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Concerning ~~A PROPOS~~ THE SERIES OF WORKS BY R. VERNIĆ ON THE REGULARIZATION AND THE PERIODICAL SOLUTIONS OF THE PROBLEM OF THREE BODIES

by

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CONCERNING ~~A PROPOS~~ THE SERIES OF WORKS BY R. VERNIĆ ON THE REGULARIZATION AND THE PERIODICAL SOLUTIONS OF THE PROBLEM OF THREE BODIES

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by G. A. Merman

SUMMARY

In a series of his works (Vernić, 1952; Vernić, 1953a; Vernić 1953; Vernić, 1954; Vernić, 1955), the author asserts that he succeeded in finding the transformation of the independent variable, which fully eliminates all the singular points of the solution of the three body problem over the entire complex plane. Besides, the author obtains a paradoxical result, consisting in the absence in the problem of three bodies of solutions, aside from the Lagrange. The inconsistency of both these assertions by Vernić is shown in the present paper.

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1. Vernić formulates the following theorem (Vernić, 1953a, p. 86, Theorem 5): "Transformation of the Sundman-type independent time variable

$$dt = S\,(r)\,du,$$

where $S\,(r) = S\,(r_0, r_1, r_2)$ is a symmetrical homogenous first power function of all distances, simultaneously regularizing all double and triple collisions in the problem of three bodies".

This transformation is a generalization of another Vernic transformation

$$dt = R\,du,$$

where $R^2 = \sum_{i=0}^{2} \frac{r_i^2}{m_i}$, which must regularize the triple collisions.

In order to demonstrate the last assertion, the author takes advantage of the limit relations obtained by Sundman in 1907 for triple real collisions:

$$\sqrt{R}\dot{R} = -\sqrt{2M\left(\sum_{i=0}^{2}\frac{1}{m_i}\right)^{\frac{3}{2}}} + \varepsilon, \quad r_i = R\left(\sum_{i=0}^{2}\frac{1}{m_i}\right)^{-\frac{1}{2}} + \varepsilon,$$

where $\varepsilon \to 0$ during the approaching of the triple collision. From these relations it follows

$$\sqrt{r_i}\dot{r}_i = -\sqrt{2M} + \varepsilon.^{*} \qquad (1)$$

With reference to these limit relations the author writes (Vernić, 1953a, p.84):

$$R = \sum_{\nu=2}^{\infty} c_\nu t^{\frac{\nu}{3}}.$$

Having shown by which method one may find

$$c_2 = \sqrt[3]{\frac{9}{2} M \left(\sum_{i=0}^{3} \frac{1}{m_i}\right)^{\frac{3}{2}}},$$

the author then reaches the conclusion that R, as well as all other unknowns of the problem of three bodies (coordinates and velocities) are represented by series, disposed by whole positive powers of u ot $t^{1/3}$.

Note in reference to this demonstration, first of all, that all this refers only to real triple collisions, inasmuch as the limit Sundman relations are obtained only for them. Further, it follows from these relations only that

$$R = c_2 + \varepsilon)\, t^{2/3},$$

but not, by any means, the expansion of R by whole powers $t^{1/3}$ and, so much the more, the regularization of the triple real collision. In order to establish the legality (validity) of coordinate expansion by whole powers of u, one ought to consider the differential equations of motion transformed to a new independent variable. In vectorial form these equations may be written in the following form:

$$\frac{d^2 r_\nu}{dt^2} = -M \frac{r_\nu}{r_\nu^3} + m_\nu \sum_{i=0}^{2} \frac{r_i}{r_i^3} \qquad (\nu = 0, 1, 2).$$

After transition to the variable u they will take the form:

$$\frac{d^2 r_\nu}{du^2} = (\sqrt{R}\dot{R})(\sqrt{R}\dot{r}_\nu) - M\left(\frac{R}{r_\nu}\right)^2 \frac{r_\nu}{r_\nu} + m_\nu \sum_{i=0}^{2} \left(\frac{R}{r_i}\right)^2 \frac{r_i}{r_i} \qquad (\nu = 0, 1, 2).$$

The limit Sundman relations show that at time of triple collision the

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* The author derives this formula without demonstration, though it is not evident, for it cannot be obtained by simple differentiation of formula

$$r_i = R\left(\sum_{i=0}^{2} \frac{1}{m_i}\right)^{-\frac{1}{2}} + \varepsilon,$$

since the derivative from an infinitely small quantity must not necessarily be infinitely small. However, the formula for $\dot{r}_i$, sought for, may be obtained with the aid of the 1907 Sundman formula $R\dot{r}_i - \dot{R}r_i = \varepsilon\sqrt{R}$, whence

..\/..

coefficients of these equations $\sqrt{R}\cdot\dot{R}$, $\frac{R}{r_\nu}$ $(\nu=0, 1, 2)$ approach specific limits. If it could be demonstrated that vectors $\sqrt{R}\,\dot{r}_\nu$, $\frac{r_\nu}{r_\nu}$ (direction cosines) also approach specific limits, one might assert on the basis of the Cauchy Picard theorem the existence of a solution $\bar{r}_\nu(u)$, regular at $u = 0$, whence the validity of the expansion by whole positive powers of $t^{1\,3}$ would be stemming.

For certain particular types of motions, and namely for Lagrange motions, the above-indicated properties are known a priori, for in these cases the problem of three bodies degenerates into 2 problems of two bodies and the triple collisions disintegrate into double collisions, for which these properties have been established by Sundman. But in the general case it is impossible to establish these properties, since for the problem of three bodies the coordinates in the vicinity of the triple collision are represented by series, disposed by irrational powers these being algebraic mass functions (Wintner, 1941). This means that if we attempt to formally satisfy the differential equations by series of the form

$$\bar{r}_\nu = \sum_{n=2}^{\infty} \bar{a}_n^{(\nu)} t^{\frac{n}{3}} \quad (\nu=0, 1, 2),$$

we would obtain for the factors $\bar{a}_n^{(\nu)}$ particular values corresponding to Lagrange motions (i. e. to the problem of two bodies at $e = 1$), and, therefore, its generalization too, that is, the above-mentioned theorem 5, is valid only for Lagrange motions, and also for such particular values of mass ratios, for which the above algebraic mass functions are reduced to rational numbers of the form $n/3$.

Vernić further asserts that the independent variable u, defined in the theorem 5, regularizes also all the imaginary collisions of the problem of three bodies. The author's line of reasonings is approximately as follows. The real collisions may be characterized as taking place in one and the same point of space, for example, at coordinate origin if the latter is placed at one of the moving points. For imaginary collisions, this may not take place either, for, the expression $r^2 = x^2 + y^2$ may, for example, contrary to the real case, become zero not only at $x = y = 0$ (collisions of first kind), but also for $x = iy \neq 0$ (collisions of 2nd kind). As to the imaginary collisions of first kind, their analyticial structure does not differ in essence from the real case, and since this last one is already considered by Vernić as regularized for double as well as triple collisions, the imaginary collision of 1st kind must, by the same token, be considered as regularized. Further, the author asserts that any collision of 2nd kind may be transformed into a collision of 1st kind by simple transformation of coordinates, and namely, by transition to the inertial system by way of transfer of the origin into the system's center of gravity.

This final result may not be correct, already because the demonstration of the regularization of triple real collisions is inconsistent, as we have already seen. But the author's assertion of the possibility of annihilation of imaginary collisions of 2nd kind by way of transition to the inertial system of coordinates is also incorrect. Indeed, here attempt is made first of all to conduct

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* ... continued from the preceding page... (whence) $\dot{r}_i = \frac{r_i}{R}\dot{R} + \frac{\varepsilon}{\sqrt{R}}$.

Eliminating R with the aid of the limit value for $\sqrt{R}\dot{R}$, we shall find $\dot{r}_i =$

$-\sqrt{2M\left(\sum_{i=0}^{2}\frac{1}{m_i}\right)^{\frac{1}{3}} R^{-\frac{1}{3}} + \varepsilon R^{-\frac{1}{2}}}$ which, by virtue of $\frac{r_i}{R} = \left(\sum_{i=0}^{2}\frac{1}{m_i}\right)^{-\frac{1}{3}} + \varepsilon R^{-1}$ formula (1) gives/

the demonstration for the problem of two bodies and then to extend this result to Lagrange motions, taking advantage of the fact that in the latter the problem of three bodies degenerates into several problems of two bodies, and finally, to derive therefrom, as a corollary, the general case of collisions, on the basis of the case, whereby for double collisions we have two problems of two bodies at limit, whereas in case of triple collisions we have the Lagrange configuration.

Let us analyze these reasonings, starting with the last one. As was shown by Sundman, the Lagrange configurations (equilateral triangle or collinear motions with constant distance ratio) are really limit for triple collisions. However, the Sundman demonstration refers only to the real motion. Inasmuch as we are interested here by imaginary triple collisions, the Lagrange configurations have with them no relation of any kind.

Further, as we shall see below, the author's demonstration utilizes for the problem of two bodies essentially the case, when after transferring the origin of coordinates to the center of gravity, the longitudes of moving bodies differ from one another by 180°. In the case of Lagrange motions this will no longer take place, for there the longitude difference will depend upon the mass ratio of three bodies (thus, for example, for equal masses it will 120°).

Finally, the starting point of this chain of reasonings, the demonstration for the problem of two bodies, is also based upon a misunderstanding.

Indeed, the author writes in the problem of two bodies the trajectory equations after transferring the origin of the coordinates to the system's center of gravity as follows (Vernic, 1953a, Theorem 13, pp 95-96):

$$\left.\begin{array}{lll} x'=r'\cos v, & y'=r'\sin v, & r'=\frac{p'}{1+\epsilon\cos v} \\ x''=-r''\cos v, & y''=-r''\sin v, & r''=\frac{p''}{1-\epsilon\cos v} \end{array}\right\}, \qquad (2)$$

whereupon the sign $(-)$ in the last formula is probably determined by the circumstance that the center of gravity is the left-hand focus of the ellipse described by the second body. The condition for collision consists in that $r_0 \equiv r_0' + r_0'' = 0$. Hence we obtain $-r_0'' = r_0'$ and consequently, $x_0' = x_0''$, $y_0' = y_0''$, which means that any collision, whether real or imaginary, is a collision of 1st kind. From the condition

$$r_0'+r_0''=\frac{(p'+p'')-(p'-p'')\epsilon\cos v_0}{1-\epsilon^2\cos^2 v_0}=0$$

the author obtains

$$r_0'=\frac{p'-p''}{2}=-r_0'', \qquad x_0'=\frac{p'+p''}{2\epsilon}=x_0'', \qquad y_0'=y_0''.$$

However, all this has been computed erroneously. Note first of all that formula (2) is incorrect.

In reality if r', u', r'', u'' are the polar coordinates of two bodies and if we take for the polar axis the common major axis of both ellipses, we may write the relationship between the Descartes and polar coordinates

$$x' = r' \cos u', \qquad x'' = r'' \cos u''$$
$$y' = r' \sin u', \qquad y'' = r'' \sin u''$$

and the equations of their trajectories:

$$r' = \frac{p'}{1 + \varepsilon \cos u'}, \quad r'' = \frac{p''}{1 - \varepsilon \cos u''}.$$

Taking into account that both bodies are always located on one straight line on either side from coordinate origin, we may write

$$u' = v, \quad u'' = v + 180°,$$

where $\underline{v}$ is the angle, counted from perihelion in both ellipses. Then we obtain

$$x' = r' \cos v, \quad x'' = -r'' \cos v, \qquad r' = \frac{p'}{1 + \varepsilon \cos v}, \quad r'' = \frac{p''}{1 + \varepsilon \cos v}.$$
$$y' = r' \sin v, \quad y'' = -r'' \sin v,$$

The condition for collision consists indeed in that $r'_0 + r''_0 = 0$, whence $-r''_0 = r'_0$. However, hence it does not follow yet that $x'_0 = x''_0$ and $y'_0 = y''_0$, since now $r'_0 = r''_0 = 0$, while $\cos v_0$ and $\sin v_0$ become infinite, so that x'_0, y'_0, x''_0, y''_0 acquire then an indeterminate form of the type $0, \infty$ which must be evaluated. Taking into account that

$$\sin v = \pm\sqrt{1 - \cos^2 v} = \pm i \cos v \sqrt{1 - \frac{1}{\cos^2 v}}$$

we have

$$x' = \frac{p' \cos v}{1 + \varepsilon \cos v}, \qquad x'' = -\frac{p'' \cos v}{1 + \varepsilon \cos v},$$
$$y' = \pm \frac{ip' \cos v}{1 + \varepsilon \cos v} \sqrt{1 - \frac{1}{\cos^2 v}}, \quad y'' = \mp \frac{ip'' \cos v}{1 + \varepsilon \cos v} \sqrt{1 - \frac{1}{\cos^2 v}}.$$

Directing $\cos v \to \infty$, we find:

$$x'_0 = \frac{p'}{\varepsilon}, \qquad x''_0 = -\frac{p''}{\varepsilon},$$
$$y'_0 = \pm i \frac{p'}{\varepsilon}, \quad y''_0 = \mp i \frac{p''}{\varepsilon},$$

whence it may be seen that $x'_0 \neq x''_0$, $y'_0 \neq y''_0$ i.e., we are again confronted with imaginary collisions of second kind.

Therefore, we may only assert that the Vernić transformation $dt = S\,(r_i)du$ regularizes only double collisions, just as does the Sundman transformation.

2. In his work 1953b) Vernić formulates the following fundamental theorem (Theorem 10, pp.259-263). "The Lagrange solutions are the unique periodical solutions of the general problem of three bodies".

The author's demonstration consists in the following. An arbitrary periodic solution of the problem of three bodies is considered with period T without collisions. If a new independent variable u is introduced according to formula

$$dt = \frac{du}{V},$$

where V is a potential function, our solution will obviously be also periodical by u with a certain period ω, whereupon $T = T(\omega)$. Our solution, as any other solution of the problem of three bodies, must satisfy the Lagrange equation

$$\frac{d^2J}{dt^2} = 2V + 4h,$$

where J is the moment of inertia and h is the kinetic energy constant. If we pass in this equation to the new Vernič variable, it will be written in the form:

$$\frac{d}{du}\left(V\frac{dJ}{du}\right) = 2\left(1 + \frac{2h}{V}\right).$$

In our periodical solution V may be considered as a known function: $V = V(u)$. Then the Lagrange equation will acquire the form of a linear inhomogenous equation of second order in self-adjoint form which must admit the periodical solution. Considering the finding of this periodical solution as a boundary value problem, Vernić reaches the following necessary condition of periodicity:

$$\omega + 2hT(\omega) = 0^* \tag{3}$$

The author further asserts that, utilizing this equation, one may demonstrate that T (u) is an odd function, and V (u) and J (u) are even functions of u. The author demonstrates, still further, that in this condition, and provided we assume for the independent variable $\frac{1}{V}$, J will be a holomorphic function of this new variable in the neighborhood of zero, i. e., for the periodical solution under consideration the following expansion will take place:

$$J = \sum_{\nu=0}^{\infty} a_\nu \left(\frac{1}{V}\right)^\nu.$$

Subsequently the case is utilized, whereby J and $\frac{1}{V}$ are homogenous functions of mutual distances respectively of second and first power. By the Euler theorem on homogenous functions we obtain:

$$2\sum_{\nu=0}^{\infty} a_\nu\left(\frac{1}{V}\right)^\nu = 2J = \sum_{i=0}^{2}\frac{\partial J}{\partial r_i} r_i = \sum_{i=0}^{2} r_i \sum_{\nu=0}^{\infty} \nu a_\nu \left(\frac{1}{V}\right)^{\nu-1} \frac{\partial\left(\frac{1}{V}\right)}{\partial r_i} = \sum_{\nu=0}^{\infty} \nu a_\nu\left(\frac{1}{V}\right)^\nu,$$

whence $2a_\nu = \nu a_\nu$, $(\nu - 2)a_\nu = 0$; hence it follows that for $\nu \neq 2$ $a_\nu = 0$ and, therefore

$$J = \frac{a_2}{V^2}.$$

It is easy to see that for the Lagrange solutions ($r_0 = r_1 = r_2 = r$) this relation is fulfilled:

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* [see the infrapaginal note next page].

$$J=\sum_{i=0}^{2}\frac{r_i^2}{m_i}=r^2\sum_{i=0}^{2}\frac{1}{m_i},\quad V=\sum_{i=0}^{2}\frac{M}{m_i r_i}=\frac{M}{r}\sum_{i=0}^{2}\frac{1}{m_i},\quad JV^2=M^2\left(\sum_{i=0}^{2}\frac{1}{m_i}\right)^3=a_2.$$

Apparently, this relation is fulfilled only for the Lagrange solutions, though the author fails to demonstrate it. But this circumstance is not so essential, for the author's theorem would not lose any of its importance if in its formulation the words "Lagrange solutions" were replaced by "solutions, for which $J \cdot V^2 \equiv$ const.".

The legitimacy of the above-written expansion is derived by the author from the monotonic and unambiguous properties of the quantity J, considered as a real function of $\frac{1}{V}$, which he tends to establish beforehand. Let us note in this connection that these properties are by no means sufficient for the holomorphic state. Thus, for example, the function $f(x) = \sqrt[3]{x}$ monotonic and single-valued over the entire real axis, but it still is not holomorphic in the neighborhood of zero. However, it is quite probable that in conditions of function's T(u) oddity the expansion of J by whole positive powers of $\frac{1}{V}$ really takes place. Indeed, by virtue of absence of triple collisions (the double collisions are regularized by the transfornation $dt = du/V$) all the three functions are holomorphic by u. Because of oddity of function T (u), the latter is represented by MacLaurin series containing only the odd powers of u. Then 1/V, as a derivative of T (u) will include in its expansion only even powers of u, that is, whole powers of u^2. If only

$$\frac{d^2}{du^2}\left(\frac{1}{V}\right)\Big|_{u=0} \neq 0$$

(and this is the only point which we still would be compelled to demonstrate), this series could be tranformed and represent u^2 in the form of series by whole positive powers of 1/V. On the other hand, on the strength of Lagrange equation,

and function's 1/V parity, J will be represented in the form of a series, containing only even powers of u, if we choose the initial moment $t = 0$ in such a way that

$$\frac{dJ}{du}\Big|_{u=0} = 0,$$

which is always possible, since function J, as is well known, always has at least one minimum. Substituting the above-obtained expression in this series instead of u^2, we shall arrive at the representation for J sought for.

But in the demonstration of function's T (u) oddity, brought up by the author, there is an error which it is no longer possible to correct. This demonstration

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* (From the preceding page). The presence of such a condition can be perceived from the following. Let us integrate the Lagrange equation from $t = 0$ to $t = T$:

$$\frac{1}{2}[J'(T)-J'(0)]=\int_0^T V\,dt+2hT=\int_0^\omega du+2hT=\omega+2hT.$$

By virtue of periodicity $J'(T) = J'(0)$, whence follows condition (3).

consists in the following. Let us compose a function $\psi(u) = T(u) + T(-u)$. Then, by virtue of determination of the variable u,

$$\frac{d\psi(u)}{du} = \frac{1}{V(u)} - \frac{1}{V(-u)}$$

Because of the periodicity condition (3) $T(0) = 0$, whence $\psi(0) = 0$. Further we have

$$\left.\frac{d\psi(u)}{du}\right|_{u=0} = 0.$$

It is easy to see that $J = T(u)$ is a partial solution of a homogenous linear equation of second order corressonding to the Lagrange equation

$$\frac{d}{du}\left(V\frac{dJ}{du}\right) = 0.$$

The author further states: " as a linear combination of partial solutions T(u), T(--u)...., (u) is also... a solution of that equation". But since this function and its first derivative satisfy the initial zero conditions, $\psi(u) \equiv 0$, whence the oddity of T(u) precisely follows. However, though T(u) is indeed a solution of this equation, the same cannot be asserted about T(—u), for that effect it would be necessary for the function V(u), as a coefficient of that equation, to be even, i. e., the author utilizes here precisely what ought to be demonstrated.

Therefore, the Vernić theorem, formulated in his work of 1953 (b), may not be considered as demonstrated by the author. But hence it does not obviously follow that the theorem itself is incorrect. Vernić himself points to the fact speaking in favor of the existence of such a theorem, that after Lagrange no one had constructed concrete periodical solutions in the general problem of three bodies. In respect to the generally well known periodical solutions by Poincare the author notes that periodical solutions were really constructed by the Poincare and also by numerical methods (Copenhagen school) only in the limited three-body problem, and, at the same time, only in a rotating coordinate system. But, at passage to the inertial system of coordinates these solutions lose their periodicity (Vernić, 1952). Leaving these author's doubts aside, namely concerning the existence of Poincare periodical solutions, let us note, however, that the latters' existence may be easily and with sufficient simplicity established in the limited three-body problem; they will remain as such even after transition to the inertial system of coordinates, basing ourselves upon the following Lyapunov theorem (1950):

If there exists a canonical system

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i},$$

where

$$H = \sum_{m=2}^{\infty} H_m$$

(H_m being the m-th power mode from $x_1, \ldots, x_n; y_1, \ldots, y_n$ with constant coefficients), to each pair of purely imaginary conjugated roots $\pm\lambda_k i$ of the characteristic equation

$$\begin{vmatrix} \frac{\partial^2 H_2}{\partial x_1^2} - \lambda, & \ldots, & \frac{\partial^2 H_2}{\partial x_1 \partial y_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial^2 H_2}{\partial y_n \partial x_1}, & \ldots, & \frac{\partial^2 H_2}{\partial y_n^2} - \lambda \end{vmatrix} = 0$$

(whereupon $\frac{\lambda_k}{\lambda_j}$ is not equal to either a whole number or zero) corresponds a family of periodical solutions dependent on two arbitrary constants c and t_0, whereupon the period is a continuous (and even holomorphic) function of c in the neighborhood of $c = 0$. If we introduce in a plane limited three-body problem in a rotating system of coordinates as new unknown functions the differences between the old unknown ones and the coordinates of triangular libration points, we shall obtain exactly a Lyapunov system, whereupon the characteristic equation will have a pair of purely imaginary and a pair of real roots, dependent on mass ratios. By virtue of continuous dependence on c of the period of the thus obtained solutions, there will exist an innumerable multiplicity of solutions, of which the period will be commensurate with the rotation period of the coordinate system. This is why such solutions will remain periodical even after transition to the inertial system of coordinates. The existence of analogous periodical solutions in the neighborhood of the Lagrange solutions may also be established in the general three-body problem. The difference from the limited problem will only consist in that the corresponding characteristic equation will have zero roots, which in the given case is an obstacle for the applicability of the Lyapunov theorem, for with the aid of area integrals the order of the system may be lowered by as many units as there are zero roots. After that the problem will in no way differ from the case analyzed. Therefore, it may be considered as demonstrated that the assertion, brought forth by the author about the nonexistence of periodical solutions of the three-body problem, other than the Lagrange solutions, is erroneous.

**** THE END ****

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