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ENGINEERING ASPECTS OF CONTROL SYSTEM DESIGN VIA THE "DIRECT METHOD" OF LIAPUNOV

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TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF ILLUSTRATIONS

Page

ENGINEERING ASPECTS OF CONTROL SYSTEM DESIGN
VIA THE "DIRECT METHOD" OF LIAPUNOV*

Richard V. Monopoli, Ph.D.

The University of Connecticut, 1965

A control system synthesis procedure for linear or nonlinear, time varying, single-input, single-output plants is developed into a useful engineering design technique. In contrast to many others, it can be applied systematically even though plant parameter variations are large and rapid. Controller design is based on Liapunov's "direct method." It results in control action which guarantees that the plant output approaches the output of a model reference. The model is such that its output for a given reference input is the desired plant output. Information required for design is a knowledge of the plant equations, the form of its nonlinearities, and the bounds on its parameter variations. It is usually necessary to assume that the control signal magnitude is unconstrained.

Problems entailed in practical application of the synthesis technique are investigated, and some solutions are found. Among the most serious problems are plant gain saturations, transducer noise, and disturbance inputs.

Methods are developed which allow the technique to be used for a class of plants with soft saturation gain characteristics, under any operating conditions, and for plants with hard saturation gain characteristics with some restrictions on operating conditions.

Though transducer noise cannot be eliminated, exact and approximate techniques are developed which substantially reduce its undesirable effects on system performance. The former include techniques which obviate the use of higher-order plant output derivatives in generating the control signal. One of these, called the reduction-of-order technique, allows design to be based on a lower-order equation than the plant equation. It is applicable to linear plants with slowly varying parameters. The approximate techniques are based on approximating system equations by neglecting instrument dynamics, and using approximate values for certain signals and controller parameters. Designs using the approximate techniques yield acceptable performance if initial conditions, and the magnitude and power spectral density of input signals are suitably restricted.

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It is shown that the synthesis technique does not lend itself to designing for a specific disturbance response, but under certain conditions disturbance rejection can be guaranteed. However, if transducer noise is present, this guarantee of disturbance rejection is at the expense of an increased noise level into the plant.

An extension of the technique is introduced which allows designing for a specified convergence time, i.e. the time required for the plant output to reach that of the model if the system is started with different initial conditions for each. It is shown that results pertaining to convergence time design apply as well to the design of a class of quasi optimal systems. Systems designed using these results are shown to achieve performance closer to the true optimal than those designed using a previously reported technique.

The techniques developed in the report, shown to be effective by computer simulation, enhance the utility of the synthesis procedure in practical control problems.

CHAPTER I

INTRODUCTION

Statement of the Problem (1.1.1)

Modern day plants, such as high-speed aircraft and missiles, operate in an extremely wide range of environmental conditions. As a result, plant parameters undergo large and rapid variations during operation. Control systems for these plants are required, but in some cases design techniques used in the past are inadequate for such systems because of the nature of the parameter variations.

"Classical" feedback techniques for linear systems, sometimes called passive adaptive techniques[1*], may not be applicable if parameter variations are either too large or too rapid or both. An approach to solving the control problem for plants with large parameter variation is the use of so called active self-adaptive techniques[2-7]. In this approach, certain controller parameters are adjusted to compensate for changes in plant parameters. Though successfully applied to the large parameter variation problem[3,4,8], these techniques have several disadvantages. First, it is often difficult, if not impossible, to analytically determine the stability properties of the adaptive loop[9,10]. Next, plant identification schemes requiring complicated instrumentation are necessary. And finally, instrument noise limits application of the technique to plants with relatively slow parameter variations[11].

The controller synthesis procedure developed in this report has been shown to apply to linear plants with large and rapid parameter variations[12,13], and to nonlinear rapidly time varying plants as well[14,15]. The procedure is based on Liapunov's "direct method". Prior to its introduction, there were no design techniques suitable for plants with rapid parameter variations, and no systematic procedures for the design of controllers for nonlinear plants. Thus, this technique helps to fill a very definite gap in control system theory.

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*Superscript numbers in text refer to references listed in Bibliography.

Consideration is restricted to controller design for single-input, single-output plants as shown in figure 1-1. The design technique is generally valid only if the control signal magnitude is unconstrained. The only information required for design, besides the knowledge of the differential equations describing plant behavior is the form of the nonlinearites and the bounds on the parameter variations. The design technique yields a nonlinear controller which insures that the plant states approach the states of the model reference. The model is stable, and generally of the same order as the plant. Its behavior is governed by a linear, constant coefficient differential equation, and its output is the desired plant output.

Nonlinear compensation schemes are not new, there being many instances of such schemes reported in the literature[16-19]. However, the design technique for the nonlinear controller being considered here has an important advantage over many of those cited. It is systematic, whereas many of the others are of a cut and try nature. In this technique, design is carried out in the time domain. The desired dynamic performance of the plant is achieved through specifications on the model. The form of the nonlinear controller is in turn completely specified by the relatively straightforward design procedure. In contrast, by other techniques, design is performed in the frequency domain and it is difficult at best to interpret requirements on the nonlinear compensating element in terms of performance specifications[20].

A major objective in this report has been to help fill the gap referred to above in practice as well as in theory, i.e. to develop the synthesis technique into one of practical engineering significance which may be applied successfully to design of controllers for the class of plants described. The extent to which this objective is achieved is described below.

The synthesis technique is generalized to include plants with types of nonlinearities commonly occurring in practice. It could not be made generally applicable to plants with a hard saturation gain, but it is shown that a suitable design for such plants may be found for a limited range of operating conditions.

An equation to design for a specified convergence time is derived for a second-order plant. Though similar results are not obtained for higher-order plants due to the complexity of the algebraic problem involved, it is shown how insight obtained from solution of the second-order problem is useful in decreasing convergence time for a third-order plant. Results obtained for convergence time design in the second-order case are shown to be directly applicable to design of quasi time optimal systems. When applied to one such system and compared to a design using previously existing techniques, the improvement in speed of response was on the order of two to one.

Transducer noise can lead to an excessive noise level into the plant. Several very effective techniques are developed for reducing this noise level. These techniques are based on deemphasizing or entirely eliminating higher derivatives of the plant output in the control signal. An additional

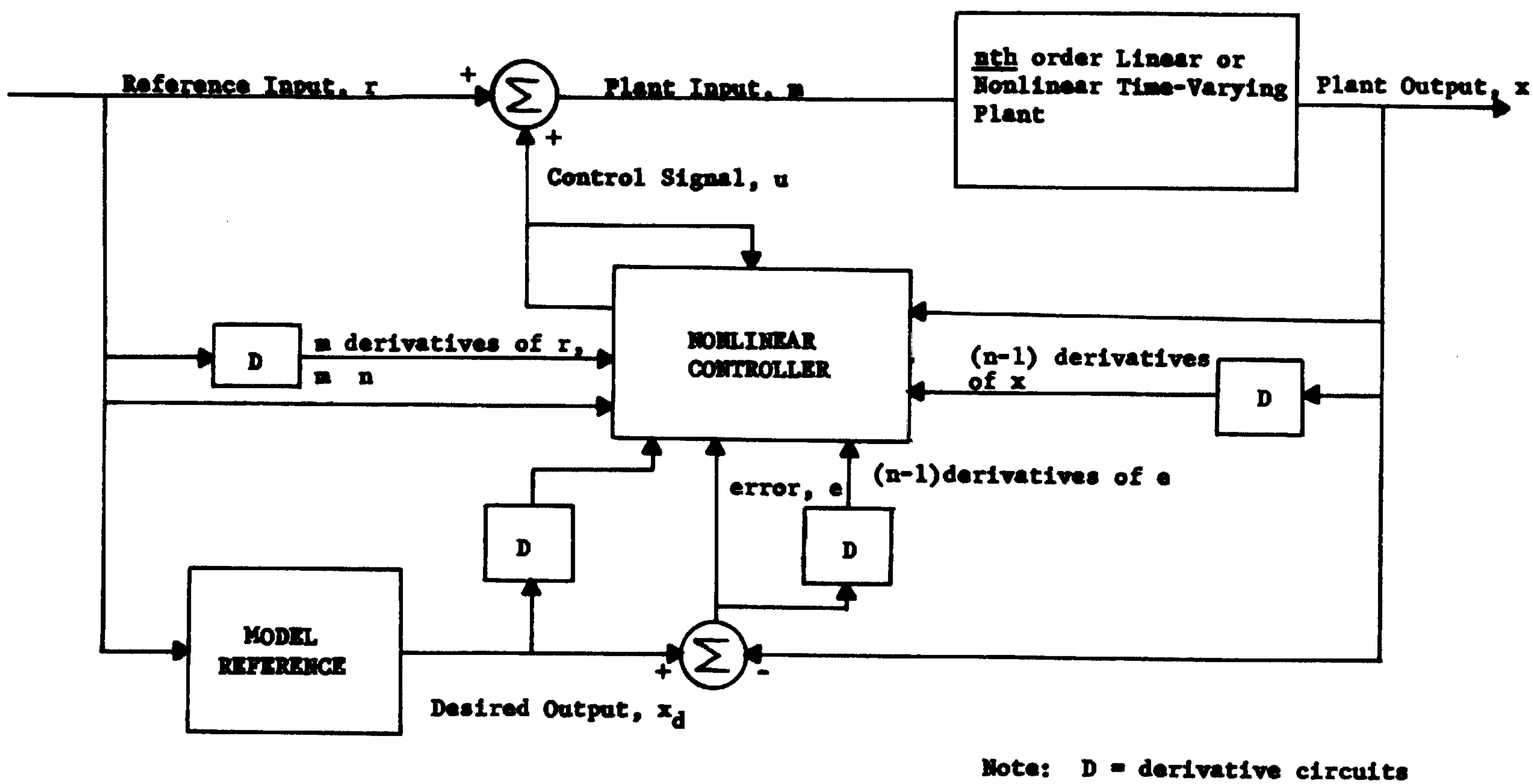


Figure 1-1: General System Configuration

technique for reducing noise power into the plant, based on lowering the gain in part of the control signal, may be employed at the expense of accuracy in tracking low level reference inputs.

Problems arising from neglecting instrument dynamics in the design of a controller for pitch axis stability augmentation of the X-15 manned re-entry vehicle are studied. With far out complex instrument poles neglected in design, an instability can be excited by any combination of initial conditions, reference input, and disturbance signals that is too large. The reason for this is that the closed loop system gain is a nonlinear function of these signals. However, it is shown that an adequate design can always be achieved if linear compensating networks are used to move the neglected poles far enough to the left in the complex s plane. Other techniques to help minimize this stability problem are also presented. Their use requires that some accuracy be sacrificed in tracking low level or rapidly varying reference inputs.

Before the development of the synthesis technique is presented, some brief comments pertaining to the "direct method" and control system synthesis techniques based on it are made in the following section. The way in which the report is organized is discussed in section 1.3.1.

Liapunov's "Direct Method" and its Application to Control System Synthesis (1.2.1)

Alexander Mikhailov Liapunov was a Russian mathematician who presented a conceptually new approach to the theory of stability of dynamic systems. His work, published in a Russian journal in 1892, was later translated to French in 1907, and reprinted in America in 1949[21]. Two approaches to the stability problem were taken by Liapunov, one quantitative and the other qualitative, referred to as the "first method" and the "second" or "direct method"* respectively. In the "first method," the study of stability proceeds from an explicit solution of the equations of motion for the system. However, it is the "direct method" which offers the more general and powerful approach to the stability problem, and in fact, ". . . . has achieved virtual preeminence in the Soviet Union as the principle mathematical tool in tackling linear and nonlinear stability problems of the most varied type, particularly in the theory of control systems."[22]

The "direct method" considers the stability of differential equations when the form of the equation is known, but not explicit solutions. It is actually a qualitative approach to the stability problem rather than a

*Appendix A includes definitions and theorems required for an understanding of the "direct method."

systematic method. Though qualitative, an ingenious user of the approach is able to gain much useful knowledge about the stability of the dynamical system under investigation. In the approach, a "fictitious" energy function is employed. This function, named a "Liapunov function," plays a role similar to that of the true energy function of a stable physical system, i.e. it assumes its minimum value at the equilibrium state, and its time rate of change is negative for all possible states of the system except the equilibrium, where it is zero.

Though the literature is replete with material involving the "direct method," little of it is directed toward the synthesis of control systems. Much emphasis has been placed on analysis of stability problems and the search for Liapunov functions to use in such problems. Works dealing explicitly with the synthesis problem include those by Bass[23], Grayson[12, 13], Johnson[24,25], Nahi[26], and the author [14,15]. Of these all but references 12, 13, 14, and 15 employ the "direct method" to design optimum or quasi-optimum control systems. Grayson and the author use the "direct method" in the design of controllers which force linear or nonlinear time varying plants to behave in a desired manner.

Optimum design of systems has received considerable impetus of late largely due to the pioneering works of Pontryagin et al.,[27] and Bellman[28]. Bass first suggested a merger of the "direct method" and optimization theory. A detailed presentation of his utilization of the "direct method" in achieving a quasi optimum control system design is also included in reference 22. His design was restricted to linear time invariant plants with the control variables subject to magnitude constraints. Though the method of design does not yield a true optimum, it has the advantage of leading to a system in which only linear feedback preceeds a simple switching type nonlinearity, and one which is guaranteed to be stable. Johnson extends this method by generalizing the cost function, and treats the quasi time optimal, quasi fuel optimal and quasi energy optimal problems in detail. Nahi presents a method of design which leads to true time optimal systems for a certain class of problems.

Grayson employed the "direct method" in a design context for the control of linear plants with large, rapid and bounded parameter variations. A model reference is employed and the design leads to a nonlinear controller which is not objectionably complex, a charge sometimes levelled at active adaptive systems. Though not related to the optimization problem, this design is roughly speaking, an outgrowth of the Bass design. In this case, however, the control variable is unconstrained and has a magnitude which is a function of the size of the parameter variations. The model is used to give the system a reference for desired plant behavior. The difference between the plant and model outputs is defined as the error. A vector differential equation in the error and its derivatives is obtained by subtracting the plant equation from the model equation. Then a quadratic form Liapunov Function of the error states is formed. The time derivative of this Liapunov Function is maint[illegible]d negative definite by selecting the control variable to have sufficient magnitude and the correct sign. Consequently, plant states are forced to approach those of the model. Though Grayson's work is theoretically appealing, it fails to treat the engineering design problems which arise in practice such as plant nonlinearities, transducer noise, disturbances, instrument dynamics, and design for a specified convergence time. These problems are dealt with in this report.

Organization of Report (1.3.1)

Chapter II, sections 2.1.1 through 2.2.3, includes a statement of the general problem and the controller design procedure for linear time-varying plants. This material is based on the technique given in references 12 and 13, but modifications are introduced which make it more attractive from an engineering point of view. The modifications make it possible to avoid impulses in the control signal in plants with zeroes, and also to reduce the gains required in the control signal. The advantages attendant to these modifications are examined in section 2.2.2 by comparison to the previously reported technique. A modification of the technique necessary for plants without integrators is introduced in section 2.2.3. Sections 2.3.1 through 2.3.3 treat the extension of the controller design technique to include a wide class of nonlinear plants. Of particular importance in this category is the problem of gain saturation exhibited to some degree by all physical plants. The technique can sometimes be applied to nonlinear plants even though the exact form of the nonlinearity is not known. This can be done if a bound on the argument of the nonlinear function can be determined from physical considerations. In these cases it is only necessary to know that the nonlinearity lies within certain bounds. Since this is often the case in physical problems, this aspect of the design technique is particularly appealing.

In section 2.3.3 several examples dealing with the application of the design technique to nonlinear plants are given. In section 2.4.1 the problems of disturbance inputs and transducer noise are examined from a theoretical viewpoint. There it is shown that generally the method does not lend itself to designing for a specified disturbance response, but under certain conditions disturbance rejection can be guaranteed. The interrelationship between disturbance rejection and transducer noise is discussed, and a full treatment of this problem from an engineering viewpoint in included in Chapter V.

Chapter III introduces a technique for including convergence time as part of the design problem. The close relationship between the convergence time problem and the quasi time optimal control problem is discussed. A change in the definition of quasi optimal from that given in reference 25 is introduced. It is shown that a design for a quasi time optimal system based on the revised definition results in performance closer to the true time optimal than the design based on the original definition.

In Chapter IV and V an investigation is made into the design problems arising due to transducer noise and instrument dynamics. Several techniques are developed in IV for eliminating plant state signals from the control law. By such elimination, especially of higher order plant states, those most corrupted by noise, the problems associated with transducer noise can be reduced. In section 4.3.1 it is shown how the model matrix can be manipulated to aid in the noise reduction problem. Section 4.5.1 deals with the very powerful reduction-of-order technique applicable to linear, slowly time varying plants with zeroes. A theorem pertaining to the control of such plants is given in section 4.5.1. The reduction-of-order technique is an ideal solution to the transducer noise problem in that it allows controller design to proceed from a lower order description of the plant, thereby avoiding the need for some of the higher-order plant states completely. In section 4.6.1 the extension of the reduction-of-order technique to plants without zeroes is considered.

In Chapter V, the reduction-of-order technique is applied to an engineering problem, the design of a controller for pitch axis stability augmentation of the X-15 manned re-entry vehicle, which has parameter variations on the order of a thousand to one. The advantages of the design using the reduction-of-order technique are brought out through a comparison with a design which does not use it. An extensive analog computer study of transducer noise, disturbance response, and instrument dynamics problems is made. Through this study, design difficulties are clarified and some solutions to these problems are obtained.

CHAPTER II

CONTROLLER SYNTHESIS TECHNIQUE

The General Problem (2.1.1)

The controller synthesis technique presented in this report is applicable to single input, single output plants as shown in figure 1-1 which can be described by the set of n first order differential equations of the form

$$\dot{x}_i = x_{i=1} \qquad (i = 1, 2, \ldots\ldots n-1) \tag{2-1}$$

$$\dot{x}_n = f(x_1, x_2 \ldots x_n, u, u^1, \ldots u^m, r^1, \ldots\ldots r^m, t)$$

or the equivalent vector differential equation form

$$\underline{\dot{x}} = A\underline{x} + B(\underline{u} + \underline{r}) + \underline{f}(\underline{x}, \underline{u}, \underline{r}, t) \tag{2-2}$$

where r is the reference input, u the control signal, x_i the plant output, and x_{i+1} the ith time derivative of x_1. The sum of u and r forms the single input signal to the plant. Superscripts on u and r denote derivatives with respect to time. The presence of these derivatives allows for the possibility of zeroes in the transfer function representation of the plant. Unknown parameter variations prohibit use of a transformation to remove the zeroes. From physical considerations, $m < n$. The first two terms on the right hand side of (2-2) include all linear terms in $\underline{x}$, $\underline{u}$, and $\underline{r}$. The matrices A and B are nxn whose elements, in general, may be time varying in an unknown fashion within known, finite bounds. The function $\underline{f}$ includes all nonlinear terms in $\underline{x}$, $\underline{u}$, and $\underline{r}$. Equations (2-1) and (2-2) represent the open loop plant unmodified by external linear feedback. Reasons for introducing linear feedback prior to generating the control signal, u, with the nonlinear controller will be discussed subsequently.

The form of (2-1) leads to an A matrix of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & 1 \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_n \end{bmatrix} \qquad (2\text{-}2\text{-}a)$$

and a B matrix, all of whose elements are zero except for those in the last row,

$$B = \begin{bmatrix} & & \bigcirc & \\ b_1 & b_2 & \cdots\cdots & b_n \end{bmatrix} \qquad (2\text{-}2\text{-}b)$$

The importance of the form of A and B to the design procedure will become evident in the discussion which follows.

The problem to be considered is under what conditions the plant of (2-2) can be made to behave like a linear model reference described by the vector differential equation

$$\underline{x}_d = A_o \underline{x}_d + B_o \underline{r} \qquad (2\text{-}3)$$

where $\underline{x}_d$ is a column n vector with the model output and its first n-1 time derivatives as its components, A_o is a stable nxn constant matrix of the same form as A, and B_o is an nxn constant matrix of the same form as B. The elements of the last rows of A_o and B_o are a_{oi} and b_{oj} respectively, where i and j take on values from i through n. The reference input r is as defined for (2-1) and (2-2).

The problem as it relates to linear time varying plants is treated first; and then plants with various forms of nonlinearities are considered.

Linear Time-Varying Plants (2.2.1)

For a linear time-varying plant, $\underline{f} \equiv \underline{0}$ in (2-2). Let the A and B matrices be separated into constant and time varying parts as follows:

$$A = A_o + \Delta A(t) \tag{2-4-a}$$

$$B = B_o + \Delta B(t) \tag{2-4-b}$$

where A_o and B_o are defined as in (2-3)

It follows from these definitions and what has been said previously about the forms of A, B, A_o, and B_o that $\Delta A(t)$ and $\Delta B(t)$ have all zero elements except for those in their last rows. Let these elements be denoted by α_i and β_i where $\alpha_i = a_i - a_{oi}$ and $\beta_i = b_i - b_{oi}$.

The controller design technique proceeds by defining an error vector differential equation as the difference between (2-3) and (2-2), i.e.

$$\dot{\underline{e}} = A_o\underline{e} - (B\underline{u} + \Delta B\underline{r} + \Delta A\underline{x}) \tag{2-5}$$

where $\underline{e} = \underline{x_d} - \underline{x}$, the error vector.

When $\underline{u} = \underline{r} = \underline{x} = \underline{0}$, (2-5) reduces to the homogeneous equation

$$\dot{\underline{e}} = A_o\underline{e} \tag{2-6}$$

Since A_o is stable by assumption, then a Liapunov function of quadratic form exists for (2-6), and its equilibrium, $\underline{e} = \underline{0}$, is asymptotically stable in the whole. Let this Liapunov function be

$$V(\underline{e}) = \underline{e}^T P \underline{e} \tag{2-7}$$

where P is a positive definite matrix to be determined. The time derivative of, (2-7) is

$$\dot{V}(\underline{e}) = \underline{e}^T(A_o^T P + PA_o)\underline{e} \tag{2-8}$$

In (2-8), let

$$A_o^T P + PA_o = -Q \tag{2-9}$$

where Q is chosen to be a symmetric positive definite matrix. Criteria for selecting the elements of the Q matrix are considered in Chapters III and IV relative to convergence time and the transducer noise reduction problem. At this time, it suffices to arbitrarily chose Q as the identity matrix, I, as is usually done for convenience in the literature. Because A_o is stable, the P matrix found from the solution of (2-9) will be positive definite, and symmetric since Q is symmetric.

When the terms in parenthesis on the right hand side of (2-5) are not zero, an additional term appears in the time derivative of (2-7), such that

$$\dot{V}(\underline{e}, \underline{u}, \underline{r}, \underline{x}, t) = -\underline{e}^T Q\underline{e} - 2\underline{e}^T P(B\underline{u} + \Delta B\underline{r} + \Delta A\underline{x}) \tag{2-10}$$

This form of $\dot{V}$ is a consequence of the fact that P is symmetric.

The control problem now reduces to determining when and how a control vector, $\underline{u}$, can be generated which will cause the inequality

$$\underline{e}^T P(B\underline{u} + \Delta B\underline{r} + \Delta A\underline{x}) \geqq 0 \tag{2-11}$$

to be satisfied. If (2-11) can be satisfied by a suitable choice of $\underline{u}$, then (2-5) will be asymptotically stable in the whole under all variations of the parameters, i.e. the plant will be forced to track the model. The basis for choosing $\underline{u}$ is considered in detail below. In Appendix A is a discussion pertaining to the mathematical justification of the synthesis technique developed here.

To examine this problem in more detail, it is necessary to write (2-11) in expanded form, which is

$$\left(\sum_{i=1}^{n} p_{in}e_i\right) \left[\sum_{j=0}^{m} (b_{j+1}u^j + \beta_{j+1}r^j) + \sum_{k=1}^{n} \alpha_k x_k\right] \geq 0 \qquad (2\text{-}12)$$

where p_{in} is an element of the last column of the P matrix. This form, which will be referred to as the "factored" form, is a consequence of the fact that B, ΔA, and ΔB have nonzero elements in their last rows only. The term "factored" refers to the fact that all of the terms involving components of $\underline{u}$, $\underline{r}$, and $\underline{x}$, are multiplied by the same factor, the summation in $p_{in}e_i$. It is this fact which makes it possible to generate a $\underline{u}$ which will cause (2-11) to be satisfied. Thus, one condition required for generating $\underline{u}$, is that A, B, A_o, and B_o have the forms previously specified. Before discussing how $\underline{u}$ is generated, further consideration is given to other conditions which must be met.

In (2-12) p_{1n} and b_{m+1} are factored yielding

$$p_{1n}b_{m+1}\left(\sum_{i=1}^{n} \rho_{in}e_i\right)\left[u^m + U(m-1)\sum_{j=0}^{m-1} c_{j+1}u^j + \sum_{\ell=0}^{m} d_{\ell+1}r^\ell + \sum_{k=1}^{n} g_k x_k\right] \geq 0 \quad (2\text{-}13)$$

where

$$\rho_{in} = \frac{p_{in}}{p_{1n}} \qquad g_k = \frac{\alpha_k}{b_{m+1}}$$

$$c_{j+1} = \frac{b_{j+1}}{b_{m+1}} \qquad U(m-1) = \begin{cases} 1 & \text{for } (m-1) \geq 0 \\ 0 & \text{for } (m-1) < 0 \end{cases}$$

$$d_{\ell+1} = \frac{\beta_{\ell+1}}{b_{m+1}}$$

It is shown in Appendix B that $p_{1n} > 0$ for a diagonal Q matrix. A condition on b_{m+1} is evident from (2-13), i.e. $b_{m+1} \neq 0$. This is required in order that c_{j+1}, d_{j+1}, and g_k be finite. The coefficient b_{m+1} may be greater than or less than zero.

Consider the case when $b_{m+1} > 0$. Then the control signal must be such that (2-13) is non negative. A control law, from which a control signal can be generated, which keeps (2-13) ≥ 0 will be given, and the reason for the choice will follow. The control law is

$$u^m = [U(m-1) \sum_{j=0}^{m-1} |c_{j+1}|_m |u^j| + \sum_{\ell=0}^{m} |d_{\ell+1}|_m |r^\ell| + \sum_{k=1}^{n} |g_k|_m |x_k|] \text{ sign } \gamma \qquad (2\text{-}14)$$

where subscript m denotes maximum value and $\gamma = \sum_{i=1}^{n} \rho_{in} e_i$. Hereafter, γ will be referred to as the switching function.

The rationale for choosing this control law is that in causes u^m to have a magnitude which is greater than or equal to the magnitude of the sum of all of the other terms in the square brackets of (2-13) for all variations of the parameters c, d, and g. This being so, the sign of the square bracket term is determined by the sign of u^m. By giving u^m the sign of γ, (2-13) is made greater than or equal to zero.

The complete $\underline{u}$ vector, and hence the control signal u, can be generated by successive integrations of u^m. The control signal so generated will force the plant output to track the model output. In the control law given by (2-14), the terms within the braces will be referred to as the magnitude function, M, since it is composed of magnitudes of variables only, and also it determines the magnitude of u^m. The restriction previously imposed that the elements of the A and B matrices be finite is required in order that the coefficients in M be finite.

If $b_{m+1} < 0$, then u^m would be chosen of opposite sign from that of (2-14). If b_{m+1} varied between positive and negative values, then a requirement of the system would be an identification scheme to determine its sign so that u^m could be generated accordingly. In order to avoid this requirement, it is assumed throughout that $b_{m+1} > 0$.

Flügge-Lotz[29] has shown that for equations which have sign functions as forcing terms, solutions may not always exist. For this reason, the discontinuous sign function is replaced by a continuous function, this being either the saturation function[22,12] or the hyperbolic tangent[25] in previously reported work. The saturation function is defined as

$$\text{sat } k\gamma = \begin{cases} +1 & \text{for} \quad \gamma > 1/k \\ k\gamma & \text{for} \quad -1/k \leqq \gamma \leqq 1/k \\ -1 & \text{for} \quad \gamma < -1/k \end{cases} \qquad (2\text{-}15)$$

where $k > 0$. It can be made to approximate the sign function as closely as desired by choosing k large enough. An investigation of the results of Flügge-Lotz shows that solutions do exist in some cases. For example, in a plant described by

$$\ddot{x} + a\dot{x} + bx = C\text{sign}(x + d\dot{x}) \qquad (2\text{-}16)$$

in which $a > 0$ and $b > 0$, solutions exist if $C > 0$ and $d > 0$ which bring the system to one or the other of the "rest points," $\pm C/b$, in figure 2-1. Such solutions are unsatisfactory, however, if the desired final state is $x = \dot{x} = 0$. When $C < 0$ and $d > 0$, there is the possibility that the trajectory intersects the switching line at two successive points on the same side of the origin such as points P_2 and P_3 in figure 2-1. The second point, P_3, is called an "end point," and motion is undefined beyond such a point because motion cannot continue along either of the dashed curves P_3A or P_3B since the former pertains only for $x + d\dot{x} > 0$ and the latter for $x + d\dot{x} < 0$. A phase plane analysis using isoclines shows that "end points" do not result for $C > 0$ and $d > 0$. Because neither the "rest points" nor the "end point" situation is desirable, both are avoided by replacing the sign function with the saturation function of (2-15) throughout the remainder of this report.

Use of the saturation function avoids the problems discussed above, but it introduces another not previosuly discussed in the literature, i.e. asymptotic stability of (2-5) is no longer assured in the region $|\gamma| \leqq 1/k$. The reason for this is that $Mk\gamma$ may not have sufficient magnitude to maintain (2-13) ≥ 0. There are two practical consequences of this fact. First, tracking of the model by the plant may be poor for low amplitude reference inputs, r. A full discussion of this problem is included in 5.3.2. Second, limit cycles or constant steady state errors may develop near $\underline{e} = \underline{0}$. It is shown in Appendix C that such limit cycles or steady state errors can be confined to an arbitrarily small region about $\underline{e} = \underline{0}$ by making k large enough.

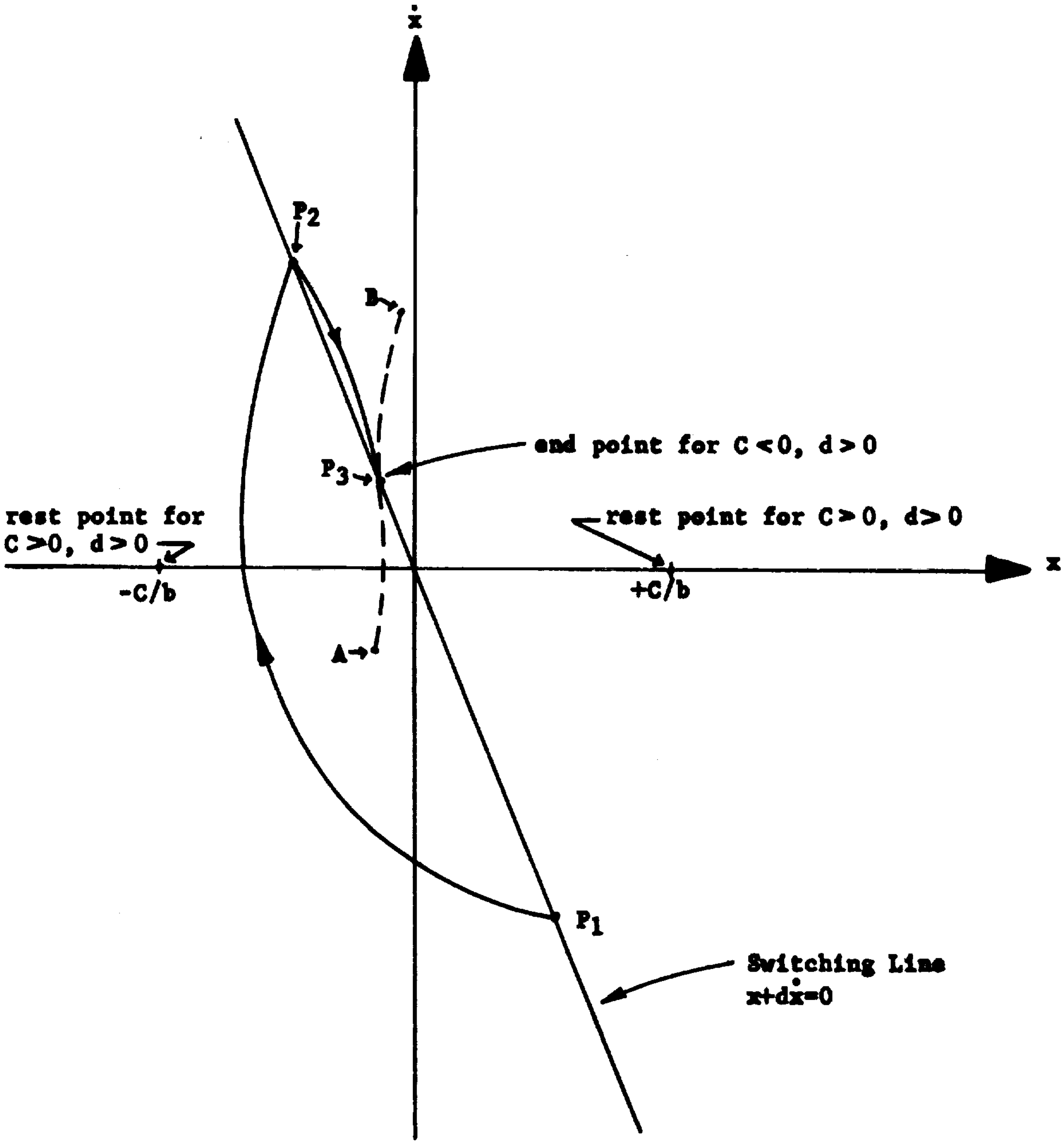


Figure 2-1: Trajectory With an End Point

For a given k, a conservative bound on the size of the region is established in terms of the components of $\underline{r}$ and $\underline{x}_d$. Such a region for the second order case is shown as the cross hatched area of figure 2-2. It is defined by the intersection of the regions $|\gamma| \leq 1/k$ and R(k). The latter region is defined in Appendix C.

Comparison to Previously Reported Technique (2.2.2)

The controller design technique for linear time-varying plants given in section 2.2.1 is quite similar to that presented in reference 12. However, several modifications can be introduced which should be pointed out. These modifications offer several advantages from an engineering design viewpoint.

The first modification is achieved by factoring b_{m+1}, as was done in (2-13), and by defining coefficients c, d, and g. If some relationship exists and is known between numerators and denominators of these coefficients, it may be used to advantage in selecting smaller values for the coefficients of (2-14). This leads to lower saturation level requirements in the controller amplifiers. For example, $|c_{j+1}|_{max}$ should be chosen as $|(b_{j+1})/(b_{m+1})|_{max}$ rather than $|(b_{j+1})|_{max}/|(b_{m+1})|_{min}$. In general it is sufficient to make the magnitude of u^m equal to the magnitude of the sum of all terms in the square brackets of (2-13) other than itself. All available information should be used to reduce the magnitude of u^m to this sufficient value.

Another modification applies to plants in which $m \neq 0$, i.e. derivatives of input signals appear in the equations. In this situation, the reference input r is not to be used as an input to the plant, but only to the model reference. Only the control signal u is to be applied to the plant input. In addition, the model should be such that its equation does not include derivatives of r. If both these conditions are satisfied, then derivatives of r do not appear in (2-14). This is especially important if r contains step functions for then impulses are avoided in generating u^m. If r is not an input to the plant, then $\Delta B\underline{r}$ of (2-10) and (2-11) is replaced by $B_o\underline{r}$, and the coefficients of r^j in (2-12), (2-13), and (2-14) must be changed accordingly.

Plants Without Pure Integrators (2.2.3)

If m = 0 in (2-1), and if the plant has no pure integrator, i.e. $a_1 \neq 0$ in the A matrix, a problem may arise due to the nature of the control signal given by (2-14). This problem is most easily examined by considering the reference input to be a unit step. As $\underline{e}$ approaches zero, u does too. However, since the plant has no pure integrator, a steady state input of unknown

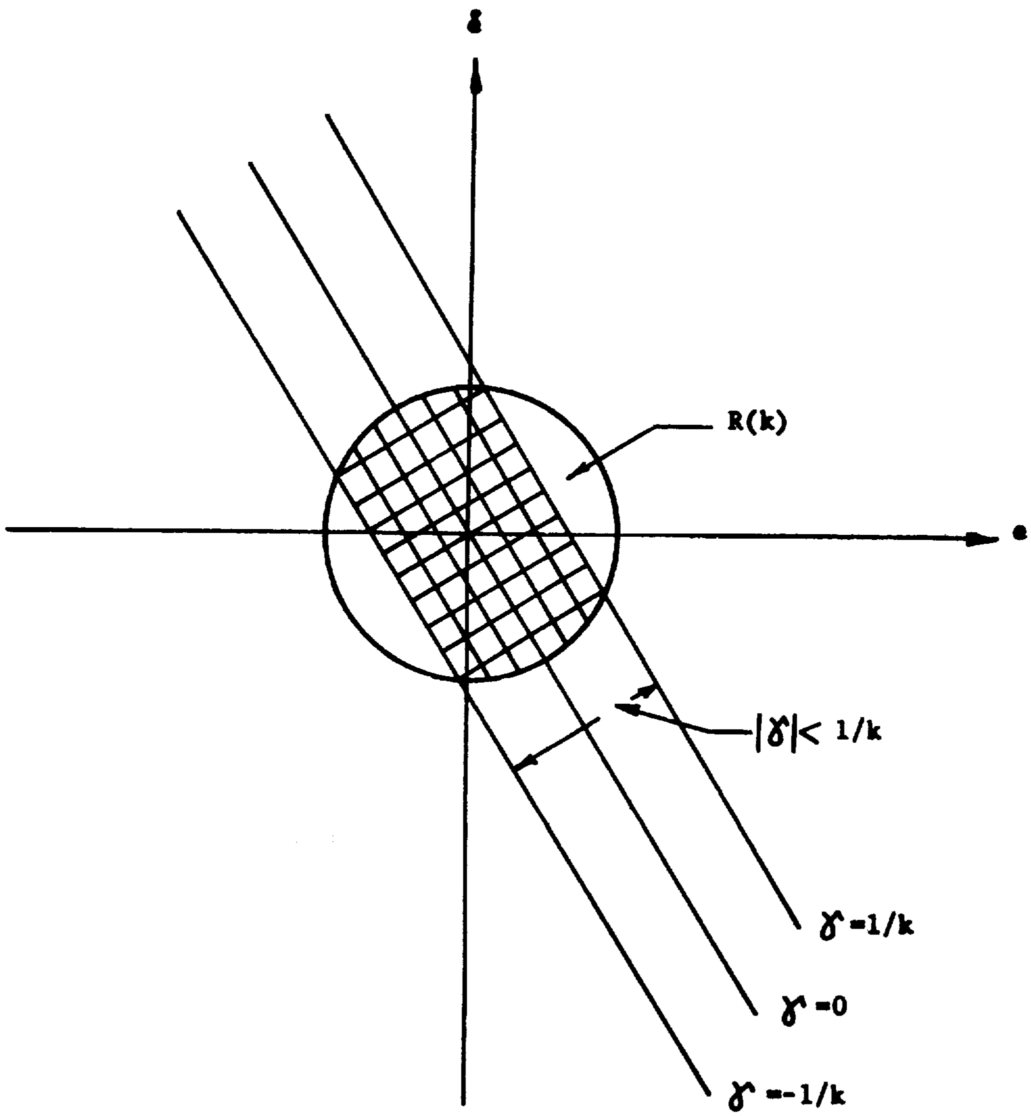


Figure 2-2: Cross Hatched Area Indicating Where $\dot{V}$ May Be Positive

magnitude is required to maintain $x_1 = 1$ in the steady state. This situation will lead to limit cycling or a constant steady state error. Note that this problem does not exist if $m \neq 0$, for then u^m goes to zero when $\underline{e}$ does, but a steady state value of u can develop which will maintain $x_1 = 1$.

In order to circumvent this problem, the following procedure is suggested. Rather than have only u as the plant input signal, let the input be $u + \int_0^t u dt$. With this input to the plant, the design procedure is unchanged. The only modification to be made is that the version of (2-14) for this case becomes

$$u = [|\int_0^t u dt| + |b_{01}/b_1|_{max}|r| + \sum_{k=1}^{n} |g_k|_{max}|x_k|] \text{ satk } \gamma \qquad (2\text{-}17)$$

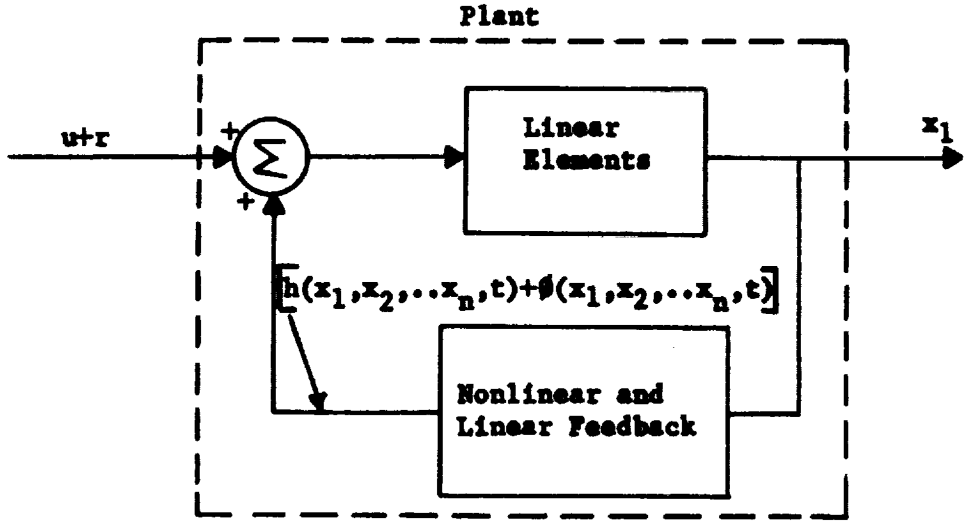
With the integral of u included as an input, a steady state value can develop at the input to the plant which maintains $x_1 = 1$.

Nonlinearities In The Feedback Path (2.3.1)

If $\underline{f} \neq 0$ in (2-2), then the design procedure as given for linear time varying plants must be modified somewhat in a way which depends on whether or not the nonlinear element is in the forward path or the feedback path. The two possibilities are depicted in figure 2-3. In either case, there is a requirement that $\underline{f}$ be a column vector with all components zero except the last, in order to obtain the "factored" form of (2-12).

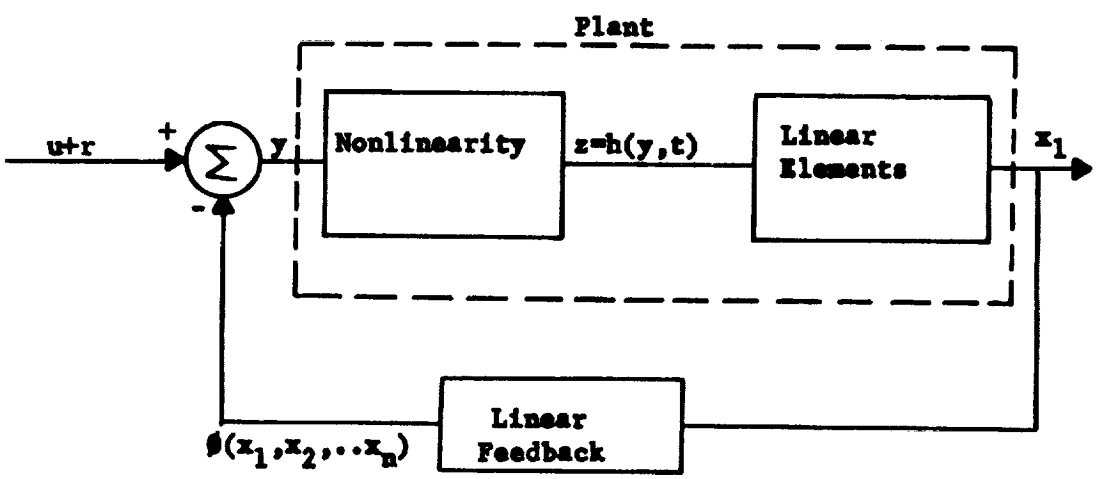
An important distinction between the two cases is the amount of information required about the nonlinearity. When the nonlinearity is in the feedback path, only bounds on $\underline{f}$ are required. If it is in the forward path, and $m \neq 0$, it is necessary to know bounds on partial derivatives of the nonlinear function with respect to all of its arguments. First, consideration is restricted to the case $\underline{f} = \underline{f}(\underline{x}, t)$, i.e. the nonlinearity is in the feedback path. In this situation, the error equation, analogous to (2-5) for the linear time varying case, becomes

$$\dot{\underline{e}} = A_o\underline{e} - [B\underline{u}+\Delta B\underline{r}+\Delta A\underline{x}+\underline{f}(\underline{x},t)] \qquad (2\text{-}18)$$



Note: ϕ is a linear function of plant states

a) Nonlinearities in the feedback path



b) Nonlinearities in the forward path

Figure 2-3: Two Classes of Nonlinear Plants

The procedure developed from (2-7) through (2-11) is again followed, so that now a $\underline{u}$ must be found such that

$$\underline{e}^T P\ [B\underline{u}+\Delta B\underline{r}+\Delta A\underline{x}+\underline{f}(\underline{x},t)] \geq 0 \tag{2-19}$$

Because $\underline{f}$ has all components zero except that in the last row, (2-19) expands to

$$p_{1n} b_{m+1} \gamma\ [u^m+U(m-1) \sum_{j=0}^{m-1} c_{j+1} u^j + \sum_{\ell=0}^{m} d_{\ell+1} r^\ell + \sum_{k=1}^{n} g_k x_k + h(x_1,x_2,\ldots x_n,t)/b_{m+1}] \geq 0 \tag{2-20}$$

where h, the component of the last row of the $\underline{f}$ vector, is a nonlinear time varying function. It is seen that (2-20) is in "factored" form as was (2-13). Therefore, with the same restrictions applied to the parameters as applied in the linear time varying case, it is again possible to choose u^m as in (2-14) provided the form of h is known and coefficients in h are bounded functions of time. For example, if

$$h = k_1(t)x_1^2+k_2(t)x_1x_2 \tag{2-21-a}$$

the following additional terms must be included in the braces of (2-14):

$$|k_1(t)/b_{m+1}|_m x_1^2+|k_2(t)/b_{m+1}|_m |x_1x_2| \tag{2-21-b}$$

where subscript m denotes maximum value.

From (2-21-b) it is seen that the requirements $|k_1(t)| < \infty$ and $|k_2(t)| < \infty$ must be imposed in order that it be possible to generate u^m with finite gain amplifiers. It is important to note that the exact form of h need not always be known. This is discussed in relation to example 2-1.

Nonlinearities In The Forward Path (2.3.2)

The following discussion pertains to nonlinearities in the forward path which precede the plant as shown in figure 2-3b. In the figure let

$$y = u + r + \sum_{i=1}^{n} \delta_i x_i \qquad (2\text{-}22)$$

where δ_i's are constants.

Again it is necessary to assume that the column vector $\underline{f}$ has all zero components except the last in order that the "factored" form result. This last component is designated as h(y, t). In contrast to the previous case, knowledge about partial derivatives of h(y, t) with respect to y and t are required here if $m \neq 0$. Therefore, attention is directed to the case when $m = 0$, i.e. no derivatives of z appear in the equations. It is assumed that h can be expressed as

$$h(y, t) = (h(y, t)/y)y = (h(y, t)/y)\,(u + y') \qquad (2\text{-}23)$$

where $y' = y - u$

If $m \neq 0$, and for discussion let $m = 1$, then it would be required that an expression of the form

$$\frac{dh(y,t)}{dt} = \frac{\partial h}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial t} = \frac{\partial h/\partial y}{y}(\dot{u}+\dot{y}') + \frac{\partial h}{\partial t} \qquad (2\text{-}23\text{-}a)$$

could be written where $\dot{u} + \dot{y}' = \dot{y}$. In order to derive the control law it would be necessary to know bounds on both partial derivatives. In addition, $(\partial h/\partial y)/y$ would have to be of one sign (analogous to b_{m+1} in the control laws for previous cases). Since this is much more information than has been assumed known about the plant, this case is not pursued further.

The procedure used leading to (2-11) is followed here with m = 0 to give the analogous equation for this case which is

$$\underline{e}^T P\ [\Delta B\underline{r}+\Delta A\underline{x}+\underline{f}(\underline{y},t)] \geq 0 \tag{2-24}$$

The scalar form of (2-24) is

$$p_{1n}(h/y)\gamma\ [u+y' + \sum_{i=1}^{n} \frac{\alpha_i}{h/y} x_i + \frac{\beta_1 r}{h/y}] \geq 0 \tag{2-25}$$

In (2-25) h/y has replaced b_{m+1} which appeared in (2-13) and (2-20). Therefore, the restrictions placed on b_{m+1} must also be placed on h/y, i.e. h(y, t)/y > 0 uniformly in t. Also the coefficients of the variables forming y' must be bounded. If the form of h is known as well as its maximum excursion with time, then u can be generated directly to satisfy (2-25) in the same manner indicated for generating it in (2-14). The general form of u will not be written out since it is quite obvious from what has been done previously, and also an example of this type is worked in the following section.

The restriction that h(y, t)/y > 0 rules out plants with a hard saturation gain characteristic. Since this is a common form of nonlinearity, it is well to consider what can be said about stability in such cases. In 4.2.2 it is shown that it is sometimes possible to determine an upper bound $U(\underline{r}, \underline{x_d})$ such that |u| < U. Thus, for a given plant saturation level, the model and reference input can be adjusted so that |u| does not exceed this level.

Generally it is not reasonable to expect a plant with a hard saturation gain to track the model since the magnitude of the input signal required may exceed the saturation level. It is important in such cases to consider the stability of the plant without regard to its ability to track.

A condition equivalent to hard saturation is that there is a magnitude constraint on u equal to the saturation level, S, i.e.

$$|u| \leq S \tag{2-26}$$

Since tracking properties are not under consideration, it is assumed that

$$\underline{r} = \underline{x_d} = 0.$$

(2-27)

Thus, $\underline{e} = -\underline{x}$

and stability can be investigated by substituting (2-26) and (2-27) into (2-25) to give

$$p_{1n}(h/y)\gamma \; [u - \sum_{i=1}^{n} \delta_i e_i - \sum_{i=1}^{n} \frac{\alpha_i}{h/y} e_i] \geq 0 \tag{2-28}$$

From this it is seen that stability can only be assured in the region where

$$\left| \sum_{i=1}^{n} (\delta_i e_i + \frac{\alpha_i}{h/y} e_i) \right| \leq S \tag{2-29}$$

This region may not be easy to determine in view of the fact that the parameters are time varying, but one always exists around the origin $\underline{e} = \underline{0}$.

If the parameters are slowly varying so that a transfer function representation of the linear part of the plant is valid, then a describing function analysis may be applicable if the control signal is generated as

$$u = S \text{ sat } k\gamma \tag{2-30}$$

rather than as the product of the magnitude function and the saturation function. This case is shown in figure 2-4 where P(s) is the transfer function for the linear part of the plant. If the product of P(s) and the feedback transfer function has suitable low pass characteristics, then a describing function analysis can be applied. Such an analysis, of course, is complicated by the parameter variations. In spite of this, it may still be possible to determine stability over the range of these variations.

Several Examples Using Nonlinear Plants (2.3.3)

To fix ideas, some examples involving nonlinear plants are worked out in detail. Examples used are taken from references 14 and 15. An example using a linear plant is taken in Chapter IV in connection with the transducer noise problem. The operation of the controller design for the nonlinear plants considered was checked by a simulation of the system either on an Electronic Associates Inc. PACE 231-R analog computer or the IBM 7040 digital computer.

In order to permit clarification of ideas while avoiding unnecessary complications all examples are taken as plants representable by second order nonlinear and time varying differential equations. Since this is so, a second order model reference suffices for all examples, and it is chosen to be

$$\underline{\dot{x}_d} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \underline{x_d} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \underline{r} \qquad (2\text{-}31)$$

If the A_o matrix defined by (2-31) is substituted into (2-9) and Q is chosen as the identity matrix, then solution of (2-9) yields

$$P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{bmatrix} \qquad (2\text{-}32)$$

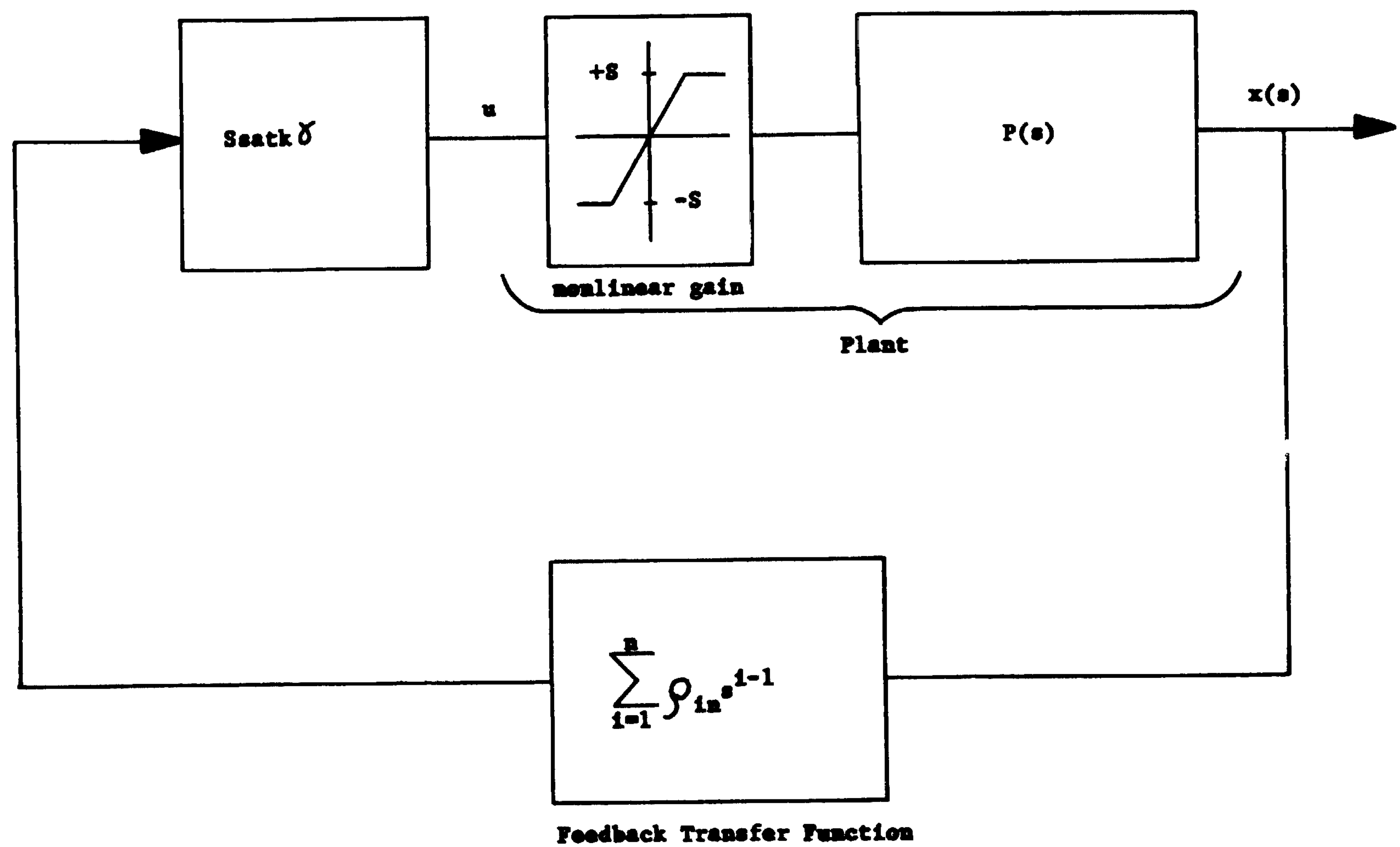


Figure 2-4: Block Diagram For Describing Function Analysis

With P matrix elements determined, the switching function can be written, and it is

$$\gamma = e_1 + 1.5e_2 \tag{2-33}$$

The P matrix and γ function above are independent of the plant. Therefore, they will be applicable in all examples which follow.

Example 2-1: This example and the one following deal with plants that have nonlinearities in the feedback path. Accordingly, (2-20) is used in each for deriving the control law.

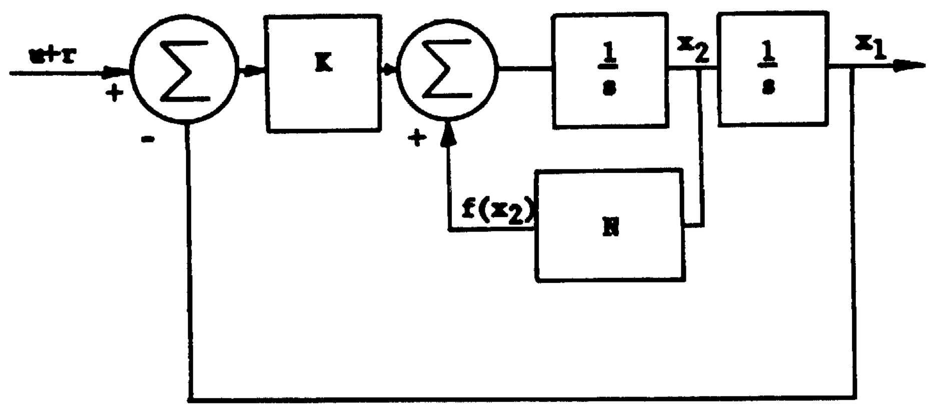
Consider the plant shown in figure 2-5 which has a square law damping characteristic. The vector differential equation describing this plant is

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -K & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} (\underline{u}+\underline{r}) + \begin{bmatrix} 0 \\ a(t)x_2^{\,2} \end{bmatrix} \tag{2-34}$$

For this plant, $m = 0$, i.e. no differentiation of u or r is involved. Because of this, and also because of the fact that the plant has a pure integrator, unity linear feedback of x is used, and r is made an input to the plant The ΔA and ΔB matrices can be found from (2-31) and (2-34) to be

$$\Delta A = \begin{bmatrix} 0 & 0 \\ -(K-2) & 2 \end{bmatrix} \quad ; \quad \Delta B = \begin{bmatrix} 0 & 0 \\ (K-2) & 0 \end{bmatrix} \tag{2-35}$$

From (2-34) it is seen that $a_1 = b_1 - K$ and $b_{m+1} = b_1 = K$. The last term on the right hand side of (2-34) is $\underline{f}$. Relating this to (2-20) shows that $h(x_2, t) = a(t)x_2^{\,2}$. The bounds on parameter variations are taken as $K \geq 1$



(a) Plant

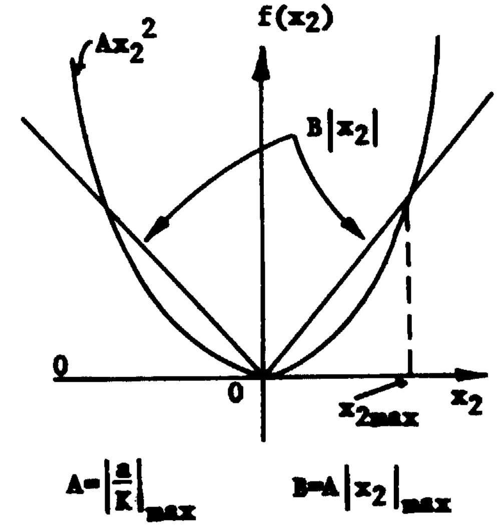


Figure 2-5: Plant With Square Law Damping

and $|a(t)| \leq 1$. With the form of h and the bounds on parameter variations known, application of the design technique is straight forward. The control law derived by applying (2-20) and (2-21-b) is

$$u = [|r-x_1| + 2|x_2| + x_2^2] \text{ sat} k\gamma \qquad (2\text{-}36)$$

Since m = 0, no integrations are required, and the control law as given by (2-36) generates the control signal, u, directly. A plot of e and u versus time is shown in figure 2-6. These results were obtained using a digital computer simulation with a(t) = sint, r(t) = sin0.1t and k = 20. The error, in general quite small, has the largest percentage value near t = o where x_d is small.

It should be noted that if physical considerations allow an upper bound for $|x_2|$ to be established, then x_2^2 in (2-36) can be replaced by $B|x_2|$ (see figure 2-5b). In some instances, this procedure may significantly simplify instrumentation by avoiding a complicated nonlinear function generator. It is also of importance in the practical case where the damping is approximately square law rather than an exact square law. Thus, any nonlinearity with magnitude less than $B|x_2|$ for $|x_2| < |x_2|_{max}$ can be handled by using $B|x_2|$ in (2-36) rather than the nonlinear function itself.

Example 2-2: In this example, as in example 2-1, the nonlinearity is in the feedback path. Here, however, the plant, shown in figure 2-7 and described by equation (2-37) below has no integrator.

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ K & K \end{bmatrix} \int_0^t \underline{u} dt + \begin{bmatrix} 0 \\ -cx_1^3 \end{bmatrix} \qquad (2\text{-}37)$$

In (2-37), c > 0, c = 0, and c <0 corresponds to a hard spring, linear spring, and soft spring respectively. Since the plant has no integrator, r is not used as an input, and the technique discussed in 2.3.1 is employed as indicated by the second term on the right hand side of (2-37) which involves the integral of u rather than u. Employing (2-31) and (2-37) leads to

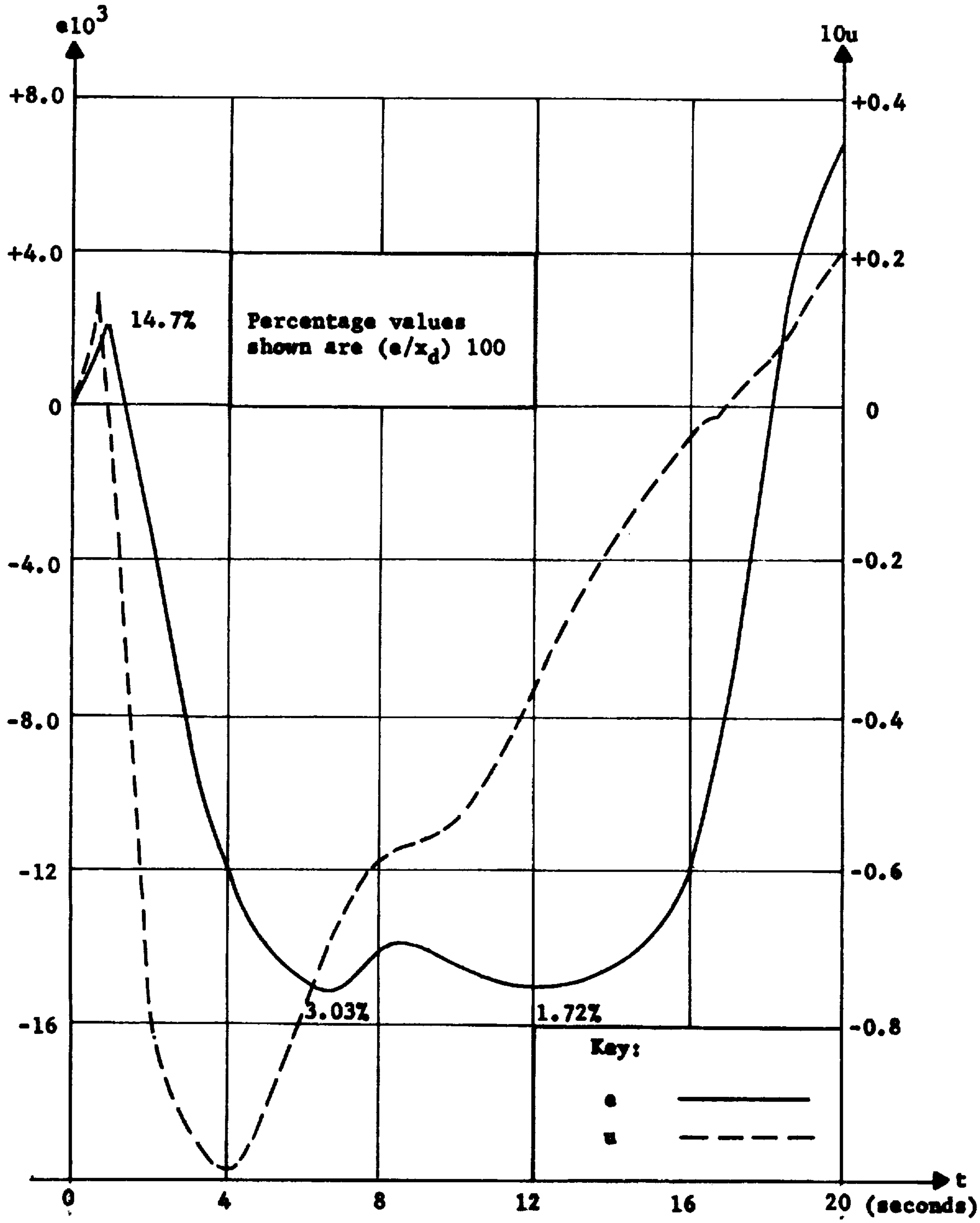


Figure 2-6: Error and Control Signal For Plant of Figure 2-5

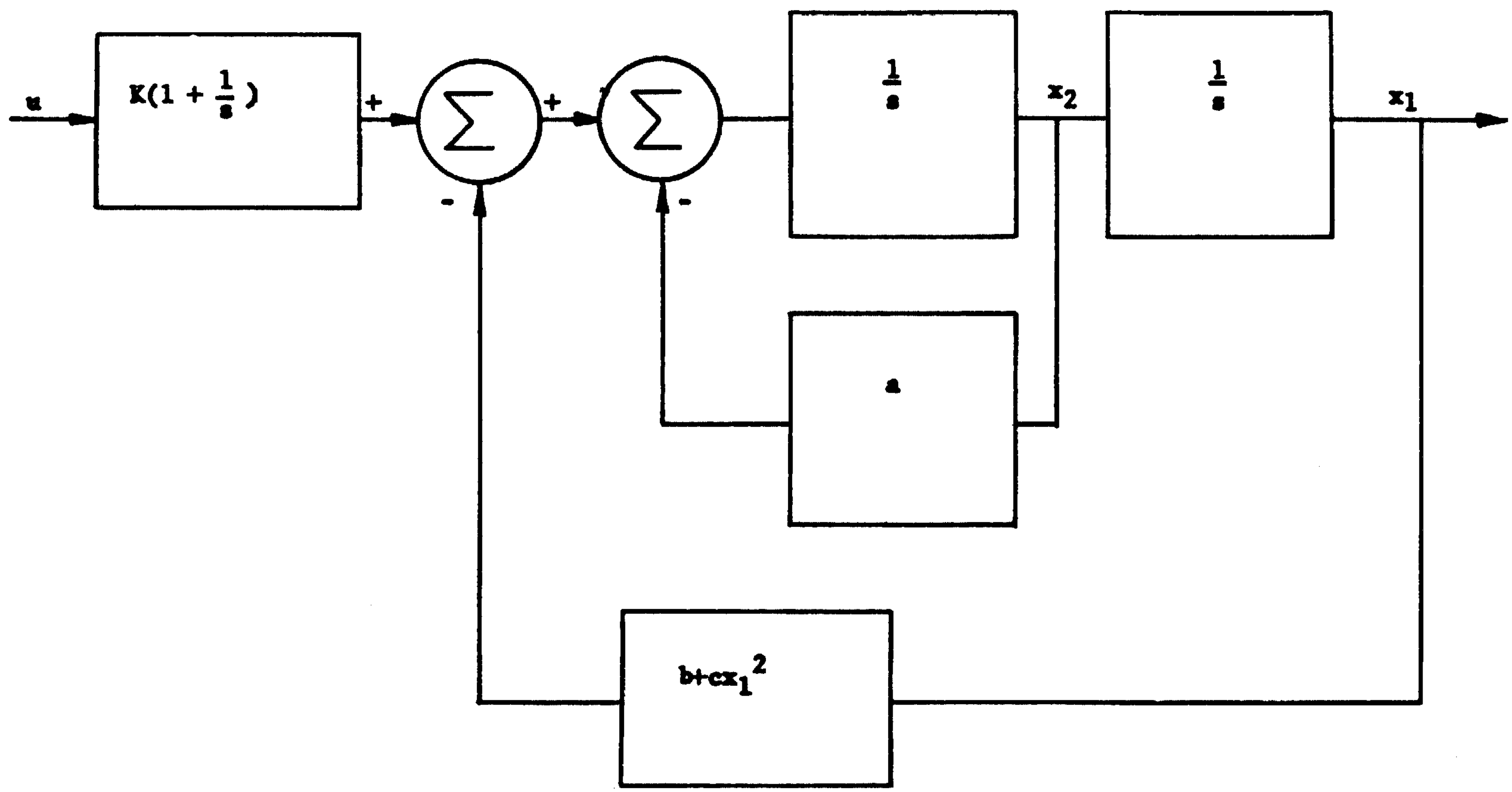


Figure 2-7: Second-Order Plant With Hard and Soft Spring Type Nonlinearity

$$\Delta A = \begin{bmatrix} 0 & 0 \\ (2-b) & (2-a) \end{bmatrix} \; ; \quad h(x_1,t) = -c_1 x_1^{\,3}$$

Since r is not an input to the plant, B_o replaces ΔB in the design equations. With parameters taken as $a(t) = \sin t$, $|b| \leq 1$, $|c| \leq 1$, and $K \geq 1$ the control law becomes

$$u = [2|r-x_1| + |x_1| + 3|x_2| + |x_1|^3 + |\int_0^t u dt|] \text{ sat} k\,\gamma \qquad (2\text{-}38)$$

Digital computer results for this example are shown in figure 2-8 for r = U(t) and k = 20. The error is seen to be very small for all time.

Example 2-3: The plant depicted in figure 2-9 has a nonlinearity in the forward path, a pure integrator, and a pole in the right half plane. Because the integrator is present and there are no zeroes, unity linear feedback is employed, and r is used as an input to the plant. Linear rate feedback is introduced which stabilizes the plant for small inputs, but the linearly compensated plant, without benefit of the nonlinear controller, is conditionally stable due to its nonlinear gain and right half plane pole. For step inputs greater than 2 volts, the output increases without bound. The function of the nonlinear controller is to circumvent the conditional stability problem, and cause the linearly compensated plant to follow the model for any input.

The equation for the plant with linear feedback is

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ 0 & a(t) \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ h(x_1,x_2,u,r) \end{bmatrix} \qquad (2\text{-}39)$$

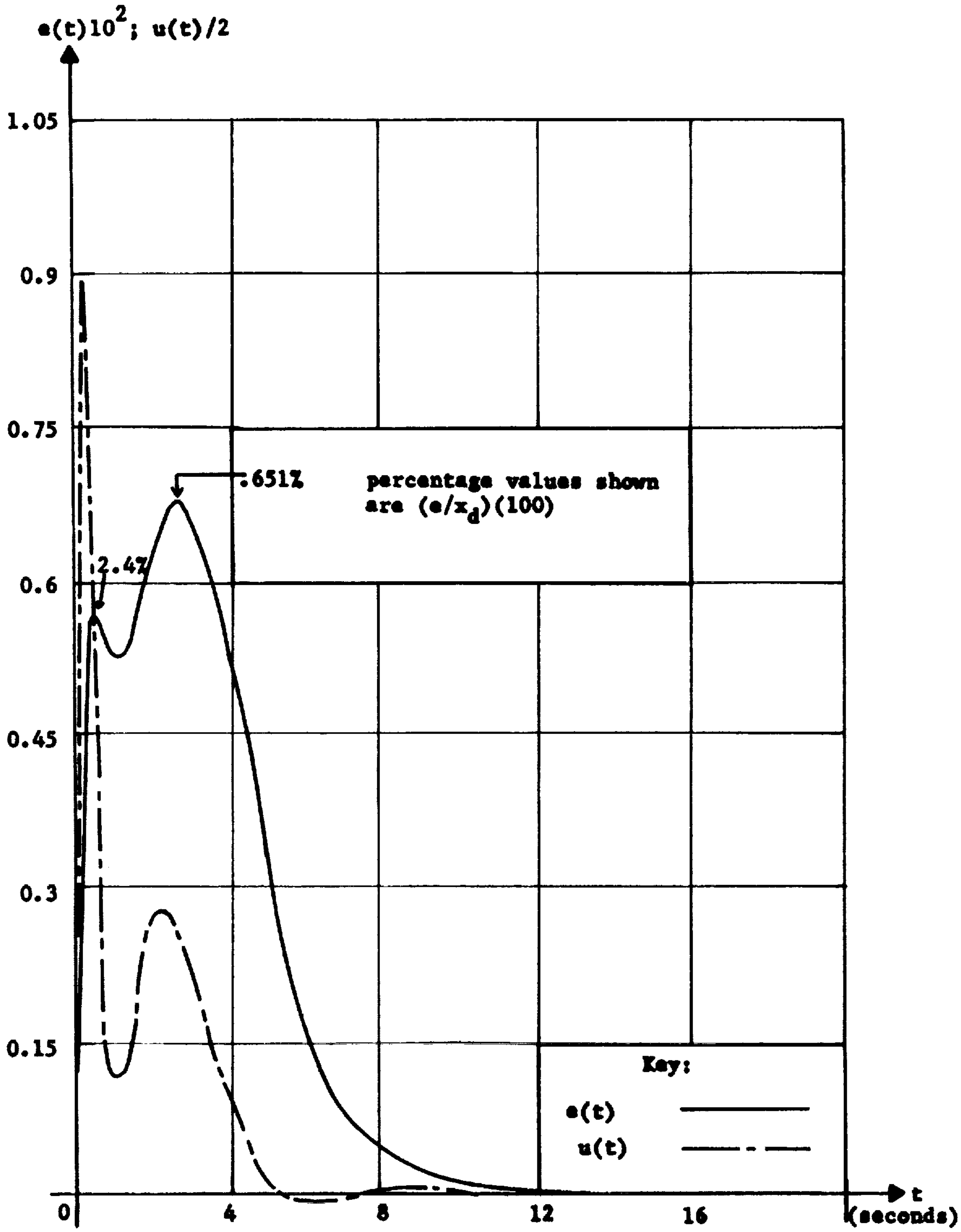


Figure 2-8: Error and Control Signal For Example 2-3

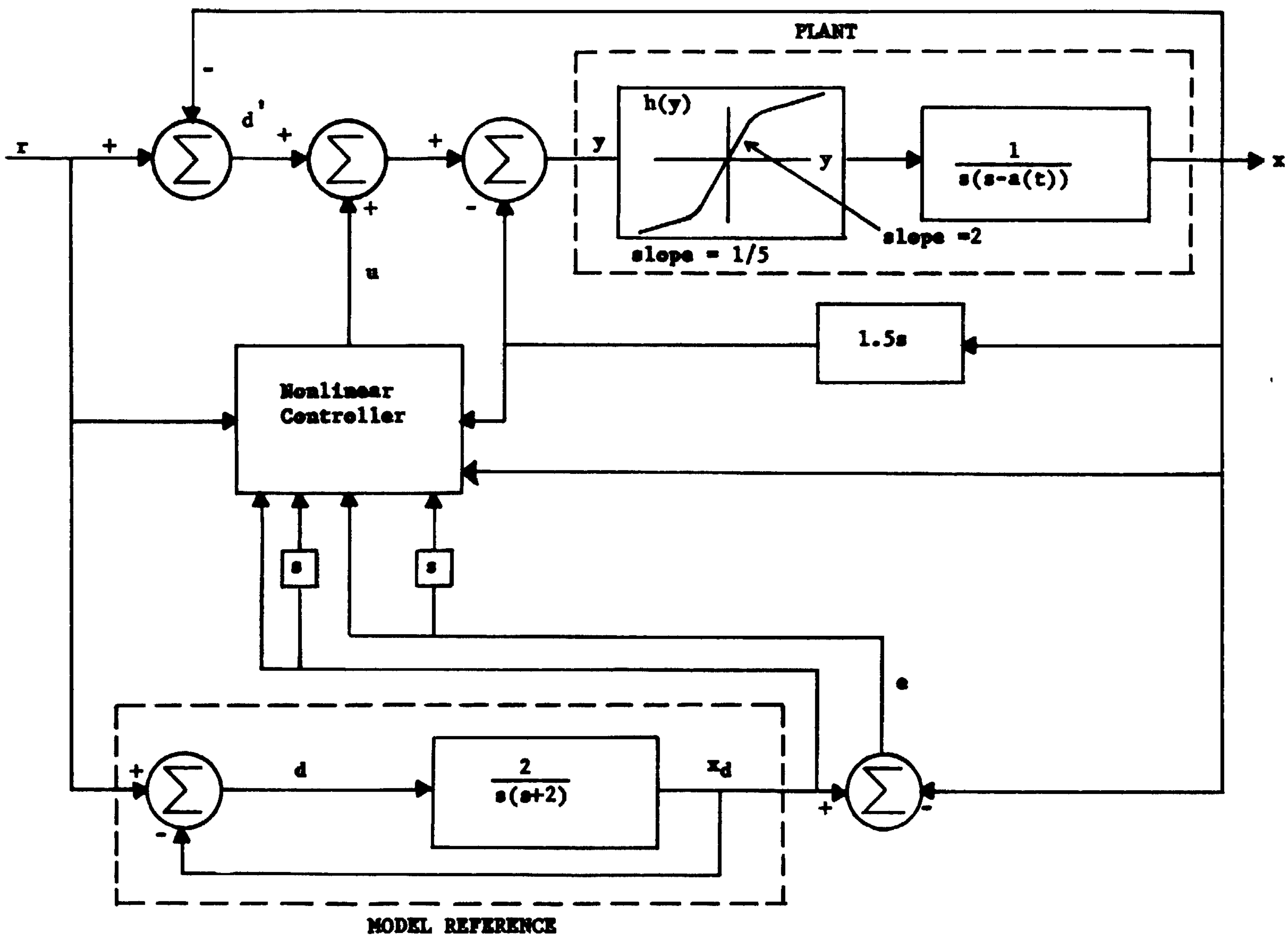


Figure 2-9: Controller for Conditionally Stable Linearly Compensated Plant

where the nonlinear term h is defined in figure 2-9, and $|a(t)| \leq 1$. Since the nonlinearity is in the forward path, the method of section 2.4.2 is used to derive the control law. There are no plant zeroes, so m = 0. The argument of the nonlinear function h is

$$y = r + u - x_1 - 1.5x_2 \tag{2-40}$$

Since y involves a linear term in u, and since $h(y)/y > 0$, the conditions discussed in 2.4.2 are met and the technique can be applied directly to obtain the control signal, u. Equation (2-25) for this problem becomes

$$p_{12}(h/y)\ \gamma \left[u+(r-x_1-1.5x_2) - \frac{2(r-x_1)+(2+a)x_2}{h/y} \right] \geq 0 \tag{2-41}$$

where $r - x_1 - 1.5x_2 = y'$

Utilizing (2-41) and the facts that $|h/y|_{min} = 1/5$, and $|a(t)| \leq 1$ the control law becomes

$$u = [9|r-x_1| + 13.5|x_2|]\ \text{satk}\ \gamma \tag{2.42}$$

where k = 200 was used in an analog computer simulation. Computer results of controller operation are shown in figure 2-10 in the form of phase plane trajectories. The variables d and d' are defined in figure 2-9.

The results show less than one percent error between plant and model variables d' and d, and less than 2.5% error in the derivatives of these variables. Trajectories shown are for a 4 volt step input, but similar results were observed for inputs up to 10 volts. The system was stable for any magnitude of input signal.

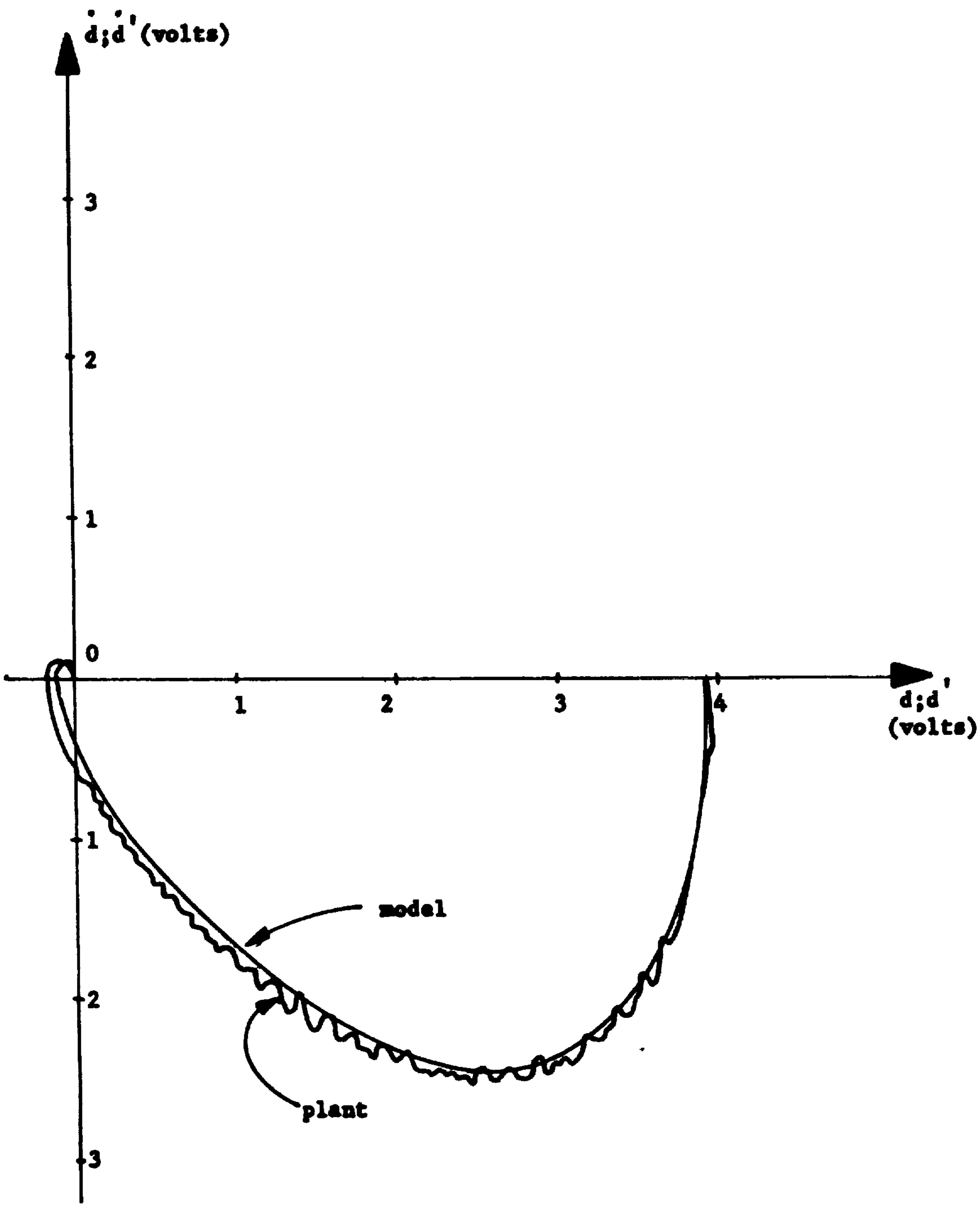


Figure 2-10: Phase Plane Trajectories for System of Figure 2-9

Disturbances and Transducer Noise (2.4.1)

A representation of the problem with transducer noise and disturbance inputs present is shown in figure 2-11. It would be a desirable adjunct to the design technique to be able to design for a specific disturbance response. However, as will be shown, this is not possible. The best that can be done is to insure that $|e_c|$ is as small as desired. For disturbance inputs this is a desirable result in that it implies disturbance rejection. For noise, however, the implication is that the plant tends to track the noise, which is undesirable.

For the purposes of this discussion, it can be assumed that the plant is a linear time varying one without loss of generality. As is seen from figure 2-11, the available signal z is not the true output but is corrupted by disturbance and noise, i.e.

$$\underline{z} = \underline{x} + \underline{d} + \underline{n} \tag{2-43}$$

where $d = d_1$, $n = n_1$, and $\dot{d}_i = d_{i+1}$, $\dot{n}_i = n_{i+1}$ for $i = 1,2,\ldots n - 1$

The plant equation is

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{2-44}$$

From (2-43)

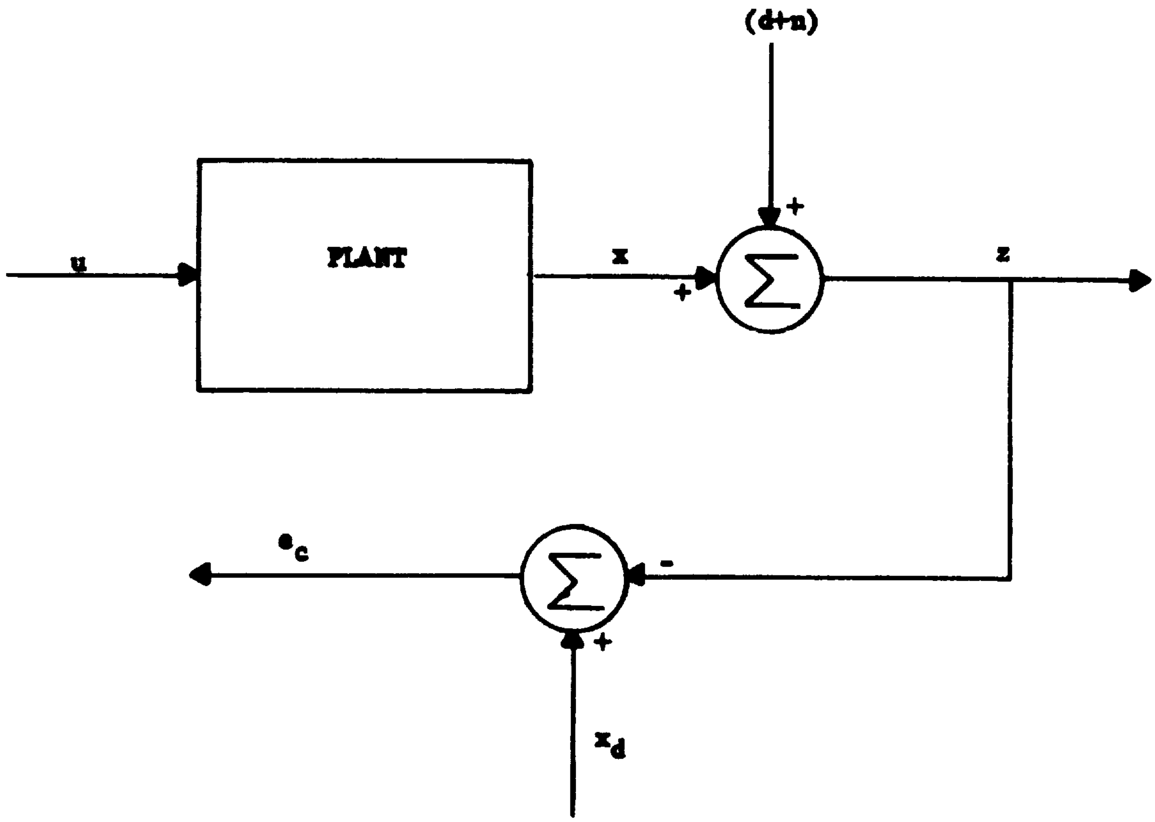
$$\dot{\underline{x}} = \dot{\underline{z}} - \dot{\underline{d}} - \dot{\underline{n}} \tag{2-45}$$

Substituting $\dot{\underline{x}}$ from (2-45) and $\underline{x}$ from (2-43) into (2-44) gives

$$\dot{\underline{z}} = A\underline{z} - A(\underline{d} + \underline{n}) + B\underline{u} + \dot{\underline{d}} + \dot{\underline{n}} \tag{2-46}$$

Subtracting (2-46) from the model equation (2-3) gives

$$\dot{e}_c = A_o e_c - Bu + A(\underline{d} + \underline{n}) - \dot{\underline{d}} - \dot{\underline{n}} - \Delta A\underline{z} + B_o\underline{r} \tag{2-47}$$



d= disturbance
n= transducer noise
e_c= corrupted error signal

Figure 2-11: System With Disturbance and Transducer Noise Present

The time derivative of

$$V(\underline{e}_c) = \underline{e}_c^T P \underline{e}_c \tag{2-48}$$

is

$$\dot{V} = -\underline{e}_c^T Q \underline{e}_c - 2\underline{e}_c^T P \; [B\underline{u} - A(\underline{n}+\underline{d}) + \dot{\underline{d}} + \dot{\underline{n}} + \Delta A \underline{z} - B_o \underline{r}] \tag{2-49}$$

As a consequence of the specified form of the A matrix, terms in d_i and n_i for i = 1,2...n do not appear in the first n - 1 equations of (2-46), i.e. $\dot{z}_i = z_{i+1}$ for i = 1,2,...n - 1. All the d_i and n_i terms as well as $\dot{d}_n$ and $\dot{n}_n$ may appear in the equation for $\dot{z}_n$. At least the expansion of (2-49) is in "factored" form. However, since the terms referred to are not generally measurable, appropriate terms to add to the magnitude function cannot be generated. This problem can be overcome if bounds for these terms are known. In this case, a constant D may be added to the magnitude function such that

$$D \geq \left| \sum_{i=1}^{n} a_i(d_i+n_i)+\dot{d}_n+\dot{n}_n \right| \max \tag{2-50}$$

Addition of D to M and generation of u^m in the usual way guarantees that $|e_c|$ can be made arbitrarily small. No control over the form of disturbance response is possible, however, since the model is chosen for desired response to the reference input. In conclusion, addition of D to M insures disturbance rejection, but forces the plant to track transducer noise in the process. The disadvantages of including a steady state term in the magnitude function when transducer noise is present are discussed in Chapter V. Also discussed there is the fact that the controller may have a disturbance rejection capability without the constant term above included as part of the magnitude function. This is a consequence of the sufficiency nature of Liapunov's theorems. Whether or not to include the constant term must be decided on the basis of factors present in the particular problem being considered.

CHAPTER III

CONVERGENCE TIME DESIGN AND ITS RELATION TO THE QUASI TIME OPTIMAL PROBLEM

General Comments (3.1.1)

In starting systems with large initial errors, the question of convergence time, i.e. the time for plant states to become equal to model states, assume importance. This problem, previously considered only from the point of view of analysis, is treated in a synthesis context in this chapter.

If the plant is tracking the model, control action is such that plant and model outputs differ only slightly, and the difference is reduced to zero quite rapidly. A detailed examination of convergence time in this situation is not essential. However, in starting systems with a large initial error, it would be desirable to know how design parameters can be selected to reduce the error to zero within a specified time.

The convergence time problem dealt with here is in essence the quasi time optimal problem which has received attention in the recent literature[25]. $\dot{V}$ is the cost function for the latter problem. Because of this, results obtained are carried over and applied to the quasi time optimal problem. It is shown that an improved quasi time optimal system can be achieved by using these results. To demonstrate this quantitatively, techniques developed in this thesis are applied to an example taken from reference 25.

Convergence Time (3.2.1)

In order to deal with convergence time quantitatively and to determine an upper bound on its magnitude, a parameter η is defined as

$$\eta = \min \left[- \frac{\dot{V}(\underline{e},\underline{r},\underline{u},\underline{x},t)}{V(\underline{e})} \right]; \; \underline{e} \neq \underline{0} \qquad (3\text{-}1)$$

From (3-1) it follows that

$$\dot{V}(\underline{e},\underline{r},\underline{u},\underline{x},t) \leq - \eta\, V(e) \tag{3-2}$$

This last equation can be solved to yield

$$V(\underline{e}) \leq V(\underline{e_o})\, \varepsilon^{-\eta(t-t_o)} \tag{3-3}$$

where

$$V(\underline{e_o}) = V(\underline{e}\big|_{t=t_o}) \tag{3-4}$$

From (3-3) it is seen that the parameter η is the reciprocal of the time constant for (3-2). In order to minimize convergence time, the design should be directed toward maximizing η. Because (3-2) is an inequality, the best that can be achieved is an upper bound on convergence time, and not an exact value.

Since $\dot{V}$ is a nonlinear, non algebraic function of u, the actual value of η is extremely difficult to compute. To avoid this difficult computation, the following quantities are defined:

$$\dot{V}_o(\underline{e}) \triangleq -\underline{e}^T Q \underline{e} \geq \dot{V}(\underline{e},\underline{r},\underline{u},\underline{x},t) \tag{3-5}$$

and

$$\eta_o \triangleq \left[- \frac{\dot{V}_o(\underline{e})}{V(\underline{e})} \right] \leq \eta; \quad \underline{e} \neq \underline{0} \tag{3-6}$$

Since η_o is the ratio of two algebraic functions of the components of the error vector, computing it is much easier than computing η. However, since (3-6) is an inequality as well as (3-2), the bound on convergence time based on η_o is even more conservative than that based on η.

If (3-6) is written as

$$\eta_o = \frac{\underline{e}^T Q \underline{e}}{\underline{e}^T P \underline{e}} \tag{3-7}$$

then it is clear that convergence time depends explicitly on the Q matrix elements, and implicitly on A_o, through the dependence of P on A_o (see equation (2-9)). Thus, the problem of minimizing convergence time reduces to the proper selection of the elements of the Q and A_o matrices. As will be seen below, this selection is generally not easy to make.

If one considers the analysis problem rather than the synthesis problem, computation of a convervative bound on convergence time is relatively straightforward. Methods are available for such computations, once Q and A_o have been chosen. One such method is based on the fact that a positive definite quadratic form is bounded above and below by the inequality

$$\min_i \lambda_i (P) \; ||\underline{e}||^2 \leq \underline{e}^T P \underline{e} \leq \max_i \lambda_i (P) \; ||\underline{e}||^2 \tag{3-8}$$

where $\lambda_i(P)$ are the eigenvalues of P for i = 1, 2, ... n. These eigenvalues are positive since P is a positive definite matrix[30]. The double vertical lines on either side of $\underline{e}$ symbolize the Euclidian norm of $\underline{e}$. A simple estimate for η is obtained through use of (3-8). This is

$$\eta \geq \eta_o \geq \frac{\min_j \lambda_j(Q)}{\max_i \lambda_i(P)} \; ; \quad \text{for } i, j = 1, 2 \ldots n \tag{3-9}$$

Although (3-9) allows easy computation for η, and thus an upper bound on convergence time, it does not afford an explicit relation from which design parameters can be chosen to yield a specified upper bound on convergence time. It is not at all clear how this bound relates to the physical problem since it derives from an arbitrary mathematical function.

The synthesis problem, i.e. designing to minimize convergence time, is one of maximizing the minimum value of η_o. To find explicit expressions for doing this in terms of the elements of the Q and A_o matrices is not an easy task. This problem is solved below for a second-order system. Though the algebraic problem becomes too unwieldy for third and higher order systems, insight can be gained from the results of the second-order case which is useful in the higher-order problem.

Design for Convergence Time in Second-Order Plants (3.2.2)

In second-order plants, the algebraic problem of maximizing the minimum value of η_o is affected by the design parameters, it is expressed in terms of these parameters as

$$\eta_o = \frac{q_{11}e_1^2+q_{22}e_2^2}{p_{11}e_1^2+2p_{12}e_1e_2+p_{22}e_2^2} \tag{3-10}$$

In (3-10) a diagonal Q matrix has been assumed with elements q_{11} and q_{22}. If a second-order model is used with parameters $a_{01} = K_o$ and $a_{02} = -a_0$, then solution of (2-9) for P yields the following elements of the P matrix

$$p_{11} = \frac{K_o}{2a_o}\left(\frac{q_{11}}{K_o} + q_{22}\right) + \frac{a_o\, q_{11}}{2K_o} \tag{3-11-a}$$

$$p_{12} = \frac{q_{11}}{2K_o} \tag{3-11-b}$$

$$p_{22} = \frac{q_{11}}{2K_o a_o} + \frac{q_{22}}{2a_o} \tag{3-11-c}$$

With these equations for the P matrix elements substituted into (3-10), η_o becomes

$$\eta_o = \frac{e_1^{\,2} + \beta e_2^{\,2}}{\left(\frac{1+\beta K_o}{2a_o} + \frac{a_o}{2K_o}\right)e_1^{\,2} + \frac{e_1 e_2}{K_o} + \left(\frac{1+\beta K_o}{2K_o a_o}\right)e_2^{\,2}} \qquad (3\text{-}12)$$

where $\beta = \dfrac{q_{22}}{q_{11}}$

The parameter β indicates the relative weighting of $e_1^{\,2}$ and $e_2^{\,2}$ in $\dot{V}_o(\underline{e})$.

To maximize the minimum value of η_o, its minimum value is found first by taking the derivatives of (3-12) with respect to e_1 and e_2, and setting these derivatives equal to zero. Following this procedure with either variable leads to the equation

$$e_1^{\,2} + \left[\frac{1}{a_o} - \frac{(\beta K_o)^2}{a_o} - \beta a_o\right] e_1 e_2 - \beta e_2^{\,2} = 0 \qquad (3\text{-}13)$$

Treating (3-13) as a quadratic in e_1, and solving yields

$$e_1 = k_1 e_2 \qquad (3\text{-}14)$$

where

$$k_1 = -1/2\left[\frac{1-\beta^2 K_o^{\,2}}{a_o} - \beta a_o\right] \pm 1/2\sqrt{\left[\frac{1-\beta^2 K_o^{\,2}}{a_o} - \beta a_o\right]^2 + 4\beta}$$

If (3-14) is substituted into (3-12) the result is

$$\eta_o = \frac{(k_1^2+\beta)2K_o a_o}{(\beta K_o^2+K_o+a_o^2)k_1^2+2a_o k_1+1+\beta K_o} \qquad (3\text{-}15)$$

All of the design parameters, i.e. K_o, a_o, and β, are brought out explicitly in (3-15). However, since η_o is a function of three parameters, it is not an easy task to maximize its minimum value by choice of these parameters. At this point the assumption is made that

$$\beta K_o \ll 1 \qquad (3\text{-}16)$$

Since K_o is dependent on the model used, it generally is not very much less than one. Therefore, (3-16) implies that $\beta \ll 1$, i.e. less weighting is given e_2 than e_1 in $\dot{V}(\underline{e})$. Use of (3-16) leads to the following simplifications

$$k_1 \approx -1/a_o \qquad (3\text{-}17\text{-}a)$$

$$k_1 \approx +\beta a_o \qquad (3\text{-}17\text{-}b)$$

The value for k_1 given by (3-17-b) gives the minimum for η_o, which is,

$$\eta_o = \frac{(1+\beta a_o^2)2a_o\beta K_o}{(1+\beta a_o^2)^2+\beta K_o+\beta^2 a_o^2 K_o(1+\beta K_o)} \qquad (3\text{-}18)$$

If the model parameters are given in terms of the standard damping ratio, natural frequency nomenclature for second-order systems, i.e.

$$K_o = \omega_o^2 \qquad (3\text{-}19\text{-}a)$$

and

$$a_o = 2\ \xi\ \omega_o \qquad (3\text{-}19\text{-}b)$$

then (3-18) becomes

$$\eta_o = \frac{(1+4\ \xi^2\ \beta\ \omega_o^{\ 2})(4\ \xi\ \beta\ \omega_o^{\ 3})}{(1+4\ \xi^2\ \beta\ \omega_o^{\ 2})^2 + \beta\ \omega_o^{\ 2} + 4(\xi\ \beta\ \omega_o^{\ 2})^2\ (1 + \beta\ \omega_o^{\ 2})} \qquad (3\text{-}20)$$

This constitutes the design equation for convergence time for the second-order system. From it can be seen that by keeping the model damping ratio constant, the convergence time can be made as small as desired (η_o can be made as large as desired) by increasing ω_o while keeping $\beta\,\omega_o^{\ 2}$ constant at a value which satisfies (3-16).

It should be noted that there is no need to actually change the model behavior to accomplish the desired result. For example, let the model be described by

$$\ddot{x}_d + a_o\dot{x}_d + (K_o + a'_{o1})x_d = K_o r + a'_{o1}x_d \qquad (3\text{-}21)$$

With the model described in this way, η_o can be increased by increasing a'_{o1}. The model behavior is as though a'_{o1} were absent. However, the term $a'_{o1}x_d$ must be included as an additional term in the magnitude function of the control law. Clearly, the decreased convergence time is gained at the expense of an increase in the control signal magnitude.

The weighting of the error states in the $\dot{V}$ function can be interpreted in terms of the switching line in the phase plane defined by the switching function of (2-14). The equation for this line is

$$e_2 = -\rho_{22}e_1 = -\frac{a_o}{1+\beta K_o}e_1 \qquad (3\text{-}22)$$

Using (3-16) in (3-22) leads to

$$e_2 \approx -a_o e_1 \qquad (3\text{-}23)$$

From (3-23) it is seen that the effect of choosing βK_o to satisfy (3-16) is to rotate the switching line toward the e_2 axis, thereby giving its slope the maximum possible magnitude. The effect of this rotation in decreasing convergence time is illustrated in figure 3-1. The implications of this switching line rotation in the quasi time optimal problem is considered in 3.3.1.

Higher Order Plants (3.2.3)

The algebraic munipulation required to arrive at (3-20) would be extremely unwieldy for a third-order system and practically impossible for systems of order higher than third. This being the case, an attempt is made here to show how results derived for the second-order system may be carried over to a third-order one. The arguments justifying this approach are intuitive rather than analytical.

Since weighing higher order states less in the $\dot{V}$ function effected a shorter convergence time in the second-order case, perhaps the same result can be attained in the third-order case by the same means. For the third-order case, this means deemphasizing acceleration and velocity terms in $\dot{V}$. Intuitively, one would expect that such a deemphasis should lead to a faster transient. To test this intuitive notion, the transient response for the error was compared using two different Q matrices and a plant described by the equation

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \underline{x} + \begin{bmatrix} & \bigcirc & \\ k_1 & k_2 & 0 \end{bmatrix} \underline{u} \qquad (3\text{-}24)$$

Equation (3-24) is the vector form of the simplified short period approximation for the transfer function of the X-15 manned re-entry vehicle[8,31] with an integrator representing a hydraulic motor actuator. This plant is considered again in Chapter V.

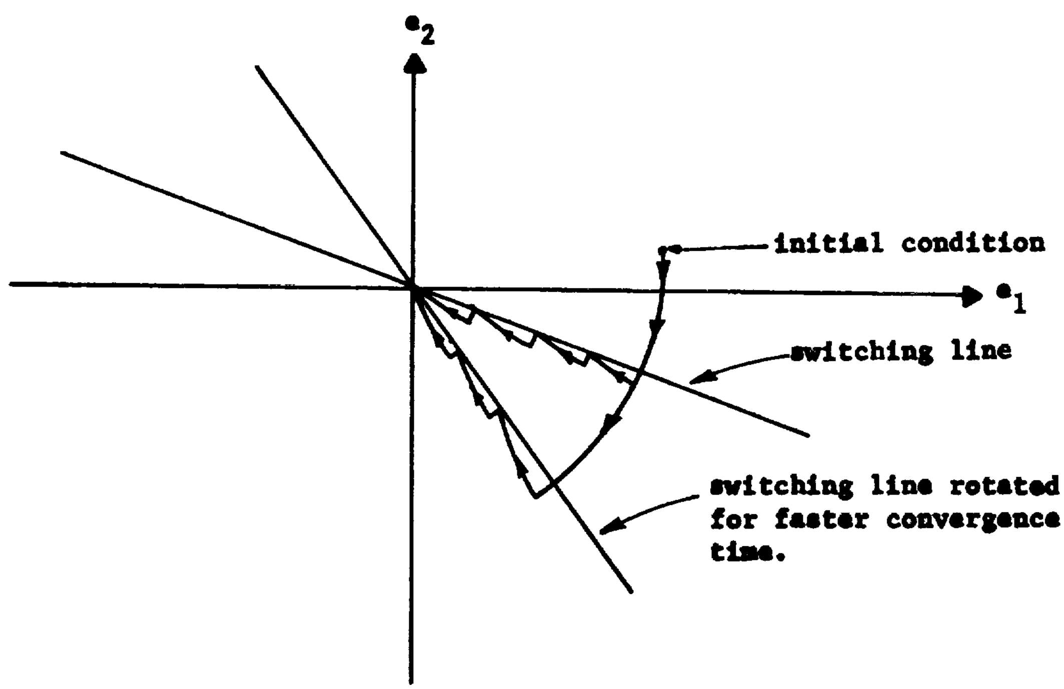


Figure 3-1: Relation of Switching Line Slope to Convergence Time

The desired behavior is given by the model equation

$$\underline{\dot{x}_d} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.2 & -2.15 & -1.3 \end{bmatrix} \underline{x_d} + \begin{bmatrix} & & \\ & \bigcirc & \\ +1.2 & 0 & 0 \end{bmatrix} \underline{r} \qquad (3\text{-}25)$$

The control law for the system given by (3-24) and (3-25) was derived using two different Q matrices. These were

$$Q_1 = (1,1,1) \qquad (3\text{-}26\text{-a})$$

$$Q_2 = (1,0.1,0.01) \qquad (3\text{-}26\text{-b})$$

where the numbers in parenthesis are the values of the diagonal elements and all other elements are zero.

The switching functions corresponding to each of these Q matrices are

$$\gamma_1 = 0.414e_1 + 1.23e_2 + 1.33e_3 \qquad (3\text{-}27\text{-a})$$

and

$$\gamma_2 = 0.415e_1 + 0.486e_2 + 0.377e_3 \qquad (3\text{-}27\text{-b})$$

where γ_1 derives from Q_1, and γ_2 from Q_2. It should be noted that further reduction of q_{22} and q_{33} results in negligible additional decrease in coefficients of e_2 and e_3 from their values in (3-27-b). The transient response for e_1 is shown in figure 3-2, Results in the figure are for initial conditions $e_1 = 5$, and $e_2 = e_3 = 0$. It is clear that the transient for Q_2 is considerably faster than for Q_1. Thus, insight gained from the second-order problem is useful and can be applied to the third-order case.

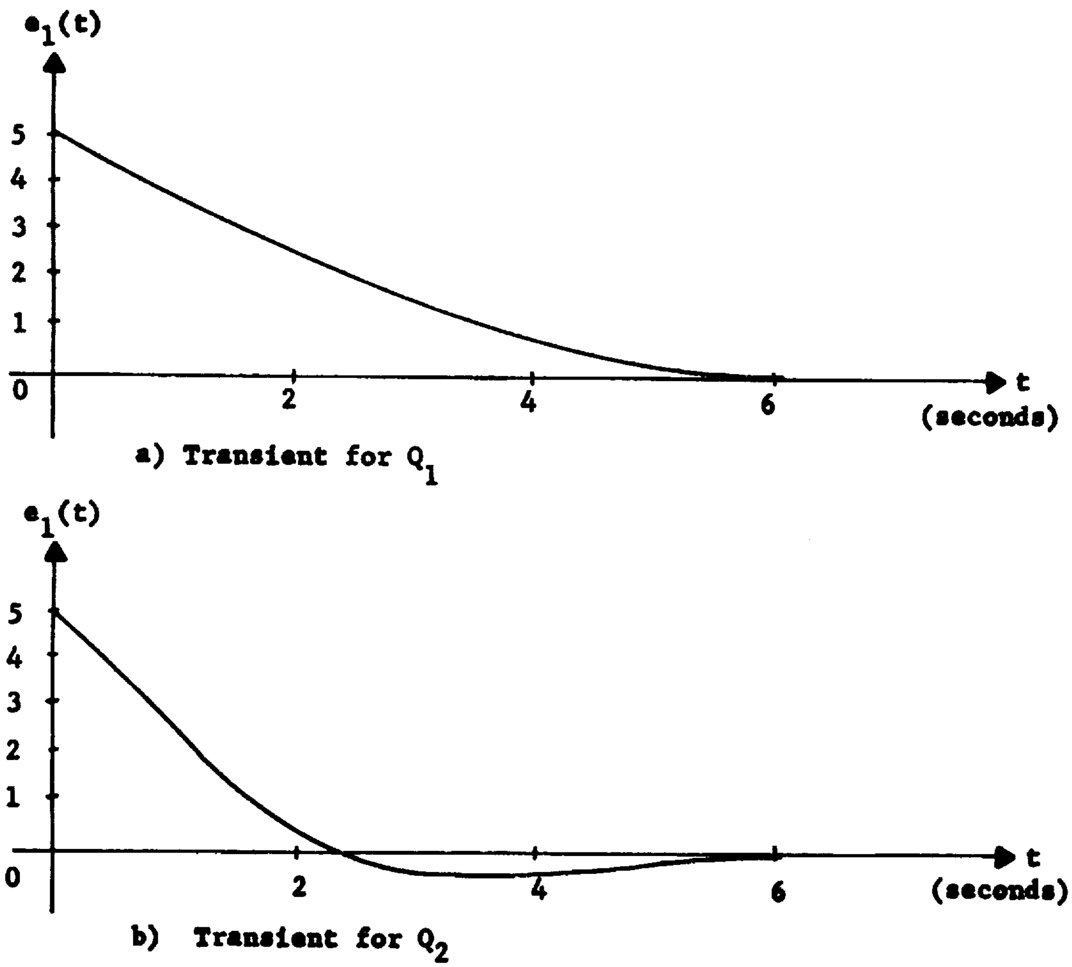


Figure 3-2: Transient Responses For Third-Order System

Relation to Quasi Time Optimal Control (3.3.1)

The results above relating to convergence time design relate as well to some recent work[24,25] done in the application of the "direct method" to design of quasi time optimal systems. In the references cited, it is pointed out that an advantage of such a design, though not pressing true time optimal properties, is that asymptotic stability of the overall system is assured. Also, the system is quasi optimal according to the following definition: "Given a linear, stationary plant $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ and a rate of cost function L (x, u, t) where L is independent of sign u_i, then the quasi optimal feedback control policy, $u^Q(\underline{x}, t)$ should minimize absolutely the modified cost function $K(\underline{x}, \underline{u}, t) = \dot{V}(\underline{x}, \underline{u}) + L(\underline{x}, \underline{u}, t)$ for a given $V(\underline{x}) = \underline{x}^T P \underline{x}$ and $\lambda \geq 0$." In the definition it is understood that P is a positive definite matrix and $V(\underline{x})$ a Liapunov function.

The link between what has been previously said concerning minimizing convergence time and a quasi time optimal design is provided by rewording the last part of the definition above as follows: ".......for a $V(\underline{x}) = \underline{x}^T P \underline{x}$ chosen in a way which aids said minimization and $\lambda > 0$." This wording seems more appropriate in that it leads the designer to consider advantages which may accrue through proper selection of the design parameters implicit in the P matrix. It will be seen in the discussion which follows that proper selection of the Q matrix elements does lead to such an advantage in the quasi time optimal design.

The following example is taken from reference 25. It illustrates how use of the reworded definition leads to a quasi time optimal design which is closer to the true time optimal one than the design based on the original definition. No disadvantage is suffered. The resultant system is still asymptotically stable. The only difference in the design procedure will be to choose P (indirectly through choice of Q) keeping in mind the desired objective of achieving a design as close to the true time optimal as possible. The example follows.

Example 3-1: Given a plant with transfer function

$$\frac{X}{U}(s) = \frac{1}{s^2 + s + 1} \tag{3-28}$$

where $|u| \leq 1$, design a controller which yields quasi time optimal performance in the sense of the original definition given above. A phase space decomposition of (3-28) with $x = x_1$ gives

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_2 + u \qquad (3\text{-}29)$$

If $V(\underline{x}) = \underline{x}^T P \underline{x}$, then

$$\dot{V}(\underline{x}, \underline{u}) = \underline{x}^T (A^T P + PA)\underline{x} + 2\underline{x}^T P\underline{bu} \qquad (3\text{-}30)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} ; \text{ and } \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In the reference, $A^T P + PA$ is arbitrarily set equal to $-I$. It is at this point that the procedure suggested by the reworded definition departs from that of the original. The reworded definition would not allow a completely arbitrary choice, but instead, $A^T P + PA$ would be set equal to $-Q$, with Q a symmetric positive definite matrix. This point is pursued further below. Proceeding with the $-I$ choice, one finds the P matrix to be

$$P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \qquad (3\text{-}31)$$

This P matrix leads to the control policy

$$u^Q = -\,\text{sign}(0.5x_1 + x_2) \qquad (3\text{-}32)$$

Consider now use of the reworded definition in deriving the control policy. As mentioned above $-Q$ is chosen in place of $-I$. The cost function is $K(\underline{x}, \underline{u}) = \dot{V}(\underline{x}, \underline{u})$, λ being zero in the case of quasi time optimal control. How is selection of the Q matrix elements to be made to aid in minimizing this cost function? In order to answer this question, the control signal given by (3-32) is written in general terms as follows

$$u = - \text{sign}(p_{12}x_1 + p_{22}x_2) \tag{3-33}$$

From (3-33) it is seen that the switching line equation is

$$x_2 = - \frac{p_{12}}{p_{22}} x_1 \tag{3-34}$$

If use is made of (3-11), (3-34) can be written

$$x_2 = - \frac{q_{11}}{q_{11}+q_{22}} x_1 = - \frac{1}{1+\beta} x_1 \tag{3-35}$$

where β is as defined for (3-12), and the parameter values $k_o = a_{01} = 1$, and $a_o = -a_{o2} = 1$ pertain. From phase plane considerations, it is argued that to achieve minimum time in traversing from any initial condition to the origin, it is advisable to have $|x_2|$ increase as much as possible before switching occurs. To accomplish this, the magnitude of the slope of the switching line should be as large as possible. Thus, β in (3-35) should be as small as possible. Conclusions reached by this line of reasoning are in agreement with the discussion pertaining to (3-23).

A basis for selection of q_{11} and q_{22} has been established. The value of q_{12} can be taken as zero without loss of generality. It should be noted that with the constraint on $|u|$, the origin will be reached without overshoot only for initial conditions in certain regions. In this regard, the switching line with the larger magnitude slope leads to greater overshoot for a given set of initial conditions. These factors must be taken into consideration in each particular problem.

For the case under consideration, the initial conditions are $x_1(0) = 0.5$ and $x_2(0) = 1.0$. Use of the reworded definition leads one to choose $\beta << 1$

to come closer to the true optimum solution. With $\beta = 0.01$, the control policy is

$$u^Q = - \operatorname{sign}(0.5x_1 + 0.505x_2) \qquad (3\text{-}36)$$

rather than (3-32) where $\beta = 1.0$ was used. A comparison of trajectories for the control policies given by (3-32) and (3-36) is shown in figure 3-3. The time to the origin is approximately 5 seconds for the latter and 9 for the former. The saturation function rather than the sign function was used in both cases. It is clear that for the given initial conditions, $\beta = .01$ is a better design choice. An additional benefit is that less fuel is consummed with $\beta = .01$. The measure used for this is $\int_0^t |u| dt$ and for $\beta = 1.0$ it is 1.68 whereas for $\beta = .01$ it is 1.47. It should be noted that making β less than .01 only increases the slope of the switching line slightly. From (3-22) it can be seen that the maximum magnitude for the slope of the switching line is $a_o = 1$, and this value is approached as β approaches zero.

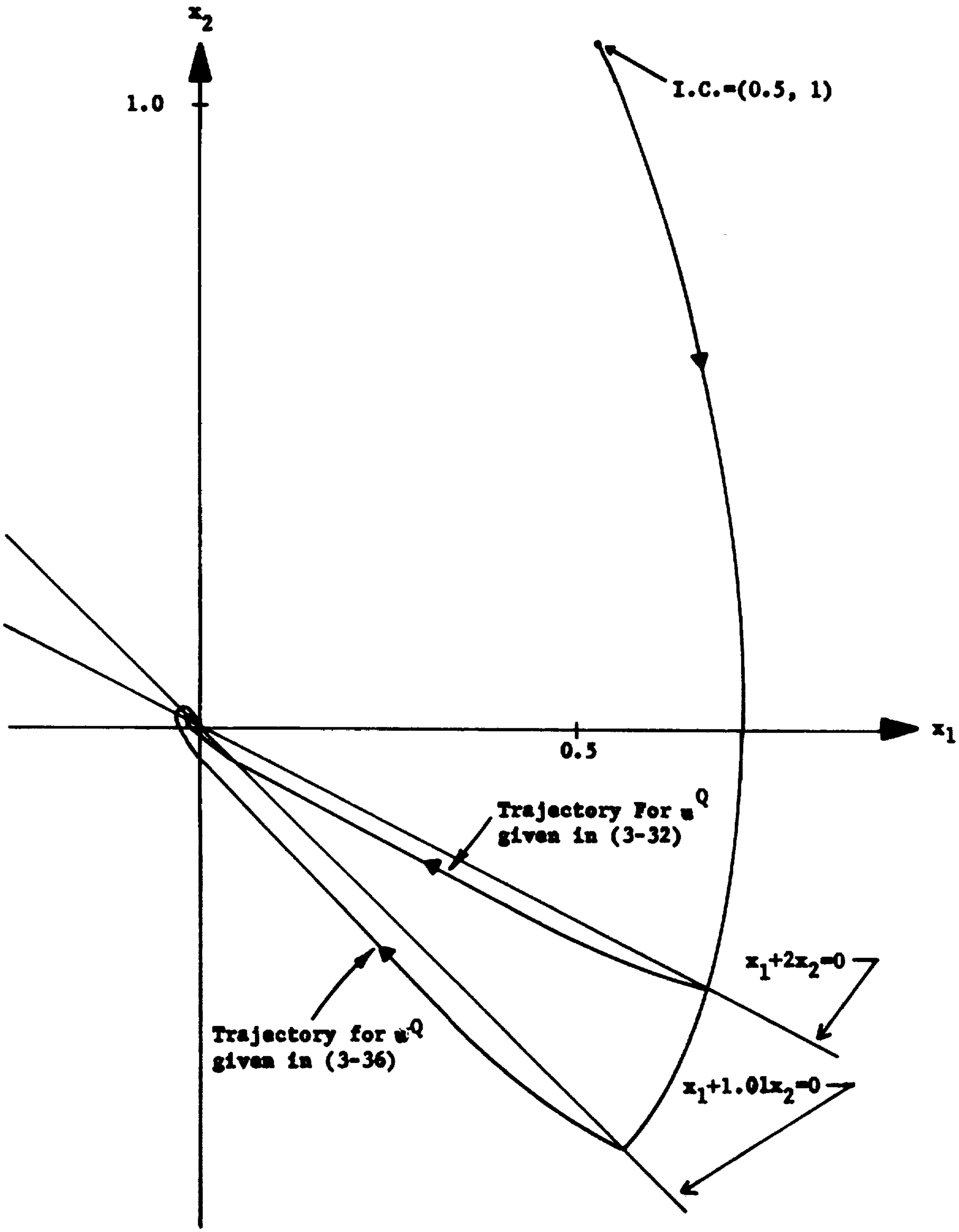


Figure 3-3: Comparison of Trajectories For Two Control Laws

CHAPTER IV

REDUCTION OF THE TRANSDUCER NOISE PROBLEM

The Transducer Noise Problem (4.1.1)

From an engineering viewpoint, the need to use (n-1) derivatives of the output in generating the control signal is undesirable since these derivative signals most likely will be corrupted by noise. Problems which might arise due to transducer noise are best understood by considering the form of the control law which is, as seen in (2-14), the product of the switching function and the magnitude function. Transducer noise in either of these signals may adversely affect system behavior in the following ways:

(i) Saturation due to noise may occur at the plant input because of a large gain, k, in the saturation function or large gains associated with the signals of the magnitude function.

(ii) The control signal may have an incorrect sign due to the corruption of the saturation-function signals by noise.

(iii) The control signal may have insufficient amplitude due to noise in the signals of the magnitude function.

(iv) A d.c. bias and/or a beat signal may occur at the plant input due to modulation effects produced by multiplication of noise in the sat function by noise in the magnitude function.

The amplitude level and frequency range of transducer noise which cause these effects to be objectionable in a given control problem depend on the performance specifications and on the input saturation level of the plant. In any event, since these noise problems are likely to arise, it would be advantageous for the system designer to have at his disposal some theoretical guidelines for dealing with them. In this chapter, such guidelines are developed. In Chapter V, some aspects of the noise problem which fall outside the realm of exact theoretical analysis are considered.

Since transducer noise becomes progressively worse in higher derivative signals, the attack on the problem is directed toward eliminating these signals from the control law entirely, or at least reducing the gains associated with them. The general problem is considered first, and then a very useful theorem for linear time varying plants with zeroes is developed.

Elimination or Reduction of Plant States (4.2.1)

Except for plants with zeroes, which will be considered later, no way has been found to eliminate higher order plant states from the switching function. In fact, since the coefficient of the highest order error state, e_n, in the switching function is ρ_{nn}, then it cannot be zero since the P matrix must be positive definite. Though higher order states generally cannot be eliminated from the switching function, their effect may be substantially reduced through judicious choice of the Q matrix elements (and hence on the coefficients of the switching function) was considered in Chaper III relative to the convergence time problem. There, reduction of the coefficients of the higher order error states was introduced in order to decrease convergence time. Here, it is seen that an additional advantage accrues, that of reducing problems resulting from transducer noise. In an analog computer study of a second order system, it was found that the mean squared noise level into the plant could be reduced by a factor of one half if β (defined in conjunction with (3-12)) is chosen as 0.1 rather than 10.

Possibilities for eliminating the higher order plant states from the magnitude function of the control law do exist. In order to examine these, the equation for a linear time varying plant with linear feedback is considered. This equation, similar to (2-2) with $\underline{f} = 0$ and F introduced as the feedback matrix, is

$$\dot{\underline{x}} = (A + BF)\underline{x} + B(\underline{u} + \underline{r}) \tag{4-1}$$

Following the controller design procedure as given in section 2.2.1 leads to the following $\dot{V}$ function:

$$\dot{V}(\underline{e}) = \underline{e}^T(A_o{}^T P + PA_o)\underline{e} - 2\underline{e}^T P\,[B\underline{u} + \Delta B\underline{r} + (\Delta A + BF)\underline{x}] \tag{4-2}$$

Before deriving the control law from (4-2) in the usual manner, it is examined for possible ways to eliminate or reduce some or all of the components of $\underline{x}$ in the resulting control law. One is led to suspect that such ways exist because the first term on the right hand side of (4-2) possesses "excess negative definiteness" when $Q = -(A_o{}^T P + PA_o)$ is chosen, as it usually is, to be a diagonal matrix. The phrase "excess negative definiteness" refers to the fact that the cross-product terms are absent from the resultant quadratic form. To give a quantitative meaning to this phrase, consider the quadratic form in two variables $a_{11}e_1{}^2 + 2a_{12}e_1e_2 + a_{22}e_2{}^2$.

This quadratic form is negative definite if $a_{11} < 0$, and $a_{11}a_{22} - a_{12}^2 = M > 0$. The quantity M is the measure of "excess negative definiteness." The larger M is, the more "excess negative definiteness" the quadratic form possesses.

The question, therefore, arises, can this "excess negative definiteness" be used to absorb some of the other terms in the equations? This might be possible if the terms to be absorbed are of the type which fit into the quadratic form, i.e. products of two and only two components of the $\underline{e}$ vector. As (4-2) stands, no terms of this type arise other than those resulting from $-\underline{e}^T Q \underline{e}$. In order to generate other terms of suitable type, $H_1 \underline{x_d}$ is added and subtracted within the square brackets of (4-2) to give

$$\dot{V}(\underline{e}) = -\underline{e}^T Q \underline{e} - 2\underline{e}^T P \, [-H_1\underline{e} + B\underline{u} + \Delta B\underline{r} + H_2\underline{x} + H_1\underline{x_d}] \qquad (4\text{-}3)$$

where $H_1 + H_2 = \Delta A + BF$

In (4-3), the term to be absorbed is $2\underline{e}^T PH_1\underline{e}$. Since PH_1 is generally not a symmetric matrix, the following relationship is used:

$$2\underline{e}^T PH_1\underline{e} = \underline{e}^T \, [PH_1 + (PH_1)^T] \, \underline{e} \qquad (4\text{-}4)$$

where $PH_1 + (PH_1)^T$ is symmetric.

For convenience, let $PH_1 + (PH_1)^T = -Q'$ Then, in order to absorb (4-4) into $-\underline{e}^T Q \underline{e}$ it is necessary that

$$Q + Q' \qquad (4\text{-}5)$$

be positive definite for all variations in ΔA and B.

Another possibility which is explored is to handle Q' alone rather than absorb it with Q. When treated separately, it suffices to make Q' positive semidefinite for all plant parameter variations. When either of these approaches is successful through suitable choices for F, Q, and H_1, then all or some of the plant state variables can be replaced by model state variables in the magnitude function of the control law. This assumes that H_1 is chosen such that $H_2\underline{x}$ is lacking some components of $\underline{x}$. With the Q' term either absorbed or handled separately, the $\underline{u}$ vector must be chosen to satisfy

$$-2\underline{e}^T P\ [B\underline{u}+\Delta B\underline{r}+H_2\underline{x}+H_1\underline{x}_d] \geq 0 \tag{4-6}$$

Elimination of All Plant States from the Magnitude Function (4.2.2)

If it is possible to choose $H_1 = \Delta A + BF$, then $H_2 = 0$ and all components of the plant state vector can be eliminated from the magnitude function and replaced by those of the model state vector. This procedure does not automatically assure a reduction of noise level at the plant input, however. Whether or not noise reduction is achieved depends to a large degree on the required linear feedback matrix, F, if indeed an F exists at all which allows elimination of all plant states. It may turn out that the magnitudes of linear feedback gains required are such that no improvement in performance is achieved over the design with plant states in the magnitude function. However, this is a matter to be investigated for each specific design problem. In general, it can be stated that necessary and sufficient conditions for replacing all plant states by model states in the magnitude function of the control law are:

(a) $H_1 = \Delta A + BF$; $(H_2 = 0)$

(b) Q + Q' is positive definite or Q' is positive semidefinite.

It is only fair to warn at this point that it may be quite difficult to determine when these conditions are satisfied for higher-order systems. An example of this technique applied to a second-order problem is given in section 4.4.1.

In cases where all components of the plant state vector can be replaced by corresponding components of the model state vector in the magnitude function of the control law, some freedom is gained in controlling the size of the magnitude function by selection of the model. This was alluded to in 2.3.2 in connection with plants having hard saturation gain characteristics. Since the magnitude function then only depends on $\underline{r}$ and $\underline{x}_d$, an upper bound on $|u|$ can be determined such that $|u| \leq U(\underline{r}, \underline{x}_d)$. If the plant has a hard saturation gain with saturation level, S, then by selecting the model properly it may be possible to maintain $U(\underline{r}, \underline{x}_d) < S$, and thereby avoid operation in the saturation region.

Elimination of the Highest Order Plant State from the Magnitude Function (4.2.3)

If it is not possible or desirable using a linear feedback matrix to eliminate all plant states it may be possible, with a less objectionable linear feedback policy, to eliminate at least the highest order plant state from the magnitude function. To achieve this goal, all elements of H_2 in the last column must be zero. Then, an attempt can be made to absorb (4-4) into the $-\underline{e}^T Q \underline{e}$ term, or the alternate approach can be taken of making Q' positive semidefinite.

A procedure for choosing H_1 so that Q' is positive semidefinite is as follows: If the last column of H_2 has all zeroes, then the element in the last row and last column of H_1 is the same as the corresponding element of $\Delta A + BF$. Let this element be g_{nn}, and then define H_1 as

$$H_1 = g_{nn} \begin{bmatrix} & \bigcirc & & & \\ h_{1n} & h_{2n} & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad (4\text{-}7)$$

The procedure is based on the proper selection of $h_{1n}, h_{2n}, \ldots h_{(n-1)n}$. It can always be applied provided $g_{nn} \leq 0$. With this requirement on g_{nn} met, then Q' will be positive semidefinite if, as shown in Appendix D, the h_{in} terms are selected as follows:

$$h_{in} = \frac{p_{in}}{p_{nn}} \qquad i = 1, 2, \ldots n-1 \qquad (4\text{-}8)$$

where the p's are elements of the P matrix. With H_1 chosen to satisfy (4-7) and (4-8), and $g_{nn} \leq 0$, the control law will not involve a term in x_n as part of the magnitude function.

A choice of H_1 useful for eliminating the highest order component from the magnitude function is

$$H_1 = \begin{bmatrix} & & \bigcirc & & \\ 0 & 0 & \cdot & \cdot & \cdot & g_{nn} \end{bmatrix} \tag{4-9}$$

where g_{nn} has the same definition as in (4-7). With this choice, conditions are sought for which Q' can be absorbed by Q and (4-5) satisfied.

Manipulation of the Model Matrix (4.3.1)

In certain cases, the model matrix can be manipulated to effect a reduction in the magnitudes of the coefficients of the plant states in the magnitude function or to extend the range of parameter variations for which the conditions necessary for elimination of plant states can be satisfied. The manipulation performed is to write the model equation as

$$\underline{\dot{x}}_d = (D_o + G_o)\underline{x}_d + B_o\underline{r} \tag{4-10}$$

where $A_o = D_o + G_o$

Note that A_o is still the model matrix, therefore, model behavior is not affected by this maneuver. The manipulation is carried through the time derivative of the Liapunov function which is written as:

$$\dot{V}(\underline{e}) = \underline{e}^T(D_oP+PD_o)\underline{e}-2\underline{e}^TP\;[B\underline{u}+\Delta B\underline{r}+(\Delta A'+BF)\underline{x}-G_o\underline{x}_d] \tag{4-11}$$

where $\Delta A' = A - D_o$

The possibility that an advantage might be gained by this technique lies in the fact that the magnitude of the elements of $\Delta A'$ might be smaller than those of $\Delta A = A - A_o$. This will lead to lower gains for the magnitude function variables which may contain noise. Also, this manipulation technique can be employed in conjunction with the techniques discussed in 4.2.1 through 4.4.1. Here the purpose would be to make it possible to satisfy (4-5) or to make Q' positive semidefinite when it might not be possible with $D_o = A_o$. In this case the following definitions would be used in place of previous ones

$$D_o^T P + P D_o = -Q$$

and

$$H_1 + H_2 = \Delta A' + BF$$

Application to a Second-Order Plant (4.4.1)

All of the techniques discussed above will now be applied to the same second-order plant in order that they may be compared. The plant and model chosen are ones used in an example in reference 13 so that the control laws found here can be compared to that in the reference. The plant equation with linear feedback is

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} (\underline{u}+\underline{r}) + \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} \begin{bmatrix} -1 & f_{12} \\ 0 & 0 \end{bmatrix} \underline{x} \tag{4-12}$$

where unity linear feedback is indicated by $f_{11} = -1$ in the F matrix. There is no loss of generality in letting $f_{21} = f_{22} = 0$ because the second column of the B matrix is zero. The equation for the model used is

$$\underline{\dot{x}_d} = \begin{bmatrix} 0 & 0 \\ -2 & -2 \end{bmatrix} \underline{x_d} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \underline{r} \tag{4-13}$$

Thus

$$\Delta A = A - A_o = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -\alpha \end{bmatrix} \qquad (4\text{-}14)$$

where $\alpha = a - 2$.

In reference 13, the plant parameter variations where taken as $1 \leq K \leq 5$, and $1 \leq a \leq 10$ or $-1 \leq \alpha \leq 8$. The control law derived there was of the form

$$u = [\,|r - x_1| + 8|x_2|\,] \text{ satk } \gamma \qquad (4\text{-}15)$$

where $k = 20$ and $\gamma = e_1 + 1.5e_2$

The coefficient of $|r - x_1|$ used in reference 13 is three. If use is made of the fact that time variations in numerator and denominator of this coefficient are related, the procedure discussed in section 2.3.1, then a coefficient of one is seen to be sufficient.

For this plant, the controller indicated by (4-15) was implemented on the PACE 231-R Analog computer. A random noise generator with power spectrum flat to 30 c.p.s. was used to simulate noise in the transducer measuring x_1. The mean squared transducer noise level was taken to be 1.45×10^{-4} volts^2. This led to a mean squared noise level into the plant of 62.5 volts^2 and a steady state d.c. error in x_1 of 0.15 volts. These results are to be compared below to those using state elimination techniques. A considerable reduction of noise level into the plant will be noted.

Example 4-1: Elimination of all plant States from the Magnitude Function.-For the problem under consideration

$$H_1 + H_2 = \Delta A + BF = \begin{bmatrix} 0 & 0 \\ (2-K) & (Kf_{12} - \alpha) \end{bmatrix} \qquad (4\text{-}16)$$

For elimination of all plant states, $H_2 = 0$ is chosen. The elements of the Q' matrix are found to be

$$q'_{11} = -2p_{12}(2-K)$$

$$q'_{12} = q'_{21} = -p_{12}(Kf_{12}-\alpha)-p_{22}(2-K) \qquad (4\text{-}17)$$

$$q'_{22} = -2p_{22}(Kf_{12}-\alpha)$$

where p_{12} and p_{22} are elements of the P matrix with values 1/4 and 3/8 respectively found from the solution of

$$A_o^T P + PA_o = -I \qquad (4\text{-}18)$$

where I is the identity matrix

With Q = I, the elements of Q + Q' Become

$$q_{11}+q'_{11} = K/2$$

$$q_{12}+q'_{12} = q_{21}+q'_{21} = (1/4)(\alpha-Kf_{12})+(3/8)(K-2)$$

$$q_{22}+q'_{22} = 1+(3/4)(\alpha-Kf_{12}) \qquad (4\text{-}19)$$

Conditions required for positive definiteness of (4-19) are

$$K > 0 \qquad (4\text{-}20\text{-}a)$$

$$(K/2)\,[1+(3/4)(\alpha-Kf_{12})] - [(1/4)(\alpha-Kf_{12})+(3/8)(K-2)]^2 > 0 \qquad (4\text{-}20\text{-}b)$$

The first of these, (4-20-a), is always satisfied if $K > 0$ is assumed. Because it would be most desirable not to use rate feedback, a check is made to see if (4-20-b) can be satisfied with $f_{12} = 0$. Unfortunately, it cannot be if the parameter variations are taken to be the same as those used in connection with (4-15). Therefore, $f_{12} = -1$ is tried. With this amount of linear rate feedback (4-20-b) can be satisfied for $K \geq 1$ and $-1.5 \; \alpha \leq 8.5$ which includes the range used for (4-15). Thus, with $f_{12} = -1$, all plant states can be eliminated from the control law which then becomes

$$u = [|r-x_{d1}| + 8|x_{d2}|] \text{ satk } \gamma \tag{4-21}$$

instead of (4-15). Since this result is achieved using a gain of -1 only for x_2 as compared to a gain of 8 for $|x_2|$ in (4-15), a significant reduction of the noise level into the plant seems likely. For the same transducer noise used with control law (4-15), the mean squared noise level into the plant using (4-21) was only 3.5 volts^2 instead of 62.5 volts^2. In addition, no measurable d.c. error in x_1 appeared as it did when control law (4-15) was used.

Example 4-2: Elimination of Highest Order Plant State from the Magnitude Function.- Here the two techniques for eliminating the highest order plant state are applied. Results obtained will be compared to those of the example 4-1. The first technique considered is that relating to (4-7). The parameter g_{nn} is

$$g_{nn} = g_{22} = Kf_{12} - \alpha \tag{4-22}$$

and according to (4-7) and (4-8),

$$H_1 = g_{22} \begin{bmatrix} 0 & 0 \\ \frac{p_{12}}{p_{22}} & 1 \end{bmatrix} = (Kf_{12}-\alpha) \begin{bmatrix} 0 & 0 \\ 2/3 & 1 \end{bmatrix} \tag{4-23}$$

Use of (4-23) and (4-16) yield

$$H_2 = \begin{bmatrix} 0 & 0 \\ (2-K)+(2/3)(\alpha-Kf_{12}) & 0 \end{bmatrix} \quad (4\text{-}24)$$

With H_1 chosen as in (4-23), Q' is positive semidefinite if

$$g_{22} \leq 0 \quad (4\text{-}25)$$

For the same range of parameters previously considered, (4-25) can only be satisfied if $f_{12} = -1$. Thus, this approach requires the use of linear rate feedback. The control law resulting is

$$u = [|r-x_1| + 6|e_1| + 9|x_{d2}|] \text{ satk } \gamma \quad (4\text{-}26)$$

Since it has been previously determined in example 4-1 that use of unity linear rate feedback allows removal of all the plant states, there is no advantage of using $f_{12} = -1$, and only removing $|x_2|$ from the magnitude function.

In order to see if the highest order plant state can be eliminated from the magnitude function without resorting to linear rate feedback, the alternate approach indicated by (4-9) is tried. Thus,

$$H_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha \end{bmatrix} ; \quad H_2 = \begin{bmatrix} 0 & 0 \\ (2-K) & 0 \end{bmatrix} \quad (4\text{-}27)$$

where it has been assumed that $f_{12} = 0$. The Q' matrix resulting from this choice of H_1 is

$$Q' = \begin{bmatrix} 0 & p_{12}\alpha \\ p_{12}\alpha & 2p_{22}\alpha \end{bmatrix} \qquad (4\text{-}28)$$

Thus, $|x_2|$ can be replaced by $|x_{d2}|$ if

$$Q+Q' = \begin{bmatrix} 1 & p_{12}\alpha \\ p_{12}\alpha & (1+2p_{22}\alpha) \end{bmatrix} \qquad (4\text{-}29)$$

is positive definite. The requirement for this is that

$$-(p_{12}\alpha)^2 + 2p_{22}\alpha + 1 = -(1/16)\alpha^2 + (3/4)\alpha + 1 > 0 \qquad (4\text{-}30)$$

This is satisfied for $-1.2 \leq \alpha \leq 13.2$, a range which exceeds that specified for the control law of (4-15). Therefore, this choice of H_1 allows $|x_2|$ to be replaced by $|x_{d2}|$ even without use of linear rate feedback. The control law resulting is

$$u = [\,|r-x_1| + 8|x_{d2}|\,] \text{ satk } \gamma \qquad (4\text{-}31)$$

Since neither linear rate feedback nor $|x_2|$ is required in this approach, it is reasonable to expect that noise levels into the plant would be less using

(4-31) for control rather than (4-15). Computer results bear this out, and do show a significant reduction in this noise level. In fact the mean squared noise level into the plant was 0.45 volts^2 with a transducer mean squared noise of 1.45×10^{-4} volts^2. This is considerably less than the noise levels into the plant obtained using either control law (4-15) or (4-21) of example 4-1. In addition to the reduction of noise level into the plant, no d.c. error in x_1 appeared.

Example 4-3: Manipulation of Model Matix.- The range of α for which (4-30) can be satisfied may be extended for different values of p_{12} and p_{22}. To obtain different values for these elements, the model matrix is manipulated as discussed in relation to (4-10). Here D_o and G_o are taken as

$$D_o = \begin{bmatrix} 0 & 0 \\ -4 & -2 \end{bmatrix} ; \quad G_o = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \qquad (4\text{-}32)$$

Solving $D_o^T P + PD_o = -I$ for P gives $p_{12} = 1/8$ and $p_{22} = 5/16$. With these values used in (4-30), the range of allowable variation in α is $-1.55 \leq \alpha \leq 41.55$.

Thus, use of this model matrix manipulation technique has greatly extended the range over which α can vary and still not violate conditions which allow $|x_2|$ to be eliminated from the magnitude function. Note that this range is valid without using linear rate feedback. For $K \geq 1$, the control law resulting is

$$u = [\,|r-x_1| + 2|e_1| + |\alpha|_{max}|x_{d2}|\,] \text{ satk } \gamma \qquad (4\text{-}33)$$

where this γ differs from those of previous examples since the P matrix elements are different. Here $\gamma = e_1 + 2.5e_2$. Note that the coefficient of e_2 is 2.5 instead of the previous 1.5. This may be disadvantageous from a noise standpoint.

It is not intended that results of this example by compared to previous results on the basis of noise level reduction, but merely to show that a much greater range of α is attainable using the technique.

Reduction of Order for Linear, Slowly Time Varying Plants with Zeroes (4.5.1)

An ideal way to minimize the adverse effects of transducer noise on system performance is to use a reduced plant and model which are of lower order than the actual plant. In this way, higher order derivatives can be eliminated from the switching function as well as from the magnitude function. Such a reduction of order is possible for linear, slowly time varying plants with zeroes. Parameter variations may be large, but within known, finite bounds. It is necessary to assume that parameter variations are slow so that the "frozen system" concept applies and a Laplace transform, transfer function representation of the plant is valid.

In addition to plants with zeroes, the technique can be applied on an approximate basis to linear, slowly time varying plants without zeroes. Approximations involved for accomplishing this, and performance to be expected are treated in section 4.6.1.

Reduction of order is accomplished by applying the design technique to the reduced plant transfer function shown in figure 4-1, which is of lower order than the complete plant transfer function. The prefilter contains known, fixed plant poles. The number of poles contained in the prefilter is at most equal to the number of plant zeroes. The presence of zeroes in the transfer function allows u to be generated in a way which guarantees that the states of the reduced plant are arbitrarily close to states of a model reference of order equal to that of the reduced plant. This result is stated in the following theorem:

Theorem I: Given:

(i) a plant with transfer function

$$\frac{X}{U}(s) = \frac{K \prod_{d=1}^{m} (s+z_d)}{s^q \left[\prod_{f=1}^{j} (s+p_f)\right]\left[\prod_{h=j+1}^{n} (s+p_h)\right]} \qquad (4\text{-}35)$$

in which X(s) and U(s) are Laplace transforms of the plant output and input respectively, K is of one sign and K_{min} is known, the z_d's and p_h's may be slowly varying in an unknown fashion within known, finite bounds, $m < n + q$, $q \geq 0$, and the p_f's are known constants; and

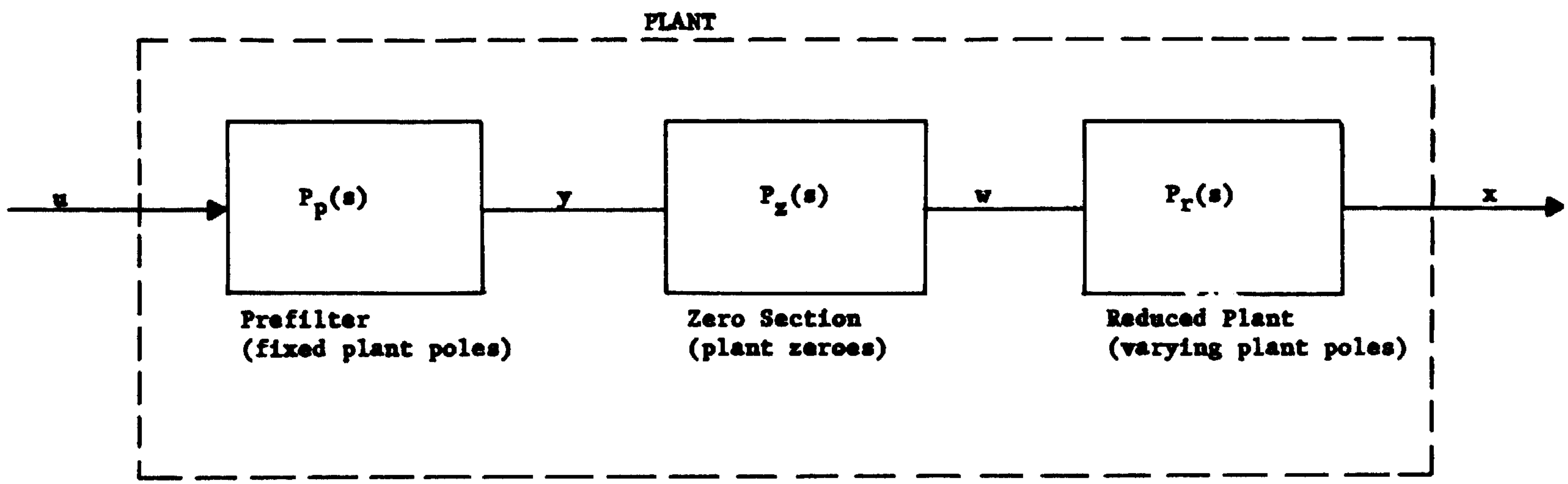


Figure 4-1: Plant Decomposition for Reduction-of-Order Technique

(ii) a stable model reference (all roots of its characteristic equation have negative real parts) with transfer function

$$\frac{X_d(s)}{R(s)} = \frac{a_o}{\sum_{i=0}^{k} a_i s^i} \qquad (4\text{-}36)$$

where $a_k = 1$, $X_d(s)$ and $R(s)$ are Laplace transforms of the model output and input respectively, and $k = q + n - m$ for $q + j \geq m$, or $k = n - j$ for $q + j < m$, and a_i = constant for $i = 0, 1, 2...(k - 1)$; there exists an input, $u = \mathcal{L}^{-1}U(s)$, which can be generated using model and plant outputs, x_d and x, and their first $k - 1$ derivatives only such that $|\underline{e}| = \sqrt{\underline{e}^T\underline{e}} < \epsilon$ for $t \rightarrow \infty$, where the error vector is defined by $e_1 = x_d - x$, and $e_{i+1} = \dot{e}_i$, for $i = 1, 2, .. k - 1$. Proof: Consider the plant transfer function to be devided into three parts as shown in figure 4-1. The transfer functions for those parts are as follows,

$$P_p(s) = \frac{Y}{U}(s) = \frac{1}{s^{q-t}\left[\prod_{f=1}^{\ell} (s+p_f)\right]} \qquad (4\text{-}37)$$

$$P_z(s) = \frac{W}{Y}(s) = \prod_{d=1}^{m} (s+z_d) \qquad (4\text{-}38)$$

$$P_r(s) = \frac{X}{W}(s) = \frac{K}{s^t\left[\prod_{f=\ell+1}^{j} (s+p_f)\right]\left[\prod_{h=j+1}^{n} (s+p_h)\right]} \qquad (4\text{-}39)$$

and are called the prefilter, zero section, and reduced plant respectively. The constants t and l are such that

$$q - t \geq 0 \qquad (4\text{-}40\text{-}a)$$

$$q - t + l \leq m \qquad (4\text{-}40\text{-}b)$$

$$l \leq j \qquad (4\text{-}40\text{-}c)$$

Inequality (4-40-b) restricts the number of fixed poles of known location which can be included in the prefilter to be at most equal to m. Constants t and l will always be chosen such that equality pertains provided there are sufficient known, fixed plant poles to do so.

The differential equations resulting from the three transfer functions (4-37) through (4-39) are

$$u = y^{(q-t+\ell)} + a_1 y^{(q-t+\ell-1)} + \ldots . a_\ell y^{(q-t)} \qquad (4\text{-}41)$$

where $a_1, a_2 \ldots a_\ell$ are constant coefficients of the polynominal resulting from expansion of the product $\prod_{f=1}^{l} (s + p_f)$.

$$w = y^m + b_{1y}^{(m-1)} + \ldots b_m y \qquad (4\text{-}42)$$

where $b_1, b_2 \ldots b_m$ are coefficients of the polynomial resulting from expansion of the product $\prod_{d=1}^{m} (s + z_d)$.

$$x^k + c_k x^{(k-1)} + c_{k-1} x^{(k-2)} + \ldots c_1 x^t = Kw \qquad (4\text{-}43)$$

where $c_1, c_2 \ldots c_k$ are coefficients of the polynomial resulting from expansion of the product

$$s^t \left[\prod_{f=\ell+1}^{j} (s+p_f) \right] \left[\prod_{h=j+1}^{n} (s+p_h) \right]$$

Let (4-43) be expressed as

$$\dot{\underline{x}} = (A_o + \Delta A)\underline{x} + \underline{b}w \qquad (4\text{-}44)$$

where $A_o + \Delta A$ is a kxk matrix, $\underline{b}^T$ is the k column vector [0, 0, ...k], and A_o is the model matrix defined below. Let the model reference be given by the vector differential equation corresponding to (4-36), i.e.

$$\dot{\underline{x}}_d = A_o \underline{x}_d + \underline{b}_o r \qquad (4\text{-}45)$$

where A_o is a stable kxk matrix, $\underline{b}_o^T$ the k column vector [0, 0, ... a_o], and $\dot{x}_{di} = x_{d(i+1)}$ for i = 1, 2, .. k-1

If an error is defined as

$$e = x_d - x \qquad (4\text{-}46)$$

then (4-44) can be subtracted from (4-45) to give

$$\dot{\underline{e}} = A_o\underline{e} + (\underline{b}_o r - \underline{b}w - \Delta A\underline{x}) \qquad (4\text{-}47)$$

Substituting for w from (4-42) into (4-47) gives

$$\dot{\underline{e}} = A_o\underline{e} - [\underline{b}(y^m + b_1 y^{m-1} + \ldots b_m y) - \underline{b}_o r + \Delta A\underline{x}] \qquad (4\text{-}48)$$

A Liapunov function, $V(\underline{e}) = \underline{e}^T P \underline{e}$ as in (2-7), is associated with (4-48). It follows from the development in section 2.2.1 that $|\underline{e}|$ is ultimately bounded, i.e., $|\underline{e}| < \epsilon$ for $t \to \infty$ if

$$y^m \left[\sum_{j=0}^{m-1} |b_{m-j}|_m |y^j| + \left|\frac{a_o}{K}\right|_m |r| + \sum_{i=1}^{k} \left|\frac{\Delta a_i}{K}\right|_m |x_i| \right] \text{sat} k\gamma \qquad (4\text{-}49)$$

where subscript m indicates maximum value and

$$\gamma = \sum_{i=1}^{k} \rho ik^e i$$

With y^m generated as in (4-49), u can be synthesized as in (4-41) since all of the required signals besides y^m can be obtained through successive integrations of y^m. Thus, the theorem is proved.

Application of Reduction of Order Technique (4.5.2)

The technique introduced in 4.5.1 is here applied to a plant with transfer function

$$\frac{X}{U}(s) = \frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)} \qquad (4\text{-}50)$$

in which p_1 and p_2 are known constants, K, z_1 and z_2 are slowly varying within known finite bounds, and $K > 0$. The model transfer function can be first order since two zeroes allow two poles to be placed in the prefilter position. The reduced plant equation is

$$P_r(s) = \frac{X}{W}(s) = \frac{K}{s} \qquad (4\text{-}51)$$

and the first order model chosen has transfer function

$$\frac{X_d}{R}(s) = \frac{a_o}{s+a_o} \tag{4-52}$$

The procedure leading to (4-48) is followed to yield for this problem

$$\dot{e} + a_o e = a_o(r-x) - K[\ddot{y} + (z_1+z_2)\dot{y} + z_1 z_2 y] \tag{4-53}$$

A suitable Liapunov function for (4-53) is

$$V(e) = (1/2)e^2 \tag{4-54}$$

for which

$$\dot{V} = -a_o e^2 - Ke[\ddot{y} + (z_1+z_2)\dot{y} + z_1 z_2 y - \frac{a_o}{K}(r-x)] \tag{4-55}$$

To insure that $|\underline{e}|$ is ultimately bounded, $\ddot{y}$ is chosen as

$$\ddot{y} = [|z_1+z_2|_m \; |\dot{y}| + |z_1 z_2|_m \; |y| + |\frac{a_o}{K}|_m \; |r-x|] \text{ satke} \tag{4-56}$$

where subscript m indicates maximum value.

The control signal derived using (4-41) is

$$u = \ddot{y} + (p_1 + p_2)\dot{y} + p_1 p_2 y \tag{4-57}$$

An analog computer study of controller operation was made using the following parameter values:

$$K \geq 1,\ a_o = 1,\ k = 100,\ |z_1+z_2|_{max} = 2,\ |z_1 z_2|_{max} = 1,$$

$p_1 = p_2 = 0.5 + j\sqrt{3/2}.$ For these values, the control signal becomes

$$u = \ddot{y} + \dot{y} + y \qquad (4\text{-}58)$$

With this control signal and with $r = U(t)$, a unit step function, the plant output followed the model output very closely for the several different zero locations including complex conjugate zeroes on the imaginary axis. The peak error in all cases was less than one percent of the step amplitude.

An important consideration from an engineering viewpoint is the sensitivity of the response to variations in the location of the poles at $s = -p_1$ and $s = -p_2$. In many cases, location of these poles may not be known exactly or may vary slightly during operation. In order for the technique to be useful, the response must be fairly insensitive to these variations. Several computer runs were made with these poles moved from their nominal positions to the positions indicated in figure 4-2. It is to be emphasized that the control signal remained unchanged for these runs, i.e. u as given by (4-58) was used. Even under these conditions, the plant followed the model so closely that the peak error again did not exceed one percent of the input step amplitude, indicating a low sensitivity to variations in the location of assumed known, fixed poles.

The Non-Minimum Phase Zero Problem (4.5.3)

The bounds for z_1 and z_2 given in section 4.5.2 do not restrict the plant zeroes to lie in the left half of the s plane. However, when either or both zeroes are in the right half plane, certain difficulties arise because of the nature of the control signal required to cause the plant to follow a given model. Consider in (4-50) the zero at z_1 to be in the right half plane at $s = +z_1$. If $R(s) = A/s$ and the plant is to follow the model of (4-52) then letting $X(s) = X_d(s)$ gives

$$U(s) = \frac{Aa_o(s+p_1)(s+p_2)}{K(s+a_o)(s-z_1)(s+z_2)} \qquad (4\text{-}59)$$

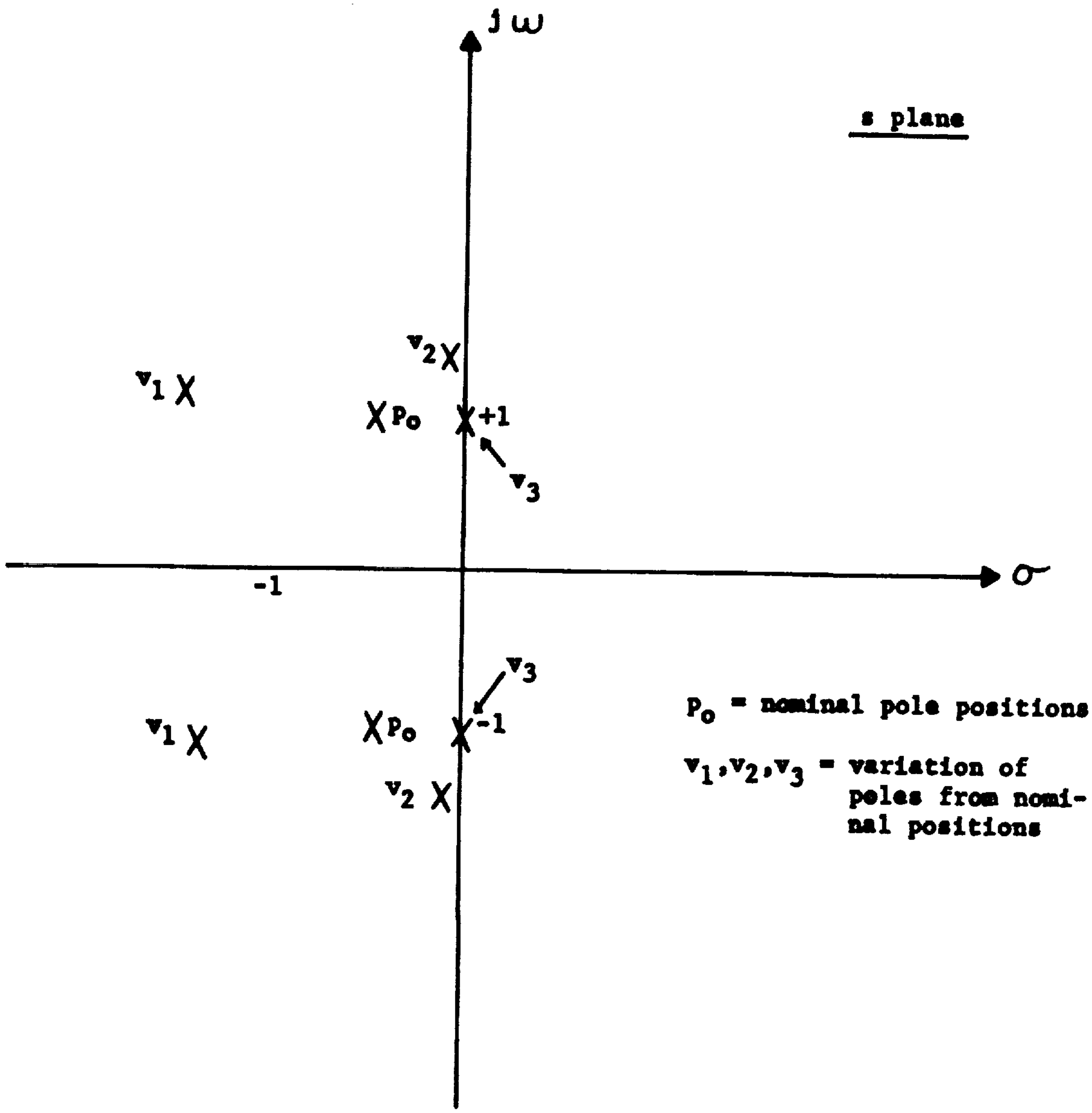


Figure 4-2: Variations of "Fixed" Poles From Nominal Positions

The inverse Laplace transform of (4-59) indicates that u(t) has a term of the form $e^{z_1 t}$, an increasing exponential. This is a consequence of the fact that the plant is to be forced to behave like a model without a right half plane zero. If the model had the same right-half plane zero as the plant, this situation would not arise. However, if the location of model and plant zeroes are not exactly the same, an exponentially increasing u(t) would still be required. Though being considered here in relation to (4-48), this problem is general and could arise in designs previously considered not related to the reduction of order technique.

In the problem under consideration, the nonlinear controller will provide the exponentially increasing control signal called for by the inverse of (4-59). However, saturation levels in the controller and plant limit the length of time over which control can be effected. When these levels are reached, the controller must be shut off.

Computer results for system behavior were obtained with zeroes as s = + .-5, s = -2.0 and also s = -0.25, s = 1.0. Other parameter values used were as indicated in section 4.5.2. In each case, the bounds previously indicated for $|z_1 = z_2|_{max}$ and $|z_1 z_2|_{max}$ are satisfied. Therefore, the control signal given by (4-58) served in these cases as well. For a step input to the model, the plant tracked the model quite closely before saturation levels were reached, and the measured control signal was almost exactly that given by the inverse transform of (4-59). The length of time required to reach saturation levels in a function of A and z_1. For instance with A = 1 volt, the controller operated for 65 seconds with $z_1 = 0.05$, but only for 9 seconds with $z_1 = +0.25$ before saturation levels within the system were reached.

Extension to Plants without Zeroes (4.6.1)

In order to broaden the class of problems for which the reduction-of-order technique can be used, consideration is given here to its extension to linear plants with slowly varying parameters but without zeroes. The ideas to be presented are applicable to plants with higher order transfer functions, but discussion here is limited to a second order plant in order not to obscure results. Let the transfer function for such a plant be

$$\frac{X}{U}(s) = \frac{K}{s(s+a)} \qquad (4\text{-}65)$$

in which K and a are slowly varying within known bounds and K > 0. In order to introduce a zero, suppose that this plant is preceeded by a unity pre-filter of the form

$$G(s) = \frac{(s + K)}{(s + K)} \qquad (4\text{-}66)$$

Note that since this prefilter has unity transfer function it need not actually be instrumented. The overall transfer function then becomes

$$\frac{X}{U}(s) = \frac{K\ (s + K)}{s(s+K)(s+a)} \qquad (4\text{-}67)$$

Since $K/(s + K)$ 1 for $0 \leq \omega << K$, then over this frequency range a good approximation to (4-67) is

$$\frac{X}{U}(s) \quad \frac{s + K}{s(s+a)} \quad : \quad \text{for } 0 \leq \omega << K \qquad (4\text{-}68)$$

Thus, with the help of the approximation, a zero has been introduced into the transfer function without raising the order of the denominator. Application of the reduction of order technique can be made directly to (4-68). If this is done using a model with equation

$$\dot{x}_d + a_o x_d = a_o r \qquad (4\text{-}69)$$

then the control law becomes

$$u = [K_{max}\ |u| + |a|_{max}\ |x| + a_o\ |r-x|]\ \text{satke} \qquad (4\text{-}70)$$

The effectiveness of this control signal is causing the plant to track a given model is dependent on K_{min} as well as the power density spectrum of the input, r. This is shown in figures 4-3 and 4-4 where $\int_0^\infty e^2 dt$ vs. K for $r = U(t)$ is plotted in the former, and $\int_0^\infty e^2 dt$ vs. ω for $r = \sin\omega t$ in the latter. These results were obtained using a digital computer simulation for the plant given by (4-65) and the model by (4-69). Parameters used were $a_o = 2$, $K_{max} = 10$, $a = 2$, and $k = 100$.

The curve in figure 4-3 serves as a design curve for a step input. It can be used to determine what K_{min} is required to meet a given specification on $\int_0^\infty e^2 dt$. If no preamplifier is needed to achieve this K_{min} (i.e. the plant gain itself is adequate), then there definitely is an advantage to be gained by having eliminated e_2 and x_2 from the control law. If preamplification is needed to achieve the required K_{min}, then an advantage may or may not be gained over using the exact control law. This is dependent on the amount of preamplification required, and whether or not this additional gain accentuates transducer noise in e_1 and x_1 to a level higher than that which would result using e_2 and x_2 to generate the exact control signal.

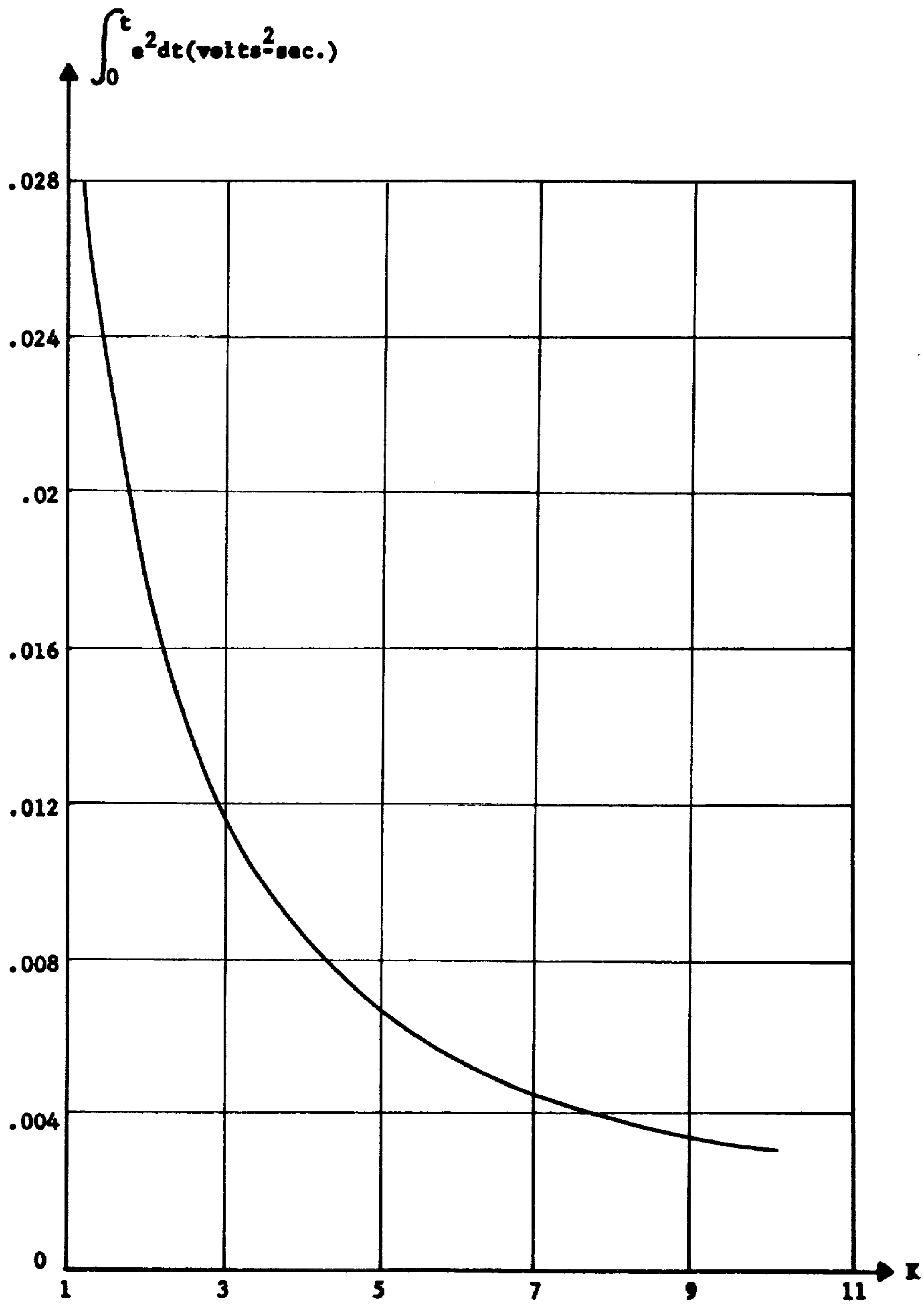


Figure 4-3: Integral Squared Error Versus Plant Gain

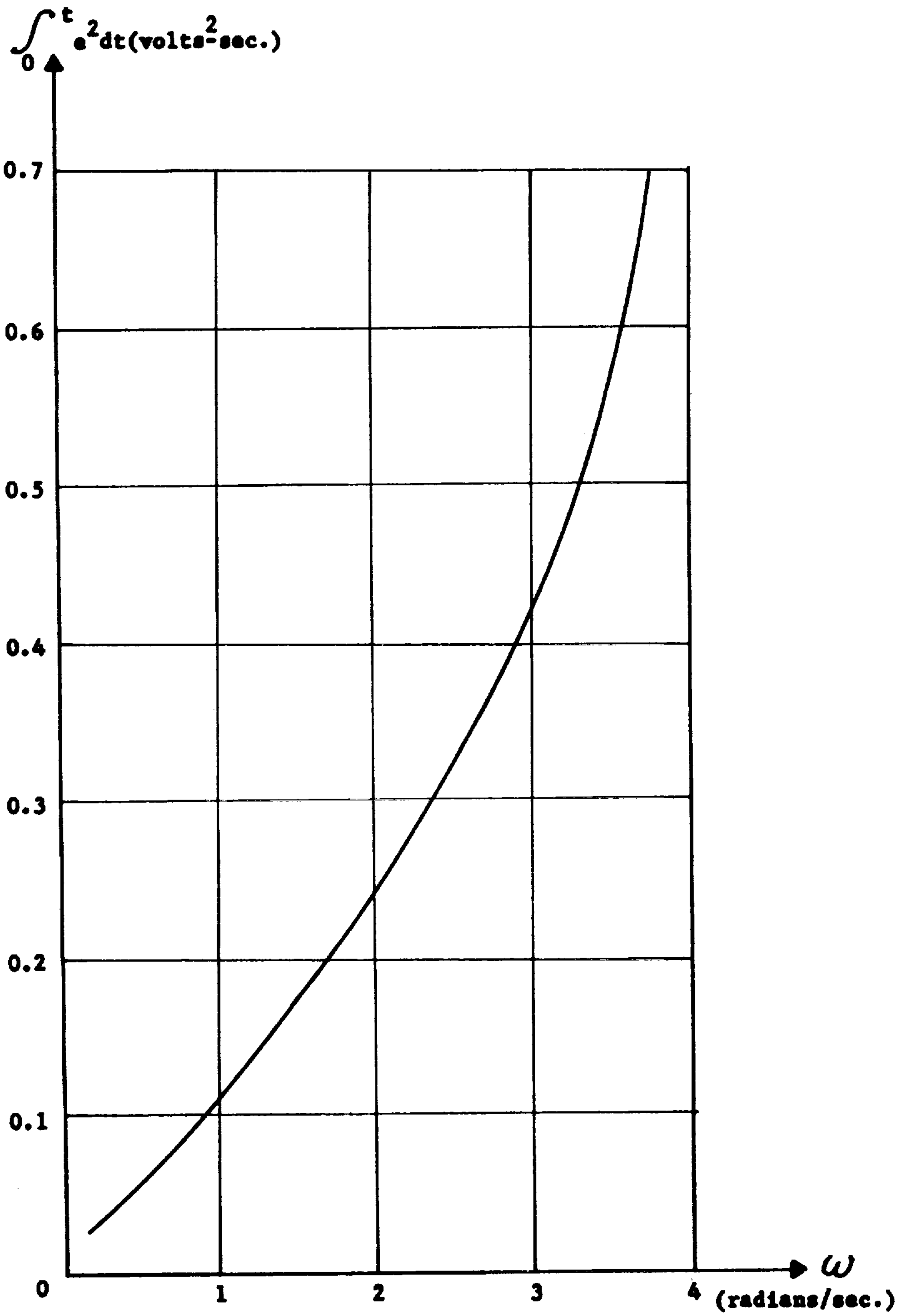


Figure 4-4: Integral Squared Error Versus Reference Input Frequency

CHAPTER V

AN ENGINEERING DESIGN PROBLEM

Introductory Comments (5.1.1)

The transducer noise problem discussed in Chapter IV is only one of the difficulties facing the design engineer in applying the controller synthesis technique to a real problem. Others which arise are brought out in this chapter by actually designing a controller for pitch axis stability augmentation of the X-15 manned re-entry vehicle[8,31], and then studying this design through an analog computer simulation of the system. This problem is chosen because of its theoretical as well as practical significance.

The extremely large range of parameter variations which are encountered over the flight regime of the vehicle are as indicated in Table 5.1[8]. The parameters listed are those of the short period approximation to the plant transfer function which is given by

$$P(s) = \frac{\dot{\theta}(s)}{\delta(s)} = \frac{K_{\dot{\theta}}\omega_a^2 T_a (s+1/T_a)}{s^2+2\xi_a\omega_a s+\omega_a^2} \qquad (5\text{-}1)$$

where $\dot{\theta}$ is pitch rate, and δ, elevator deflection angle. The actuator for the elevator is a hydraulic motor. If an approximation to its transfer function is taken as K_H/s, then this combines with (5-1) to give

$$\frac{\dot{\theta}(s)}{U(s)} = \frac{K_H K(s + z)}{s(s^2 + as + b)} \qquad (5\text{-}2)$$

TABLE 5.1 *

Variation of Parameters with Mach Number and Altitude

Plant Condition	Altitude (feet)	Mach Number	$-1/T_a$	K_δ	$\zeta_a \omega_a$	ω_a
1	35,000	0.3	-0.123	0.1066	0.2064	1.988
4	40,000	1.0	-0.282	0.312	0.445	2.972
5	40,000	1.0	-0.206	0.262	0.375	2.769
9	70,000	2.0	-0.088	0.0583	0.1052	2.631
13	100,000	4.0	0.0366	0.0223	0.0396	1.919
16	140,000	6.0	-0.00794	0.00865	0.00823	0.8067
17	120,000	6.0	-0.0184	0.0198	0.019	1.203
18	120,000	6.0	-0.02585	0.0199	0.0276	1.859
21	60,000	6.0	-0.325	0.362	0.326	4.327
28	10,000	1.2	-2.07	1.950	2.49	7.492
29	10,000	1.0	-1.975	3.42	2.31	4.915
30	10,000	0.6	-0.955	2.03	0.943	2.5708
31	5,000	0.6	-1.163	2.92	1.113	2.550
32	0	0.2	-0.0356	0.00343	0.151	1.511
33	160,000	6.0	-0.00368	0.00394	0.0038	0.555

* from reference 8

where K_H = gain of hydraulic motor

$K = K_{\dot\theta}\omega_a^2 T_a$

$z = 1/T_a$

$a = 2\zeta_a\omega_a$

$b = \omega_a^2$

U(s) - Transform of u(t), the control signal into hydraulic motor

The approximation ot the motor transfer function neglects the quadratic in s which appears in the denominator of the exact transfer function. (Refer to figure 5-10). Similarly, an approximation to the transfer function of the rate gyro used to measure $\dot\theta$ is K_g, a constant. A discussion of problems arising due to these approximations is postponed until section 5.3.3. Initially, it is assumed that the motor and gyro are exactly represented by transfer functions K_H/s and K_g respectively. Designs based on this assumption are discussed in sections 5.2.1 through 5.3.2. In section 5.2.1, the design does not make use of the reduction-of-order technique. This design is then compared to that of section 5.3.1 which is based on reduction-of-order. The comparison points out the considerable advantages of the latter design.

Design Not employing The Reduction-of-Order Technique (5.2.1)

Since the transfer function of (5-2) has a left half plane zero and a fixed pole at the origin, it is possible to apply the reduction-of-order technique introduced in section 4.5.1. If applied, the reduced plant transfer function is second-order, consequently a second-order model can be used. In order to demonstrate the advantages of the reduction-of-order technique, a design not using it is presented in this section, and then compared to the design which does use it given in section 5.3.1. An obvious advantage of the latter design is that the second derivative of pitch rate is not required in generating the control signal. Other advantages will be mentioned in section 5.3.1.

A block diagram for the design not employing the reduction-of-order technique is shown in figure 5-1. Because the plant has a zero, the reference input, $\dot\theta_r$, is not used as an input to the hydraulic motor. As discussed in section 2.2.2, this is required to avoid impulses in the control signal when $\dot\theta_r$ has finite discontinuities. The state equation derived from (5-2) is

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & 0 \end{bmatrix} \underline{u} \qquad (5\text{-}3)$$

where $k_1 = K_H K_z$, and $k_2 = K_H K$

Model selection plays an important part in the design procedure. One must be chosen which causes the plant behavior to satisfy specifications, and also makes the elements of the ΔA matrix as small as possible. Some trial and error is a necessary part of this selection procedure.

The specification on responses to step inputs is quoted from reference 8: ".... the system is required to have less than 25% overshoot and to damp to one eighth amplitude or less in one cycle when subjected to a step input. Furthermore, the response time (time to reach 90% of the command (reference input) value) shall be less than three seconds." The specification does not stipulate how much less than three seconds the response time might be, but in selecting the model it was assumed that the vehicle could not realistically be expected to perform the maneuver in too much less time than this. With this in mind, as well as the goal of keeping elements of ΔA small, the following third order model was selected:

$$\underline{\dot{x}}_d = A_o \underline{x_d} + B_o \underline{r} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.2 & -2.81 & -1.3 \end{bmatrix} \underline{x_d} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.2 & 0 & 0 \end{bmatrix} \underline{\theta}_r \qquad (5\text{-}4)$$

The model was intentionally chosen not to have derivatives of $\dot{\theta}_r$ so that impulses would be avoided in generating the control signal.

According to the design procedure, $\underline{e}$ is defined as $\underline{x_d} - \underline{x}$ and (5-3) is subtracted from (5-4) to give

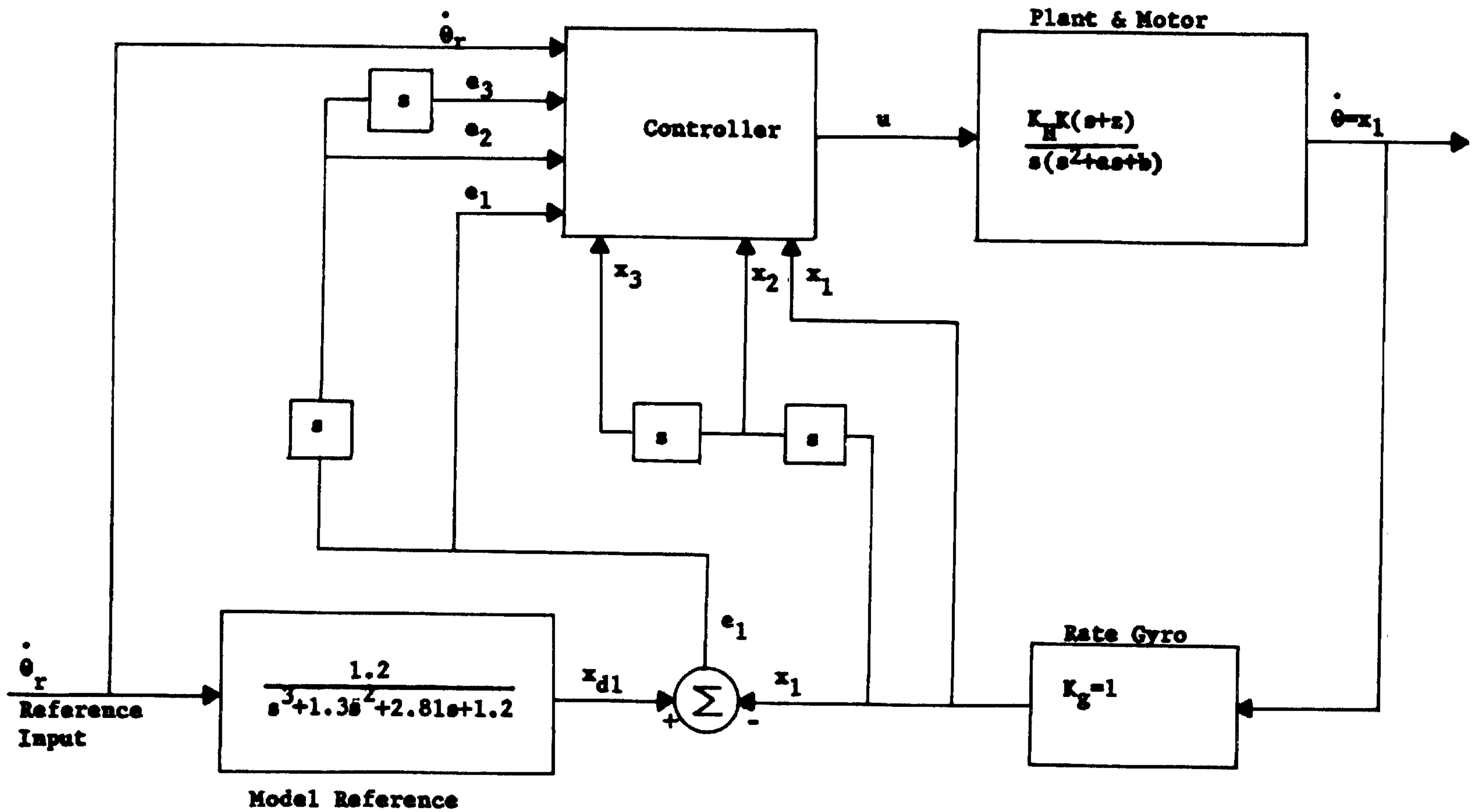


Figure 5-1: Block Diagram For Design Not Employing Reduction-of-Order Technique

$$\dot{\underline{e}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.2 & -2.81 & -1.3 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1.2 & \beta & \alpha \end{bmatrix} \underline{x}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1.2 \end{bmatrix} \dot{\theta}_r + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & 0 \end{bmatrix} \underline{u} \qquad (5\text{-}5)$$

where $\beta = b - 2.81$ and $\alpha = a - 1.3$.

The control law required to guarantee that $|\underline{e}|$ is ultimately bounded is

$$\dot{u} = [c_1^*|\dot{\theta}_r - x_1| + c_2^*|x_2| + c_3^*|x_3| + c_4^*|u|] \text{ satk}\gamma \qquad (5\text{-}6)$$

where $c_1 = |1.2/k_2|$, $c_2 = |\beta/k_2|$, $c_3 = |\alpha/k_2|$, and $c_4 = |k_1/k_2|$. The maximum value for each of these coefficients is denoted by an asterisk and may be found in Table 5.2. The coefficient c_4 is independent of K_H. Coefficients c_1, c_2, and c_3 were calculated assuming $K_H = 1$. If the hydraulic motor gain is greater or less than one, then c_1 through c_3 are decreased or increased respectively. For Q = I, the coefficients of the γ function are $p_{13} = 0.415$, $p_{23} = 1.229$. $p_{33} = 1.330$. The parameter values found above numerically specify the control law. The control signal, u, is obtained by a single integration of $\dot{u}$ in (5-6).

One engineering design problem is evident from figure 5-2 where instrumentation for generating u is shown. Although the gyro used to measure x_1 has to a first approximation been assumed to be ideal, x_2 and x_3 are generated using approximate derivative circuits. It will not be assumed that ω_d is chosen large enough to neglect, but instead, ω_d will be made as small as practically possible in order to avoid accentuating high frequency noise. Since the derivative circuit poles were not accounted for in the design, a

TABLE 5.2

Parameter Variations in Equation (5-5)

Condition	$c_1=\|1.2/k_2\|$	$c_2=\|\beta/k_2\|$	$c_3=\|\alpha/k_2\|$	$c_4=\|k_1/k_2\|$
1	0.35	0.375	0.244	0.123
4	0.132	0.75	0.0525	0.282
5	0.123	0.5	0.054	0.206
9	0.261	0.9	0.237	0.088
13	0.539	0.36	0.585	0.0366
16	1.695	3.04	1.8	0.00794
17	0.771	0.89	0.8	0.0184
18	0.448	0.24	0.465	0.02585
21	0.058	0.76	0.0308	0.325
28	0.0228	1.02	0.0625	2.07*
29	0.0287	0.5	0.0805	1.975
30	0.0854	0.16	0.025	0.955
31	0.735	0.23	0.06	1.163
32	5.45*	2.8	4.65*	0.0356
33	3.64	7.6*	3.94	0.00368

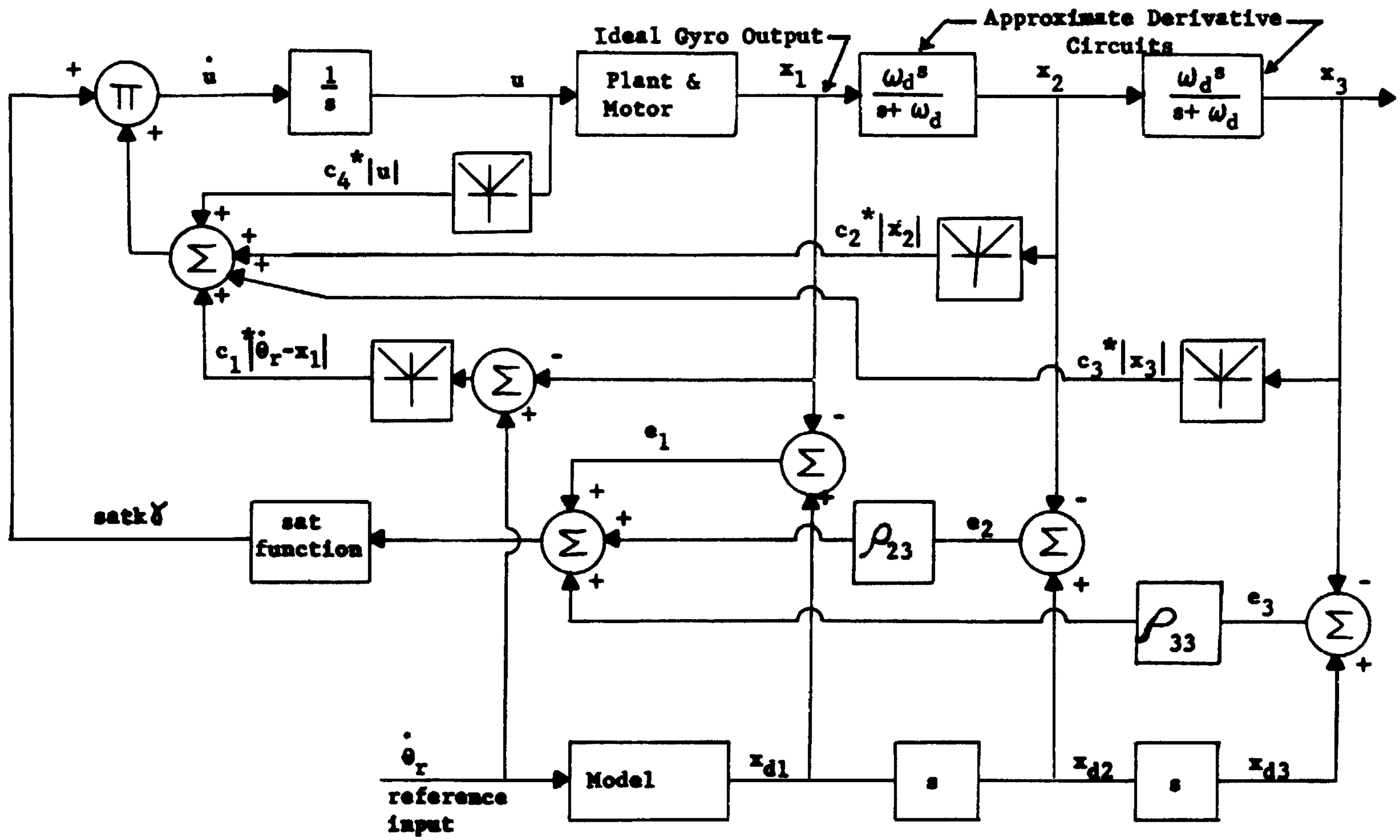


Figure 5-2: Controller Details For Design Not Employing Reduction-of-Order Technique

stability problem resulted for certain plant parameter values. In particular, even with ω_d = 100 rad/sec, a figure much larger than the model bandwidth which is only on the order of one rad/sec, the system oscillated at 16 cycles per second for plant condition 28, probably because this condition has the highest plant gain, k_2 = 52.5. The amplitude of this oscillation, though not significant in x_1, was large enough in the derivative signals to cause the steady state value of x_1 to be 30% lower than the desired value. The oscillation could be eliminated by setting c_3 to zero in (5-6). Performance with $c_3 = 0$ was still satisfactory, which points out that conditions imposed on $\dot{u}$ by the control law are sufficient, but not necessary. The magnitude function had sufficient amplitude even with $c_3 = 0$. This coefficient remains zero for the remainder of the discussion.

Controller performance was checked for values of ω_d less than 100 in anticipation of the transducer noise problem. The performance measure used was the ratio of e_p, the peak error during the transient response to a step input, to R, the amplitude of the reference step input. Results are shown in figure 5-3 for plant condition 9. It is seen that sensitivity to ω_d is quite high for $\omega_d < 50$. An undamped oscillation occurred at a frequency of 1.4 cycles per second when ω_d was reduced to 10 rad/sec. Even with ω_d = 20, a damped oscillation of 3.15 cycles per second occurred during the transient. The amplitude of this oscillation was approximately 5% of the input signal amplitude.

Results shown in figure 5-3 will be compared later in section 5.3.1 with the sensitivity to ω_d for the design based on the reduction-of-order technique. There it will be seen that ω_d for the single derivative circuit required can be reduced almost to the model bandwidth without noticeable deterioration of performance. Thus, the advantage accrued from this technique in the form of noise reduction is obvious.

The undesired oscillations and sensitivity to ω_d make the design technique impractical for this problem if reduction-of-order is not used. However, when the reduction-of-order technique is employed, much more encouraging results are obtained. This is considered in the next section.

Design Employing the Reduction-of-Order Technique (5.3.1)

As discussed in section 5.2.1, the reduction-of-order technique is applicable to the transfer function of (5-2). Employing the procedure of section 4.5.1 gives the following decomposition for that transfer function:

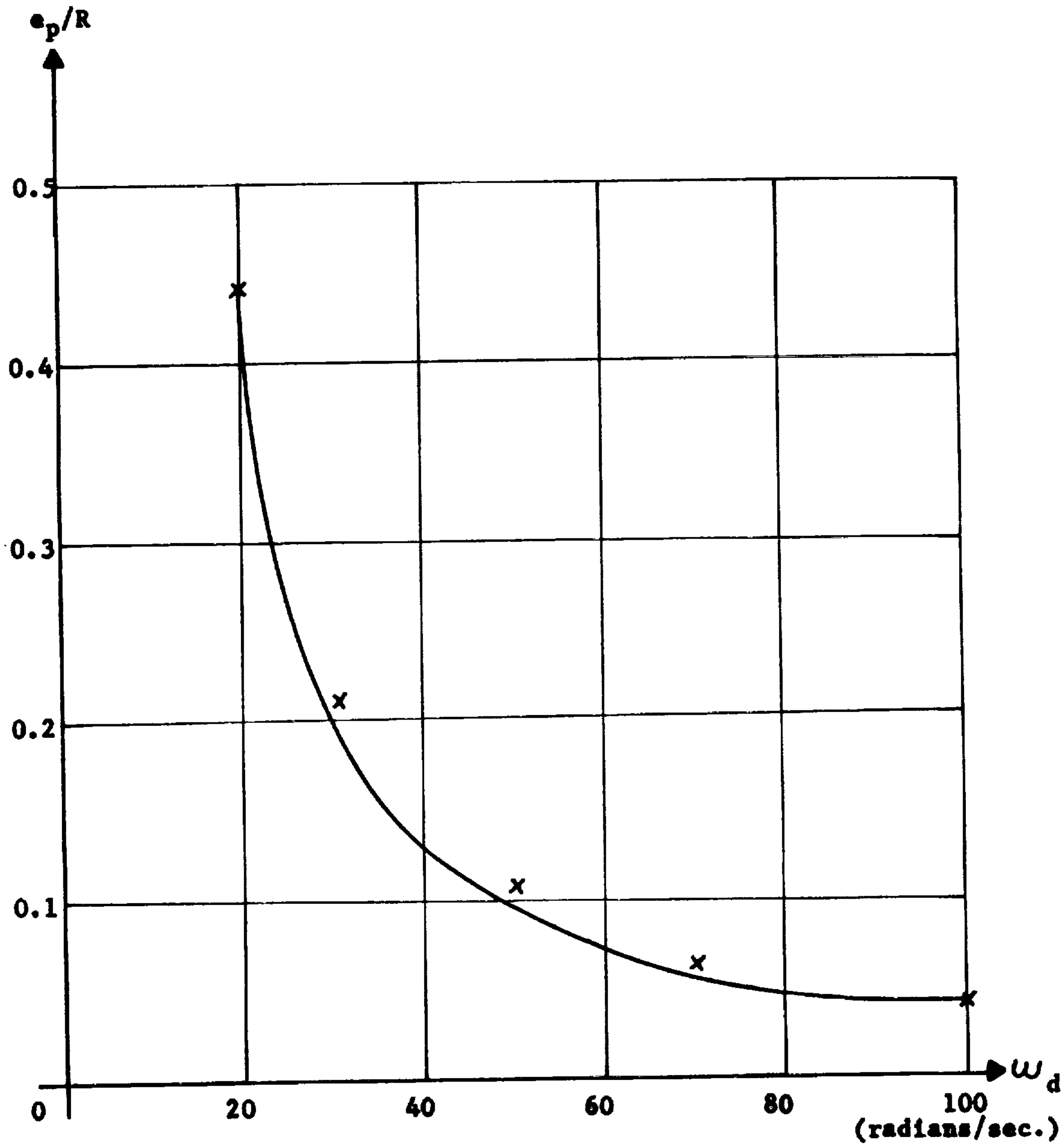


Figure 5-3: Normalized Peak Error Versus Derivative Circuit Bandwidth

$$P_p(s) = \frac{Y}{U}(s) = \frac{1}{s} \tag{5-7-a}$$

$$P_z(s) = \frac{W}{Y}(s) = s + z \tag{5-7-b}$$

$$P_r(s) = \frac{X}{W}(s) = \frac{k_2}{s^2 + as + b} \tag{5-7-c}$$

where the notation of section 5.2.1 has been employed.

A second-order model which meets the specifications regarding response to step inputs and which has been chosen properly with regard to other factors previously discussed is

$$\dot{\underline{x}}_d = A_o \underline{x}_d + \underline{b}_o r = \begin{bmatrix} 0 & 1 \\ -0.8 & -1.27 \end{bmatrix} \underline{x}_d + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \dot{\theta}_r \tag{5-8}$$

The state equation relating to the reduced plant equation, (5-7-c), is

$$\dot{\underline{x}} = A\underline{x} + \underline{d}w = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ k_2 \end{bmatrix} w \tag{5-9}$$

The error equation resulting when (5-9) is subtracted from (5-8) is

$$\underline{\dot{e}} = \begin{bmatrix} 0 & 1 \\ -0.8 & -1.27 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 & 0 \\ \beta & \alpha \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \dot{\theta}_r - \begin{bmatrix} 0 \\ k_2 \end{bmatrix} w \qquad (5\text{-}10)$$

where $\alpha = a - 1.27$ and $\beta = b - 0.8$. Forming the Liapunov function for (5-10) and taking its derivative leads to the control law

$$\dot{y} = [\ c_1^* |\dot{\theta}_r - x_1| + c_2^* |x_1| + c_3^* |x_2| + c_4^* |y|]\ \text{sat} k\gamma \qquad (5\text{-}11)$$

where $c_1 = |0.8/k_2|$, $c_2 = |b/k_2|$, $c_3 = |\alpha/k_2|$, $c_4 = |k_1/k_2|$. Asterisks denote maximum values. These may be found in Table 5.3 for c_1, c_2, and c_3. The maximum value for c_4 is found in Table 5.2.

Since the prefilter, (5-7-a), consists of an integrator, the control signal, derived directly from (5-11) and the inverse transform of (5-7-a), is $u = \dot{y}$.

The coefficients of the γ function, found from the solution of

$$A_o^T P + P A_o = - \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix} \qquad (5\text{-}12)$$

are $p_{12} = 0.625$ and $p_{22} = 0.496$. The element q_{22} was deliberately chosen as 0.01 rather than 1.0 to give a smaller coefficient for e_2 in the switching function, and thereby help to reduce errors due to noise. For the P matrix

TABLE 5.3

Parameter Variations in Equation (5-10)

Condition	$c_1 = \|0.8/k_2\|$	$c_2 = \|b/k_2\|$	$c_3 = \|\alpha/k_2\|$
1	0.234	1.55	0.219
4	0.088	0.905	0.042
5	0.082	0.786	0.0535
9	0.174	1.51	0.23
13	0.359	1.64	0.214
16	1.130	0.916	1.775
17	0.515	0.931	0.79
18	0.298	1.300	0.456
21	0.0387	0.898	0.03
28	0.0152	1.061	0.07
29	0.0192	0.578	0.08
30	0.0569	0.47	0.043
31	0.49	0.398	0.0585
32	3.64*	10.4*	4.31*
33	2.43	0.934	3.82

elements satisfying (5-12) the switching function becomes $\gamma = e_1 + 0.794\ e_2$. If $q_{22} = 1.0$ had been used, the switching function would have been $\gamma = e_1 + 1.42\ e_2$. Thus, use of $q_{22} = 0.01$ rather than 1.0 leads to a reduction of approximately 45% in the coefficient of e_2.

Using the control law derived above, the plant output followed that of the model very closely for all plant conditions listed in Table 5.1, and the problems which arose in the design not employing reduction-of-order were eliminated. Oscillations did not occur for any of the parameter values listed even though the pole in the derivative circuit transfer function (see figure 5-4) was again neglected in design. Sensitivity to ω_d was much less for this design than indicated in figure 5-3 for the design of section 5.2.1. This result might have been anticipated since the energy content of the first-derivative signal, the highest derivative required in this case, is restricted to a lower range of frequencies than the energy of the second derivative signal. Consequently, distortion in the switching function is less in this design for a given ω_d. These and other results are discussed quantitatively below.

Derivative Circuit Bandwidth, Dependence of Transient Response and Noise on Reference Input Amplitude, and Disturbance Response (5.3.2)

Introductory Comments.-Since the design of section 5.3.1 based on the reduction-of-order technique showed promise, it was subjected to an extensive analog computer investigation to point out clearly the difficulties which would arise in the transition from the theoretical to the hardware stage. This study led to the techniques presented in this section for minimizing some of the difficulties.

The sensitivity of response to derivative circuit bandwidth, ω_d, was determined. The minimum value of ω_d for which the specifications could be satisfied was chosen as the design value. It is shown that this procedure led to the lowest possible mean squared noise level into the hydraulic motor.

The form of the response to step reference inputs was dependent on the magnitude of the input, i.e. the plant did not track the model well for low level inputs. For a given low level reference input, tracking could be improved by either increasing the gain k in the linear region of the saturation function or increasing coefficients of signals in the magnitude function. The effect that increasing these gains has on the mean squared noise levels into the motor, and the possibility of a trade-off between tracking accuracy for low level inputs and this noise level are discussed.

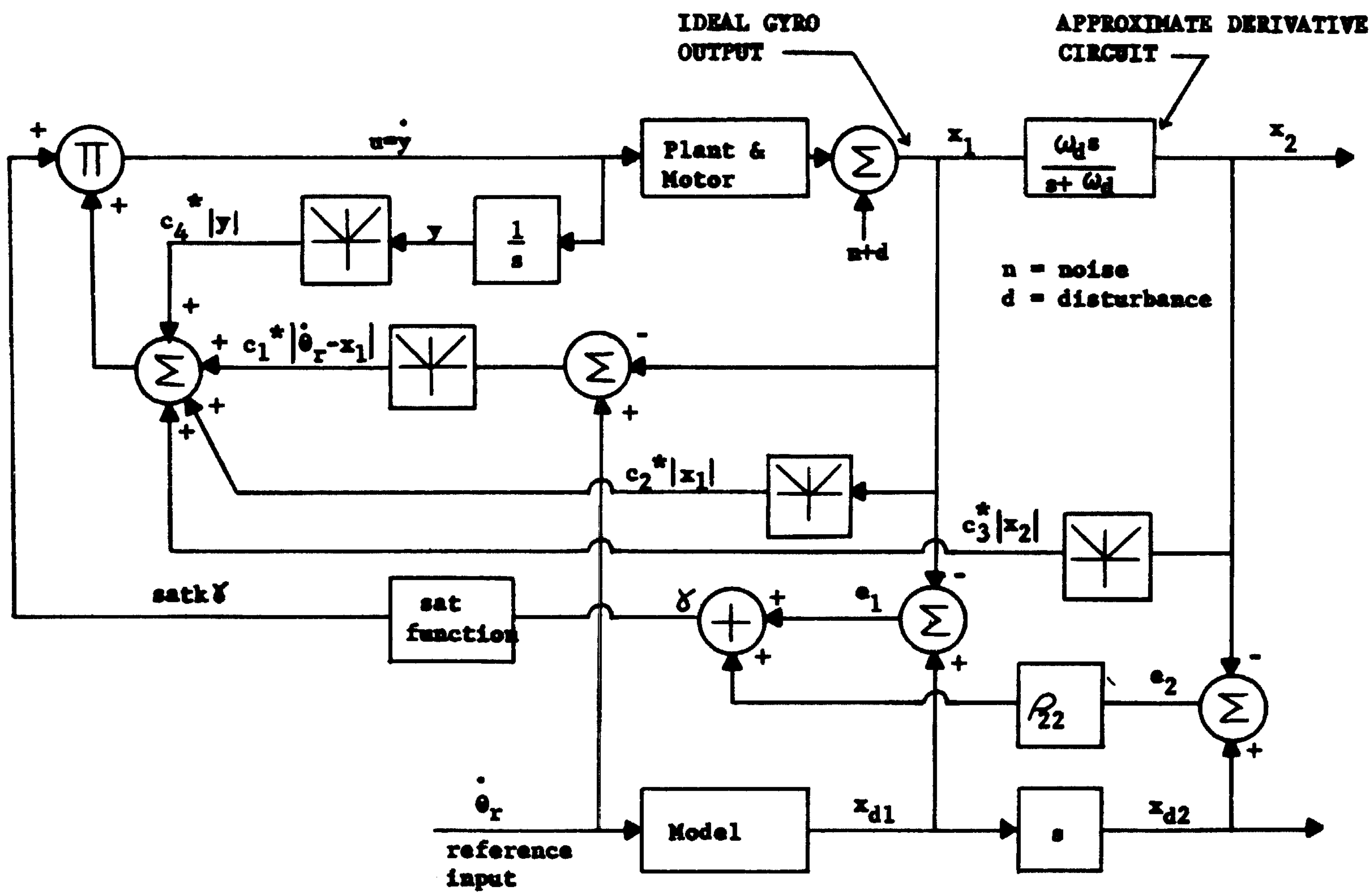


Figure 5-4: Controller Design Using Reduction-of-Order Technique

The mean squared noise level into the motor was larger for larger step reference inputs because the magnitude function of the control law was larger. A technique for using approximate signals in the magnitude function is introduced. The approximate signals are such as to reduce the steady state value of the magnitude function to zero. It is shown that this leads to considerable reduction in the noise level.

The controller was shown to have a disturbance rejection capability, i.e. for step disturbance d = U(t), (see figure 5-4) $x_1 \to 0$ in the steady state. The response to disturbance was noted for various values of the parameters ω_d, k, and c_1^* through c_4^*. This was done in order to determine what tradeoff existed between the form of disturbance response and the mean squared noise level into the motor.

<u>Derivative Circuit Bandwidth</u>.-The mean squared noise level into the hydraulic motor, u_n^2, is plotted in figure 5-5 versus the derivative circuit bandwidth, ω_d, for plant conditions 16 of Table 5.1. A one-volt step reference input signal was used. The level of the reference input is mentioned explicitly because u_n^2 is a function of $|\dot{\theta}_r|$ as is discussed below. It is clear from figure 5-5 that the lowest possible value of ω_d should be chosen which allows the specification on response to step commands to be met.

The integral squared error (hereafter referred to as ISE) for a step reference input was used as a measure for determining how the transient response was affected by reducing ω_d. The form of the response was also noted to insure that it conformed to the specifications. In contrast to results obtained in section 5.3.1 for the design not employing the reduction-of-order technique (see figure 5-3), the response in this design was insensitive to ω_d for $\omega_d > 2$ radians/sec. Even for $\omega_d = 2$, the ISE for all plant conditions given in Table 5.1 was approximately equal to that for $\omega_d > 2$, and the slight variations in the forms of the transient responses, most pronounced for plant conditions 16, 32, and 33, were not such as to fail to meet specifications. Accordingly, $\omega_d = 2$ radians/second is satisfactory for the design insofar as the specifications relating to response to command inputs is concerned. This value for ω_d will lead to considerably lower noise levels into the motor than $\omega_d > 20$, the value which would have been required in the design of section 5.3.1.

<u>Transient Response Dependence on Reference Input Amplitude</u>.-As mentioned briefly in section 2.2.1, certain difficulties arise because the sign function in the control law must be replaced by the saturation function. The origin of these difficulties is that negative definiteness of the $\dot{V}$ function is not guaranteed by the control law in the region where $|\gamma| < 1/k$. The most

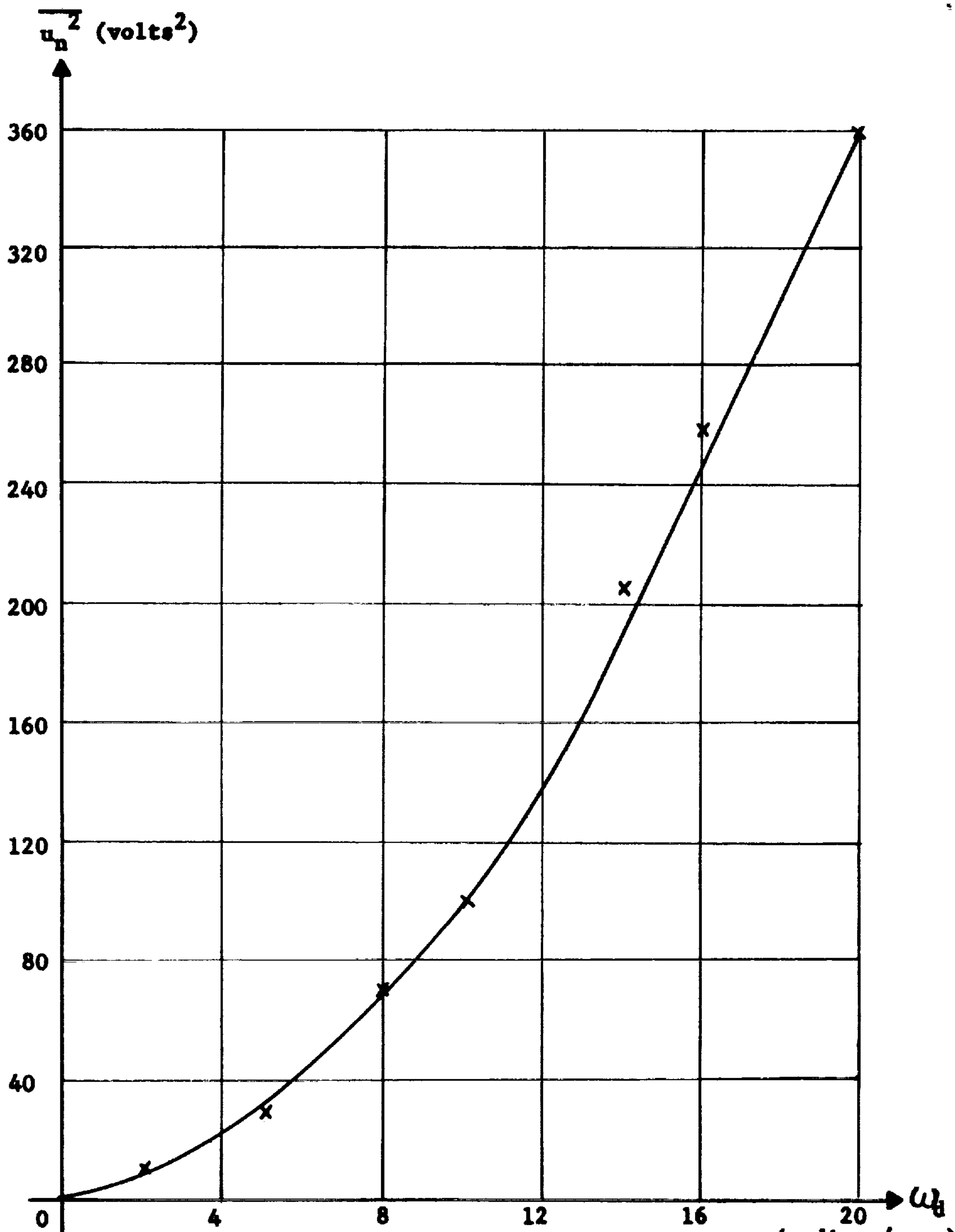


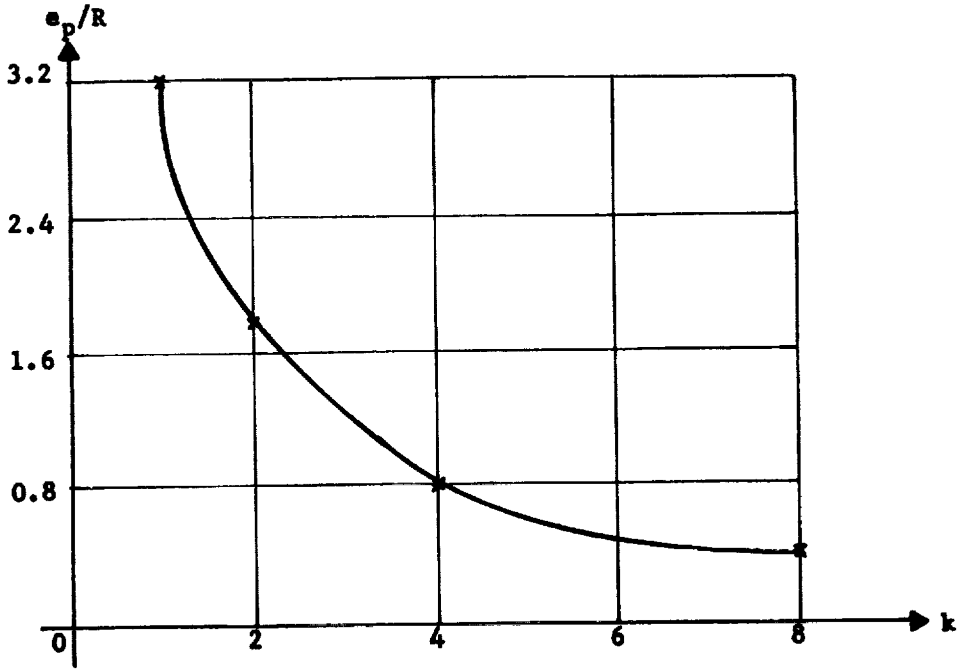
Figure 5-5: Mean Square Noise Level into Plant Versus Derviative Circuit Bandwidth

disturbing practical consequence of this is that system performance is dependent on the reference input amplitude. This dependence was manifested by a failure of the plant to track the model for small input amplitudes. An inverse relationship was found between the reference input amplitude for which failure to track occurs and the gain k in the linear region of the saturation function. The larger k was, the smaller the reference input amplitude could be before this deterioration of performance occurred. However, larger values of k caused an increase in u_n^2. Thus, a trade-off had to be made between tracking accuracy for low level inputs and noise power into the plant.

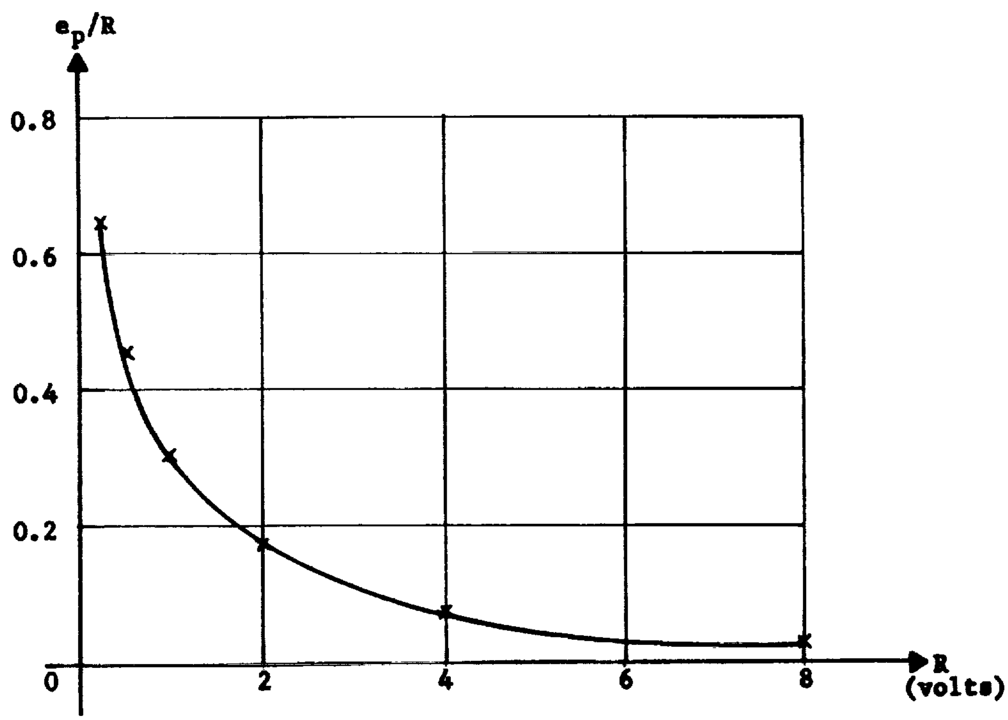
In deciding what trade-off to make, the plant exhibiting the worst degradation of performance was used. The ISE to a step input was used as a measure to determine that plant condition 16 was the most difficult to handle, giving the largest ISE. The next most difficult was plant condition 32 which has an ISE equal to one third of the previous case. Plant conditions giving the lowest value of ISE to a step input were 28, 29, and 31. All of these were approximately one-twentieth of the value for condition 16. A gain k equal to one was used in making all of these measurements.

In order to demonstrate reference input-amplitude dependence, and the dependence of the transient response on the parameter k, some computer results are shown in figure 5-6. There the normalized peak error e_p/r, where R is the amplitude of the step reference input and e_p is the peak error during the transient, is plotted versus k and R in figures 5-6a and 5-6b respectively. From these results it is clear that information concerning the minimum expected reference input amplitude must be available to the designer if he is to make an intelligent choice of k.

Before a value of k was selected based solely on the considerations above, further attempts were made to reduce the ISE to a step input by increasing the coefficients of the magnitude function variables one at a time. Replacing any one of the coefficients c_1*, through c_4* in (5-11) by larger coefficients led to a reduction in ISE. However, only increasing c_1* reduced ISE <u>without</u> increasing u_n^2. Because increasing c_1* had this effect, a trade-off could be made between c_1* and k, i.e. c_1* could be replaced by a larger value and k decreased while the ISE was held constant. The advantage which was gained by the procedure was that u_n^2 was decreased due to the reduction of k. Results of the procedure are illustrated in figure 5-7. One curve shown is for k = 0.5 and c_1^* = 100, and the other is for k = 1.0 and c_1* = 10. The ISE to a step input is the same for both sets of parameters. It is seen that for $n^2 < 0.029$ volts^2, the former values for k and c_1* lead to lower values of u_n^2. Thus, decreasing k and increasing c_1* gives a clear advantage only if the transducer mean squared noise level is less than 0.029 volts^2.



a) Normalized Peak Error Versus Saturation Function Gain



b) Normalized Peak Error Versus Magnitude of Reference Input

Figure 5-6

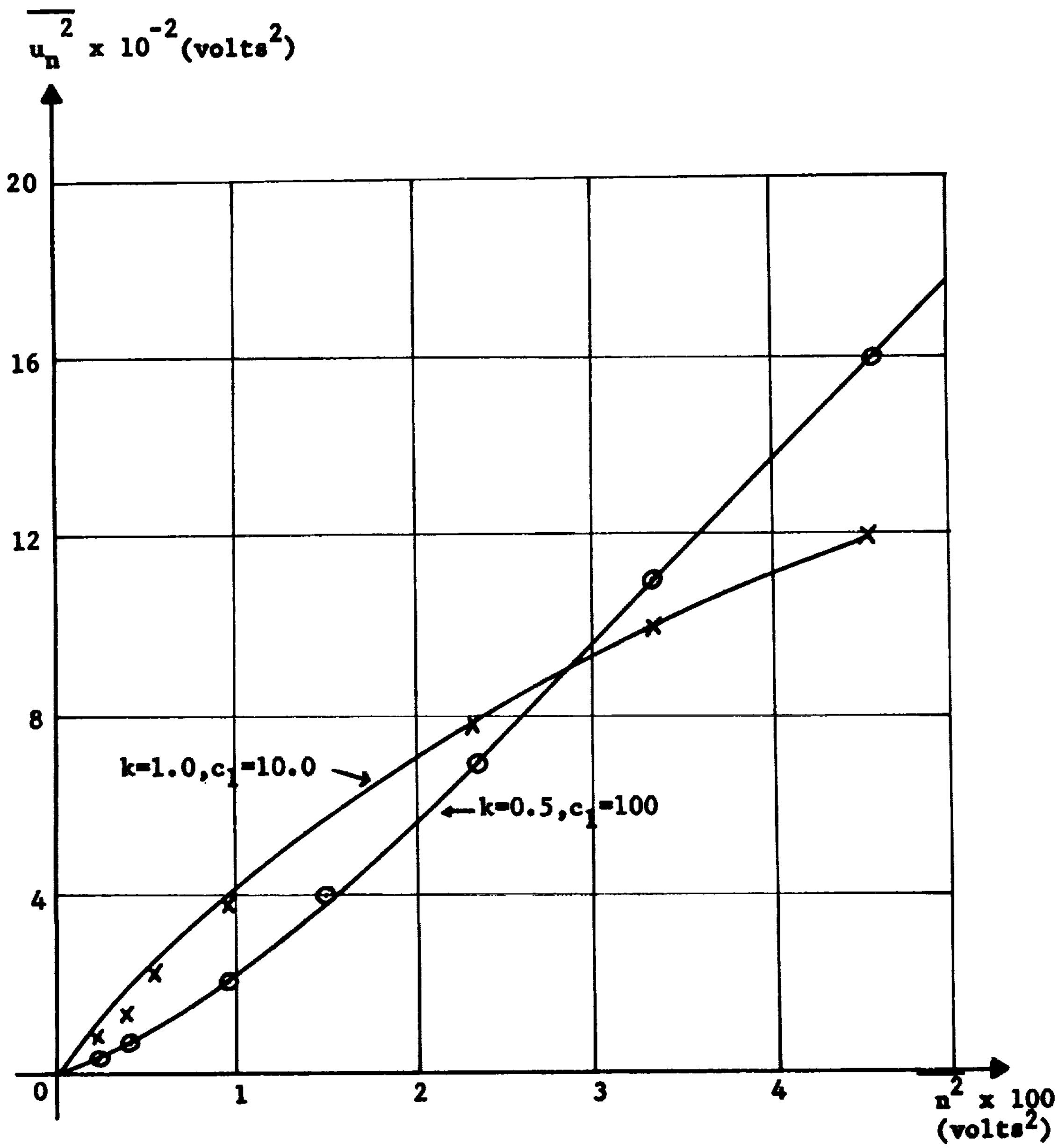


Figure 5-7: Effect of Varying k and c_1 on Mean Squared Noise Level

Each problem must be approached with the above considerations in mind to arrive at suitable values for k and c_1*. In this problem, k and c_1* cannot be optimally specified without adding specifications as to allowable noise power into the motor, minimum expected amplitude for $\dot{\theta}_r$, and transducer noise level.

Dependence of Noise Power into Motor on Reference Input Amplitude.-A disturbing practical problem in the design was that the mean squared noise level into the hydraulic motor, u_n^2, was dependent on $|\dot{\theta}_r|$. This is shown in figure 5-8 for plant condition 16. For $\dot{\theta}_r = 0$, u_n^2 was 0.59 volts^2, a value too small to be read from the curve.

The explanation of this fact is that the magnitude function of the control law is a nonlinear, time-varying gain for noisy signals in the saturation function, and this gain achieves a larger steady state value for larger step reference inputs. The terms which cause this increased value for the magnitude function are $c_2^*|x_1|$ and $c_4^*|y|$ in (5-11). A way to eliminate this problem is discussed below.

The control signal u theoretically goes to zero in the steady state for a step reference input because γ goes to zero. Since this is the case, it is reasonable to ask whether or not the magnitude function might be allowed to go to zero in the steady state as well as the switching function without serious consequences. If so, then u_n^2 would be independent of $|\dot{\theta}_r|$, and would have the value noted above for $\dot{\theta}_r = 0$. Fortunately, this procedure worked and the advantage was gained.

Means for accomplishing the results above are as follows. Instead of generating y from $\dot{y}$ by a true integration, a network for approximate integration is used which has the transfer function

$$\frac{Y_a}{Y}(s) = \frac{1}{s+b} \qquad (5\text{-}13)$$

where the subscript a is used to denote approximate value. If b in (5-13) is small enough, y_a approximates y closely enough over a long enough time to give controller behavior identical to that obtained when y is used in the magnitude function. But unlike y, y_a goes to zero in the steady state since $\dot{y}$ goes to zero. The step response of the system was unaffected when y_a replaced y with b = 0.03 in (5-13).

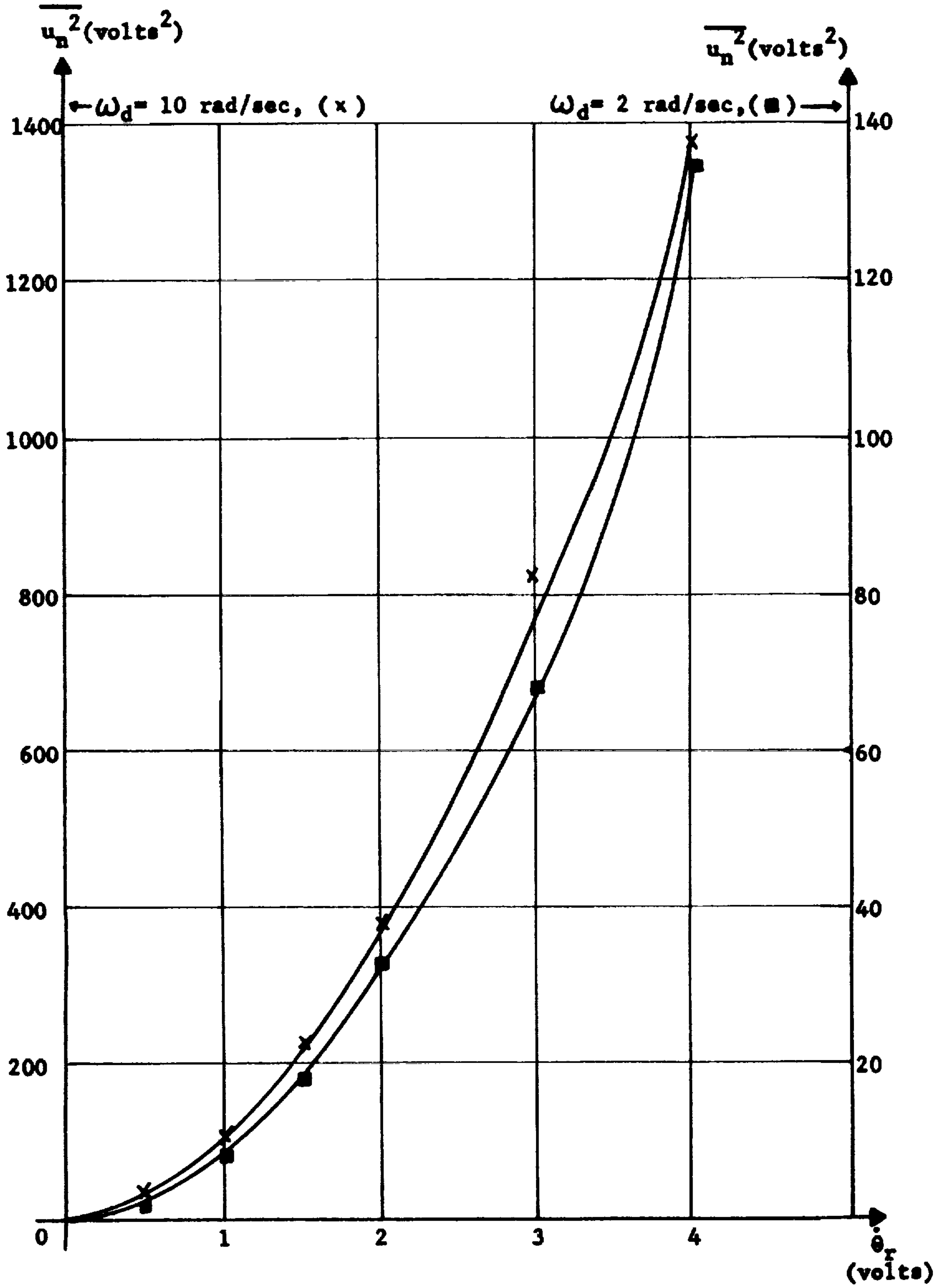


Figure 5-8: Effect of Reference Input Level on Mean Squared Noise Level

As for the signal $|x_1|$, it was replaced by $|x_{1a}|$ where x_{1a} was generated using a network with the following transfer function

$$\frac{X_{1a}}{X_1}(s) = \frac{s}{s+d} \qquad (5\text{-}14)$$

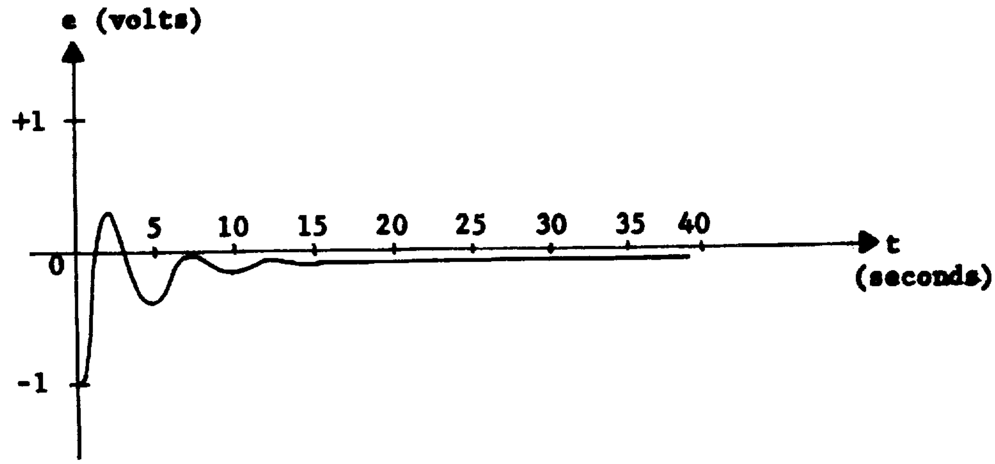
With d small enought, x_{1a} approximates x_1 closely enough during the transient period, and in the steady state it goes to zero. A satisfactory value for d was found to be 0.01.

With these approximate signals replacing the exact signals in the magnitude function of $\dot{y}$, u_n^2 was reduced to 0.54 volts^2, independent of $|\theta_r|$.

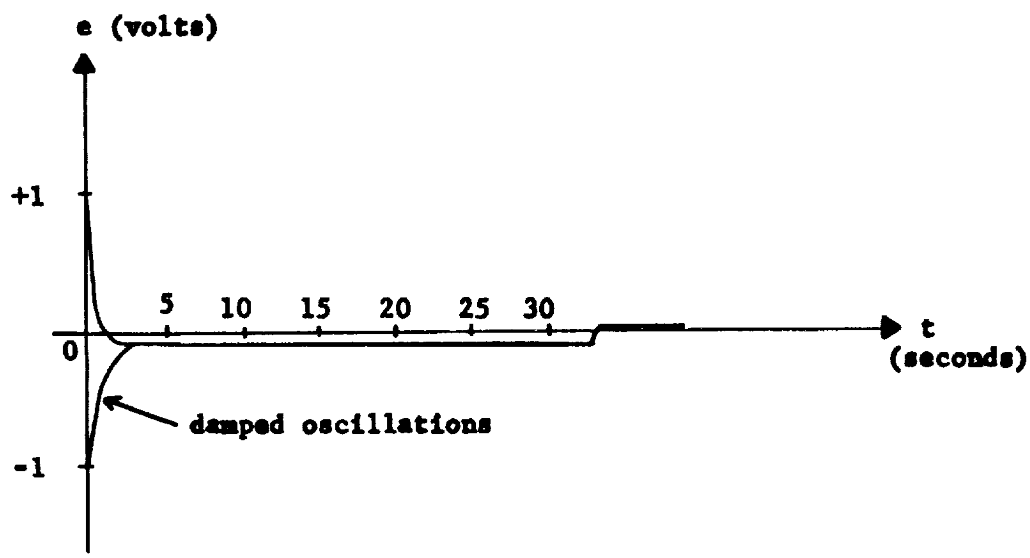
Disturbance Response.-In section 2.4.1 it was shown that one is generally unable to design for a specific disturbance response. This is an unfortunate restriction in the case of the X-15 problem since there is a specification on disturbance response given in reference 8, which is quoted here: "As far as gust disturbance response is concerned, the requirement is that such disturbances will be damped to less than one-fourth amplitude in one cycle."

Though the controller given by the control law of (5-11) could not be designed to satisfy the disturbance response specification quoted, it provided rejection to step disturbances, d = U(t) in figure 5-4, under all plant conditions of Table 5.1. For these plant conditions, the disturbance response exhibited a damped oscillatory character with $x_1 \rightarrow 0$ in the steady state. Two extremes in terms of frequency of the response to a step disturbance are shown in figure 5-9 for plant conditions 16 and 28. Table 5.4 gives values of ISE with a one-volt step disturbance for all plant conditions except 5, 17 and 30. These were excluded because of their similarity to 4, 18 and 31 respective.y In order to determine whether or not some control could be gained over the form of the disturbance response, the same parameters which were varied in connection with the noise studies were again varied. What was being sought was the basis for a trade-off between disturbance response and mean squared noise level into the plant.

For plant conditions 16 with $\dot{\theta}_r = 0$ and d = 1 volt, ω_d was varied from 2 to 20 radians. For $\omega_d > 5$, the ISE remained essentially equal to 0.86 volt^2-seconds. For $\omega_d = 5$ it was 0.88 and for $\omega_d = 2$ it was 1.15. The form of the response was almost identical in all cases. For $\omega_d = 2$, the peak overshoot was approximately 40% greater than for $\omega_d \geq 5$. Thus, though $\omega_d = 2$ was previously chosen on the basis of minimizing u_n^2, it may not be the best choice from the point of view of disturbance response because of the larger overshoot.



a) PLANT CONDITIONS 16



b) PLANT CONDITIONS 28

Figure 5-9: Disturbance Response For Two Plant Conditions

TABLE 5.4

Integral Squared Error for One Volt Step Disturbance

Condition	ISE(volts^{2}sec.)
1	0.307
4	0.26
9	0.372
13	0.40
16	0.86
18	0.403
21	0.562
28	0.140
29	0.177
31	0.22
32	0.45
33	0.52

When the gain k in the saturation function was varied from 1 to 10, the ISE was reduced by only 16%. The form of the response was generally the same for these values of k. However, for k = 10 the frequency of the damped oscillation was 0.5 c.p.s. and it persisted for three cycles, while for k = 1 the frequency was 0.134 c.p.s. and only one cycle occurred.

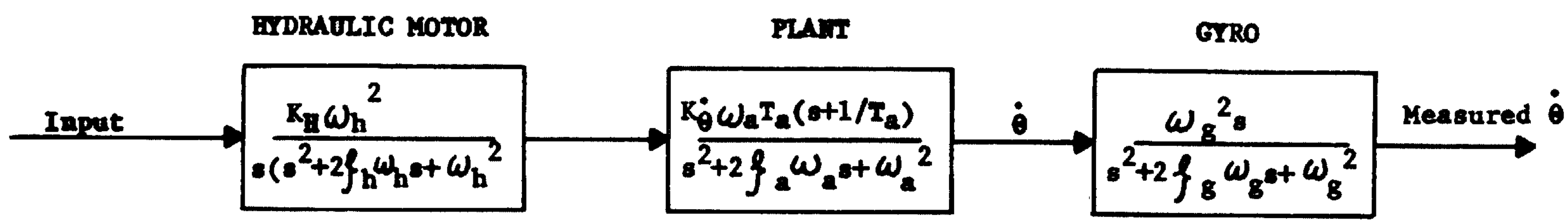
Varying the coefficient c_1^* and replacing x_1 and y by approximate values in the magnitude function changed neither the form of the disturbance response nor the ISE.

The parameter ω_d and k, then, do provide some basis for a trade-off between u_n^2 and the form and ISE of the step disturbance response.

Design Including Hydraulic Motor Dynamics and Gyro Dynamics (5.3.3)

The complete system, i.e. one including hydraulic motor and gyro complex, far out poles, is shown in figure 5-10 along with its component transfer functions and the overall transfer function. The overall transfer function is seventh-order instead of the third-order one used for the incomplete system which excluded motor and gyro far out poles. Use of the reduction-of-order technique for the complete system would lead to a sixth-order reduced plant transfer function since one zero and a fixed pole at s = 0 are involved. Even with a sixth-order description, exact application of the controller design technique would require time derivatives of x_1 up to and including the fifth. Since it is impractical to generate these higher order derivatives, a way is sought to apply the design technique without them. Necessarily, such a solution must be based on an approximation to the actual system dynamics. As will be discussed and demonstrated by computer results, it is possible through suitable linear compensating networks to make the approximate dynamics quite close to the true dynamics. Whether or not this is practical in a given situation is dependent on the power density spectrum of the transducer noise.

A look at the open loop pole zero plot of figure 5-11 leads to the conclusion that if the natural frequencies of the gyro and motor are large enough, then the system transfer function can be adequately approximated by the pole, zero cluster around the origin of the s plane, i.e. these will dominate the system's dynamics. In this case, the complex poles of the gyro and motor might be neglected, and the controller design would proceed exactly as in section 5.3.1 because the approximate plant transfer function would be just that of (5-2). The gyro and motor may have high enough natural frequencies for this approximation to be valid. If they do not, then linear compensating networks can be used to cancel gyro and motor complex poles and place poles further out in the s plane. However, if compensation must be used a problem may arise due to accentuation of high frequency transducer noise by the differentiating characteristic of the required networks. For



$$L(s) = \frac{K_H K_{\dot{\theta}} T_a \omega_h^2 \omega_a^2 \omega_g^2}{s(s^2+2\zeta_h \omega_h s+\omega_h^2)(s^2+2\zeta_a \omega_a s+\omega_a^2)(s^2+2\zeta_g \omega_g s+\omega_g^2)}$$

L(s) = Loop Transfer Function

Figure 5-10: Complete System Transfer Function

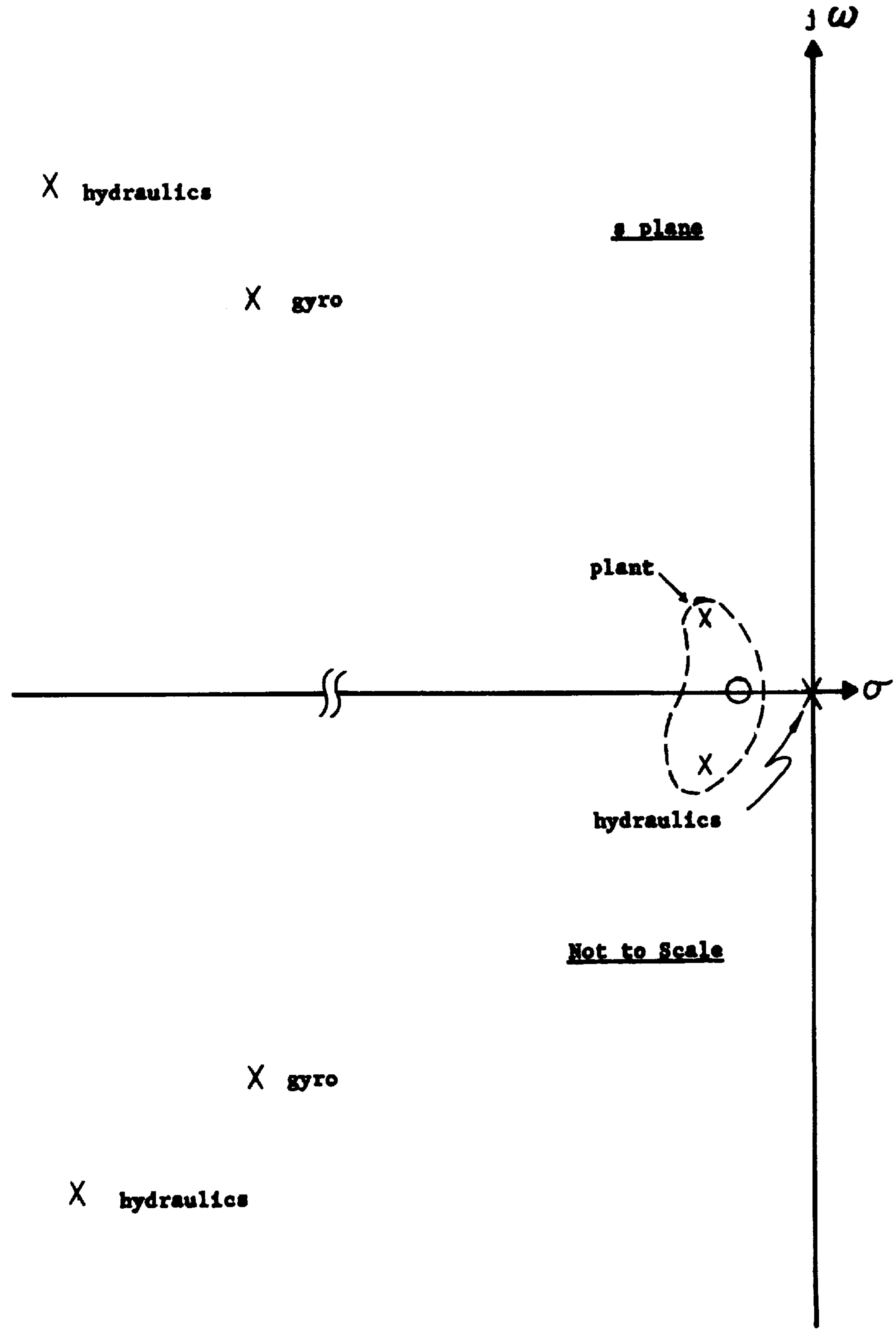


Figure 5-11: Open Loop Pole Zero Plot For L(s)

example, if ω_g is too small, then the necessary compensation has the form

$$G_c(s) = \frac{(s/\omega_g)^2 + (2\xi_g/\omega_g)s+1}{(s/\omega'_g)^2 + (s\ \xi'_g/\omega'_g)s+1} \tag{5-16}$$

where $\omega'_g > \omega_g$. The asymptotic Bode diagram for (5-16) is shown in figure 5-12. If transducer noise is restricted to a range of frequencies well below ω_g, then there is no problem. However, if the noise is not so restricted, the rising characteristic of $G_c(s)$ in the range from ω_g to ω'_g may cause excessive noise levels at the input to the hydraulic motor.

In order to examine problems which arise from approximating system dynamics by neglecting far out poles, the complete system was simulated on the PACE analog computer. With far out poles neglected, the required controller for pitch axis stability augmentation was that designed in 5.3.1 (control law (5-11)). Because the design is only approximate, the possibility exists that the system will become unstable for high closed loop gains. Factors of this closed loop gain are the magnitude function of the control law, the gain k in the linear region of the saturation function, and the plant gain, $K_{\dot{\theta}}\,\omega_a{}^2 T_a$. The first of these is time-varying and nonlinear and the last is dependent on altitude and mach number. The computer simulation verified that a stability problem did exist. For certain plant conditions and reference input signal amplitudes, sinusoidal oscillations in the output x_1 occurred. This oscillation is not a limit cycle arising from approximating the sign function by the saturation function which is discussed in section 2.2.1. It is an output mode associated with the neglected far out poles. Since the forms of the nonlinearities in the controller are not amenable to describing function analysis, computation of the amplitude and frequency of oscillations is impossible. However, some qualitative observations can be made. The loop gain is signal dependent since it is partially determined by the signal dependent magnitude function of the control law. Consequently, instability can be excited if initial conditions and/or the reference input and disturbance signal amplitudes are large enough to make the closed loop gain too high. In practice, then, an extensive computer investigation would be required to determine regions of initial conditions, $x_1(0)$ and $x_2(0)$, which do not lead to the unstable condition. The size of these regions would vary with plant conditions, reference input magnitude, and disturbance signal magnitude. Operation would have to be restricted to the most conservative region. If this region is unsatisfactory, it must be extended by using the linear compensation networks to move the neglected poles further out.

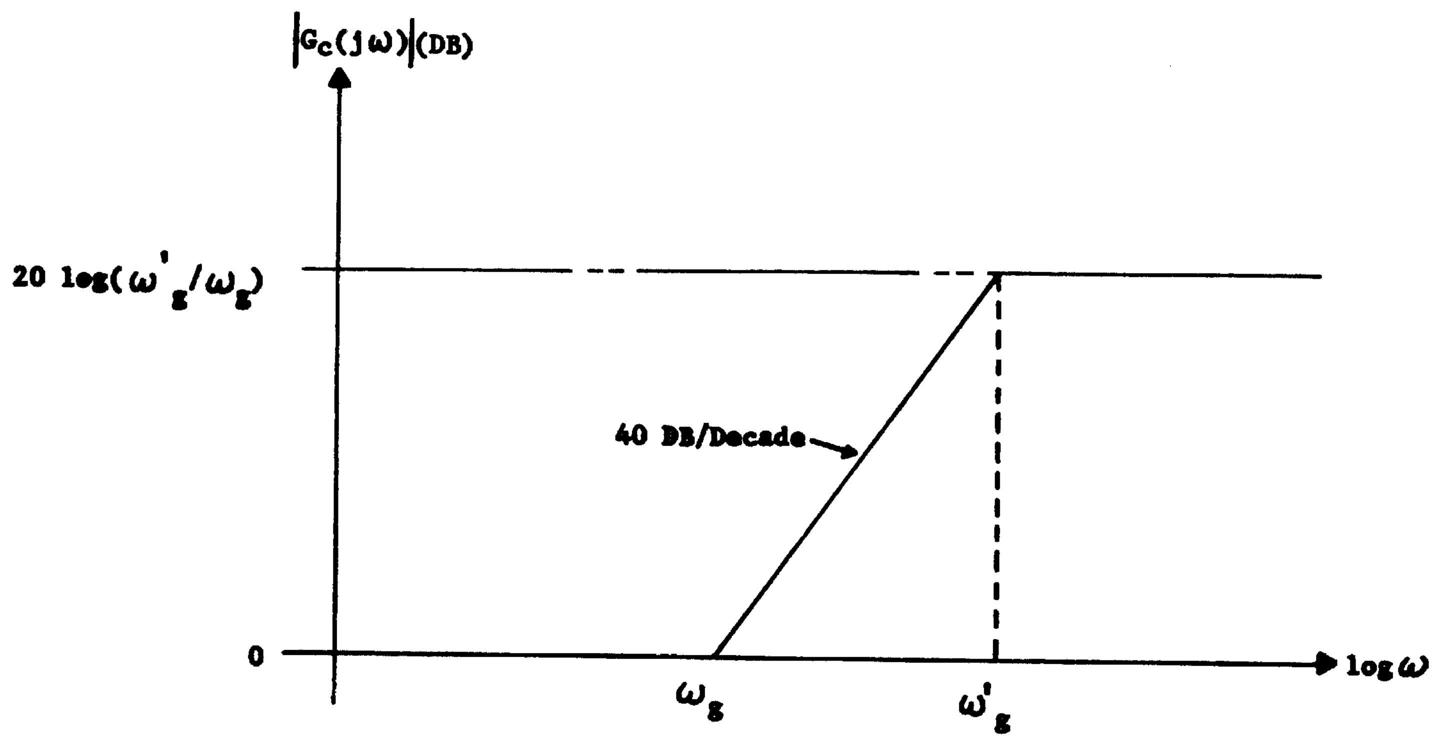


Figure 5-12: Asymptotic Bode Plot For $G_c(s)$

To put some of these ideas on a quantitative basis, the following computer study was made. With initial conditions $x_1(0) = x_2(0) = 0$, step functions of different magnitudes were applied to the system first as reference inputs and then as disturbances. The step magnitude resulting in instability was found to be a function of the far out pole location. The results of this study are presented below.

In the computer study, the gyro and hydraulic motor were assumed to have the following characteristics[31], $\omega_g = 70$, $\xi_g = 0.5$, $\omega_h = 150$, and $\xi_h = 0.5$. The gain for the linear region of the saturation function was taken as $k = 1$. It was found that the stability problem did not arise for plant conditions 16, 32, and 33, the three with lowest gains. Step inputs up to 10 volts were employed, (the maximum compatible with computer saturation levels), but no sign of instability appeared. For these three plant conditions, the response to a step reference input or step disturbance input was nearly identical with or without the far out poles in the system. One change only was required in the controller. The derivative circuit bandwidth, ω_d, had to be increased from 2 radians/second used in section 2.3.1 to 5 radians/second when far out poles were in the system.

Plant conditions other than 16, 32, and 33 were not so easily handled. Of the others, plant condition 28 was the least stable. Step inputs of 0.2 volts would excite steady oscillation for this condition. Let this value of reference input be denoted by $|\theta_r|_c$, the critical magnitude. For reference inputs less than ninety percent of the critical magnitude, the system operated as desired. Magnitudes of the reference input between ninety and one hundred percent of the critical magnitude led to damped oscillations during part of the transient period. Input magnitudes greater than critical resulted in sustained oscillations.

An appreciation of the reason for the difference between the situation for plant condition 16, 32 and 33 and plant condition 28 can be gained by comparing the Bode diagrams for each shown in figure 5-13. It is not intended here that the Bode diagram be used as a tool for an exact stability analysis of the nonlinear system. Arguments presented are qualitative rather than quantitative. However, as will be shown, computer results bear out the fact that there is a strong correlation in this case between results obtained for the nonlinear design and results which would be predicted for a linear system using the classical Bode diagram method of stability analysis. From figure 5-13 it is seen that since $|P(j\omega)/(j\omega)|$ is negligible for $\omega \geq 5$ for plant conditions 16, 32, and 33, and since gyro and motor dynamics contribute little phase angle in the region where $\omega < 5$, an effective separation exists between dominant and far out poles. On the other hand, $|P(j\omega)/(j\omega)|$ for plant condition 28 is appreciable beyond $\omega = 5$, and the motor and gyro do contribute significant phase lag in that range of frequencies. Thus, for condition 28, the separation does not exist, and one is led to suspect that a stability problem will arise when the loop is closed. Since it did arise for plant conditions other than 16, 32, and 33, a design was sought which

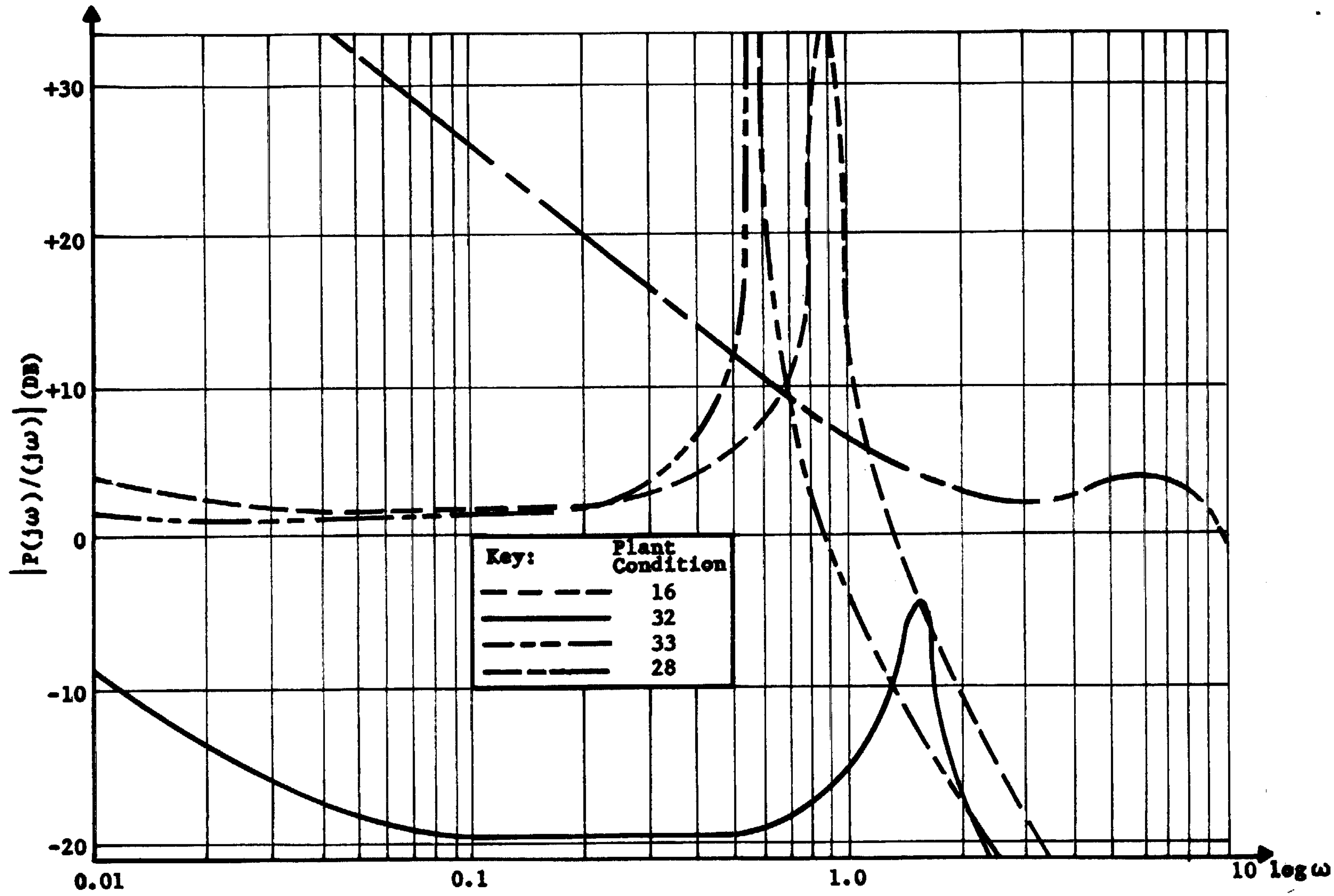


Figure 5-13: Bode Diagrams for Plant Conditions 16, 32, 33 and 28

would be suitable for these troublesome plant conditions. It is not unreasonable to have a different set of controller parameters for plant conditions 16, 32 and 33 than for the others since the former are easily identifiable by the pilot as two extremes of altitude. In fact, this was found to be the most practical approach to the design. A single design suitable for both sets of plant conditions could not be found short of moving the neglected poles extremely far out.

Since plant condition 28 was the most easily excited into oscillation, it was used in searching for an improved design. Ways were sought which would avoid oscillations and give proper controller operation for large step inputs. Because the amplitude of the magnitude function was part of the reason for the problem, the first step taken in an attempt to solve it was to reduce this amplitude. This was possible for a design excluding plant conditions 16, 32 and 33, for then the coefficients of $|\dot{\theta}_r - x_1|$, $|x_1|$, and $|x_2|$ could be reduced to 0.515, 1.64, and 0.79 respectively, (see Table 5.3.) This improved stability somewhat, but did not increase $|\dot{\theta}_r|_c$ sufficiently to be satisfactory. The reduced coefficients were retained for the remainder of the computer study.

A reduction of the gain k in the linear region of the saturation function was tried next. This also improved stability, but again not sufficiently. An approximate inverse relationship was noted between k and $|\dot{\theta}_r|_c$, i.e. if $|\dot{\theta}_r|_c$ = 1 volt produced sustained oscillations with k = 1, then for k = 0.5, $|\dot{\theta}_r|_c$ became approximately two volts. This approach could not be carried to its logical conclusion, i.e. reduce k enough to avoid oscillations for the largest expected value of $|\dot{\theta}_r|$, because then the behavior for small values of $|\dot{\theta}_r|$ would not be satisfactory. This problem was discussed in section 5.3.2.

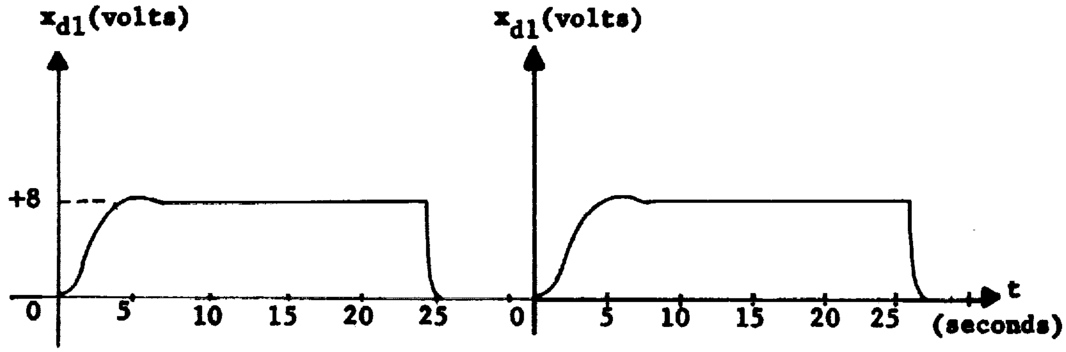
Since neither of the measures taken above solved the problem completely the poles of the gyro and hydraulic motor had to be cancelled and new ones placed further out with appropriate compensation networks. Compensators of the form of (5-16) were placed in cascade before the hydraulic motor and following the gyro. With the far out poles placed such that $\omega_g' = \omega_h' = 333$, and $\xi_g' = \xi_h' = 0.455$, then, for plant condition 28, $|\dot{\theta}_r|_c$ was 3.5 volts for k = 1. If k was reduced to 0.5, the critical magnitude was found to be 7 volts. However, with $|\dot{\theta}_r|$ = 1 volt, plant condition 17 exhibited a longer transient and larger transient and steady-state errors with k = 0.5 than with k = 1.0. Condition 17 was the worst in this regard. With k = 1.0, the steady-state error was two percent of the step magnitude, but it increased to four percent for k = 0.5. The specification does not state what the acceptable steady-state error is. However, if four percent is not objectionable, then specifications can be met for the range of input levels $1 \leq |\dot{\theta}_r| \leq (0.9)(7)$ with the initial conditions being considered.

To emphasize the usefulness of the Bode diagram as an aid in design, it is now shown how it was used to improve on the above results. The damping ratios ξ'_g and ξ'_h were reduced by a factor of one half so that phase-angle contribution of the far out poles at the crossover frequency was reduced. A considerable stabilizing influence was noted. The critical magnitude was increased to 6 volts from 3.5. Reduction of these damping ratios by a factor of one quarter increased the critical magnitude still further to a value of 7 volts. The results noted were obtained using k = 1.0. Reducing k to 0.5 approximately doubled the critical magnitudes given above.

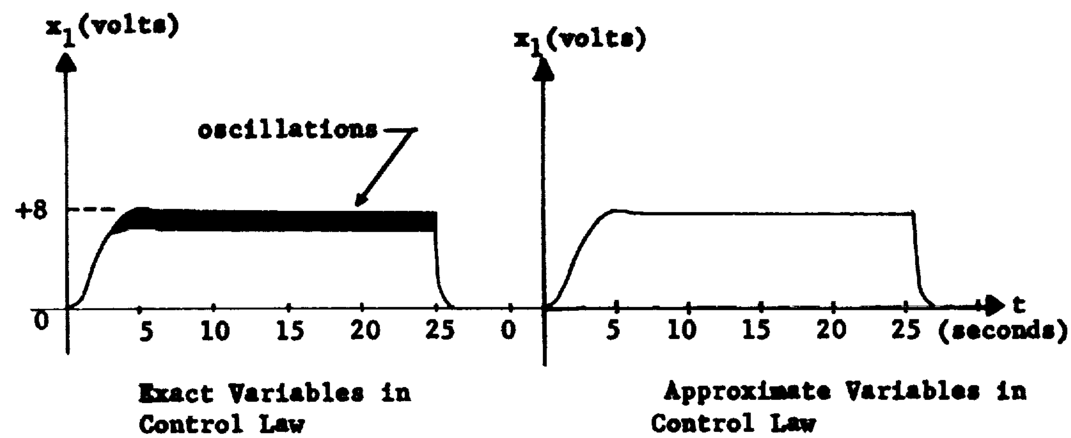
The use of approximate signals for x_1 and y, discussed in section 5.3.2 relative to noise reduction, proved to be another measure which could be used to advantage in the stability problem. The critical magnitude, $|\theta_r|_c$, could be increased by replacing $|x_1|$ and $|y|$ in the magnitude function with their approximate values as given in (5-14) and (5-13) respectively. The improved stability can be attributed to a reduced closed loop gain due to a reduction in the magnitude function amplitude. Transient responses using exact and approximate values for $|x_1|$ and $|y|$ are compared in figure 5-14. It is seen there that use of exact values of the variables x_1 and y caused a six cycle per second oscillation with a peak-to-peak amplitude equal to seven and one half percent of the eight volt reference input level. When approximate values of the variables were used, these oscillations were eliminated.

Though the stability problem has been discussed relative to step reference inputs, similar results were observed for step disturbances. For example in the design presented above in which $|\theta_r|_c$ was found to be 7 volts for k = 1, oscillations were not induced for $|d| < 7$ volts either. However, with $|d| = 8$ volts, a steady-state oscillation of nine cycle per second and 0.5 volts peak-to-peak amplitude resulted. Again plant condition 28 was the least stable. Therefore, the upper bound of seven volts is conservative for all other plant conditions. Similar to results for reference inputs, the disturbance responses with and without the far-out poles in the system were practically identical provided $|d|$ was less than seven volts.

It is well to conclude with a discussion of the nature of the stability problem which arises due to neglecting instrument dynamics in the design. Is it basic or does it arise simply because the state of the art of instrument design is not sufficiently far advanced? In the case of the gyroscope, the latter is the case. The transfer function for the rate gyro as given in figure 5-9 is that for a single degree-of-freedom (SDF) device. With the advent of new two degree-of-freedom (TDF) gyros[34] such as the electrically suspended gyro (ESG) invented by Dr. A. Nordsieck of the University of Illinois, and developed by Minneapolis Honeywell and General Electric, the problem considered above no longer exists. This is so because the TDF and SDF gyros differ in their dynamic properties. In the SDF, the rotor precesses in response to a displacement input. Thus, rotor inertia and restraining spring lead to the quadratic denominator term in the transfer function. On



a) Model Response



b) Plant Response

Figure 5-14: Effect of Approximate Variables in Reducing Oscillations the other hand, the TDF gyro rotor remains fixed in inertial space under all operating conditions; hence the displacement to voltage transfer function is simply a constant.

Development of improved hydraulic motors, or other type drives, could lead to higher natural frequencies for these devices. Even if ω_h cannot be appreciably increased through development of improved drives, the compensating network required to move the poles of the hydraulic motor to a suitable location is not nearly as objectionable as that required to move the poles of both the gyro and motor. For example, with no gyro poles present the hydraulic motor poles need only to be moved to the location where $\omega_h' = 224$, and $\xi_h' = 0.336$ to get the same results obtained above with $\omega_h' = \omega_g' = 333$ and $\xi_h' = \xi_g' = 0.114$. The compensator to do this places a high frequency gain of 2.25 between gyro output and motor input, as compared to a high-frequency gain of 106 needed when both gyro and motor must be compensated.

CHAPTER VI

CONCLUSIONS

In this report, a controller synthesis procedure based on Liapunov's "direct method" has been taken from the realm of mathematical theory and developed as a useful engineering design technique. The technique is applicable to nonlinear as well as linear plants whose parameters may be rapidly varying in an unknown fashion within known finite bounds. In transition from theory to practice, several significant modifications and extension of the procedure were made. Design problems resulting from transducer noise, disturbances, and instrument dynamics were investigated by analog and digital computer simulation of complete systems. In certain cases, ways were found to eliminate or minimize these problems. In others, the problems could not be eliminated or minimized, but the study at least revealed their existence.

Modifications of the theoretical procedure which are significant from an engineering design point of view include techniques for avoiding impulses in generating the control signal, and for reducing controller amplifier gains. Extensions of the procedure were made in the areas of design for nonlinear, time varying plants, incorporating a specification of convergence time as part of the design problem and eliminating plant state variables from the control signal to reduce adverse effects of transducer noise. Results relating to convergence time were shown to be directly applicable to improving the design of a class of quasi time optimal control systems.

The extension to allow designing for a specified convergence time is of importance when starting systems with large initial errors. Though a design equation was obtained for second-order systems, the complexity of the algebraic problem prohibited an exact solution for higher-order cases. However, it was shown by computer simulation that insight gained from the solution of the second-order case was useful in reducing convergence time for a third-order system. The design equation for the second-order case was applied to a quasi time optimal control problem. Performance was improved by a factor of two over a design not employing these results, i.e. the time to reach the origin from a given initial condition was reduced by a factor of one half.

Techniques introduced for replacing plant state variables by model state variables in the magnitude function of the control law were shown to lead to significant reduction of the transducer noise problem. In one example, the mean squared noise level into the plant was reduced by a factor of one hundred by employing these techniques.

A theorem was introduced relating to control of linear, slowly time varying plants with zeroes. Its use permits controller design to be based on a model and a reduced plant of lower order than the actual plant. For example, a third order plant with two zeroes was made to follow a first-order model by using only the plant and model output signals to generate the control signal. A third order model would have been required, along with first and second derivatives of plant and model output signals had the reduction-of-order technique not been employed. This technique is an ideal way to minimize transducer noise problems since it avoids the need for higher derivatives in generating the control signal. By example it was shown that the reduction-of-order technique can be applied on an approximate basis in some cases to linear slowly time varying plants without zeroes.

The reduction-of-order technique was used to design a controller for pitch axis stability augmentation of the X-15 manned reentry vehicle. Computer results showed that the performance of the controller for a third-order system which excluded the complex poles of both the gyro and hydraulic motor was quite good. The outline of a design procedure for the system including these poles was presented. Since the resultant transfer function is seventh-order in this case, the procedure had to be approximate since transducer noise precluded use of higher derivatives to make it exact. The approximation made was to neglect the complex poles of the gyro and motor transfer functions, and assume that the system dynamic response was dominated by the same three poles used in the design referred to above. It was shown that stability of the approximate design was dependent on initial conditions of the plant output and its first derivative as well as the magnitude of reference and disturbance inputs. Hence, part of the design procedure would be an extensive computer study to determine the stable operating conditions. Stability can always be assured for all operating conditions by using linear compensating networks to move the complex poles neglected in design far enought out to the left in the s plane. However, such compensation may result in an increased noise level into the hydraulic motor. A limited computer study demonstrated the validity of the approximate design procedure by showing that the controller performed quite well for operating conditions which did not lead to instability. Techniques were introduced which extended the range of stable operating conditions, while not increasing the noise level into the motor.

Possible fruitful areas for further research which came to mind while performing the research for this report are the following:

1. Generalizing the reduction-of-order technique to plants with rapidly varying parameters, and/or to plants without zeroes.

2. Finding additional technqiues which allow control with less than the total number of state variables.

3. Finding a way to track low level reference inputs other than increasing the gain in the linear region of the saturation function.

4. Applying the technique to control problems which are not amenable to other techniques.

Any of the above, if successfully accomplished, would greatly extend the usefulness of the technique.

APPENDIX A

DEFINITIONS AND THEOREMS PERTINENT TO THE "DIRECT METHOD"

Emphasis in this report is on application of the "direct method" to controller design rather than on the intricacies of the method itself. Consequently, only those definitions and theorems required to understand the design approach are presented. The theorems are stated without proof except for theorems A.5 and A.6. Proofs for these are given because they are the basis of the synthesis technique presented herein. A full treatment of the details of the theory, including proofs of theorems, may be found in references 22, 24, and 32.

The "direct method" can be applied to dynamical systems governed by the vector differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \qquad \text{(A-1-a)}$$

This is equivalent to the set of n scalar differential equations

$$\dot{x}_i = f_i(x_1, x_2 \ldots x_n, u, u^1, u^2, \ldots u^m, t) \qquad i = 1, 2, \ldots n \qquad \text{(A-1-b)}$$

The vector $\underline{x}$ is the plant state vector, and its components are the state variables. In this report, the components of the plant state vector are always taken as the plant output and its first n-1 derivatives. The vector $\underline{u}$ is the control vector, and it too is taken throughout as the control signal u, and its first m derivatives.

If $\underline{u}(t) \equiv \underline{0}$ for all t, then (A-1-a) is called "free" or "unforced" and it becomes

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \qquad \text{(A-2)}$$

It is assumed throughout that the function $\underline{f}$ is continuous, and is such that solutions of (A-1-a) exist, are unique, and are continuously dependent on the initial value $\underline{x}_o$ (t_o) at any t_o.

The following definitions, due to Liapunov, are quoted from reference 22.

Definition A-1: An equilibrium state $\underline{x}_e$ of a free dynamic system is stable if for every real number δ (ε, t_o) there exists a real number $\varepsilon > 0$ such that $||\underline{x}_o - \underline{x}_e|| < \delta$ implies

$$||\underline{\emptyset}(t;\underline{x}_o, t_o) - \underline{x}_e|| \leq \varepsilon \text{ for all } t \geq t_o$$

where $\emptyset$ $(t; \underline{x}_o, t_o)$ is the unique solution to (A-2) for the initial conditions $\underline{x}_o$, i.e.

$$\text{(i)} \quad \underline{\emptyset}(t_o;\underline{x}_o, t_o) = \underline{x}_o$$

$$\text{(ii)} \quad \dot{\underline{\emptyset}}(t;\underline{x}_o, t_o) = \underline{f}(\underline{\emptyset}(t;\underline{x}_o, t_o),t)$$

Definition A-2: An equilibrium state $\underline{x}_e$ of a free dynamic system is asymptotically stable if

(i) it is stable

(ii) every motion starting sufficiently near $\underline{x}_e$ converges to $\underline{x}_e$ as $t \to \infty$

The above definitions are local in nature, i.e. they refer to behavior near the equilibrium. If δ is independent of t, then the stability is termed uniform stability. In addition, the local nature of the concept of asymptotic stability can be removed if in (ii) of Definition A-2, "sufficiently near" includes all points $\underline{x}_o$ from which motion originates. In this case, the equilibrium is called asymptotically stable in the large[32] and if "sufficiently near" allows all points in the phase space, the equilibrium is said to be asymptotically stable in the whole.

Further definitions required prior to the statement of theorems are taken from Hahn.

Definition A-3: A function $V(\underline{x}, t)$ with $V(\underline{0}, t) = 0$ is called positive (negative) definite if a function $\emptyset(r)$ of class K exists such that the relation

$$V(\underline{x},t) \geq \emptyset(|\underline{x}|) \quad (\leq -\emptyset(|\underline{x}|))$$

is satisfied in K_{h,t_o}.

Remark A-3: The statement $\emptyset(r)$ belongs to the class K means that $\emptyset(r)$ is a continuous, real function defined in the close interval $0 \leq r \leq h$ and that $\emptyset(r)$ vanishes at $r = 0$ and increases strictly monotonically with r. The notion $|\underline{x}|$ indicates the absolute value of the vector $\underline{x}$, i.e. $\sqrt{\underline{x}^T\underline{x}}$. The region K_{h,t_o} is the semicylindrical domain of the motion space,

i.e. $K_{h,t_o} = [\underline{x},t \,|\, |\underline{x}| \leq h, t \geq t_o]$

Definition A-4: A function $V(\underline{k}, t)$ is called radially unbounded if inequality of definition A-3 is valid for arbitrarily large h and when $\emptyset(r)$ increases unboundedly with r.

Definition A-5: A function $V(\underline{x}, t)$ is called decrescent if a function $\psi(r)$ of the class K exists such that in K_{h,t_o}

$$|V(\underline{x}, t)| \leq \psi(\underline{x})$$

is valid.

Definitions A-1 through A-5 suffice to quote without proof the following theorem from Hahn. The theorem is due to Barbashin and Krasovskii.

Theorem A.1 The equilibrium is asymptotically stable in the whole if there exists a function $V(\underline{x}, t)$ which is everywhere positive definite, radially unbounded, and decrescent, and whose total derivative for (A-2) is negative definite.

Definition A-6: A function V which satisfies the conditions of theorem A.1 is called a Liapunov function for the differential equation (A-2).

A theorem for linear autonomous free systems which is used in the design procedure is quoted from reference 22.

Theorem A.2 (Liapunov) The equilibrium state $\underline{x}_e = \underline{0}$ of a continuous-time, free, linear, stationary dynamic system

$$\dot{\underline{x}} = A\underline{x} \tag{A-3}$$

where A is a constant matrix, is asymptotically stable (a) if and (b) only if given a symmetric positive-definite matrix Q there exists a symmetric, positive-definite matrix P which is the unique solution of the set of n(n + 1)/2 linear equations

$$A^T P + PA = -Q \tag{A-4}$$

and $\underline{x}^T P \underline{x}$ is Liapunov function for (A-3).

Further theorems for linear stationary plants which are useful in the design procedure are quoted from reference 24.

Theorem A.3 The equilibrium of (A-3), (detA≠0) is uniformly asymptotically stable in the whole if all the

$$\sigma_j = \mathrm{Re}\{s_j\} < 0 \qquad \text{for } j = 1,2,\ldots k \leq n$$

where s_j is an eigenvalue of A and k is the number of distinct eigenvalues.

Theorem A.4 If the equilibrium of (A-3) is uniformly asymptotically stable, then any real, symmetric, positive definite matrix Q, there exists a quadratic form $V(\underline{x}) = \underline{x}^T P \underline{x}$ which is a Liapunov function for (A-3) in the sense of Theorem A.1 where $-Q = A^T P + PA$.

Two theorems which pertain directly to the synthesis technique considered herein have been recently reported by Grayson[33]. The first supposedly provides a theoretical justification for the synthesis technique, and the second a basis for selecting the control vector. This author disagrees with these claims. These theorems are presented so that the reasons for this disagreement can be discussed.

Theorem A.5 If, for the systems

$$\dot{\underline{x}} = \underline{g}(\underline{x}, \underline{z}, \underline{u}) \qquad \text{(A-5)}$$

and

$$\dot{\underline{y}} = \underline{f}(\underline{y}) \qquad \text{(A-6)}$$

where $\underline{f}(\underline{0}) = \underline{g}(\underline{0}, \underline{0}, \underline{0}) = \underline{0}$, scalar functions $V(x)$ and $V_1(\underline{y})$ exist such that the following conditions are true,

1. $V(\underline{x}) = V_1(\underline{y})$ for all $\underline{x} = \underline{y}$ is an open region D about the origin
2. $V_1(\underline{y})$ is positive definite
3. $(d/dt)\ V_1(\underline{y})$ is negative definite
4. $\dot{V}(\underline{x}, \underline{z}, \underline{u}) \leq \dot{V}_1(\underline{y})$ for all $\underline{x} = \underline{y}$ in D then (A-5) is asymptotically stable; i.e. $\underline{x} \to \underline{0}$ as $t \to \infty$

Proof: System (A-6) is asymptotically stable with respect to the equilibrium $\underline{y} = 0$ by conditions 2. and 3. and Theorem A.1. Also, for $\underline{x}_o = \underline{y}_o$

$$V(\underline{x}) = V(\underline{x}_o) + \int_{t_o}^{t} \dot{V}(\underline{x},\underline{z},\underline{u})dt \leq V_1(\underline{y}_o) + \int_{t_o}^{t} \dot{V}_1(\underline{y})dt$$

$$= V_1(y) \quad \text{for } t \geq t_o$$

But $V_1(\underline{y}) \to 0$ as $t \to \infty$. Hence, since $V(\underline{x})$ is positive definite, $V(\underline{x}) \to 0$ as $t \to \infty$; therefore, $x \to 0$ as $t \to \infty$. Q.E.D.

Grayson contends that theorem A.5 is necessary to provide mathematical justification for the synthesis technique because (A-5), the form of equation arising in synthesis, "does not possess an equilibrium state at $\underline{x} = \underline{0}$ - a tacit assumption in the theorems of the second method." But since $\underline{x} \to 0$, the $\underline{g} \to 0$ also. Therefore, the theorem simply states that $\underline{x} = \underline{0}$ is an equilibrium state. It should be noted also that Grayson's proof is incorrect because the inequality used relies on condition 4 which is not necessarily true unless $\underline{x} = \underline{y}$. However, in general $\underline{x} \neq \underline{y}$ for $t > t_o$. Therefore, $\dot{V}$ may be greater than $\dot{V}_1$ for $t > t_o$. The conclusion that $\underline{x} \to 0$ as $t \to \infty$ is correct, however, since $V(\underline{x})$ is positive definite by conditions 1 and 2 and $\dot{V}(\underline{x},\underline{z},\underline{u})$ is negative definite by conditions 3 and 4.

The theorem given in reference 33 for extending theorem A.5 into a method of design for a class of systems by suitably selecting $\underline{u}$ is the following:

Theorem A.6 Given the systems

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{z}(t) + \underline{u} \tag{A-7}$$

and

$$\dot{\underline{y}} = \underline{f}(\underline{y}) \tag{A-8}$$

where $\underline{f}(\underline{0}) = 0$ and (A-8) is asymptotically stable, then (A-7) is asymptotically stable, i.e. $\underline{x} \to \underline{0}$ as $t \to \infty$, if $(\text{grad}_x V) \cdot (\underline{z} + \underline{u}) \leq 0$ or

$$u_i = - |z_i(t)|_{max} \, \text{sgn}(\text{grad}_x V)_i, \; i = 1, \ldots. n$$

This theorem is invalid since the u_i's selected are such that solutions to (A-7) may not exist[29]. This problem is discussed at length in section 2.2.1 of Chapter II. To insure that solutions exist, the discontinuous sign function is replaced by a continuous function. Here the continuous function used is the saturation function. Because of this substitution, $\underline{x}$ may not approach zero as t increases, but, as shown in Appendix C, $|\underline{x}|$ can be made arbitrarily small as $t \to \infty$.

With the control vector selected so that solutions of (A-5) exist, it is felt that this synthesis technique is mathematically justified and that the development of the technique in Chapter II and Appendix C adequately demonstrates this.

APPENDIX B

SIGN OF ELEMENT IN FIRST ROW AND LAST COLUMN OF P MATRIX

Consider the symmetric positive definite diagonal matrix, Q,

$$Q = \begin{bmatrix} q_{11} & & & & \\ & q_{22} & & \bigcirc & \\ & & \cdot & & \\ & \bigcirc & & \cdot & \\ & & & & q_{nn} \end{bmatrix} \tag{B-1}$$

and the nxn matrix A_o all of whose eigenvalues have negative real parts

$$A_o = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{01} & a_{02} & \cdot & \cdot & \cdot & a_{0n} \end{bmatrix} \tag{B-2}$$

The P matrix is defined as the solution of

$$A_o^T P + P A_o = -Q \tag{B-3}$$

The equation for p_{1n} is, therefore

$$p_{1n} = \frac{-q_{11}}{2a_{01}} \qquad \text{(B-4)}$$

Since Q is positive definite and A_o has eigenvalues with negative real parts, then $q_{11} > 0$ and $a_{01} < 0$. Therefore, from (B-4)

$$p_{1n} > 0$$

APPENDIX C

STABILITY IN THE LINEAR REGION OF THE SATURATION FUNCTION

To investigate the stability properties of (2-5) in the region where $|\gamma| < 1/k$, consider the expanded form of $\dot{V}$, (2-20), which is

$$\dot{V}=-\sum_{i=1}^{n} e_i^2-p_{1n}b_{m+1}\gamma\ [u^m+U(m-1)\sum_{j=0}^{m-1} c_{j+1}u^j+\sum_{\ell=0}^{m} d_{\ell+1}r^{\ell}+\sum_{i=1}^{n} g_i x_i] \quad \text{(C-1)}$$

where it has been assumed that Q = I, the identity matrix. The term involving u^m always makes a negative contribution to $\dot{V}$, therefore it can be neglected in the following considerations and the results obtained will be conservative. The remaining terms of $\dot{V}$ must satisfy the inequality

$$-\sum_{i=1}^{n} e_i^2-p_{1n}\gamma\ [U(m-1)\sum_{j=0}^{m-1} b_{j+1}u^j+\sum_{\ell=0}^{m} \beta_{\ell+1}r^{\ell}+\sum_{k=1}^{n} \alpha_k x_k] < 0 \quad \text{(C-2)}$$

When m = 0, (C-2) becomes

$$-\sum_{i=1}^{n} e_i^2-p_{1n}\gamma\ [\beta_1 r+\sum_{k=1}^{n} \alpha_k x_{dk}-\sum_{i=1}^{n} \alpha_i e_i] < 0 \quad \text{(C-3)}$$

in which the model states have been added and subracted within the square brackets in order to introduce terms in e_i and x_{di}. The reason for doing this is that the upper bound

$$\left|\beta_1 r+\sum_{k=1}^{n}\alpha_k x_{dk}\right| \leq M < \infty \tag{C-4}$$

can be established for a bounded input, r, whereas a similar bound could not be established in terms of plant states without making the unwarranted assumption that the plant is stable. Another upper bound which can be established and which is useful is

$$\left|\sum_{i=1}^{n}\alpha_i e_i\right| \leq (|\alpha_i|_m)(\sum_{i=1}^{n}|e_i|) \leq (|\alpha_i|_m)(\sum_{i=1}^{n} ne_i^2)^{1/2} \tag{C-5}$$

where $|\alpha_i|_m = \underset{i}{\mathrm{Max}} \{|\alpha_i|\}$ for i = 1, 2,...n

The last inequality of (C-5) can be shown as follows:

$$(\sum_{i=1}^{n} e_i)\Bigg|_{-\infty<e_i<+\infty} \leq (\sum_{i=1}^{n}|e_i|)\Bigg|_{-\infty<e_i<+\infty} =(\sum_{i=1}^{n} e_i)_{0<e_i<+\infty} \tag{C-6}$$

The last summation in (C-6) can be expressed

$$\sum_{i=1}^{n} e_i=K(\sum_{i=1}^{n} e_i^2)^{1/2}; \text{ for } e_i \geq 0,\ i=1,2...n \tag{C-7}$$

where K is dependent on e_i. To determine the maximum value of K, set

$$\frac{\partial K}{\partial e_j} = \frac{\partial}{\partial e_j}\left[\frac{\sum_{i=1}^{n} e_i}{(\sum_{i=1}^{n} e_i^2)^{1/2}}\right] = 0 \quad j=1,2...n \tag{C-8}$$

This is

$$\frac{\partial K}{\partial e_j} = \frac{\sum_{i=1}^{j-1} e_i(e_i-e_j)+ \sum_{i=j+1}^{n} e_i(e_i-e_j)}{(\sum_{i=1}^{n} e_i^2)^{3/2}} = 0 \tag{C-9}$$

Since (C-9) must be satisfied for all e_i including the smallest, it can only be satisfied if

$$e_i = e_j \quad \text{for all i, j from 1 to n} \tag{C-10}$$

Use of (C-10) leads to

$$K_{max} = \sqrt{n} \tag{C-11}$$

Hence

$$\sum_{i=1}^{n} e_i \leq (n \sum_{i=1}^{n} e_i^2)^{1/2}; \; e_i \geq 0 \text{ for } i=1,2...n \tag{C-12}$$

Use of (C-4) and (C-5) leads to the conservative inequality

$$\dot{V} \leq -R^2(k) + p_{1n}\gamma \, [M + \alpha_{i\ max} \sqrt{n} \ \ R(k)] \qquad \text{(C-13)}$$

where

$$R(k) = (\sum_{i=1}^{n} e_i^{\,2})^{1/2}$$

In the region where $|\gamma| < 1/k$

$$\dot{V} \leq -R^2(k) + aR(k) + b \qquad \text{(C-14)}$$

where

$$a = \frac{p_{1n}}{k} \ \ |\alpha_i|_{max} \sqrt{n}$$

and

$$b = \frac{p_{1n}M}{k}$$

From (C-14) it can be seen that

$$\dot{V} < 0 \qquad \text{(C-15)}$$

everywhere outside the spherical region

$$R(k) < \frac{a}{2} + \frac{1}{2} \sqrt{a^2+4b} \qquad \text{(C-16)}$$

By choosing k large enough R(k) can be made arbitrarily small. It can be concluded that as $t \to \infty$, u will cause $\underline{e}$ to be within the region common to R(k) and $|\gamma| < 1/k$. Asymptotic stability cannot be concluded. Limit cycles or a constant steady state error may exist, but can be made arbitrarily small by choosing k large enouth.

In the event $m \neq 0$, (C-3) must include the additional terms

$$\sum_{j=0}^{m-1} b_{j+1}u^j + \sum_{\ell=1}^{m} \beta_{\ell+1} r^\ell \qquad \text{(C-17)}$$

within the square brackets.

If r and its derivatives are finite, then (C-4) becomes

$$\left| \sum_{\ell=0}^{m} \beta_{\ell+1} r^\ell + \sum_{i=1}^{n} \alpha_i x_{di} \right| \leq M < \infty \qquad \text{(C-18)}$$

Again an upper bound has been established, so the additional terms arising due to derivatives of r are handled without complications. If the restriction that derivatives of r be finite is prohibitive, the problem can always be formulated in a way which avoids derivatives of r as was discussed in 2.2.2. The additional terms appearing in (C-3) due to u and its derivatives do present a problem, however.

Since these terms are functions of the plant output and its first n-1 derivatives, an upper bound for them cannot be assumed to exist. A way to handle this problem is to make k a function of these variables as follows:

$$k = k_1+k_2 \left|b_{j+1}\right|_{max} \sum_{j=0}^{m-1} |u^j| \qquad \text{(C-19)}$$

where k_1 and k_2 are constants, and $|b_{j+1}|_{max} = \underset{j}{Max} \{|b_{j+1}|\}$
for $j = 0, 1, \ldots m-1$

Thus, the equivalent of (C-14) for this case becomes

$$\dot{V} \leq -R^2(k) + aR(k) + b + c \qquad \text{(C-20)}$$

where $a = \frac{p_{1n}}{k_1} |\alpha_i|_{max} \sqrt{n}$, $b = \frac{p_{1n}M}{k_1}$, and $c = \frac{p_{1n}}{k_2}$

From (C-20), $\dot{V} <$ 0 outside the spherical region

$$R(k) < \frac{a}{2} + \frac{1}{2} \sqrt{a^2+4(b+c)} \qquad \text{(C-21)}$$

This region can be made arbitrarily small also by choosing k_1 and k_2 large enough.

A procedure similar to that directly above may be followed for nonlinear plants. For instance, consider example 2-1 in section 2.4.3. In this case

$$\dot{V} = -(e_1^2+e_2^2)+2p_{12}\gamma \left[-Ku-(K-2)(r-x_{d1})-2x_{d2}+ax_2^2-(K-2)e_1+2e_2\right] \qquad \text{(C-22)}$$

and the appropriate choice for k is seen to be

$$k = k_1 + k_2 x_2^2 \qquad \text{(C-23)}$$

For this choice

$$R(k) < a/2 + 1/2 \sqrt{a^2+4(b+c)} \qquad \text{(C-24)}$$

where

$$a = \frac{4p_{12} \sqrt{2}}{k_1}$$

$$b = \frac{2p_{12}M}{k_1}$$

$$c = \frac{1}{k_2}$$

Instrumenting the k's given by (C-19) or (C-23) considerably complicates the controller and is not justified unless a computer study of the system indicates that a problem may exist. In all of the examples used in the report, no difficulties were encountered even though a constant k was used throughout.

APPENDIX D

PROOF OF SEMIDEFINITENESS OF A QUADRATIC FORM

With the matrix H_1 chosen as

$$H_1 = \begin{bmatrix} & \bigcirc & & \\ h_{1n} & h_{2n} & \cdot & 1 \end{bmatrix} = \frac{1}{p_{nn}} \begin{bmatrix} & \bigcirc & & \\ p_{1n} & p_{2n} & \cdot & p_{nn} \end{bmatrix} \qquad \text{(D-1)}$$

where $h_{in} = \dfrac{p_{in}}{p_{nn}}$ for $i = 1,2...n-1$, and p_{in}, for $i = 1,2...n$ are the elements of a symmetric positive definite matrix, it turns out that

$$PH_1 + (PH_1)^T = 2PH_1 \qquad \text{(D-2)}$$

since PH_1 is symmetric.

Thus, in order to show that $PH_1 + (PH_1)^T$ is positive semidefinite, it suffices to show that

$$\begin{bmatrix} p_{11} & p_{12} & \cdot & p_{1n} \\ p_{12} & p_{22} & \cdot & p_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ p_{1n} & \cdot & \cdot & p_{nn} \end{bmatrix} \begin{bmatrix} & \bigcirc & & \\ & & & \\ & & & \\ p_{1n} & p_{2n} & \cdot & p_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{12} & a_{22} & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & \cdot & \cdot & a_{nn} \end{bmatrix} = A \qquad \text{(D-3)}$$

is positive semidefinite. The elements a_{ij} are given by

$$a_{ij} = p_{in}p_{nj} \tag{D-4}$$

Therefore, the quadratic form $\underline{e}^T A \underline{e}$ is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_{in}p_{jn}e_i e_j \tag{D-5}$$

which can be expressed as

$$\left(\sum_{i=1}^{n} p_{in}e_i \right)^2 \tag{D-6}$$

Since (D-6) $\geq$ 0 for all e_i, then $PH_1 + (PH_1)^T$ is positive semidefinite.

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