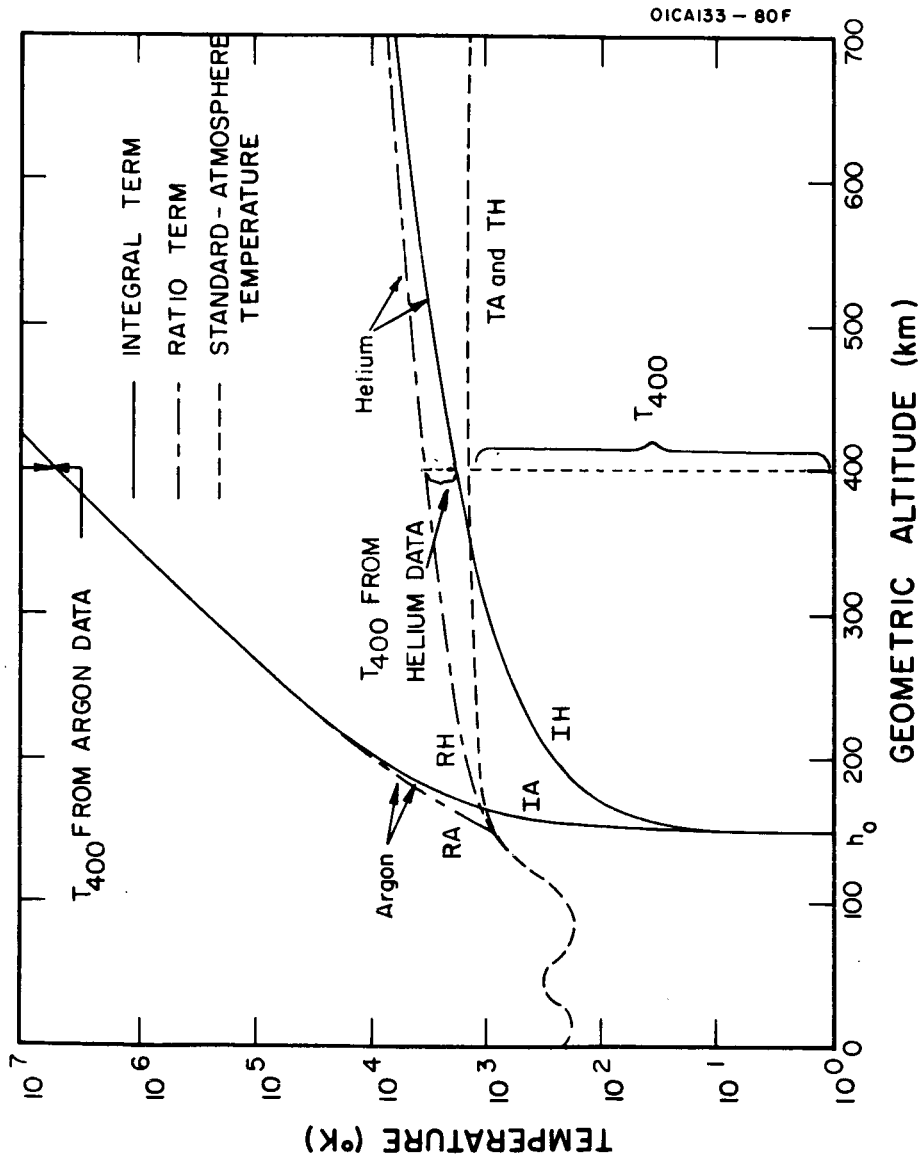
Figure 5. Three altitude-dependent sets of values (IA) the integral term of Equation (1), (RA) the ratio term of that equation and (TA) the difference between these two terms (which is the temperature), each set as evaluated from Standard-Atmosphere related argon data, compared with three similar sets of values IH, RH and TH based upon the Standard-Atmosphere related helium data, when, in both cases, the calculation is upward from a 150-km reference altitude. (The difference sets TA and TH each consisting of the temperature-altitude profile of the U. S. Standard Atmosphere are identical for the two cases).

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small increments until T_q at 450 km is found to be a reasonable value. Then, for the lower 75 percent of the altitude region 150 to 450 km, the value of T_q will have been determined with a reasonable accuracy. This latter process is, in effect, a form of Case-1 calculation previously discussed.

The influence of both the mean temperature and the molecular weight of the gas species on the relative utility of Case-2 calculations may be seen in Figure 6. This figure presents the altitude variation of the values of the normalized ratio terms computed on the basis of the four number-density profiles of Figure 1, where the normalization is accomplished by the dividsion of the ratio terms by $T_q(h)$. Using the criterian established above, that the smaller the ratio term relative to T_q the more useful the calculation method, one can state that for Case-2 upward calculations, light gases and high average temperatures yield more accurate results than heavy gases and low average temperatures. Unfortunately, T_0 must still be known accurately for a reasonably accurate temperature profile to be obtained. Since such is usually not the case the single-gas upwind calculation method even with a light gas is usually of little value by itself. When used iteratively in conjunction with a heavy-gas downwind calculation, however, the upward calculation, as discussed below, can be part of a useful approach.
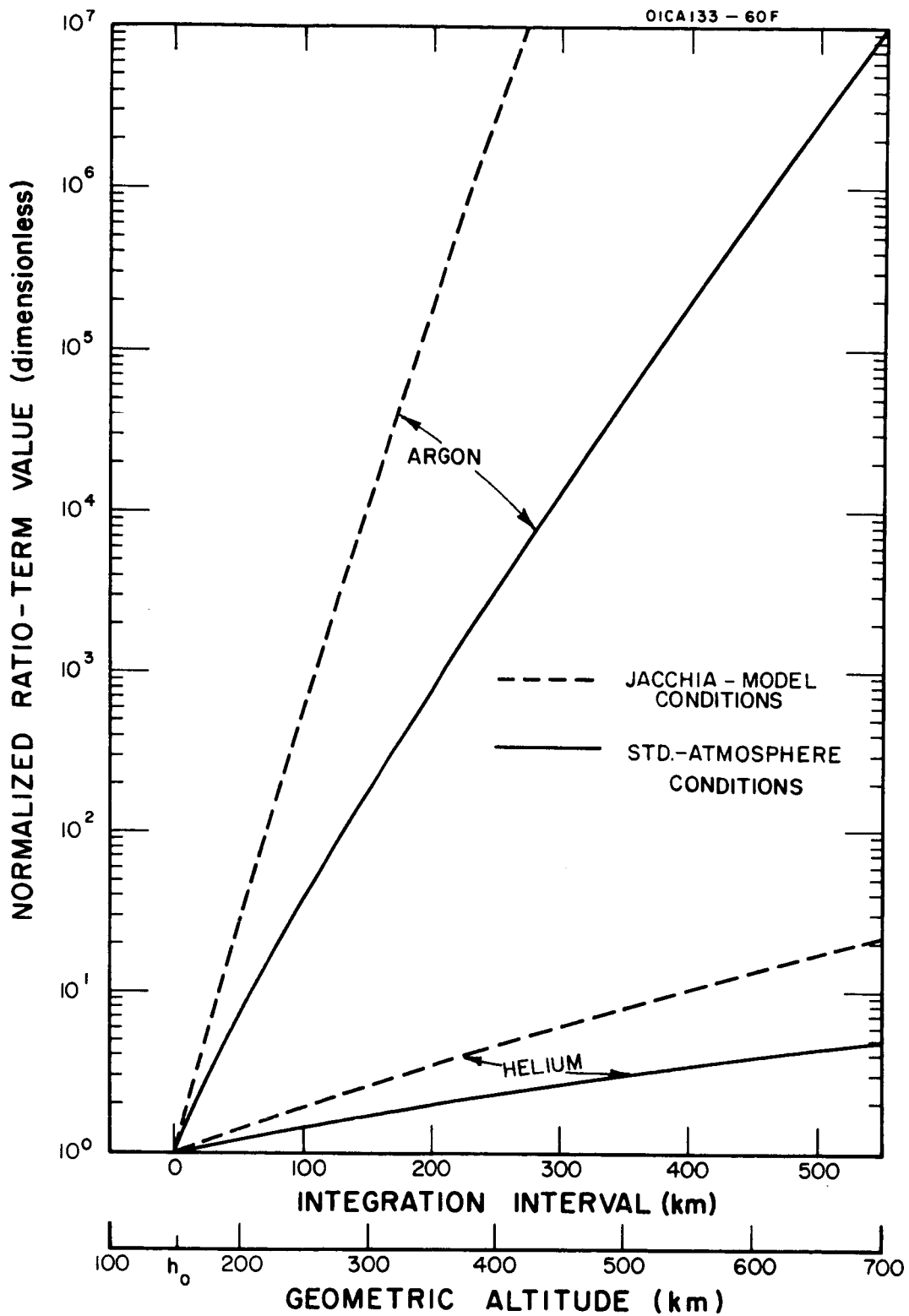
Figure 6. Values of the normalized ratio-term versus integration interval or running altitude for the type of evaluation depicted in Figure 5.

THE RELATIONSHIP OF TEMPERATURE TO THE SIMULTANEOUSLY OBSERVED
NUMBER-DENSITY PROFILES OF TWO GASES

General Considerations

Accurate temperature-versus-altitude data can be obtained without any dependence upon the assumed value of T_O from the simultaneously observed number-density profiles of two gases in diffusive equilibrium, each with different molecular weight M. Equation (1) can be written in two forms, each form reflecting the use of number densities of one of the two gases. The resulting system of two simultaneous equations contains only two unknowns, T_q and the reference temperature T_O. Consequently, when these two equations are combined, the reference temperature T_O is eliminated, leaving an expression for T_q as a function of the ratios and integrals of two sets of number-density data only. For clarity and simplicity of notation, all parameters relating to the heavy gas in the system of simultaneous equations are marked with an asterisk; that is, n^* signifies heavy-gas number density, while n signifies light-gas number density. Hence, Equation (1) as written applies henceforth to the light gas, while for the heavy gas we write

$$T_q = \frac{n_o^*}{n_q^*} T_o - \frac{G\ M^*}{R\ n_q^*} \int_{h_o}^{h_q} n^*(h)\ dh \tag{3}$$

If now we designate the light-gas integral term of Equation (1) by I_q and the heavy-gas integral term of Equation (3) by I_q^* we have the following pair of simultaneous equations:

$$T_q = \frac{n_o}{n_q} \cdot T_o - I_q \tag{4}$$

$$T_q = \frac{n_o^*}{n_q^*} \cdot T_o - I_q^* \tag{5}$$

The elimination of T_O from this pair of equations leads to a two-gas expression for T_q which may for convenience take either of two principal forms depending upon whether h_q is less than or greater than h_O, for which situations the analysis is referred to as Case 3 or Case 4, respectively.

Case 3: Double-Gas, Downward Calculation ($h_q \leq h_o$)

For case 3 when $h_q \leq h_o$ it is convenient to express the solution of Equations (4) and (5) as:

$$T_q = \frac{-I_q^*}{1 - \dfrac{n_o^* \, n_q}{n_q^* \, n_o}} - \frac{-I_q}{\dfrac{n_q^* \, n_o}{n_o^* \, n_q} - 1} \tag{6}$$

where $[-I_q^*]$ and $[-I_q]$ are both positive quantities as are the denominators of both fractions. Thus T_q is seen to be the difference between two positive quantities, and since T_q can never be negative, the first term of Equation (6) must always be greater than the second term of that equation.

For $h_q = h_o$, the ratios n_o^*/n_q^* and n_o/n_q are both equal to unity; therefore, the denominators of both fractions of Equation 6 are equal to zero. Both integral terms are also equal to zero; hence, both terms of Equation (6) are indeterminate. It may be shown, however, that as $h_q \to h_o$ each term approaches a limit such that term for term, the limit of Equation (6) may be expressed as

$$\lim_{h \to h_q} T_q = T_o \left(\frac{M^*}{M^* - M} \right) - T_o \left(\frac{M}{M^* - M} \right) \tag{7}$$

It is apparent that the sum of the two terms on the right-hand side of Equation (7) reduces to T_o.

Graphs of the altitude variations of both terms of Equation (6) evaluated for the standard-atmosphere values as well as for the Jacchia-Model values of number-density data as given in Figure 1 are presented in Figure 7, along with the graphs of both sets of computed values of T_q. These two sets of computed values of T_q are identically the temperatures of the standard atmosphere and of the Jacchia model, and these curves, along with the curves representing the values of two terms of Equation (6) for each model, are appropriately labeled.

The calculations whose results are presented in Figure 7 are limited to altitudes below 450 km by virtue of the argon number-density profile and the same mass-spectrometer-sensitivity condition applied to the calculations shown in Figures 2 and 3 (for the heavy-single-gas downward calculation). If one were to observe useful argon number densities at greater altitudes, the upper-altitude limit of calculation could be increased accordingly.

It is apparent that the first term of Equation (6) is the major term and that this term varies from $[M^*/(M^* - M)]T_o$ to T_q as h_q varies from h_o to a value very much less than h_o. It is also apparent that the second term of Equation (6) has a very small value which varies from $[M/M^* - M)]\,T_o$ to 0 as h_q varies over the specified range. For the case when the two gases are argon and helium, the values of the two terms at $h_q = h_o$ are about $1.111\,T_o$ and $0.111\,T_o$, respectively. The difference between two terms of such magnitudes presents no accuracy difficulty.

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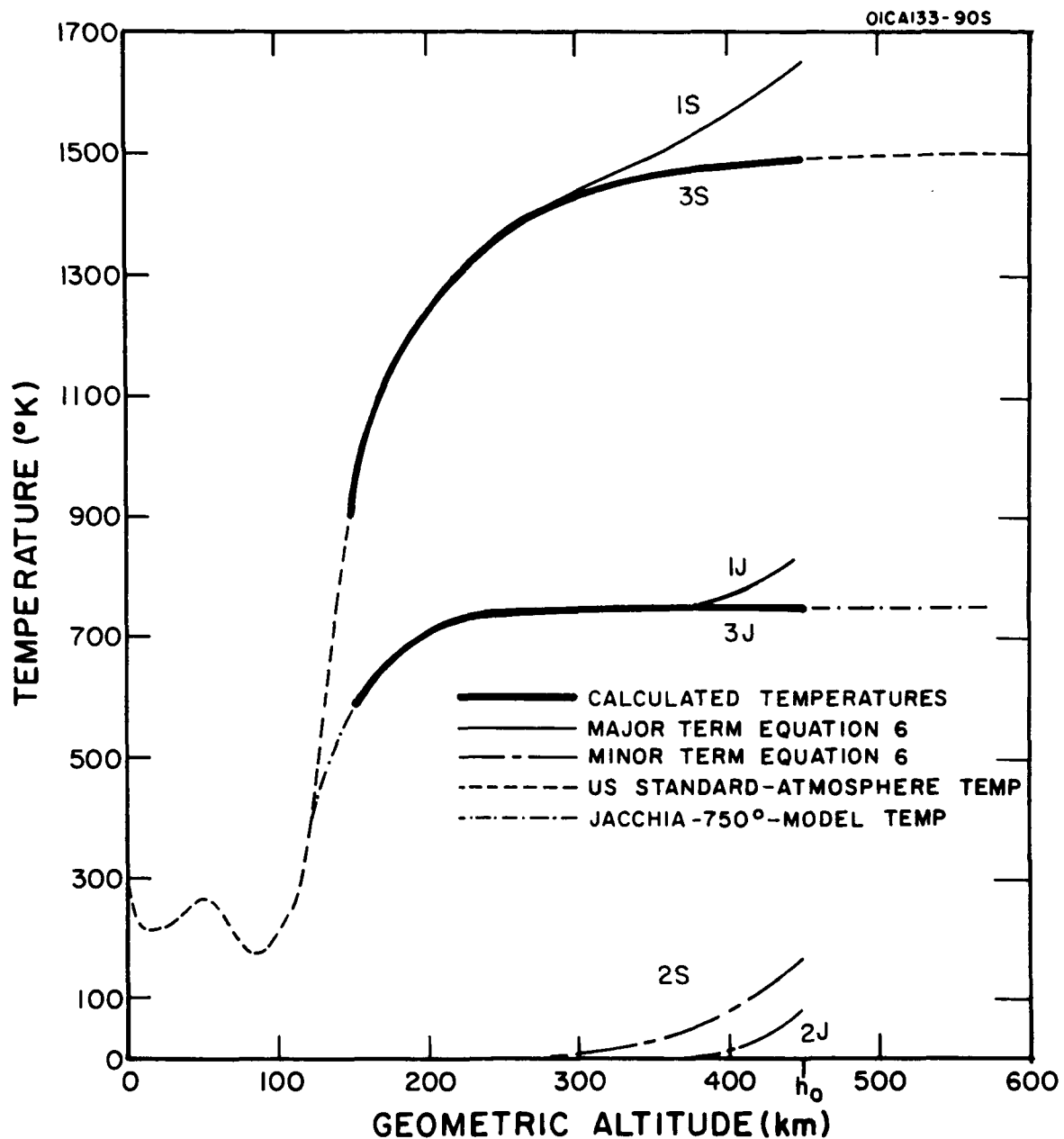
Figure 7. Three altitude-dependent sets of values; 1S, the major term of Equation (6), 2S the minor term of Equation (6), and 3S the algebraic sum of these terms (which is identically the temperature of the related model) all based upon a reference level of 450 km and upon Standard-Atmosphere related values of helium and argon number-density data compared with three similar sets of values 1J, 2J and 3J based upon helium and argon data associated with the Jacchia 750-degree model atmosphere.

The altitude range required for the minor term to approach zero within a specified value decreases as the ratio M/M^* decreases, and also decreases as the exospheric temperature of the atmosphere decreases. The convergence of the major term to T_q of necessity is also seen to be more rapid for those number-density data associated with the lower temperature model.

In both of these aspects (convergence of the major term to T_q and the more rapid convergence for lower temperatures) the double-gas downward calculation depicted in Figure 7 is similar to the single-heavy-gas downward calculation as depicted in Figures 2 and 3, except that in the double-gas case, the convergence is from greater values down to T_q while for the single-gas case, the convergence is from lower values up to T_q.

The two cases appear to differ significantly in regard to the altitude region for which temperature is determined: For the single-gas case, the values of T_q are determined only for an altitude region considerably below the greatest altitude of heavy-gas number-density data, while, for this double-gas case, the additional light-gas data in principle permits the unique determination of T_q over the entire altitude range for which two-gas density data are available. The accuracy of values of T_q determined by the double-gas down calculations relative the accuracy of values of T_q determined by the single-gas calculations for either of the two models considered is not directly apparent from the above calculations but does become evident from the error analysis in part 2 of this paper.

Significantly, two-gas number-density data need not be available for the entire altitude region of 450 to 150 km in order that accurate temperatures may be obtained for this region. If the light-gas data should be available only from h_o down to 250 km, for example, while the heavy-gas data exists from h_o down to 150 km, the calculation of T_q could be extended without interruption through this lower 100-km region by means of the single-heavy-gas down technique using Equation (5) provided only that the altitude range of the light-gas data in the higher altitude region is of sufficient extent (100 to 200 km) to establish T_q with a small uncertainty. It will be seen from the error analysis in Part II that the greatest accuracy is achieved when both gases are measured to the lowest possible altitude. The reason the heavy-gas down calculations may be continuous at altitudes immediately below the double-gas down calculations is that for h_q sufficiently less than h_o Equation (6) may be shown to approximate Equation (5), as follows:

For the situation $h_q \ll h_o$ we have

$$\frac{n_o^*}{n_q^*} \ll 1 \qquad (8)$$

and

$$\frac{n_o}{n_q} < 1 \qquad (9)$$

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such that

$$\frac{n_o^*}{n_q^*} \cdot \frac{n_q}{n_o} \ll 1 \tag{10}$$

and

$$\frac{n_q^*}{n_o^*} \cdot \frac{n_o}{n_q} \gg 1 \tag{11}$$

From inequalities (9) and (10) it follows that, for $h_q \ll h_o$, Equation (6), the double-gas equation for downward calculations, reduces to

$$T_q \simeq - I_q^*$$

This expression is equivalent to the results of Case-1 downward calculation with a heavy gas for large values of $(h_o - h_q)$ as represented by Equation (5), when T_o is kept from becomming unreasonably large.

In short, the double-gas down calculation of T_q is similar to the single-heavy-gas down calculation except that in the latter situation some plausable value of T_o must be used to cause the ratio term to approach zero, so that the integral term becomes essentially T_q, while in the double-gas case no assumptions regarding T_o are necessary.

Case 4: Double-Gas Upward Calculation $(h_q \gtrsim h_o)$

For upward calculation, that is, when $h_q \geq h_o$, it is convenient to express the solution of Equations (4) and (5) as

$$T_q = \frac{I_q^*}{\dfrac{n_o^*}{n_q^*} \cdot \dfrac{n_q}{n_o} - 1} - \frac{I_q}{1 - \dfrac{n_q^*}{n_o^*} \cdot \dfrac{n_o}{n_q}} \tag{12}$$

where, for this order of integration limits, I_q^* and I_q are now both positive quantities as are the denominators of both of these fractions. Thus, as in Equation (6), T_q is expressed as the difference between two positive quantities. In this case, however, neither term approaches zero for increasing values of the altitude interval of integration. Rather, both terms increase continually from specific initial values in accordance with some functions of the integration interval $(h_q - h_o)$, and one must be concerned with the situation relative to the accuracy of the quantity determined from the difference of two large numbers.

For $h_q = h_o$, Equation (12) is undefined, but it may be demonstrated that the limit expressed by Equation (7) applies to Equation (12) as well as to Equation (6). This situation is to be expected since Equation (12) is but a different form of Equation (6). Thus, for increasing values of $(h_q - h_o)$, the value of the first term of Equation (12) increases from $[M^*/(M^* - M)] T_o$ while the value of the second term of that equation increases from $[M/(M^* - M)] T_o$.

Figure (8) shows the value of each term of Equation (12) as well as the computed value of T_q, as a function of altitude when these terms are evaluated for each of the two sets of argon and helium number densities of Figure 1. As in Case-3 calculations, those helium and argon number densities related to the United States Standard Atmosphere yield values of T_q equal to the standard-atmosphere temperatures while the set of number densities related to the Jacchia 750-degree model atmosphere yield values of T_q equal to temperatures of that Jacchia model.

The evaluation of the two terms of Equation (12) using those number densities of helium and argon related to the standard atmosphere yield results which are similar to the results obtained from the evaluation of the terms of Equations (4) or (1) for the same helium data alone. If such Equation (12) evaluations had been plotted in Figure 5 with the graphs of the helium evaluation of the terms of Equation (1), the differences between these two sets of graphs in the semilog scale would be so small that only that portion of the line representing second-term values of Equation (12), for h_q below 250 km would be clearly distinguished from the lines already there. It follows that for all practical purposes, the terms of double-gas expression calculated upwards yields results which in magnitude are essentially equal to those of the light-single-gas expression calculated upwards when the initial temperature in the single-gas expression is known. It follows that the intrinsic uncertainties introduced by taking the difference between the two relatively large terms should be about the same for the two cases.

The real difference between the double-gas upward calculation, Equation (12), and the light-single-gas upward calculation, Equation (4), is that in the evaluation of Equation (4), the value of T_o the low-altitude reference temperature must be independently known with a high degree of certainty, while in the evaluation of Equation (12) no such information or assumption is required. Thus, again, for any particular atmospheric temperature profile, the two-gas equation appears to be favored over the closely related single-gas calculation.

Again, from Figure 8, one sees that for double-gas upward calculations, the values of individual terms for any particular integration interval are greater for data associated with low-temperature models then for data associated with high-temperature models. This comparison indicates that more accurate temperatures are obtained by the upward calculation at times of higher temperatures. This situation is similar to that observed for the single-light-gas upward calculation. Again a rigorous error analysis is required if the relative uncertainties as a function of altitude are to be determined.

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GEOMETRIC ALTITUDE (km)

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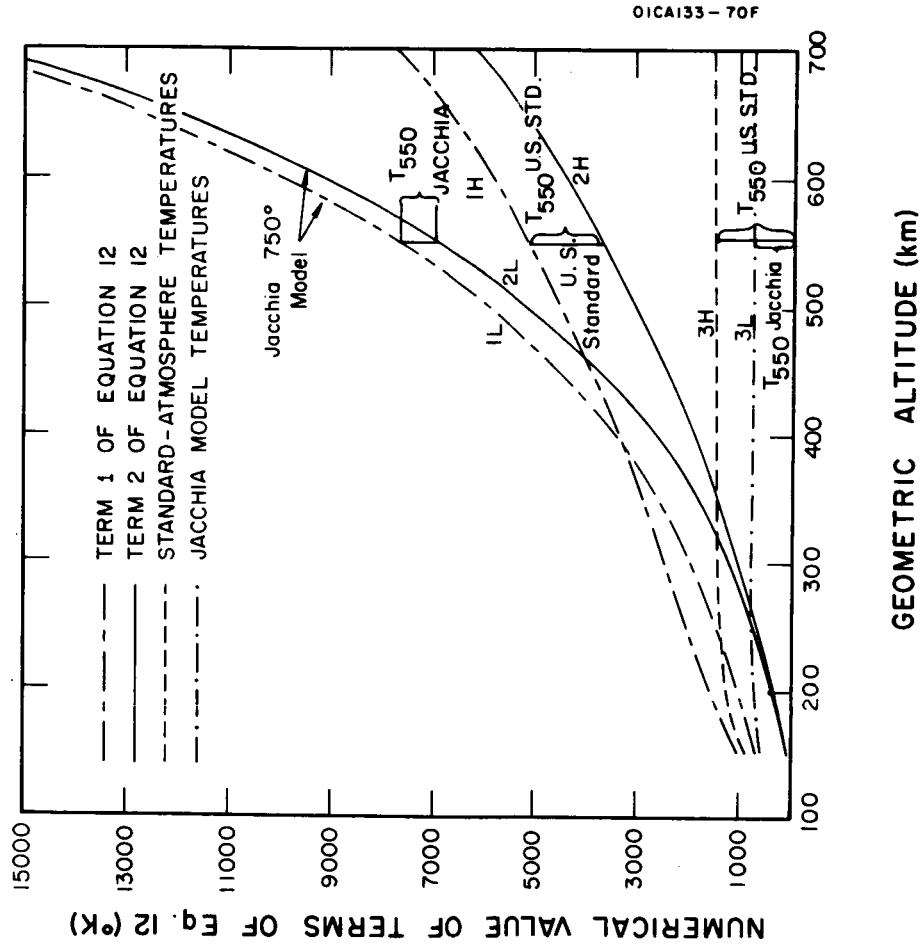
Figure 8. Three altitude-dependent sets of values; 1H, for term 1 of Equation (12); 2H, for term 2 of this equation, and 3H for the algebraic sum of these two terms (which is the temperature of the related model) all based upon a reference level of 150 km and upon Standard-Atmosphere related values of helium and argon number-density data for conditions of high solar activity compared with three similar sets of values 1L, 2L, and 3L based upon the helium and argon data associated with the Jacchia 750-degree model atmosphere for conditions of low-solar activity.

Although Equation (12) involves argon number densities, the evaluation of this equation presented in Figure 8 is carried up to 700 km. This situation is permitted since for h_q sufficiently greater than h_o, Equation (12) may be shown to degenerate essentially to the form of Equation (4), and the heavy-gas data are no longer required for extending the calculations to greater altitudes. Thus, if argon number-density data are available from 150 to 450 km, for example, and helium data are available from 150 to 700 km, Equation (12) would provide results from 150 to 700 km merely by the use of zero as the value for argon number density at and above 450 km. The difference in the calculated values of T_q at 450 km for the actual argon number density and zero would be negligible. Such a procedure would be equivalent to using Equation (4) for altitudes above 450 km. The validity of such a procedure is demonstrated analytically as follows;

For $h_q \gg h_o$ we find that

$$\frac{n_q^*}{n_o^*} \ll 1 \tag{13}$$

and

$$\frac{n_q}{n_o} < 1 \tag{14}$$

such that

$$\frac{n_q^*}{n_o^*} \cdot \frac{n_q}{n_o} \ll 1 \tag{15}$$

and

$$1 - \frac{n_q^*}{n_o^*} \cdot \frac{n_o}{n_q} \simeq 1 \tag{16}$$

It follows that

$$\frac{n_o^*}{n_q^*} \cdot \frac{n_q}{n_o} - 1 \simeq \frac{n_o^*}{n_q^*} \cdot \frac{n_q}{n_o} \tag{17}$$

The incorporation of approximation (16) and (17) into Equation (12) yeilds

$$T_q \simeq \frac{n_o}{n_q} \cdot \frac{n_q^*}{n_o^*} I_q^* - I_q \tag{18}$$

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while the incorporation of inequality (13) into Equation (5) leads to

$$\frac{n_q^*}{n_o^*} I_q^* \simeq T_o \tag{19}$$

Combining approximations (18) and (19) yields

$$T_q \simeq \frac{n_o}{n_q} T_o - I_q \tag{20}$$

which expression, except for the approximation sign, is Equation (4) for single-light-gas upward calculation. Thus, for h_q sufficiently greater than h_o, the values of the first and second terms of Equation (12) closely approach the values of the first and second terms respectively of Equation (4), or of Equation (1) applied to a light gas.

It may be shown that the difference between the first term of Equation (12) and the first term of Equation (4) is equal to the similarly ordered difference between the second terms of these two equations, and for $h_q = h_o$ this difference is found to be only about 11 percent of the related value of T_q. The ratio of these equal differences to the corresponding value of T_q decreases almost exponentially (depending upon the temperature-altitude profile governing the number-density data) with increasing values of integration interval ($h_q - h_o$), as shown in Figure 9. It is apparent from this figure that for integration intervals greater than 250 km the difference between the terms of Equation (12) and the related terms of the Equation (6) is less than 0.1 percent for nearly any realistic model atmosphere, and the effect on values of T_q caused by shifting from Equation (12) to Equation (6) at such an integration interval would be negligible.

Thus, not only is the extension of the data of Figure 8 up to 700 km justified, but it has been demonstrated that loss of the heavy-gas data in the upper portion of a set of dual-gas observations does not limit the determination of atmospheric temperature in this altitude region. The error analysis in Part II of this paper demonstrates that no loss of accuracy in T_q results from such a situation.
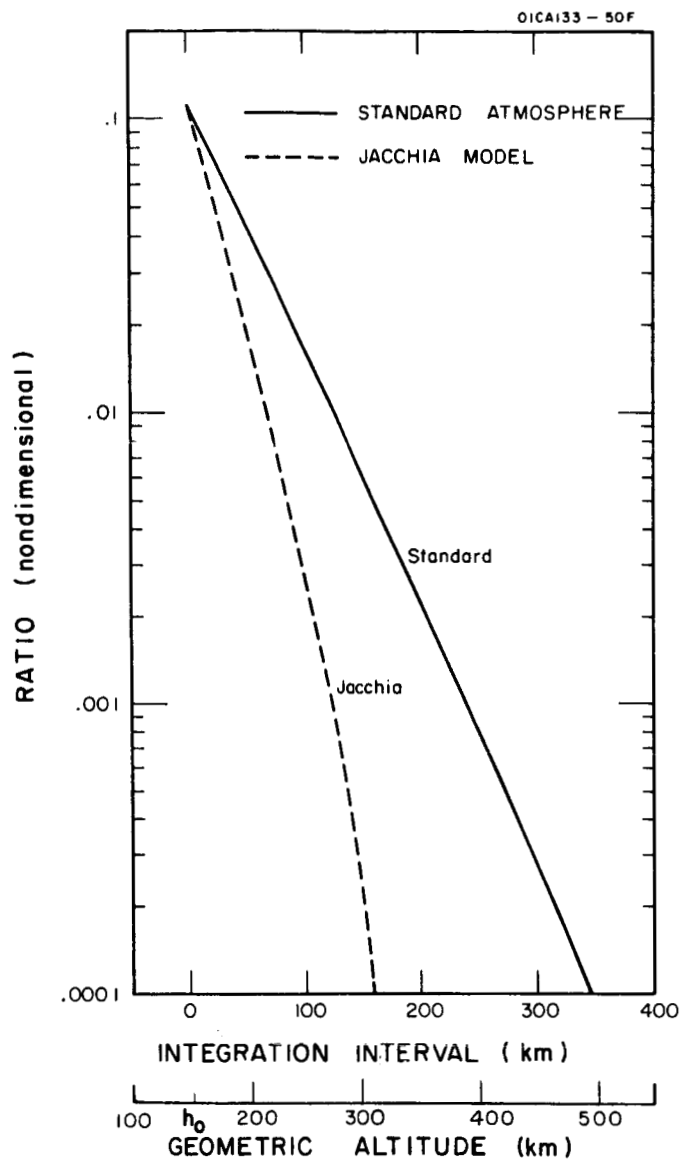
Figure 9. Altitude-dependent values of the ratio (defined as term one of
Equation (12) minus the corresponding term of Equation (4) all
divided by the related value of temperature) based upon a
reference level of 150 km and upon the standard-atmosphere
related values of helium and argon number density data compared
with a similar ratio based upon the helium and argon data
associated with the Jacchia 750-degree model atmosphere.

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PART II

ERROR ANALYSIS

GENERAL CONSIDERATIONS

In Part I of this paper, single-gas and double-gas number-density equations were examined analytically and numerically in regard to the influence of various boundary conditions on the suitability of the function for determining a temperature-altitude profile. The boundary conditions whose variations were considered included the following:

(1) Types of gases, covering a wide range of molecular weights.
(2) Altitude of reference level for a fixed direction of calculation.
(3) Direction of Calculation, i.e., $h_q < h_o$, and $h_q > h_o$.
(4) Number-density versus altitude profiles which were varied over a range to include day to night variation as well as that variation which one might expect to observe over a considerable portion of the 11-year solar cycle.

While one could make some approximate inferences concerning the accuracy of the deduced temperature-altitude profiles from Part I of this study, the ability to make precise statements regarding the relative accuracies of the various approaches depends upon a rigorous error analysis. Part II of this report discusses such an analysis.

It is intuitively apparent that the uncertainties in temperature T_q as determined by the several temperature expressions, Equations (4), (5), (6) and (12), would most likely be different for each equation evaluated at any particular altitude with a specific set of boundary conditions, and that the uncertainty in T_q evaluated at various altitudes by the same equation would also vary. The results of a rigorous error analysis in which only random errors are considered verifies this contention, and permits one to select the most suitable temperature equation for any particular altitude region of the atmosphere and for any particular set of boundary conditions. The error analysis has also shown how to apply the optimum boundary conditions to a particular set of observational data.

The error analysis employed is the gaussian method, wherein each observed variable y_i entering into the expression of a particular function of these variables $x(y_i)$ is assumed to have an observational uncertainty δy_i that meets the conditions of a gaussian distribution about y_i the true value of each variable. Thus, if the value of x is determined from the functional expression $x(y_i)$, the value of δx the implicite uncertainty in x is given by

$$(\delta x)^2 = \sum_i \left(\frac{\partial x}{\partial y_i} \delta y_i \right)^2 \tag{21}$$

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NUMERICAL INTEGRATION EXPRESSION
(LOGARITHMIC TRAPEZOIDAL RULE)

The exact form which the several expressions of temperature uncertainty will assume upon the application of the above operation depends upon the particular numerical-integration expression used to represent the perfect integrals in each of the four temperature expressions, Equations (4), (5), (6) and (12). The linear trapezoidal rule previously used by Minzner, et al. [7] in the analysis of mass-density data has been found [16] to result in an inferior approximation to the perfect integral of any function which varies nearly exponentially with linear variations of the independent variable (as in the case of values of mass density or number density versus altitude). A more correct approximation for this type of function is obtained by the application of a logarithmic trapezoidal rule [17] suggested to the author by J. F. Smith private communication, August 1965 [18]. In terms of this latter rule, the two normalized perfect-integral expressions used throughout the previous sections of this paper, and abbreviated by the symbols I_q^* and I_q respectively, may each be evaluated numerically by one of the following two approximations:

$$I_q^* = \frac{GM^*}{R\, n_q^*} \int_{h_o}^{h_q} n^*(h)\; dh \;\simeq\; \frac{GM^*}{R\, n_q^*} \sum_{j=1}^{q} \frac{(h_j - h_{j-1})\,(n_j^* - n_{j-1}^*)}{[\ln(n_j^*) - \ln(n_{j-1}^*)]} \tag{22}$$

and

$$I_q = \frac{GM}{R\, n_q} \int_{h_o}^{h_q} n(h)\; dh \;\simeq\; \frac{GM}{R\, n_q} \sum_{j=1}^{q} \frac{(h_j - h_{j-1})\,(n_j - n_{j-1})}{[\ln(n_j) - \ln(n_{j-1})]} \tag{23}$$

SPECIFIC UNCERTAINTY EQUATIONS

When approximations (22) and (23) are appropriately introduced into the temperature equations and when the resulting expressions are subjected to the operations indicated by Equation (21), a corresponding set of temperature-uncertainty equations result. Equations (24) and (25) represent the uncertainty of temperatures determined by Equations (4) and (5), respectively, while the single Equation (26) represents the uncertainty of temperatures determined by both Equations (6) and (12).

This latter situation of one uncertainty equation for two temperature equations results because Equations (6) and (12) are really identical, since these two equations represent only different forms consistent with the two different boundary conditions $h_q < h_o$ and $h_q > h_o$. The single uncertainty equation, however, does not imply equal uncertainties for T_q at any particular altitude as deduced by the two temperature equations (6) and (12), but rather demonstrates the existence of widely differing uncertainties depending upon the particular boundary condition values.

Case 1: Single-Heavy-Gas downward calculation

$$\left(\delta T_q\right)^2 = \left(\frac{\delta T_o}{T_o}\right)^2 T_o^2 + \left(\frac{\delta n^*_o}{n^*_o}\right)^2 \left(\frac{n^*_o}{n^*_q}\right)^2 \left(F^*_o\right)^2 + \left[\sum_{j=1}^{q-1}\left(\frac{\delta n^*_i}{n^*_j}\right)^2 \left(\frac{n^*_i}{n^*_q}\right)^2 \left(F^*_j\right)^2\right] + \left(\frac{\delta n^*_q}{n^*_q}\right)^2 \left(F^*_q\right)^2 \tag{24}$$

Case 2: Single-Light-Gas upward calculation

$$\left(\delta T_q\right)^2 = \left(\frac{\delta T_o}{T_o}\right)^2 T_o^2 + \left(\frac{\delta n_o}{n_o}\right)^2 \left(\frac{n_o}{n_q}\right)^2 \left(F_o\right)^2 + \left[\sum_{j=1}^{q-1}\left(\frac{\delta n_i}{n_j}\right)^2 \left(\frac{n_i}{n_q}\right)^2 \left(F_j\right)^2\right] + \left(\frac{\delta n_q}{n_q}\right)^2 \left(F_q\right)^2 \tag{25}$$

Case 3, and Case 4: Double-Gas upward and downward calculation

$$\left(\delta T_q\right)^2 = \left\{\left(\frac{\delta n^*_o}{n^*_o}\right)^2 \left(\frac{n^*_o}{n^*_q}\right)^2 \left(F^*_o\right)^2 + \left[\sum_{j=1}^{q-1}\left(\frac{\delta n^*_i}{n^*_j}\right)^2 \left(\frac{n^*_i}{n^*_o}\right)^2 \left(F^*_j\right)^2\right] + \left(\frac{\delta n^*_q}{n^*_q}\right)^2 \left(F^*_q\right)^2\right\}\left(\frac{1}{1-r_q}\right)^2 +$$

$$\left\{\left(\frac{\delta n_o}{n_o}\right)^2 \left(\frac{n_o}{n_q}\right)^2 \left(F_o\right)^2 + \left[\sum_{j=1}^{q-1}\left(\frac{\delta n_i}{n_j}\right)^2 \left(\frac{n_i}{n_q}\right)^2 \left(F_j\right)^2\right] + \left(\frac{\delta n_q}{n_q}\right)^2 \left(F_q\right)^2\right\}\left(\frac{-r_q}{1-r_q}\right)^2 \tag{26}$$

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where

$$F_o^* = \left(\frac{GM^*}{R}\right)\frac{(h_1-h_o)}{(\ln n_1^* - \ln n_o^*)^2}\left[1 - \left(\frac{n_1^*}{n_o^*}\right) + \ln\left(\frac{n_1^*}{n_o^*}\right)\right] + T_o$$

$$F_o = \left(\frac{GM}{R}\right)\frac{(h_1-h_o)}{(\ln n_1 - \ln n_o)^2}\left[1 - \left(\frac{n_1}{n_o}\right) + \ln\left(\frac{n_1}{n_o}\right)\right] + T_o$$

$$F_j^* = \left(\frac{GM^*}{R}\right)\frac{(h_i-h_{i-1})}{(\ln n_j^* - \ln n_{j-1}^*)^2}\left[1 - \left(\frac{n_{i-1}^*}{n_j^*}\right) + \ln\left(\frac{n_{i-1}^*}{n_j^*}\right)\right] + \frac{(h_{i+1}-h_i)}{(\ln n_{j+1}^* - \ln n_j^*)^2}\left[1 - \left(\frac{n_{i+1}^*}{n_j^*}\right) + \ln\left(\frac{n_{i+1}^*}{n_j^*}\right)\right]$$

$$F_j = \left(\frac{GM}{R}\right)\frac{(h_i-h_{i-1})}{(\ln n_j - \ln n_{j-1})^2}\left[1 - \left(\frac{n_{i-1}}{n_j}\right) + \ln\left(\frac{n_{i-1}}{n_j}\right)\right] + \frac{(h_{i+1}-h_i)}{(\ln n_{j+1} - \ln n_j)^2}\left[1 - \left(\frac{n_{i+1}}{n_j}\right) + \ln\left(\frac{n_{i+1}}{n_j}\right)\right]$$

$$F_q^* = \left(\frac{GM^*}{R}\right)\frac{(h_q-h_{q-1})}{(\ln n_q^* - \ln n_{q-1}^*)^2}\left[1 - \left(\frac{n_{q-1}^*}{n_q^*}\right) + \ln\left(\frac{n_{q-1}^*}{n_q^*}\right)\right] + T_q$$

$$F_q = \left(\frac{GM}{R}\right)\frac{(h_q-h_{q-1})}{(\ln n_q - \ln n_{q-1})^2}\left[1 - \left(\frac{n_{q-1}}{n_q}\right) + \ln\left(\frac{n_{q-1}}{n_q}\right)\right] + T_q$$

and where

$$r_q = \frac{n_o^*}{n_q^*} \cdot \frac{n_q}{n_o}$$

$$\frac{GM^*}{R} = 45.06843\,^{\circ}K \text{ per km' for argon, while}$$

$$\frac{GM}{R} = 4.51564\,^{\circ}K \text{ per km' for helium}$$

DERIVED EQUATIONS COMPARED WITH THOSE BASED ON LINEAR TRAPEZOIDAL RULE FOR NUMERICAL INTEGRATION

Equations (24), (25) and (26) are somewhat more complicated than those which are obtained when the perfect integral is approximated by the linear trapezoidal rule. In the latter instance the general format of these equations is unchanged, but the values of the F functions are much simpler; i.e.,

$$F_o^* = T_o + \left(\frac{GM^*}{R}\right) \frac{(h_o - h_1)}{2} , \qquad F_o = T_o + \left(\frac{GM}{R}\right) \cdot \frac{(h_o - h_1)}{2} ,$$

$$F_j^* = \left(\frac{GM^*}{R}\right) (h_{j+1} - h_{j-1}), \qquad F_j = \left(\frac{GM}{R}\right) (h_{j+1} - h_{j-1}) ,$$

$$F_q^* = \left(\frac{GM^*}{R}\right) \frac{(h_{q-1} - h_q)}{2} - T_q , \qquad F_q = \left(\frac{GM}{R}\right) \cdot \frac{(h_{q-1} - h_q)}{2} - T_q .$$

The use of the logarithmic trapezoidal rule yields significantly improved values of the integral of number density and hence of temperature over those values obtained with the use of the linear trapezoidal rule. The difference in the values of δT_q as obtained by using one or the other of the two sets of F functions, however, may be of somewhat less significance except in special situations where a precise comparison of results of the several equations is important, as in this study or where the extra complexity may be easily handled by machine calculation. Calculations using first the logarithmic form of the F functions and then the simpler form of the F functions show the difference in corresponding values of δT_q as determined by Equation (26) to vary by less than five percent for the upward type calculations of this study but by as much as 37 percent for downward calculations.

NUMBER-DENSITY UNCERTAINTIES

The evaluation of the double-gas uncertainty expression depends upon several considerations: the boundary conditions, some known physical constants, the measured number-density profiles of each of the two gases, and the number-density uncertainties which may be estimated from the accuracies of the measuring instrument and of the telemeter and recording systems. The evaluation of the single-gas uncertainty expressions require additional information concerning temperature and temperature uncertainty at the reference level. Such information is generally not available, and in this study will be used only when it is first determined by the double-gas expression, Equations (6), (12) and (26).

For the numerical error evaluations of the hypothetical experiments considered in this paper, the quantities $\delta n^*/n^*$ and $\delta/n/n$ have been arbitrarily given the values represented by the functions

$$\frac{\delta n^*}{n^*} = \frac{\Delta n^*}{n^*} + \Delta t , \qquad (27)$$

and

$$\frac{\delta n}{n} = \frac{\Delta n}{n} + \Delta t , \qquad (28)$$

where Δn^* and Δn are both taken to be 1×10^{10} m^{-3} (1×10^4 cm^{-3}) in accordance with the sensitivities of present-day mass spectrometers, and where Δt is taken to be .01 in accordance with the overall accuracy of a good telemeter system. Thus, the value of $\delta n^*/n^*$ reaches a minimum of about .01 at low altitudes where the number density is high, and increases to over 1.0 at altitudes where the number density decreases to less than 1×10^{10} m^{-3}.

NUMERICAL EVALUATION OF δT_q FOR DOUBLE-GAS CALCULATIONS

The heavy solid-line curve of Figure 10 represents δT_q the temperature uncertainties as derived for various values of h_q from Equation (26) for $h_q < h_o = 450$ km, when the helium and argon number densities are those related to the U. S. Standard Atmosphere and when the number-density uncertainties are those expressed by Equations (27) and (28). These temperature uncertainties correspond to the temperatures determined by the double-gas downward calculation of Equation (6).

For $h_q = h_o$ the value of T_q is undetermined by Equation (6) since the integration interval is zero, and Equation (26) correspondingly shows δT_q to be positive infinity for this situation. As h_q decreases to lower values, δT_q decreases very rapidly reaching a value of $7.86^\circ K$ at 150 km. The value of δT_q in the region of 300 km as well as at other altitudes is slightly dependent upon the value of h_o, although this dependence falls off very rapidly with decreasing altitude. In the vicinity of 300 km the values of δT_q are minimized when $h_o \simeq 450$ km.

The thin solid-line curve of Figure 10 represents those values of δT_q versus h_q as derived from Equation (26) for $h_q > h_o = 150$ km, using the same number densities and related uncertainties as for the heavy-line values. These temperature uncertainties correspond to the temperatures determined by the double-gas upward calculation of Equation (12). In this expression as in Equation (6), T_q is undetermined for $T_q = T_o$ because of a zero integration interval, and the corresponding value of δT_q from Equation (26) is seen to be positive infinity. For increasing values of h_q, δT_q decreases rapidly to a minimum value of about $61^\circ K$ and then rises slowly to a value of about $246^\circ K$ at 700 km.

The heavy-line and thin-line uncertainty values differ considerably from each other and indicate the desirability of using the two different sets of boundary conditions and the corresponding different dual-gas temperature expressions for the determination of T_q in different portions of the altitude region of observation, so that the minimum temperature uncertainty may be obtained for all portions of the altitude region of interest.

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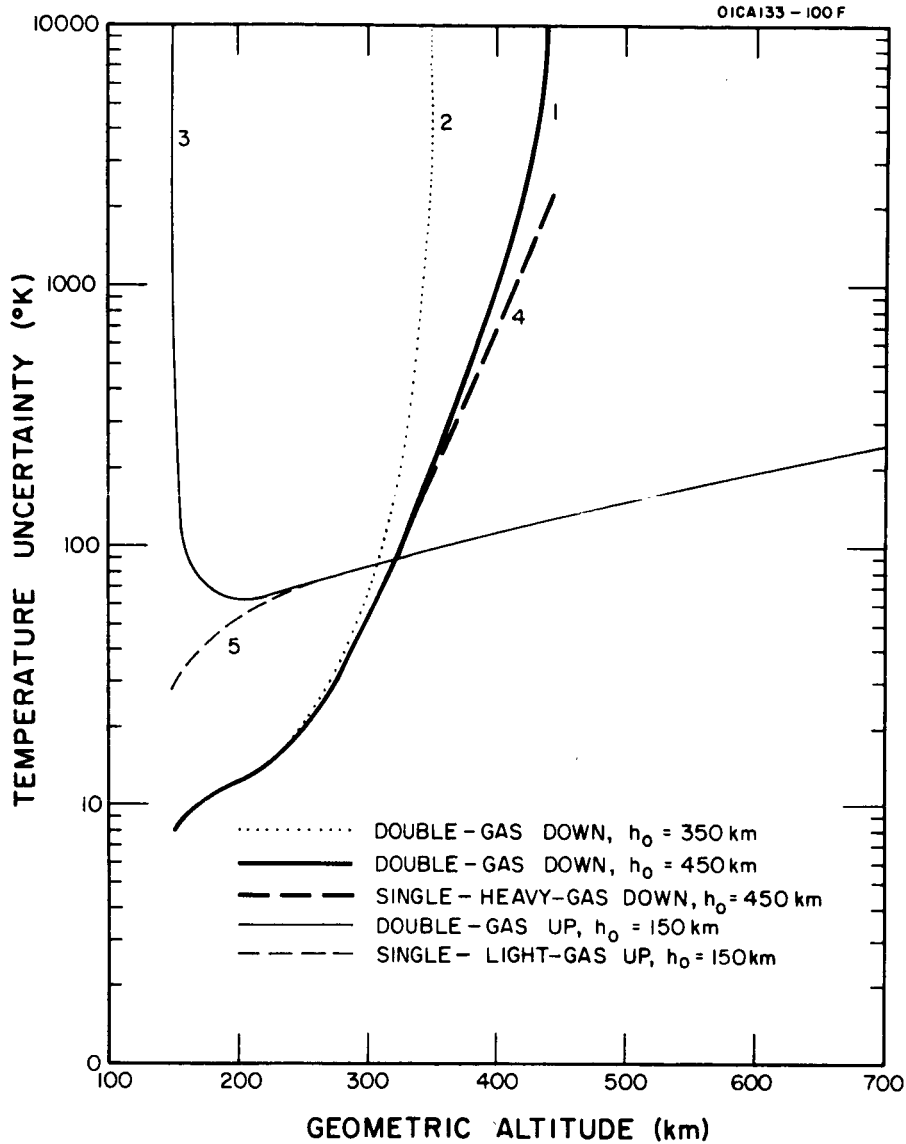
Figure 10. Five altitude-dependent sets of temperature uncertainty, 1 through 5, each for the same temperature-altitude profile determined respectively in the following five different ways:

(1) By Equation (6) from a near optimum reference altitude of 450 km.

(2) By Equation (6) from a reference altitude of 350 km.

(3) By Equation (12) from a reference altitude of 150 km.

(4) By Equation (5) computed downward from a reference altitude of 450 km when the reference-level temperature has been previously obtained from Equation (6).

(5) By Equation (4) computed upward from a reference altitude of 150 km when the reference-level temperature has been previously obtained from Equation (6).

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NUMERICAL EVALUATION OF δT FOR SINGLE-GAS CALCULATIONS

The double-gas downward calculation of temperature using number-density data related to the Standard Atmosphere has been shown in Figure 7 to yield a value of temperature at 150 km (T_{150}) of about 893°K with an uncertainty δT_{150} of the order of 7.86°K for the conditions involved. Using these values for T_{150} and δT_{150} as T_o and δT_o respectively in the single-light-gas upward-calculation expressions for T_q and δT_q, Equations (4) and (25) respectively, one obtains a temperature-altitude profile whose uncertainty profile is given by values shown as the thin dashed line in Figure 10. This thin dashed line is seen to merge with the thin solid line as the integration interval becomes sufficiently large.

This convergence of the values of the two uncertainty expressions is as expected, since the functional forms of the related temperature expression, Equations (4) and (12), have already been seen to converge to a common form under the same circumstances. In particular, the value of T_q as determined for h_q = 450 km from Equation (4) with the above specified boundary conditions is essentially identical to that determined from Equation (6). Similarly, the 450-km value of δT_q associated with each of these two upward-calculation expressions of T_q are also essentially identical.

Introducing the 450-km values of T and δT from these calculations as T_o and δT_o respectively into the single-heavy-gas downward-calculation expressions for T_q and δT_q, Equations (5) and (24) respectively, yields a temperature-altitude profile whose related uncertainty profile is given by values shown as the heavy dashed line in Figure 10. At 450 km these values are considerably lower than those given by Equation (26) for the case when h_o = 450 km. At lower altitudes, however, these two sets of values of δT_q merge in keeping with the previously demonstrated fact that the related functions for temperature determination, Equations (12) and (5), converge for a sufficiently large integration interval. Thus, for the end-point case, h_q = 150 km, the values of T and δT as determined by the single heavy-gas expression using the particular 450 km set of boundary conditions are essentially identical to those obtained from the double-gas downward-calculations begun from the same reference altitude.

It is apparent that one cycle of using the single-light-gas upward calculation followed by the single-heavy-gas downward calculation, if optimum values of T_o and δT_o were to be used for the initial calculation, would yield values of T and δT equivalent to those obtained when a single application, both the upward and downward double-gas calculation, is employed. In this case there is no concern for the value of T or δT at either reference altitude.

In order to obtain an optimum temperature-altitude profile with the single-gas equations, however, when no specific initial temperature information is available, one would have to assume some initial temperature and go through a number of iterative cycles, each successive cycle yielding a set of values which converges more closely to the optimum values, as obtained by the double-

gas equations directly. Thus, whether used iteratively with single-gas equations or directly with double-gas equations, number-density data for two gases with widely different molecular weights are required, and these data must be evaluated from both a high-altitude and low-altitude reference level if an optimum temperature-altitude profile is to be obtained.

The values of δT_q versus altitude as represented by the several curves of Figure 10 do not remain fixed for the particular sets of values of number-density versus altitude and for the particular related number-density uncertainty conditions considered when the value of either reference altitude is allowed to change. In the double-gas downward calculation for which a nearly optimum set of values of δT_q as represented by the solid-heavy-line curve, the values of δT_q versus altitude are strongly dependent upon the value of h_o particularly in the altitude region directly below h_o, since at h_o, regardless of its value, the value of δT_q is positive infinity. As h_q decreases below h_o, however, the values of δT_q decreases rapidly, approaching the values of the optimum heavy-line curve asymptotically from above, as depicted by the light dotted line of Figure 10, for the case when h_o is taken to be 350 km. The optimum curve is based on the condition that δT_q at $h_q = 300$ km is seen to vary through a very shallow minimum as h_o varies from 400 to 500 km. The value $h_o = 450$ km has been arbitrarily selected as representative of this optimum condition, although a more precise examination may show a slight shift in h_o for the true minimum value of δT_{300}.

In the case of the values δT_q associated with the single-gas downward calculations, as depicted by the heavy dashed line of Figure 10, the variation of low-altitude values of δT_q with respect to h_o also has a minimum which, in this instance, appears to coincide almost exactly with that altitude for which $\delta n^*/n^*$ equals unity, i.e., 440 km. For symmetry with the double-gas calculations, however, the single-gas downward computations depicted in Figure 10 were made for the case $h_o = 450$ km although the differences in δT_q would never be discernible on the graph.

It follows that, for an instrument of a given sensitivity, it is most desirable that the double-gas data be gathered to an altitude consistent with the conditions for minimum double-gas temperature uncertainty, i.e., to about that altitude for which instrument limitations and atmospheric number-density distribution together make the heavy-gas ratio $\delta n^*/n^* = 1.0$.

For the double-gas upward calculation as represented by the thin-line curve of Figure 10, the entire curve of δT_q versus altitude shifts toward greater or smaller uncertainties as the value of h_o increases or decreases respectively from $h_o = 150$ km, without any apparent existence of a particular set of minimum values. The same situation prevails for the single-light-gas upward calculation as long as $\delta T_o/T_o$ employed in this calculation is appropriately taken from the double-gas down calculations. Hence, for the upward-type calculation it is desirable to measure the double-gas data from the lowest possible altitude (consistent with diffusive-separation considerations). While it is possible that diffusive equilibrium completely governs the number-density distribution to altitudes as low as or lower than 120 km, the value 150 km was taken to be a conservative lower limit for this study.

DEPENDENCE OF δT_q ON ALTITUDE DISTRIBUTION OF NUMBER DENSITY
(i.e., ON THE TEMPERATURE-ALTITUDE PROFILE)

When double-gas calculations of T_q and δT_q are made on the basis of the Jacchia number densities presented in Figure 1, the sets of values of δT_q are substantially shifted from those related sets associated with the number densities of the U. S. Standard Atmosphere, as shown in Figure 11. The upward type calculations using the Jacchia data have uncertainties ranging between one-half to one and one half times those involving the standard-atmosphere data. The downward-type calculations have been approximately optimized by choosing h_o = 300 km, for it is near 300 km altitude that $\delta n^*/n^* \simeq 1.0$ with the Jacchia data. For altitudes below 210 km, the downward calculations using the Jacchia-model data have smaller uncertainties than the comparable calculations using the standard-atmosphere data, but a comparison of the same two curves at altitudes in excess of 220 km shows that the lower number densities of the Jacchia model in this altitude region yield greater values of δT_q.

These shifts in temperature uncertainty for each of the two double-gas temperature expressions are in keeping with the downward compression of the atmosphere associated with Jacchia's lower temperatures, and with the associated shift in the relative uncertainty of observed number-density data. When the minimum values of δT associated with each model are compared, they are seen to cross three times within the plotted altitude interval. When the uncertainties are expressed in terms of the percentage of the related temperatures, the curves no longer cross as illustrated in Figure 12. Instead, the percentage uncertainty is seen to be consistently lower for that model depicting conditions of high solar activity. This situation arises primarily from the greater high-altitude concentration of the heavy gas, and the related decrease in heavy gas number-density uncertainty under these solar-activity conditions. This situation suggests the desirability of a future study in which one might examine the over-all influence of using the number densities of molecular nitrogen, either in place of the argon observations or in addition to the argon observations.
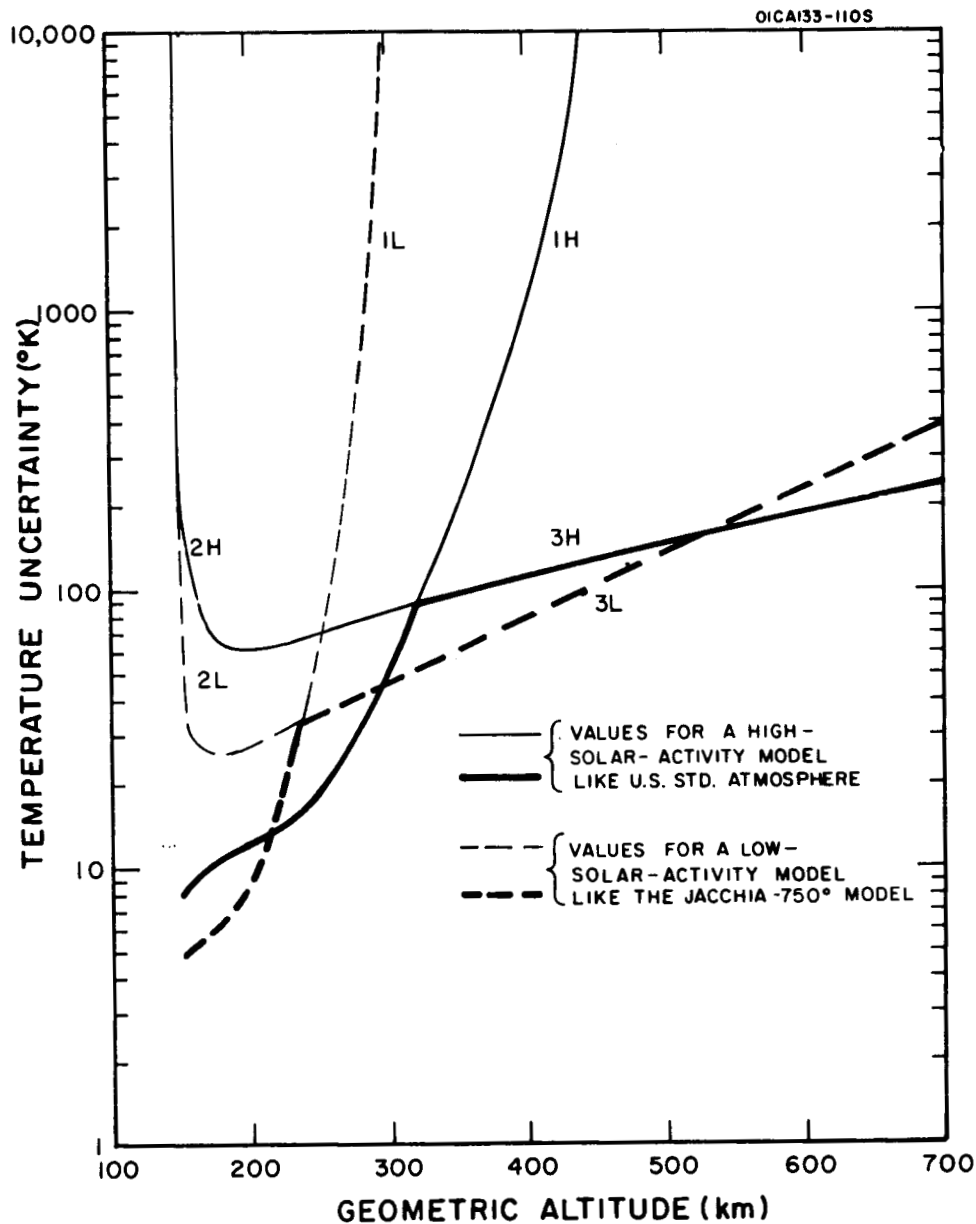
Figure 11. Three altitude-dependent set of temperature uncertainty,
(1L) that for the temperature-altitude profile based on a near
optimum application of the double-gas down equation,
(2L) that for the temperature-altitude profile based on the
application of the double-gas up equation, and
(3L) that consisting of the locus of minimum uncertainties
(made up of parts 1L and 2L), all three for a low solar-
activity model atmosphere (like the Jacchia 750-degree model)
compared with three similar altitude dependent sets of tempera-
ture uncertainty 1H, 2H, and 3H all three for a relatively high
solar-activity model atmosphere (like the U. S. Standard
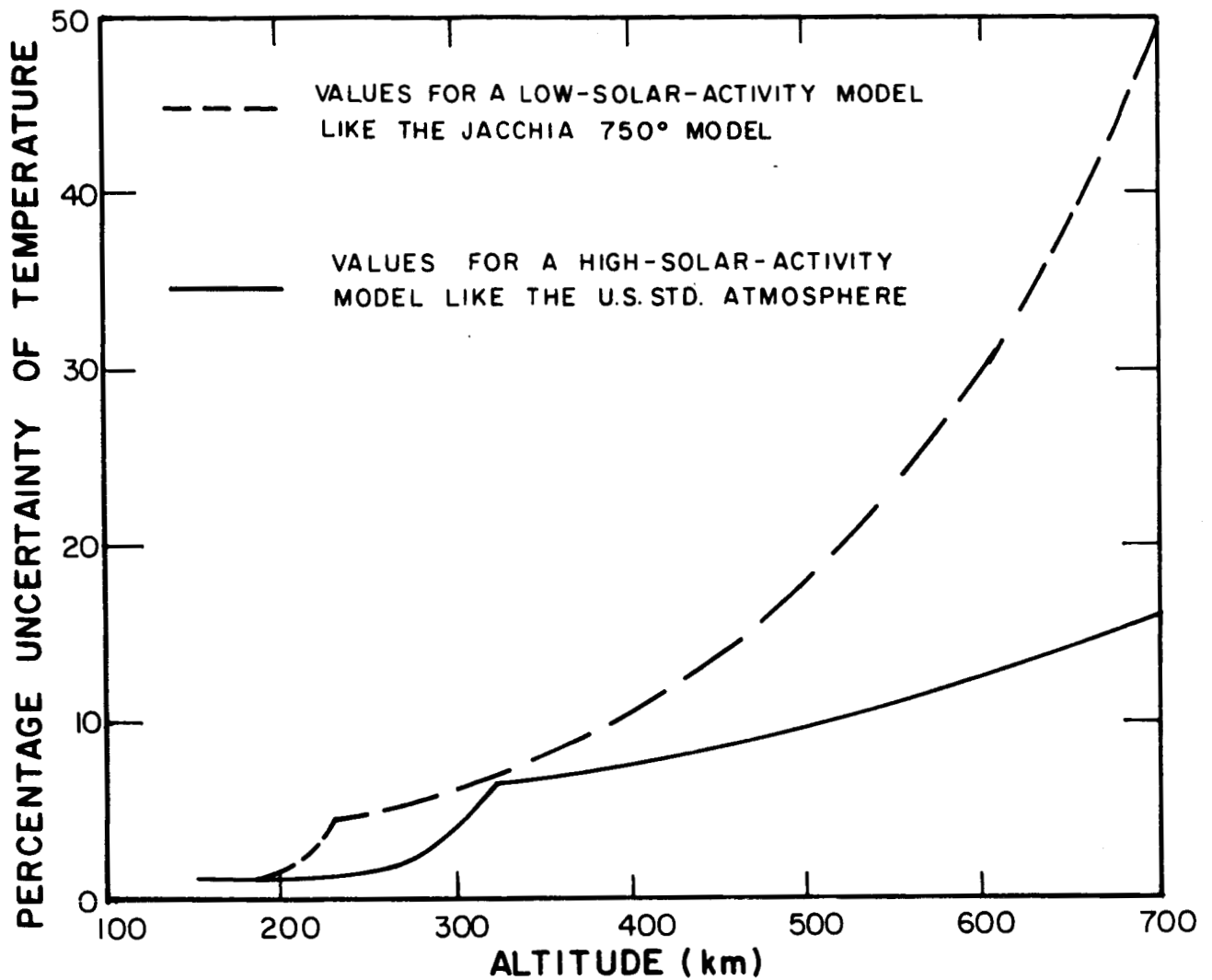Atmosphere).

Figure 12. Altitude dependence of the percentage uncertainty of temperature as deduced from the simultaneously observed values of helium and argon number-density data for both low- and high-activity conditions.

CONCLUSIONS

1. Simultaneously observed number-density data versus altitude for both a heavy and light inert gas, such as argon and helium, over a significant common altitude interval (150 km or more) may be analyzed to yield kinetic temperature versus altitude from the lowest altitude of the heavy-gas data to the greatest altitude of the light-gas data.

2. Single gas number-density data versus altitude for a heavy inert gas such as argon may be analyzed to yield accurate temperature-altitude profiles for only the lower end of a sufficiently extended density-altitude profile when a downward-type calculation is employed and when the high-altitude reference-level temperature is restrained to be within realistic bounds. No significant values of temperature are obtained for the upper portion of the altitude range.

3. Single gas number-density data versus altitude for a sufficiently light gas such as helium may be analyzed to yield temperature-altitude profiles over the entire altitude range of the number-density data, by an upward type calculation, but only if the temperature at the lowest altitude of the available data is known to a rather high degree of accuracy.

4. Sets of simultaneously measured values of light-gas and heavy-gas number density data obtained with present-day mass spectrometers may, under optimum conditions, yield temperature-altitude profiles with uncertainties ranging approximately between values of less than $10^{\circ}K$ at 150 km to values of about $300^{\circ}K$ at 700 km for number-density distributions associated with a wide range of solar activity.

5. The percentage of uncertainty of temperatures inferred from the double-gas method at low altitudes is essentially independent of the particular model atmosphere employed. At altitudes above 500 km, however, the percentage uncertainty of temperatures so determined is significantly greater for models associated with low solar-activity conditions, than for models associated with high solar-activity conditions.

6. The high-altitude temperature uncertainty can be minimized if the concentrations of the two gases is measured to the lowest possible altitude above the diffusive separation level.

7. The mid-altitude temperature uncertainty can be minimized if the heavy-gas number density is measured up to the altitude for which the measuring uncertainty equals the observed value. This situation suggests that number density measurements of molecular nitrogen (N_2) might reduce the mid-altitude uncertainty particularly during periods of low solar activity if the altitude at which the uncertainties of the upward and downward dual-gas calculations cross is raised. A greater altitude interval of N_2 number-density data would probably be required, however, for the low-altitude uncertainty to reach the same small values obtained with argon. Perhaps three gases should be measured.

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8. Essentially the same temperature-uncertainty results are obtained by the consecutive and iterative use of the single-gas temperature equations for a sufficient number of cycles, as is obtained by the single application of each of the two forms of the dual-gas temperature equations.

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