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STUDY ON DETERMINING STABILITY DOMAINS FOR NONLINEAR DYNAMICAL SYSTEMS — II

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STUDY ON DETERMINING STABILITY DOMAINS FOR NONLINEAR DYNAMICAL SYSTEMS - II

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ABSTRACT

This second quarterly progress report discusses results obtained during the period August 1, 1966 to October 31, 1966 under Contract NAS 8-20306, "Study on Determining Stability Domains for Nonlinear Dynamical Systems." Particular items discussed are: experimental results obtained via the algorithm developed during the previous quarter; reformulation of the algorithm to avoid computational difficulties encountered during the experiments; development of a parameterization of the set of positive definite $n \times n$ matrices; and analysis of a system with time dependent closed loop guidance. Plans for research during the last phase of the contract are outlined.

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INTRODUCTION

This second quarterly report discusses the work carried out during the period August 1, 1966 to October 31, 1966 under Contract NAS 8-20306, "Study on Determining Stability Domains for Nonlinear Dynamical Systems." Our activities during this period were devoted to continuing the experimental investigation of the algorithm described in Ref. 1, reformulating the algorithm to circumvent observed computational difficulties, developing a parameterization of the set of positive definite $n \times n$ matrices, and analyzing a system with time dependent closed loop guidance.

EXPERIMENTAL RESULTS

As in the previous quarterly report (Ref. 1), we will only report on the qualitative aspects of the results of our numerical experiments. During the past quarter we devised a subroutine for obtaining an initial search point on the constraint locus $\dot{V} = 0$. This alleviated some of the sensitivity of the success of the computation to the initial search point. However, in the case of the Van der Pol equation, the locus $\dot{V} = 0$ has two pairs of branches that are not symmetric, and the success of the computation depends on which pair of branches the initial point is on. In an attempt to circumvent this situation, we have reformulated the function to be minimized. (This reformulation is described in the following section.)

We also investigated whether using the quadratic part of the derivative of the optimal quadratic V as the arbitrary function in Zubov's equation would yield an improved estimate of the domain of attraction via the fourth order solution to Zubov's equation. The result was negative for the Duffing equation and positive for the Van der Pol equation.

Reformulation of the Numerical Algorithm

In Refs. 1 and 2 an algorithm for computing an estimate of the domain of attraction of a quasi-linear dynamical system was given. The algorithm consisted of two steps: 1) given a set of function parameters α (the elements of Q), compute x such that $V(x,\alpha)$ is minimum on $\dot{V}(x,\alpha) = 0$ (i.e.,

$$\ell(\alpha) = V(x^\circ,\alpha) = \min_{x \neq 0} \left\{V(x,\alpha) + k\dot{V}^2(x,\alpha)\right\} \qquad (1)$$

is used to solve this stage numerically); and 2) find α such that the figure described by $V(x,\alpha^\circ) = \ell(\alpha^\circ)$ has the largest volume (i.e., find α° such that

$$\frac{\left(\ell(\alpha^\circ)\right)^{n/2}}{\left(\det P(\alpha^\circ)\right)^{\frac{1}{2}}} > \frac{\left(\ell(\alpha)\right)^{n/2}}{\left(\det P(\alpha)\right)^{\frac{1}{2}}} \tag{2}$$

for all allowable α). This second step is carried out numerically by describing the Q-matrix in a parametric form, in terms of α, which guarantees that the matrix is positive definite (given appropriate restrictions on α) and by computing $\ell(\alpha)$ from Eq. (1) and $P(\alpha)$ from

$$A^T P(\alpha) + P(\alpha) A = - Q(\alpha) \quad , \tag{3}$$

where A is the matrix of the linear part of the dynamical equations.

This two stage optimization, in which one computes successively better values of x and α, does not work well for the Van der Pol problem. Apparently, a relative insensitivity of the area of the estimate to α and the existence of four branches (two symmetric pairs) of the locus $\dot{V}(x,\alpha) = 0$ cause the observed difficulties in the computation of an estimate of the domain of attraction. In order to alleviate these difficulties and to make the algorithm more efficient, we have reformulated it as a single minimization over x and α, simultaneously.

Consider that the problem is to compute a constrained minimum of V relative to x and a constrained minimum of the area relative to α. This can be accomplished by recognizing that the content of the figure described by $V(x,\alpha) < \ell(\alpha)$ is proportional to $\left(\ell(\alpha)\right)^{n/2}\left(\det P(\alpha)\right)^{-\frac{1}{2}}$. Then we can compute either

$$\min_{x,\ \alpha}\left\{\frac{\left(\det P(\alpha)\right)^{\frac{1}{2}}}{\left(\ell(\alpha)\right)^{n/2}} + k_1 V(x,\alpha) + k_2 \dot{V}^2(x,\alpha) + k_3 g(\alpha)\right\} \tag{4}$$

or

$$\max_{x,\ \alpha}\left\{\frac{\left(\ell(\alpha)\right)^{n/2}}{\left(\det P(\alpha)\right)^{\frac{1}{2}}} + C_1\left(V(x,\alpha) + C_2\dot{V}^2(x,\alpha)\right)^{-1} + C_3 g(\alpha)\right\}, \tag{5}$$

where we have used the penalty formulation for handling the constraints $\dot{V}(x,\alpha) = 0$ and Q positive definite (accounted for by $g(\alpha)$). Cursory examination indicates that $x = 0$, the trivial solution, is the global solution of Eq. (5); this is not true for Eq. (4), and thus only Eq. (4) is considered further.

A computer program has been written to compute Eq. (4) for the Van der Pol equation by using the Min-All algorithm developed by McGill and Taylor. Unfortunately, we do not have sufficient data to draw conclusions about this formulation at this time.

Parameterization of the Set of Positive Definite Matrices

The stability analysis algorithm described in Refs. 1 and 2 requires the generation of positive definite $n \times n$ matrices for the analysis of n-dimensional quasi-linear dynamical systems. The brute force approach is to form an $n \times n$ symmetric matrix and then apply the determinantal test (Ref. 3) to determine if it is positive definite. This procedure requires the arbitrary choice of $\frac{n(n+1)}{2}$ matrix elements and then the evaluation of the determinants of the n-principal minors of the matrix. However, it does not provide information on how to correct a candidate matrix that fails the test for positive definiteness. Therefore it would be desirable to generate the matrix by a procedure that guarantees the matrix is positive definite and spans the entire set of positive definite matrices. In this section we develop such a procedure based upon the work of Murnaghan (Ref. 4) on the parameterization of the group of unitary matrices.

It is well known (Ref. 3) that all real symmetric matrices are orthogonally similar to a diagonal matrix, and that all positive definite (pd) matrices are then orthogonally similar to a diagonal matrix with positive diagonal elements; i.e., let P be pd, then

$$P = S^T \Lambda S \quad , \tag{6}$$

where

$$\Lambda = \mathrm{diag}\left\{\lambda_1, \lambda_2, \ldots, \lambda_n\right\}$$

$$\lambda_i > 0 \quad , \quad i = 1, 2, \ldots, n \tag{7}$$

$$S^T S = I \quad .$$

Thus the parameterization of all pd matrices P is reduced to the parameterization of the group of orthogonal matrices S.

In Ref. 4 Murnaghan proves that the parameterization of the group of $n \times n$ unitary matrices U is accomplished by the factorization

$$U = D \times \prod_{k=1}^{n-1} U_k \quad , \tag{8}$$

where

$$D = \operatorname{diag}\left\{e^{i\delta_1}, e^{i\delta_2}, \ldots, e^{i\delta_{n-1}}, e^{i\varphi_n}\right\} , \tag{9}$$

$$U_k = \left(\prod_{\ell=k+1}^{n-1} U_{k\ell}(\theta_\mu, \sigma_\rho)\right) \times U_{kn}(\varphi_k, \sigma_\gamma) , \tag{10}$$

$$\gamma = \frac{(2n - k)(k - 1)}{2} + 1 ,$$

$$\rho = \frac{(2n - k)(k - 1)}{2} + 1 + n - \ell ,$$

$$\mu = \frac{(2n - k - 2)(k - 1)}{2} + (n - \ell) ,$$

$$U_{k\ell} = (u_{ij}) : \begin{cases} u_{ii} = 1 , \ i \neq k,\ell \\ u_{kk} = \cos\theta \\ u_{\ell\ell} = \cos\theta \\ u_{ij} = 0 , \ i \neq j , \ i,j \neq k,\ell \\ u_{k\ell} = - e^{-i\sigma} \sin\theta \\ u_{\ell k} = + e^{+i\sigma} \sin\theta , \end{cases} \tag{11}$$

$$-\pi \leq \phi < \pi , \quad -\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2} , \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} , \quad -\frac{\pi}{2} \leq \delta \leq \frac{\pi}{2} .$$

The factorization of the group of orthogonal matrices is immediately obtained by requiring U to be real; i.e., $\delta = \sigma = 0$, $\varphi_n = \pm\pi$, $-\pi \leq \varphi_k < \pi$, $k \neq n$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. In particular,

$$S = D_1 \times \prod_{k=1}^{n-1} S_k \quad , \tag{12}$$

$$D_1 = \text{diag}\left\{1, \ldots, 1, \pm 1\right\} \quad , \tag{13}$$

$$S_k = \left(\prod_{\ell=k+1}^{n-1} U_{k\ell}(\theta_\mu, 0) \right) \times U_{kn}(\varphi_k, 0) \quad , \tag{14}$$

$$\mu = \frac{(2n - k - 2)(k - 1)}{2} + n - \ell \quad .$$

This factorization contains $\frac{(n - 1)(n - 2)}{2}$ thetas and n-phis, or a total of $\frac{n(n - 1)}{2} + 1$ parameters. The n-lambdas in Eq. (7) raise the number of parameters to $\frac{n(n + 1)}{2} + 1$ — one more than required. Thus if we restrict S to be a rotation matrix (i.e., choose $\varphi_n = 0$), the number of parameters will be $\frac{n(n + 1)}{2}$; the number required to represent an arbitrary symmetric matrix. The choice $\varphi_n = 0$ is intuitively motivated by the consideration that we wish to rotate and scale the ellipsoid associated with the pd matrix and do not want to reflect coordinates or change the handedness of the coordinate system.

The factorization of a pd matrix of dimension three is thus given by

$$P = S^T \Lambda S \quad , \tag{15}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} , \tag{16}$$

$$\lambda_1, \lambda_2, \lambda_3 > 0 \quad ,$$

and

$$S = S_{23}(\varphi_2)\ S_{12}(\theta_1)\ S_{13}(\varphi_1) \quad , \tag{17}$$

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\varphi_2 & -s\varphi_2 \\ 0 & s\varphi_2 & c\varphi_2 \end{pmatrix} ,$$

$$S_{12} = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & \theta \\ 0 & 0 & 1 \end{pmatrix} , \tag{18}$$

$$S_{13} = \begin{pmatrix} c\varphi_1 & 0 & -s\varphi_1 \\ 0 & 1 & 0 \\ s\varphi_1 & 0 & c\varphi_1 \end{pmatrix} ,$$

$$-\pi \leq \varphi_1 < \pi \ , \ -\pi \leq \varphi_2 < \pi \ , \ -\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2} \quad ,$$

$$c\varphi_1 = \cos\ \varphi_1 \ , \ s\varphi_1 = \sin\ \varphi_1 \quad .$$

The factorization given here is similar to that conjectured in Ref. 1.

<u>Stability Analysis of a System with Time Dependent Closed Loop Guidance</u>

In their simplest form, booster guidance control systems are described for finite intervals of time by nonlinear, nonautonomous differential equations. In the more general case, the control laws are determined by iterative procedures and cannot be expressed as simple functions of the state. Because the standard well-known techniques of stability analysis are not readily applicable to such systems, new techniques, or variations of present Liapunov methods are being sought.

Initial research in this area has consisted of determining the stability of a simple control system that contains characteristics representative of the more complex booster guidance systems. Specifically, we are concerned with guiding the motion of a particle moving at constant speed in a plane in the presence of a constant disturbance. The dynamics of the system are given by

$$\dot{x}_1 = v \cos u$$
$$\dot{x}_2 = V_o + v \sin u \quad , \tag{19}$$

where v is the magnitude of the velocity of the particle relative to the disturbance, V_o is the disturbance and u is the direction of the velocity. The control law $u(t)$ is such that the particle is guided from the initial point (x_1^o, x_2^o) to the final point (x_1^f, x_2^f) in minimum time in the face of disturbances in the initial conditions. The problem has been made more specific by letting $v = 1$, $V_o = 1/2$, $x_1^o = x_2^o = 0$ and $x_1^f = 2$, $x_2^f = 1$.

In the absence of any disturbance in the initial conditions, the optimum control law for minimum time is $u^*(t) = 0$, with the corresponding trajectory $x_1^* = t$, $x_2^* = t/2$, and nominal time $T_f = 2$. The initial disturbance is assumed to be randomly distributed with a bivariate normal distribution of errors in initial conditions with mean value 0 and standard deviation 0.1. The control law is assumed to be linear, time varying, and of the form

$$u(x) = u^*(t) + p_1(t)\left[x_1(t) - x_1^*(t)\right] + p_2(t)\left[x_2(t) - x_2^*(t)\right] , \tag{20}$$

where

$$p_1 = p_{10} + p_{11}t \quad , \quad \text{and}$$

$$p_2 = p_{10} + p_{21}t \quad . \tag{21}$$

For the specific problem considered, with

$$\begin{aligned} p_{10} &= 0.153 \qquad p_{11} = 0.090 \\ p_{20} &= -0.305 \qquad p_{21} = -0.195 \quad , \end{aligned} \tag{22}$$

the mean square miss of the target point (2,1) is 0.00139; the resulting expected final time is 2.00287.

The stability problem for the system described above may be stated as follows: From what set of initial states (x_1^o, x_2^o) will the particle reach a point in some ϵ-neighborhood of the final state (2,1), i.e.,

$$(x_1^f - 2)^2 + (x_2^f - 1)^2 < \epsilon \quad , \tag{23}$$

subject to the constraint $T_f \leq M$, where M is some constant.

In order to determine any characteristic of the problem that might be useful in the development of a general stability technique, an analysis of the system response was undertaken. It was initially assumed that the control law $u(t)$ was time invariant, i.e.,

$$u(t) = p_{10}\left(x_1(t) - x_1^*(t)\right) + p_{20}\left(x_2(t) - x_2^*(t)\right) \quad . \tag{24}$$

Thus the system is described by

$$\begin{aligned} \dot{x}_1(t) &= \cos\left[p_{10}\ (x_1(t) - t) + p_{20}\ (x_2(t) - t/2)\right] \\ \dot{x}_2(t) &= \frac{1}{2} + \sin\left[p_{10}\ (x_1(t) - t) + p_{20}\ (x_2(t) - t/2\right] \ . \end{aligned} \tag{25}$$

Under the translation

$$\begin{aligned} y_1(t) &= x_1(t) - t \\ y_2(t) &= x_2(t) - t/2 \ , \end{aligned} \tag{26}$$

Eqs. (25) become

$$\begin{aligned} \dot{y}_1 &= -\ 1 + \cos\left[p_{10}y_1 + p_{20}y_2\right] \\ \dot{y}_2 &= \sin\left[p_{10}y_1 + p_{20}y_2\right] \end{aligned} \tag{27}$$

with the equilibrium solution $y_1 = y_2 = 0$.

The loci of equilibrium points for Eqs. (27) are given by

$$y_2 = -\frac{p_{10}}{p_{20}}\, y_1 + \frac{2n\pi}{p_{20}} \ . \tag{28}$$

Thus if the initial point $(y_1^o = x_1^o\ ,\ y_2^o = x_2^o)$ lies on the locus, the solution for $y_1(t)$ and $y_2(t)$ will be constant in time, and the trajectories in the x_1, x_2-plane will be given by

$$\begin{aligned} x_1(t) &= t + x_1(0) \\ x_2(t) &= t/2 + x_2(0) \ . \end{aligned} \tag{29}$$

These trajectories are parallel to the optimum trajectory and do not converge to the target point.

To find the complete trajectories in the y_1, y_2-plane, let

$$z(t) = p_{10}y_1(t) + p_{20}y_2(t) \quad , \tag{30}$$

then from Eqs. (27)

$$\dot{z}(t) = ay_1^o(t) + by_2^o(t) = -a + a\cos z + b\sin z \quad . \tag{31}$$

Solving for $z(t)$,

$$z(t) = 2\tan^{-1}\left[\frac{p_{20}}{p_{10}}\,\frac{1}{1 - ke^{-p_{20}t}}\right] \quad , \tag{32}$$

where

$$k = 1 - \frac{1}{\dfrac{p_{10}}{p_{20}}\tan\dfrac{z(0)}{2}} \quad . \tag{33}$$

By substituting Eq. (32) into Eqs. (27), and integrating,

$$\begin{aligned} y_1(t) = {} & \frac{2p_{10}}{p_{10}^2 + p_{20}^2}\left[\tan^{-1}\frac{p_{10}}{p_{20}}\left(ke^{-p_{20}t} - 1\right) - \tan^{-1}\frac{p_{10}}{p_{20}}(k - 1)\right] \\ & + \frac{p_{20}}{p_{10}^2 + p_{20}^2}\ln\frac{\left\{(k-1)^2 + \dfrac{p_{20}^2}{p_{10}^2}\right\}e^{-2p_{20}t}}{\left(ke^{-p_{20}t} - 1\right)^2 + \dfrac{p_{20}^2}{p_{10}^2}} + y_1(0) \end{aligned} \tag{34}$$

$$y_2(t) = \frac{2p_{20}}{p_{10}^2 + p_{20}^2} \left[\tan^{-1} \frac{p_{10}}{p_{20}} \left(ke^{-p_{20}t} - 1 \right) - \tan^{-1} \frac{p_{10}}{p_{20}} (k - 1) \right]$$

(34 Cont.)

$$- \frac{p_{20}}{p_{10}^2 + p_{20}^2} \ln \frac{\left\{ (k - 1)^2 + \frac{p_{20}^2}{p_{10}^2} \right\} e^{-2p_{20}t}}{\left(ke^{-2p_{20}t} - 1 \right)^2 + \frac{p_{20}^2}{p_{10}^2}} + y_2(0) \quad .$$

It is evident from Eqs. (33) and (34) that if the initial point is on the equilibrium locus, Eqs. (34) reduce to

$$y_1(t) = y_1(0) \quad , \quad y_2(t) = y_2(0) \quad .$$

It can also be seen from Eqs. (34) or Eq. (32) that if the initial point $\left(y_1(0), y_2(0) \right)$ does not lie on the equilibrium locus, and if p_{20} is real and negative, then the final point $\left(y_1(\infty), y_2(\infty) \right)$ will lie on the equilibrium locus. Thus if p_{20} is real and negative, every solution $y(t)$ tends to the set of points where

$$y_2 = - \frac{p_{10}}{p_{20}} y_1 + \frac{2n\pi}{p_{20}} \quad .$$

An approximate analysis of the response of the system when the control law is time varying, as given by Eqs. (20) and (21), can be carried out if $p_1(t)$ and $p_2(t)$ are approximated by "staircase" functions such that

$$p_1(nT) = p_{10} + p_{11}nT$$
$$p_2(nT) = p_{20} + p_{21}nT \quad . \tag{35}$$

In the n^{th} interval, i.e., $(n - 1)T < t < nT$,

$$z_n(\tau) = (p_{10} + p_{11}nT)\, y_{1n}(\tau) + (p_{20} + p_{21}nT)\, y_{2n}(\tau)$$
$$= 2 \tan^{-1}\left[\frac{p_2(nT)}{p_1(nT)} \cdot \frac{1}{1 - k_n e^{-p_2(nT)\tau}}\right] \tag{36}$$

for $0 \leq \tau \leq T$. The constant k_n is given by

$$k_n = 1 - \frac{1}{\dfrac{p_1(nT)}{p_2(nT)} \tan \dfrac{z_{n-1}(T)}{2}} \quad . \tag{37}$$

By using Eqs. (36) and (37), the following linear difference equation for k_n results:

$$k_n - F_n k_{n-1} = \frac{p_{11}p_{20} - p_{10}p_{21}}{p_1(nT)\; p_2[(n-1)T]} \quad , \tag{38}$$

where

$$F_n = \frac{p_1[(n-1)T]\; p_2(nT)}{p_1(nT)\; p_2[(n-1)T]}\, e^{-p_2[(n-1)T]T} \quad . \tag{39}$$

Solving Eq. (38) yields

$$k_n = \left[\prod_1^n F_i\right]\left[k_0 + \sum_1^n \frac{R_i}{\prod_1^i F_j}\right] \tag{40A}$$

or

$$k_n = \left[\frac{p_{10}p_2(nT)}{p_{20}p_1(nT)}\, e^{-\left(p_{20} + \frac{p_{21}(n-1)T}{2}\right)nT}\right] \cdot \left[k_0 + \frac{p_{20}(p_{11}p_{20} - p_{10}p_{21})}{p_{10}}\right.$$

(40B)

$$\left.\sum_{i=1}^{n} \frac{e^{\left(p_{20} + p_{21}\frac{(i-1)T}{2}\right)iT}}{p_2(nT)\; p_2\left((n-1)T\right)}\right] \cdot$$

The approximate location of the particle at any instant of time is then specified by

$$y_{in}(\tau) = P_{in}(\tau) + y_{io}(0) + \sum_{j=1}^{n-1} P_{ij}(1) \quad , \; i = 1,\, 2 \;, \tag{41}$$

where

$$P_{in}(\tau) = \frac{2p_i(nT)}{p_1^2(nT) + p_2^2(nT)}\left[\tan^{-1} \frac{p_1(nT)}{p_2(nT)}\left(k_n e^{-p_2(nT)\tau} - 1\right) - \tan^{-1} \frac{p_1(nT)}{p_2(nT)}(k_n - 1)\right]$$

(42)

$$- (-1)^i \frac{p_2(nT)}{p_1^2(nT) + p_2^2(nT)} \ln\left\{\frac{(k_n-1)^2 + \frac{p_2^2(nT)}{p_1^2(nT)}}{\left(k_n e^{-p_2(nT)\tau} - 1\right)^2 + \frac{p_2^2(nT)}{p_1^2(nT)}}\right\} e^{-2p_2(nT)\tau}$$

The work carried out to date has essentially consisted of analyzing the given system and has resulted in Eqs. (41) and (42). It will now be necessary to study these equations along with the trajectories that result from a computer simulation in order to determine characteristics that will enable development of a stability analysis technique that does not require a solution of the system equations. To achieve this, our efforts in the next quarter will be directed toward examining the possibility of applying standard Liapunov stability techniques over each interval $(n-1)T \leq t \leq nT$, and using the information obtained from the sub-intervals to conclude stability for the entire interval $0 \leq t \leq T_f$.

PLANS FOR FUTURE WORK

The emphasis during the last phase of this contract will be upon formulating other examples that are representative of the booster guidance stability problem. There are, however, other particular areas that will receive some attention, viz.:

1. Formulation of a function whose global minimum is achieved for the values of state x and parameters α that produce the optimal quadratic estimate of the domain of attraction.

2. Development of a procedure for successively improving a given quadratic estimate by modification of the Liapunov function via geometric considerations.

3. Development of an efficient procedure for solving the Liapunov matrix equation (Eq. (4) of Ref. 2).

4. Numerical investigation of the guidance example discussed earlier, and an analytical investigation of whether stability results over segments of the interval of operation of the nonautonomous system can be combined into a result for the entire interval of concern.

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