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Prepared under Contract No. NAS 8-11259 by

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ABSTRACT AND SUMMARY

Background is supplied and a theorem is proved that gives the existence of a linear approximation of a real valued function, defined on Euclidean n-space, by a class of functions within a Tchebycheff norm. The hypotheses of the theorem do not require any special conditions on the characteristics of the domain, the basic approximation functions, or the approximation bounds associated with any point of the domain.

LIST OF SYMBOLS

$f(x)$	function of a variable x
D	domain of a function
$\{\ \}$	set consisting of the listed elements
$\{x:\ c(x)\}$	set consisting of the elements x satisfying the condition c(x)
I	closed interval
$\vec{x}$	1 x n vector $(x_1, x_2, \cdots, x_n)$
$\phi_i(x)$	a basic approximating function
F(A, x)	an approximation established by the parameter set A
$\overline{pq}$ or [p, q]	the closed line segment determined by the points p and q
E^k	k-dimensional euclidean space
ε	is an element of
(x, y)	point in E^2
$(\vec{x}, y)$	point in E^{n+1} with first n coordinates identical with coordinates of vector $\vec{x}$

1. Introduction and Statement of the Problem.

An important problem to computer technology is the representation of a function when the values of the function were known only within approximations. That is, data inputs specify that the unknown function $f(x)$ has values within an interval for each of the points of the domain.

In the finite domain situation, we would have a domain, $D = \{c_1, c_2, \ldots, c_n\}$, consisting of a finite number of points and information that would specify that the $f(c_i)$ values are within closed intervals $I_i = [\ell(c_i), u(c_i)]$. Here $\ell(c_i)$ are the lower endpoint values of the interval and $u(c_i)$ are the upper endpoints values.

For domains consisting of an infinite number of points, we would have a specified set of independent argument values, D, over which the function values $f(x)$ were known to be within the closed intervals $I(x) = [\ell(x), u(x)]$ for every x in the domain. Infinite domains could consist of intervals of finite length, collections of intervals of finite length, or an infinite number of discrete points. Note figure 1 and figure 2.

The previous discussion was directed toward functions of a single variable. In general we will be interested in functions of n variables, and we will simply represent them as functions of $1 \times n$ vectors $\vec{x}$. That is, $f(x)$ will represent a function $f(x_1, \ldots, x_n)$ of the n variables $x_1, x_2, \ldots, x_n$.

We now have the problem of treating such a function in computational schemes. One approach is to represent the function as being within a certain class that is inherently easy to process

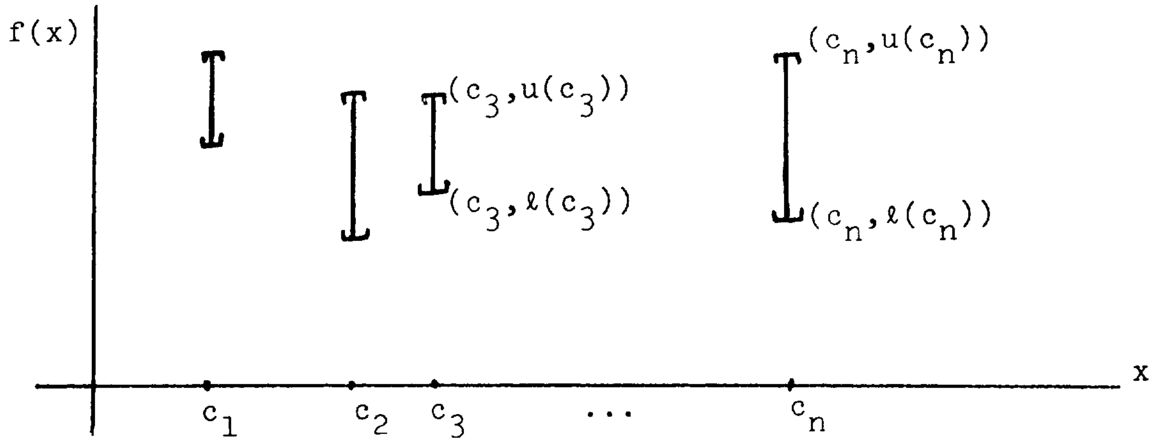

Figure 1

Approximate function values over a Domain comsisting of a finite number of points

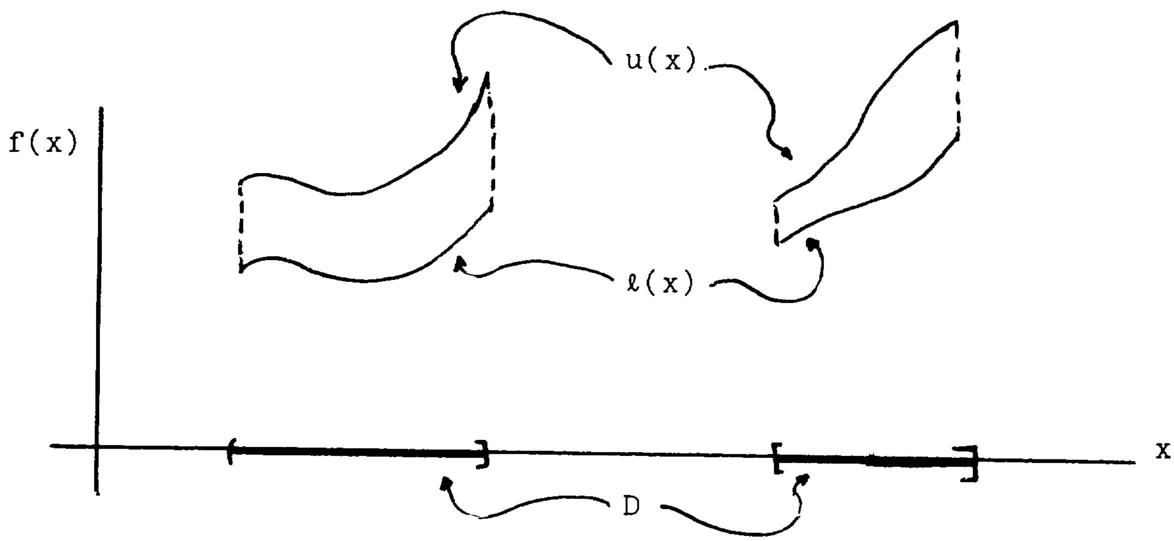

Figure 2

Approximate function values over a Domain consisting of an infinite number of points through the computation. For example, we might elect to represent $f(x)$ as a polynomial of a certain degree, or we might elect to represent the function as a truncated Fourier series. In most schemes, the process calls for expressing the function as a linear combination of specified approximation functions. That is, $\Phi_1(x), \ldots, \Phi_k(x)$ are specified and we want to fit $a_1\Phi_1(x) + a_2\Phi_2(x) + \ldots + a_k\Phi_k(x)$ within the approximation intervals by determining the $a_1, a_2, \ldots, a_k$ values. In case we require that $f(x)$ be represented as a polynomial, the approximating Φ functions would be $\Phi_1(x) = 1$, $\Phi_2(x) = x$, $\Phi_3(x) = x^2$, $\ldots$, $\Phi_k(x) = x^{k-1}$. Then values $a_1, \ldots, a_k$ would give us a specific polynomial $a_1 + a_2x + \ldots + a_kx^{k-1}$. Likewise, in the truncated Fourier series, the function $f(x)$ would be expressed as a linear combination of the first k Fourier functions; and the a_i values would be the Fourier coefficients. See Rice [1] for detailed treatment of approximation problems.

The general approximation structure may be stated in the following manner. For a specified approximating class, given by linear combinations of the basic functions $\Phi_1(x), \ldots, \Phi_k(x)$; we will note $\sum_{i=1}^{i=k} a_i\Phi_i(x)$ as $F(A,x)$. Where A represents the set of parameter values $a_1, \ldots, a_k$ that establish the particular approximation $F(A,x)$.

One problem basic to approximation theory and important to computer applications is the existence of an approximation function $F(A,x)$ within the error bounds or approximation intervals of $f(x)$. This report is directed toward a theorem that is applicable to this problem.

2. Basic Theorems of the Helly type.

The details of the main result of this paper are built around an important theorem due to Eduard Helly (1884-1943) [2,pp.101-103]. The following statement and proof of Helly's Theorem are essentially that of Yaglom and Boltyanskii [3,p.16 and p.121].

Definition: A set S is <u>convex</u> if and only if for any two points p and q of S, the line segment determined by p and q, $\overline{pq}$, is contained in S.

Theorem 1 (Helly's): If n ($n \geq 4$) convex sets are given in the plane with the condition that any three of the sets have a common point, then there is a point common to all of the sets.

Proof by induction on $n \geq 4$:

(i) If $n = 4$, then denote the sets as S_1, S_2, S_3, S_4. The point known to be common to S_1, S_2, S_3 denote as p_4. Likewise, p_3 is common to S_1, S_2, S_4; p_2 to S_1, S_3, S_4; p_1 to S_2, S_3, S_4. Now consider the triangle formed by p_1, p_2, and p_3. Denote it as T_4. It is an immediate consequence that triangle T_4 is contained in S_4. Likewise, triangle T_1 established by p_2, p_3, p_4; T_2 established by p_1, p_3, p_4; and T_3 established by p_1, p_2, p_4 are respectively contained in S_1, S_2, and S_3. If one of the p_i points is contained in T_i, then it is contained in S_i and it must be common to all four of the convex sets. Otherwise, the four points form the vertices of a convex quadrilateral. The intersection of the diagonals of this quadrilateral is a point that is common to all four of the triangles T_1, T_2, T_3, T_4. Thus it is common to all of the four convex sets S_1, S_2, S_3, and S_4.

(ii) Now assume an inductive hypothesis that the theorem is true for $n = k$, and note the convex sets $S_1, S_2, \ldots, S_k, S_{k+1}$. If we denote S_k^* as the intersection of S_k and S_{k+1}, then we see that $S_1, S_2, \ldots, S_{k-1}, S_k^*$ form a collection of k convex sets satisfying the conditions of the theorem. For example, S_1, S_2, S_k^* have a common point since S_1, S_2, S_k, and S_{k+1} have a common point. By the inductive hypothesis $S_1, S_2, \ldots, S_k^*$ have a common point, so $S_1, S_2, \ldots, S_k, S_{k+1}$ must and we see that the theorem is proved.

The statement of various Helly type theoremsis listed below for convenience. The proofs are available in the references listed. The proofs are not given here since they are somewhat complicated. In particular, the proofs related to infinite collections of sets involve topological concepts like finite intersection properties.

Theorem 2. If n ($n \geq k+2$) convex sets are given in Euclidean k-space, E^k, with the condition that any $k + 1$ of the sets have a common point; then there is a point common to all n of the sets]2,p.102].

Theorem 3. If more than $k + 2$ compact convex sets are given in Euclidean k-space, E^k, with the condition that any $k + 1$ of the sets have a common point; then there is a point common to all of the sets [2,p.102].

Definition: A <u>slab</u> is the unbounded convex set bounded by two parallel hyperplanes.

Theorem 4. If more than $k + 2$ closed slabs are given in Euclidean k-space, with the condition that any $k + 1$ of the slabs have a common point; then there is a point common to all of the slabs [4,p.178] and [2,p.350].

3. Applications to line approximations.

The application of Helly type theorems to line approximations through a finite number of parallel line segments is detailed in part by Yaglom and Boltyanskii [3,p.20]. A similar discussion is given here as an aid to understanding the main results of the next section.

Theorem 5. If n ($n \geq 4$) parallel line segments are given in the plane, with the condition that any three of the segments have a common line transversal; then there is a line transversal intersecting all of the line segments.

Proof: If we establish an x-y coordinate structure on the plane in such a manner that the y-axis is parallel to the given line segments; then the coordinates of the endpoints of the i-th segment would be (x_i,y_i') and (x_i,y_i''). Note the upper endpoints with the double prime and the lower endpoints with a single prime, i.e., $y_i' < y_i''$.

Any line in the x-y coordinate plane has a representation of $y = kx + b$. Thus knowledge that $y = kx + b$ intersects the segment $[(x_i,y_i'),(x_i,y_i'')]$ is expressed as $y_i' \leq kx_i + b \leq y_i''$. Since a line is uniquelly determined by the values k and b; we note that for a given x_i, the set of values of k and b that satisfy $y_i' \leq x_ik + b \leq y_i''$ precisely describe the collection of lines that intersect the segment above the horizontal coordinate x_i. This collection of k and b values is representable as a slab between two parallel lines in a k-b coordinate plane. See figures 3 and 4, and note the line $y = k_0x + b_0$ in the x-y plane is associated with the point (k_0,b_0) in the k-b plane. That is, $y_i' \leq k_0x_i + b_0 \leq y_i''$ implies that (k_0,b_0) is between the parallel lines $b = -x_ik + y_i'$ and $b = -x_ik + y_i''$ in the k-b plane. Observe that the point (k_0,b_0) would be in the slab established by the segment above the x_2 value, $(k_0,b_0) \in \{(k,b): y_2' \leq x_2k + b \leq y_2''\}$; and it would not be in the slab established by the x_1 value, $(k_0,b_0) \notin \{(k,b): y_1' \leq x_1k + b \leq y_1''\}$.

The hypothesis of the theorem states that any three of the parallel line segments are such that some line intersects the three. Since the lines through any given segment are identified as a slab in the k-b plane, this hypothesis translates to say that the n slabs in the k-b plane are such that any three have a common point. Employ Theorem 4 to conclude that the slabs all must contain some point (k^*,b^*). Thus the line $y = k^*x + b^*$ intersects all of the segments.

We can extend the above concepts to more complicated approximations. That is, the x_i values and the segments above the x_i values could be taken to be infinite in number. For example, a line approximation of a function over an interval. Refer to figure 5 and note that the above theorem in its extension would say that f(x), known to be between $\ell(x)$ and u(x) over the interval [a,b], has a line intersecting the three segments $[(x_1,\ell(x)),(x_1,u(x_1))]$, $[(x_2,\ell(x_2)),(x_2,u(x_2))]$, and $[(x_3,\ell(x_3)),(x_3,u(x_3))]$.

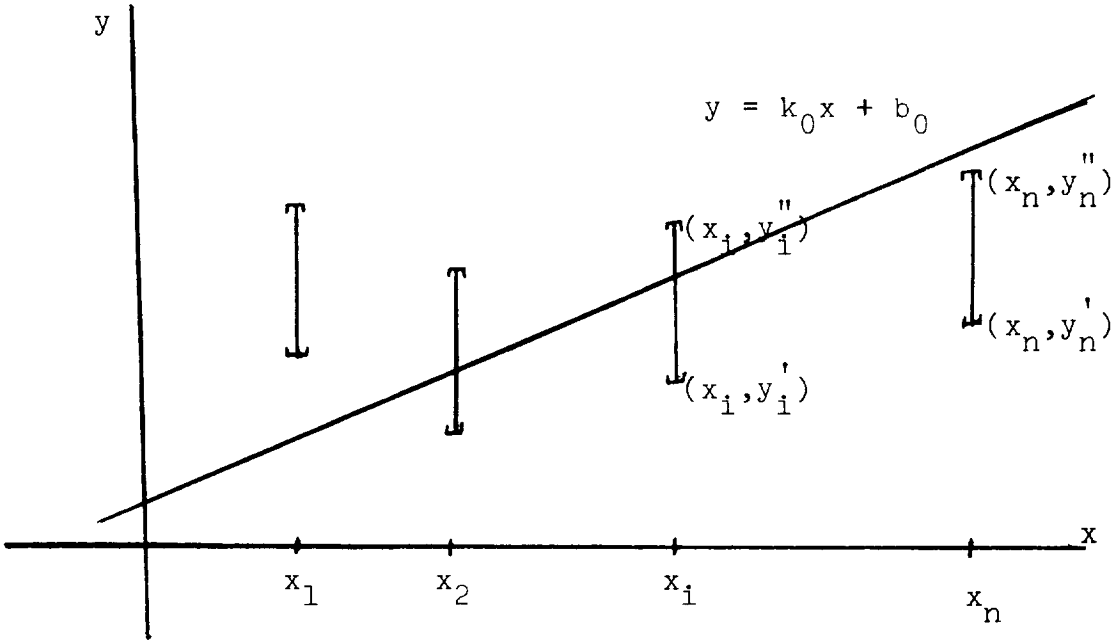

Figure 3

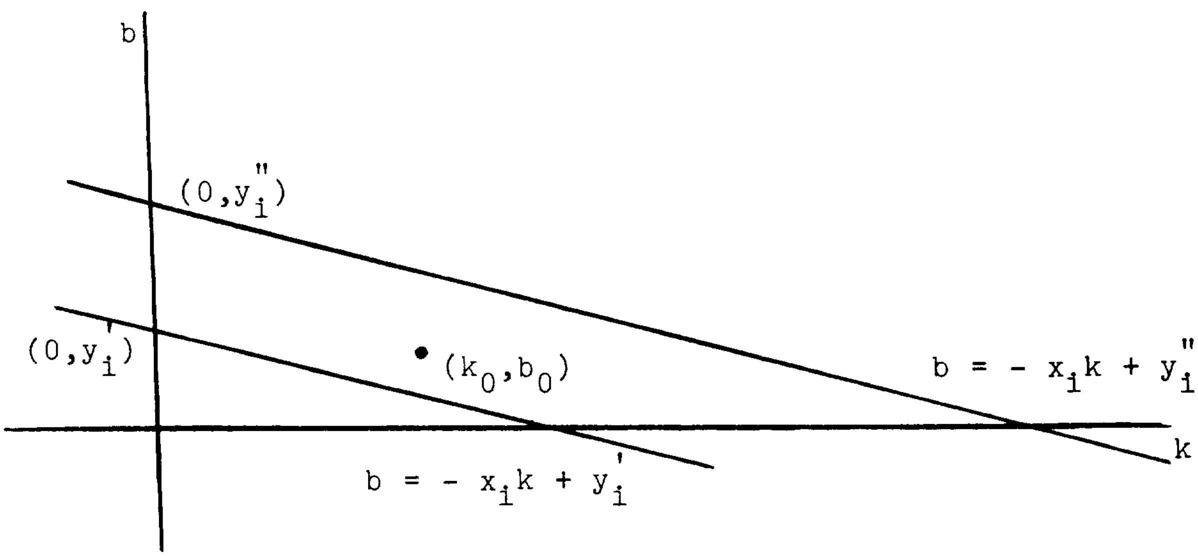

Figure 4

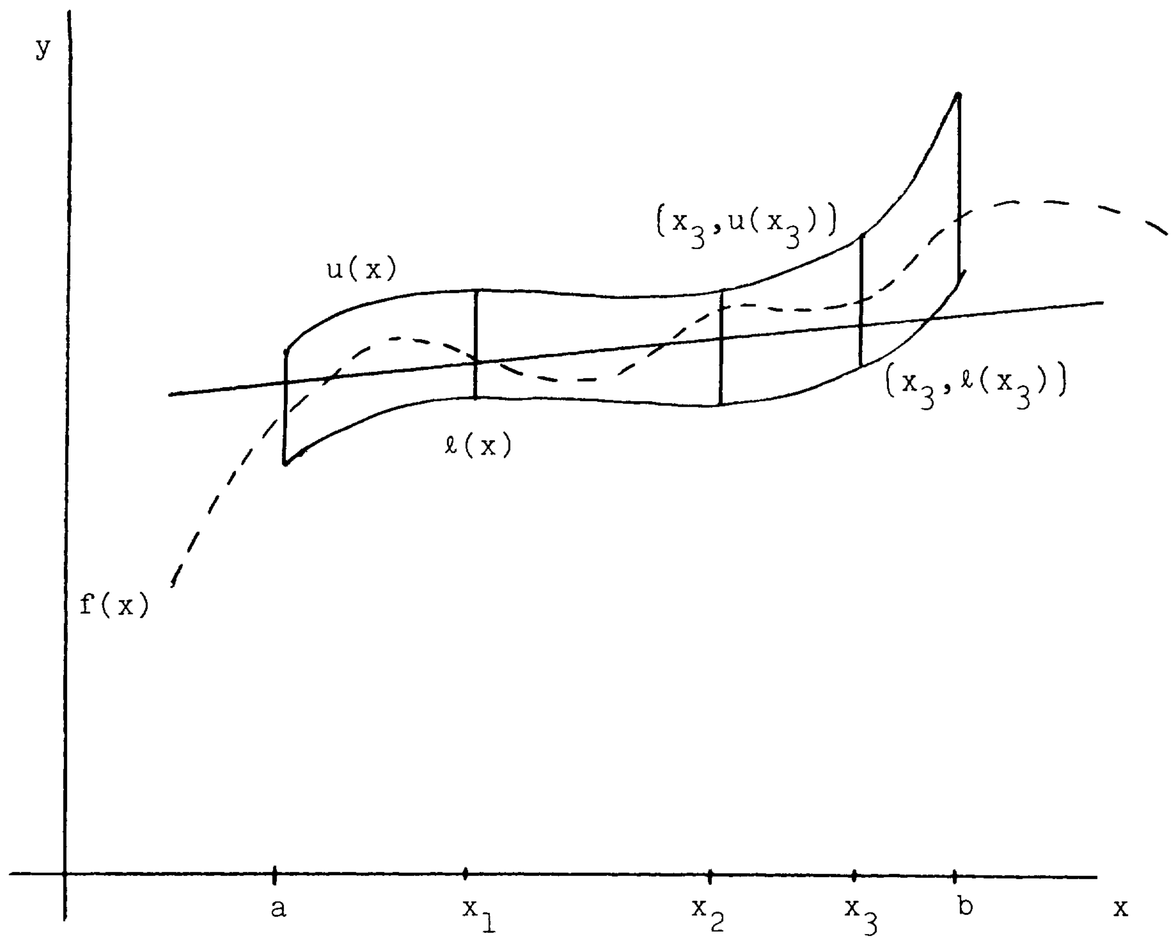

Figure 5

Polynomial approximations are also connected with an extension of Theorem 5. A polynomial of k-th degree, $\rho(x) = a_0+a_1x^1+\ldots+a_kx^k$, is uniquelly determined by its k + 1 coefficients. Thus the graph of the polynomial $\rho(x)$ intersects the segment $[(x_i,y_i'),(x_i,y_i'')]$ if and only if $y_i' \leq a_0+x_ia_1+\ldots+x_i^ka_k \leq y_i''$. The collection of a_j $(j = 0,1,\ldots,k)$ values, related to polynomials that in their graph intersect a segment $[(x_i,y_i'),(x_i,y_i'')]$, can be associated with the slab in Euclidean k + 1 space. The slab is between the two parallel hyperplanes $a_0+x_ia_1+\ldots+x_i^ka_k = y_i'$ and $a_0+x_ia_1+\ldots+x_i^ka_k = y_i''$ in a k + 1 dimensional space with coordinate variables $a_0,a_1,\ldots,a_k$. Thus f(x) in figure 5 admits a polynomial approximation of degree k, between $\ell(x)$ and u(x), if and only if for any k + 2 points $x_1,\ldots,x_{k+2}$ the segments $[(x_i,\ell(x_i)),(x_i,u(x_i))]$, $i = 1,\ldots,k+2$, are such that the graph of some polynomial of degree k intersects the segments.

This analysis leads to the following general theorem.

4. The General Theorem.

Theorem 6. Given a class of linearly independent approximation functions $\Phi_1(\vec{x}),\Phi_2(\vec{x}),\ldots,\Phi_n(\vec{x})$ and a function $f(\vec{x})$ of n variables $(x_1,\ldots,x_n)$, known to be within bounds $\ell(\vec{x})$ and $u(\vec{x})$, $\ell(\vec{x}) \leq u(\vec{x})$, over a domain D. Then $f(\vec{x})$ admits an approximation $L(A',\vec{x}) = \sum_{j=1}^{k} a_j'\Phi_j(\vec{x})$ within the bound functions over the domain D if and only if for any k + 1 points of D, $\vec{x}_1,\ldots,\vec{x}_{k+1}$, there is some set of values $A = \{a_1,\ldots,a_k\}$ such that $\sum_{j=1}^{k} a_j\Phi_j(\vec{x})$ intersects the k + 1 closed intervals $[(\vec{x}_i,\ell(\vec{x}_i)),(\vec{x}_i,u(\vec{x}_i))]$, $i = 1,2,\ldots,k+1$, in Euclidean n + 1 space.

Proof: The collection of functions $L(A,\vec{x})$ that intersect the interval above $\vec{x}_i$ is described by the condition:

$$\ell(x_i) \leq a_1\Phi_1(\vec{x}_i)+\ldots+a_k\Phi_k(\vec{x}_i) \leq u(\vec{x}_i)$$

Associate with $L(A,\vec{x})$ the point $A = (a_1,\ldots,a_k)$ in Euclidean k space. Then $L(A,\vec{x})$ intersects the interval above $\vec{x}_i$ if and only if A is an element of the closed slab in E^k space between the following two parallel hyperplanes:

$$\Phi_1(\vec{x}_i)a_1+\ldots+\Phi_k(\vec{x}_i)a_k = u(\vec{x}_i)$$

$$\Phi_1(\vec{x}_i)a_1+\ldots+\Phi_k(\vec{x}_i)a_k = \ell(\vec{x}_i)$$

The hypothesis that any k + 1 points $\vec{x}_1,\ldots,\vec{x}_k$ are such that some $L(A,\vec{x})$ intersects the intervals above $\vec{x}_1,\ldots,\vec{x}_k$, translates to say that any k + 1 of the slabs have a common point. Thus Theorem 4 implies that all of the slabs intersect in a common point $A' = (a_1',\ldots,a_k')$. Restated in terms of approximation functions, we see that $L(A',\vec{x}) = \sum_{j=1}^{k} a_j'\Phi_j(\vec{x})$ intersects all of the intervals above points $\vec{x}$ in the domain D.

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