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by G. W. Garrison, Jr., H. A. Hassan, and R. K. Seals, Jr.

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STUDY OF INSTABILITIES IN LINEAR HALL CURRENT ACCELERATORS

G. W. Garrison, Jr., H. A. Hassan and R. K. Seals, Jr.
North Carolina State University

SUMMARY

The stability of a linear Hall current accelerator with applied axial electric and radial magnetic fields against a screw type perturbation is investigated. The analysis is based on the ion and electron conservation equations and takes into consideration the effects of ionization. It is shown that the observed instability cannot be explained when the axial magnetic field is identically zero even when the unperturbed potential and density distributions have both radial and axial dependence. On the other hand, when the axial component of the magnetic field is small, but not zero, the column is unstable, in the absence of an axial density gradient, to a right-handed helical perturbation $(m = -1)$. The resulting instability, which is of the Kadomtsev-Nedospasov type, is analyzed using a normal mode analysis.

INTRODUCTION

The instabilities observed in linear Hall current accelerators with applied axial electric and radial magnetic fields (fig. 1) are typical of a number of instabilities observed in devices which employ crossed electric and magnetic fields (refs. 1, 2 and 3). In these accelerators, the ions are accelerated more or less in the direction of the axial electric field while the electron motion in the axial direction is retarded by the radial magnetic field as a result of the large differences between the electron and ion radii of gyration. Thus, a linear Hall accelerator avoids, to a large extent, the space charge limitations of conventional ion rockets. Conservation of mass requires the ion density to decrease with increasing velocity. Thus, with the possible exception of a region next to the anode, the density gradient and the applied axial electric field are antiparallel over much of the accelerator length. Also, since the density at the wall is zero (Shottky condition), the density and potential gradients in the radial direction are not zero.

The systems considered by Simon (ref. 1), Hoh (ref. 2), and Morse (ref. 3) have no density and potential gradients in the direction of the applied magnetic field. Therefore, their results are not directly applicable to a linear Hall accelerator. In an attempt to understand the nature of the experimentally observed screw instability and associated anomalous diffusion (refs. 4, 5, 6, 7, and 8), an analysis is performed in which the steady state solution possesses both radial and axial dependence and the axial component of the magnetic field is identically zero. A normal mode analysis similar to that employed by Johnson and Jerde (ref. 9) shows that the system is unstable only when the sign of the product of the axial electric field and the density gradient and its direction is positive, which is the Simon-Hoh criterion. Since in the linear Hall accelerator the axial density gradient in the axial electric field are antiparallel over much of the accelerator length, it is concluded that such analysis cannot explain the observed instability.

The experiments of Hess et. al. (ref. 6) showed that the spectrum of the instability was strikingly similar to that observed in the positive column for magnetic fields above the critical value. Also, the instability was found to exist all along the accelerator length. This, and the results of the above analysis suggest that the origin of the instability may not be dependent on the existence of an axial density gradient but may be a result of the small, but non-vanishing axial component of the magnetic field. Therefore, an analysis is performed which employs a magnetic field with both radial and axial components. As a result of this analysis, it is shown that the column is unstable, in the absence of axial density gradients, to a right-handed helical perturbation $(m = -1)$. The resulting instability is of the Kadomtsev and Nedospasov type (ref. 10).

SYMBOLS

a	defined in equation (21)
$a_o \ldots a_6$	defined in Appendix C
a^*_{jp}	defined in equation (24A)
$\mathbf{A}$	defined in equation (24)
A_1, A_2, A_3	defined in Appendix C
A_{pj}	defined in equation (33)
A^*_{pj}	defined in equation (27A)
b	defined in equation (21)
b_1, b_2	defined in equation (35)
$b_1 \ldots b_{10}$	defined in equation (4B) (Appendix B only)

b^*_{jp}	defined in equation (24A)
$\vec{B}$	magnetic flux density
$\bar{B}_x$	B_x/B_r
B_{jp}	defined in equation (33)
B^*_{jp}	defined in equation (27A)
c	defined in equation (25)
$c_o \cdots c_3$	defined in Appendix C
c^*_{jp}	defined in equation (24A)
C_{jp}	defined in equation (33)
C^*_{jp}	defined in equation (27A)
d_1, d_2	defined in equation (35)
d^*_{jp}	defined in equation (24A)
D	diffusion coefficient
D_{pj}	defined in equation (33)
D^*_{pj}	defined in equation (24A)
e	electronic charge
e^*_{jp}	defined in equation (26A)
$\vec{E}$	electric field
E^*_x	defined in equation (51)
E_{pj}	defined in equation (33)
E^*_{pj}	defined in equation (27A)
f	density radial perturbation
F	integral transform of f
g	potential radial perturbation
h_o	defined in equation (47)
h_j	defined in equation (2A)

H	defined in equation (28)
H^*	defined in equation (30)
I	total current
j	current density
J	Bessel function
k	wave number or Boltzmann constant
k_o	dimensionless wave number
$k_{1i}, k_{2i}, k_{1e}, k_{2e}$	defined in equation (35)
ℓ	defined in equation (28) or equation (55)
L	integral transform of ℓ or length of the device
m	particle mass
M	confluent hypergeometric function
$M_1 - M_{10}$	defined in equation (11B)
n	particle number density
$\bar{n}_x$	defined in equation (28)
N_o	defined in equation (21)
$N_1 - N_{10}$	defined in equation (11B)
p^*, q^*, r^*	defined in equation (20A)
q	defined in equation (14)
q_1, q_2	defined in equation (50)
q_3, q_4, q_5	defined in equation (53)
r, θ, x	cylindrical coordinates
R	radius of the device
s^*_{jp}	defined in equation (26A)
S^*_{jp}	defined in equation (27A)

$t_1, \ldots t_{10}$	defined in equation (8B)
T	temperature
u^*_{jp}	defined in equation (26A)
U	electric potential
U^*_{jp}	defined in equation (27A)
V_1, V_2	defined in equation (7B)
V	temperature in electron volts
$\vec{w}$	mean velocity
y	defined in equation (36)
α	defined in equation (46)
β	defined in equation (46)
β_o	defined in equation (22)
β_j	defined in equation (31)
γ_o	defined in equation (47)
Γ	Gamma function
$\vec{\Gamma}$	particle flux vector
δ	ratio of mobilities
δ_1	defined in equation (50)
δ_{jp}	Kronecker delta
Δ	defined in Appendix C
Δ_1, Δ_2	defined in equation (2B)
$\bar{\theta}$	V_i/V_e
η	defined in equation (47)
η_1, η_2	defined in equation (3B)
λ	wave length

λ_o	defined in equation (47)
λ^*	defined in equation (16)
μ	particle mobility
μ^*	defined in equation (44)
ν	collision frequency
ξ	rate of production of charged particles per electron per sec.
$\sigma_{1i}, \sigma_{1e}, \sigma_{2i}, \sigma_{2e}$	defined in equation (35)
τ	$1/\nu$
ψ_1, ψ_2	defined in equation (3B)
ω_1	frequency of perturbation
ω_2	growth rate of perturbation

Subscripts

e,i	refers to electrons and ions, respectively
o	refers to steady state
1	refers to perturbed state
r,x,θ	refers to radial, axial and azimuthal direction, respectively
℄	refers to center line

Superscripts

$\rightarrow$	vector quantity
$'$	denotes derivative or, use defined in equations (8) and (9)
$''$	use defined in equation (53)
-	dimensionless quantity

GOVERNING EQUATIONS

A collision dominated cylindrical plasma immersed in applied electric and magnetic fields is considered as a model for the previously described device. The frictional coupling between charged and neutral particles does not set the netural gas in motion. The degree of ionization is assumed small so that electron-ion collisions are not appreciable. The electron and ion mean free paths are small compared to the tube radius. The production rate of charged particles is assumed to be of the form ξn_e, where n_e is the electron number density and ξ is the number of ionizing collisions per electron per second and depends upon the electron temperature, T_e, and the neutral gas density. Two stage ionization, volume recombination, and electron attachment are neglected. The electron and ion gases are assumed to be isothermal at temperatures T_e and T_i respectively.

The governing equations are those of a slightly ionized gas in the presence of electric and magnetic fields and may be derived from the moments of the Boltzmann equation. The conservation of mass and momentum can be written as

$$\frac{\partial n_{i,e}}{\partial t} + \nabla \cdot n_{i,e}\vec{w}_{i,e} = n_{i,e}\xi \tag{1}$$

$$\pm e n_{i,e}(\vec{E} + \vec{w}_{i,e} \times \vec{B}) - kT_{i,e}\nabla n_{i,e} - n_{i,e}\nu_{i,e}\vec{w}_{i,e} = 0 \tag{2}$$

where n is the particle number density, e is the electronic charge, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\vec{w}$ is the average particle velocity, k is Boltzmann's constant, m is the particle mass, ν is the collision frequency of charged particles with neutral particles, and the subscripts i and e refer to ions and electrons respectively. In writing these equations it is assumed that the collision frequency, ν, is large compared to the characteristic frequency of the system thus the inertia terms may be neglected. Also, it has been assumed that electron and ion temperatures are constants. These temperatures are usually calculated from energy considerations.

Defining the particle flux vector as $\vec{\Gamma}_{i,e} = n\,\vec{w}_{i,e}$, the equations may be written in the form

$$\frac{\partial n}{\partial t} + \nabla \cdot \vec{\Gamma}_{i,e} = n\xi \tag{3}$$

$$\pm\mu_{i,e} n\vec{E} \pm \mu_{i,e}\vec{\Gamma}_{i,e} \times \vec{B} - D_{i,e}\nabla n - \vec{\Gamma}_{i,e} = 0 \tag{4}$$

where

$$\mu_{i,e} = \frac{e}{m_{i,e}\nu_{i,e}} \tag{5}$$

$$D_{i,e} = \frac{kT_{i,e}}{m_{i,e}\nu_{i,e}} \tag{6}$$

are the mobilities and diffusion coefficients respectively for the ions and electrons. In writing the above equations it has been assumed that $n_e \approx n_i = n$.

An explicit expression for the particle flux vector can be obtained by taking the dot and cross product of equation (4) with the magnetic field. The resulting epxression can be written as

$$\begin{aligned} \vec{\Gamma}_{i,e} &= \pm \mu'_{i,e} n\vec{E} - D'_{i,e}\nabla n \mp \mu_{i,e} D'_{i,e} \nabla n \times \vec{B} \\ &\pm \mu^2_{i,e}\mu'_{i,e}(n\vec{E}\cdot\vec{B})\vec{B} - \mu^2_{i,e}D_{i,e}(\nabla n\cdot\vec{B})\vec{B} \\ &+ \mu_{i,e}\mu'_{i,e} n\vec{E}' \times \vec{B} \quad , \end{aligned} \tag{7}$$

where

$$\mu'_{i,e} = \frac{\mu_{i,e}}{1 + \mu^2_{i,e}B^2} \tag{8}$$

$$D'_{i,e} = \frac{D_{i,e}}{1 + \mu^2_{i,e}B^2} \tag{9}$$

Combining equation (7) with equation (3) and noting that the assumption of constant temperatures and mobilities permits $D_{i,e}$ to be taken outside the differential operators, the final equation is

$$\frac{\partial n}{\partial t} - n\xi \pm \mu'_{i,e}\nabla\cdot n\vec{E} - D'_{i,e}\nabla^2 n \mp \mu_{i,e}D'_{i,e}\nabla\cdot(\nabla n \times \vec{B})$$

$$+ \mu^2_{i,e}\mu'_{i,e}\nabla\cdot(n\vec{E}\cdot\vec{B})\vec{B} - \mu^2_{i,e}D'_{i,e}\nabla\cdot(\nabla n\cdot\vec{B})\vec{B}$$

$$+ \mu_{i,e}\mu'_{i,e}\nabla\cdot(n\vec{E} \times \vec{B}) = 0 \qquad (10)$$

In deriving this result it is assumed that any induced magnetic fields are much smaller than the externally applied magnetic field, i.e., $\nabla \times \vec{B} = 0$. Also the condition $\nabla(\vec{B}\cdot\vec{B}) = 0$ was employed along with Maxwell's equation, $\nabla\cdot\vec{B} = 0$. The former justifies the factoring out of $\mu'_{i,e}$ and $D'_{i,e}$.

Equation (10) forms the basis for the stability analysis. A steady state solution is first obtained and then the resulting solution is perturbed. Thus, letting

$$n = n_o + n_1 \ , \quad \vec{E} = \vec{E}_o + \vec{E}_1 \quad , \qquad (11)$$

where n_o and $\vec{E}_o$ represent the steady state solution, substituting into equation (10) and ignoring higher order terms, one finds

$$\frac{\partial n_1}{\partial t} - n_1 \xi \pm \mu'_{i,e}\nabla\cdot(n_o\vec{E}_1 + n_1\vec{E}_o) - D'_{i,e}\nabla^2 n_1 \mp \mu_{i,e}D'_{i,e}\nabla\cdot(\nabla n_1 \times \vec{B})$$

$$\pm \mu^2_{i,e}\mu'_{i,e}\nabla\cdot\ [\vec{B}\cdot(n_o\vec{E}_1 + n_1\vec{E}_o)] - \mu^2_{i,e}D'_{i,e}\nabla\cdot[(\nabla n_1\cdot\vec{B})\vec{B}]$$

$$+ \mu_{i,e}\mu'_{i,e}\nabla\cdot\ [(n_o\vec{E}_1 + n_1\vec{E}_o) \times \vec{B}] = 0 \quad . \qquad (12)$$

A normal mode analysis is used to obtain the dispersion relation from equation (12) and, hence, the conditions for instability.

STABILITY ANALYSIS FOR A PURELY RADIAL MAGNETIC FIELD

Steady State Solution

The steady state for the case of a uniform radial magnetic field can be obtained from equations (3) and (7) or from equation (10). Using the relation (ref. 11)

$$\Gamma_{ir} = \Gamma_{er} \qquad (13)$$

equation (7) gives

$$n_o E_{or} = \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\partial n_o}{\partial r} \equiv q \frac{\partial n_o}{\partial r} \tag{14}$$

and therefore

$$\Gamma_{ir} = \Gamma_{er} = - \frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e} \frac{\partial n_o}{\partial r} \quad . \tag{15}$$

Similarly, equations (3) and (13) give

$$\frac{\partial}{\partial x} (\Gamma_{ix} - \Gamma_{ex}) = 0 \quad \text{or} \quad \Gamma_{ix} - \Gamma_{ex} = \lambda^* \quad . \tag{16}$$

Combining equations (7) and (16) one obtains

$$n_o E_{ox} = \frac{\lambda^*}{\mu_i' + \mu_e'} + \frac{D_i' - D_e'}{\mu_i' + \mu_e'} \frac{\partial n_o}{\partial x} \tag{17}$$

and

$$\Gamma_{ix} = \Gamma_{ex} + \lambda^* = \frac{\mu_i'}{\mu_i' + \mu_e'} \lambda^* - \frac{\mu_e' D_i' + \mu_i' D_e'}{\mu_i' + \mu_e'} \frac{\partial n_o}{\partial x} \quad . \tag{18}$$

The governing equation for the steady state density distribution follows from equations (3), (15) and (18) as

$$\frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n_o}{\partial r}\right) + \frac{\mu_e' D_i' + \mu_i' D_e'}{\mu_i' + \mu_e'} \frac{\partial^2 n_o}{\partial x^2} + n_o \xi = 0 \ . \tag{19}$$

The boundary conditions are

$$n_o = 0 \quad \text{at} \quad r = R \quad \text{and finite at} \quad r = 0 \quad \text{for all} \quad x \ . \tag{20}$$

Because of the anode and cathode sheaths, the boundary conditions in the x direction are not known. As a result of this, it will be assumed that the number density variation in the axial direction is linear. In this case, the solution of equation (19) is

$$n_o = N_o J_o(\beta_o r)(ax + b) \tag{21}$$

where

$$\beta_o^2 = \frac{(\mu_i + \mu_e)}{\mu_e D_i + \mu_i D_e} = \frac{(2.4)^2}{R} \quad . \tag{22}$$

N_o, a, b are constants and R is the radius of the device. In a given problem, a and b should be given while λ^* and N_o are determined from the voltage and total current relations which can be written as

$$U = \int_0^L E_{ox} dx \quad , \quad I = \int_0^R 2\pi re \, (\Gamma_{ix} - \Gamma_{ex}) dr \tag{23}$$

where e is the electronic charge. Since E_{ox} is independent of r , equation (17) shows that λ^* may be chosen as

$$\lambda^* = A \, N_o \, (\mu_i' + \mu_e') \, \frac{V_e + V_i}{R} \, J_o(\beta_o r) \tag{24}$$

where V_e, V_i are the electron and ion temperatures in electron volts and A is constant. With this choice of λ^* , one finds

$$(ax + b) E_{ox} = c = \text{const.} \tag{25}$$

Normal Mode Analysis

In order to study the stability of the steady state solution, the perturbations n_1 and $\vec{E}_1$ will be chosen as

$$n_1 = f(r)(ax+b)\exp\,[i(\omega t + m\theta + kx)]$$

$$E_1 = -\nabla U_1 , \; U_1 = g(r)\exp\,[i(\omega t + m\theta + kx)] \tag{26}$$

with m and k real and ω complex. Substituting equations (26) into equations (12), one finds

$$D'_{i,e}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r^2} f\right] + \{\xi - i\omega \mp ik\,\mu'_{i,e}E_{ox} - k^2 D'_{i,e} + 2ika\, D'_{i,e}/(ax+b)$$

$$- \frac{im}{r}\mu_{i,e}\mu'_{i,e}B\,E_{ox}\}\,f + \mu^2_{i,e}D'_{i,e}B^2\frac{d^2 f}{dr^2} \mp \mu'_{i,e}\left[\frac{1}{r}\frac{d}{dr}(rfE_{or})\right.$$

$$\left. + \mu^2_{i,e}B^2\frac{d}{dr}(fE_{or})\right] = \frac{1}{ax+b}\left\{\mp \mu'_{i,e}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rn_o\frac{dg}{dr}\right) - \frac{m^2}{r^2}n_o g\right]\right.$$

$$\mp \mu^2_{i,e}\mu'_{i,e}B^2\frac{\partial}{\partial r}\left(n_o\frac{dg}{dr}\right) \mp ik\,\mu'_{i,e}\,g\,\frac{\partial n_o}{\partial x} \pm \mu'_{i,e}n_o g k^2$$

$$\left. + i\frac{m}{r}\mu_{i,e}\mu'_{i,e}B\,g\,\frac{\partial n_o}{\partial x}\right\} \quad . \tag{27}$$

Integrating equation (27) with respect to x from 0 to L, where L is the length of the device, and introducing the quantities

$$H = \frac{aR}{\bar{n}_x} \quad , \quad \bar{n}_x = \frac{1}{L}\int_0^L (ax+b)dx \quad , \quad \bar{E}_{ox} = c/\bar{n}_x$$

$$\bar{\omega} = \frac{\beta_o^2 R^2 \omega}{\xi} = \bar{\omega}_1 + i\bar{\omega}_2 \ , \quad k_o = kR \quad , \quad E_x = \frac{\beta_o^2 R \mu_i \bar{E}_{ox}}{\xi}$$

$$\ell(r) = qf(r) + N_o J_o(\beta_o r)\, g(r) \tag{28}$$

equation (27) reduces to

$$\frac{\xi}{\beta_o^2 R^2} \left\{ [\frac{R^2}{r} \frac{d}{dr}(r\frac{df}{dr}) - \frac{m^2}{r^2} f] + [(\beta_o^2 R^2 + \bar{\omega}_2 - i\bar{\omega}_1)(1 + \mu_{i,e}^2 B^2) \right.$$

$$\mp ik_o \frac{\mu_{i,e}}{\mu_i}(E_x + H^*) - k_o^2 - im\frac{R}{r}\frac{\mu_{i,e}^2}{\mu_i} B(E_x - H^*)$$

$$\left. + 2ik_o \frac{D_{i,e}}{q\mu_i} H^*] f + \mu_{i,e}^2 B^2 \frac{d^2 f}{dr^2} \right\} = \mu_{i,e} \left\{ \mp [\frac{1}{r}\frac{d}{dr}(r\frac{d\ell}{dr}) - \frac{m^2}{r^2}\ell] \right.$$

$$\pm (1 + \mu_{i,e}^2 B^2)\frac{1}{r}\frac{d}{dr}(\frac{r}{n_o}\frac{\partial n_o}{\partial r}\ell) \pm \mu_{i,e}^2 B^2 \frac{d^2\ell}{dr^2}$$

$$\left. \pm k_o^2 \mp ik_o H + im\frac{R}{r}\mu_{i,e} BH \right\} \tag{29}$$

where

$$H^* = q\,\mu_i \frac{\beta_o^2}{\xi} H \quad . \tag{30}$$

Equation (29) does not have a closed form solution and, therefore, transform techniques will be employed to derive the dispersion relationship. Since the steady state solution is given in terms of Bessel functions, it is convenient to transform equation (29) by using a finite Hankel transform. Thus, letting

$$F(\beta_j) = \int_0^R f(r) r\, J_m(\beta_j r) dr \quad , \quad f(r) = \frac{2}{R^2}\sum_{p=1}^{\infty} \frac{F(\beta_p) J_m(\beta_p r)}{[J_{m+1}(\beta_p R)]^2}$$

$$L(\beta_j) = \int_0^R \ell(r) r\, J_m(\beta_j r) dr, \quad \ell(r) = \frac{2}{R^2}\sum_{p=1}^{\infty} \frac{L(\beta_p) J_m(\beta_p r)}{[J_{m+1}(\beta_p R)]^2} \tag{31}$$

where $J_m(\beta_j R) = 0$ for all j ; multiplying equation (29) by $rJ_m(\beta_j r)$ and integrating, one obtains

$$\frac{\xi}{\beta_o^2 R^2 \mu_{i,e}} \Big\{ F(\beta_j)[(1 + \mu_{i,e}^2 B^2)(-\beta_j^2 R^2 + \beta_o^2 R^2 - i\bar{\omega}_1 + \bar{\omega}_2) - k_o^2 \mp i \frac{\mu_{i,e}}{\mu_i} k_o(E_x + H^*)$$

$$+ 2ik_o \frac{D_{i,e}}{q\mu_i} H^*] - im \frac{\mu_{i,e}}{\mu_i} \mu_{i,e} B(E_x - H^*) \sum_{p=1}^{\infty} A_{pj} F(\beta_p)$$

$$+ \mu_{i,e}^2 B^2 \sum_{p=1}^{\infty} D_{pj} F(\beta_p) \Big\}$$

$$= \pm [k_o^2 + \beta_j^2 R^2 (1 + \mu_{i,e}^2 B^2) - ik_o H] L(\beta_j)$$

$$+ im\, \mu_{i,e} BH \sum_{p=1}^{\infty} A_{pj} L(\beta_p) \mp \mu_{i,e}^2 B^2 \Sigma\, D_{pj} L(\beta_p)$$

$$\pm \mu_{i,e}^2 B^2 \sum_{p=1}^{\infty} E_{pj} L(\beta_p) \pm \sum_{p=1}^{\infty} C_{pj} L(\beta_p) \qquad (32)$$

where

$$A_{pj} = \frac{2}{[J_{m+1}(\beta_p R)]^2} \int_0^R J_m(\beta_p r)\, J_m(\beta_j r) \frac{dr}{R}$$

$$D_{pj} = \frac{2}{[J_{m+1}(\beta_p R)]^2} \Big\{ m(m-1) \int_0^R J_m(\beta_j r) J_m(\beta_p r) \frac{dr}{r}$$

$$+ \beta_j \int_0^R J_m(\beta_p r)\, J_{m-1}(\beta_j r) dr \Big\}$$

$$B_{pj} = \frac{2\beta_o}{[J_{m+1}(\beta_p R)]^2} \int_0^R J_m(\beta_p r) \frac{J_1(\beta_o r)}{J_o(\beta_o r)} J_m(\beta_j r) dr$$

$$E_{pj} = \frac{2\beta_o \beta_j}{[J_{m+1}(\beta_p R)]^2} \int_0^R r\, J_m(\beta_p r) \frac{J_1(\beta_o r)}{J_o(\beta_o r)} J_{m-1}(\beta_j r) dr$$

$$+ (1-m) B_{pj}$$

$$C_{pj} = E_{pj} - B_{pj} \tag{33}$$

The desired dispersion relation is obtained by setting the determinant of the infinite system of equations represented by equation (32) equal to zero.

The Stability of a Helical Perturbation

A first approximation to the general dispersion relation can be obtained by choosing $p = j = 1$ in equation (32), (ref. 9). The resulting expression may be written as

$$[(\bar{\omega}_2 - i\bar{\omega}_1)(1 + y/\delta) + \sigma_{1i} + ik_{1i}](\sigma_{2e} + ik_{2e}) +$$

$$[(\bar{\omega}_2 - i\bar{\omega}_1)(1 + \delta y) + \sigma_{1e} + ik_{1e}](\sigma_{2i} + ik_{2i}) = 0 \tag{34}$$

where

$$\sigma_{1i} = -k_o^2 + b_1 + \frac{y}{\delta} b_2 \ , \quad \sigma_{2i} = -k_o^2 + d_1 + \frac{yd_2}{\delta}$$

$$\sigma_{1e} = -k_o^2 + b_1 + \delta y b_2 \ , \quad \sigma_{2e} = \delta(-k_o^2 + d_1 + \delta y d_2)$$

$$b_1 = -(\beta_1^2 + \beta_o^2)R^2, \ b_2 = b_1 + D_{11}, \ d_1 = -\beta_1^2 R^2 - C_{11}, \ d_2 = -\beta_1^2 R^2 + D_{11} - E_{11}$$

$$k_{1i} = -(k_o + m\sqrt{y/\delta}\ A_{11})\ E_x + (-k_o + 2k_o \frac{D_i}{q\mu_i} + m\sqrt{y/\delta}\ A_{11})\ H^*$$

$$k_{1e} = (k_o\delta - m\delta\sqrt{\delta y}\ A_{11})\ E_x + (k_o\delta + 2\frac{k_o D_i}{q\mu_i} + m\ \delta\sqrt{\delta y}\ A_{11})\ H^*$$

$$k_{2i} = (k_o - m\sqrt{y/\delta}\ A_{11})H \ , \quad k_{2e} = (k_o\delta + m\ \delta\sqrt{\delta y}\ A_{11})\ H \ , \tag{35}$$

and

$$\delta = \frac{\mu_e}{\mu_i} \ , \quad y = \mu_e\mu_i B^2 \quad . \tag{36}$$

Equation (34) gives

$$\bar{\omega}_1 = -\frac{1}{A_1}[A_2 E_x + A_3 H] \tag{37}$$

where A_1, A_2 and A_3 are defined in Appendix C. The requirement for stability is

$$\bar{\omega}_2 \geq 0 \quad , \quad \text{or}$$

$$(\sigma_{1i}\sigma_{2e}+ \sigma_{1e}\sigma_{2i})[(1+\delta y)\sigma_{2i}+ (1+y/\delta)\sigma_{2e}] + (\sigma_{1i}k_{2e}+ \sigma_{1e}k_{2i})x$$

$$[(1+\delta y)k_{2i}+ (1+y/\delta)k_{2e}] + (\sigma_{2e}k_{2i}- \sigma_{2i}k_{2e})[(1+\delta y)k_{1i}- (1+\delta/\delta)k_{1e}] \leq 0 \ . \tag{38}$$

As an application of the above analysis we shall consider the case of $m = -1$ which corresponds to a right-handed helical perturbation. In this case

$$b_1 = -8.85, \ b_2 = -3.68, \ d_1 = -6.03 \text{ and } d_2 = -5.18 \tag{39}$$

hence, for the case where $H = 0$, $\bar{\omega}_2 > 0$ and the system is always stable. However, when H is different from zero, the complexity of equation (38) makes it necessary to resort to numerical calculations to show whether the system is stable or not. To investigate this, we note that, at the stability boundary $\bar{\omega}_2 = 0$ and the equality in equation (38) must hold. Also, the derivative of the resulting equation with respect to k_o must be zero. This latter requirement follows from a qualitative plot of $\bar{\omega}_2$ vs. k_o which is shown in fig. 2. It is evident from fig. 2 that $\bar{\omega}_2 = 0$ and $\partial\bar{\omega}_2/\partial k_o = 0$ at the stability boundary.

Expanding equation (38) in powers of k_o, one finds

$$a_6k_o^6 + a_4k_o^4 + (a_3+ c_3H\ E_x)k_o^3 + (a_2+ c_2H\ E_x)k_o^2$$

$$+ k_o(a_1+ c_1H\ E_x) + a_o+ c_oH\ E_x = 0 \tag{40}$$

where a_o, $a_1,\ldots$, c_o, $c_1,\ldots$, which are given in Appendix C, are functions of y, H^2, δ, V_e/V_i . Setting the derivative of equation (40) to zero gives

$$HE_x = - \frac{6a_6k_o^5+ 4a_4k_o^3 + 3a_3k_o^2 + 2a_2k_o + a_1}{3c_3k_o^2 + 2c_2k_o+ c_1} \tag{41}$$

Solution of equation (40), with HE_x given by equation (41) gives a relation between y and k_o at the critical point for a given H, δ, V_e/V_i. The electric field vs. y follows from equation (41) while the frequency follows from equation (39). The results show that solutions can be obtained only when $HE_x > 0$. Thus, it is concluded that an instability occurs only when $HE_x > 0$, which is the Simon-Hoh criterion. Since in a linear Hall accelerator the axial density gradient and the axial electric field are anti-parallel over much of the accelerator length, it is concluded that the above analysis cannot explain the observed instability.

STABILITY ANALYSIS FOR AN OBLIQUE MAGNETIC FIELD

Steady State Solution

The steady state solution for the case of constant applied electric field and constant radial and axial magnetic fields and no axial density gradients follows from equation (10). Expanding equation (10), one finds that the equations governing the steady state solution can be written as

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[\mu_e^* n_o E_{or} + D_e^* \frac{dn_o}{dr} + \mu_e' \mu_e^2 B_r^2 \bar{B}_x E_{ox} n_o\right]\right\} + n_o \xi = 0 \tag{42}$$

and

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[-\mu_i^* n_o E_{or} + D_i^* \frac{dn_o}{dr} - \mu_i' \mu_i^2 B_r^2 \bar{B}_x E_{ox} n_o\right]\right\} + n_o \xi = 0 \tag{43}$$

where

$$\bar{B}_x = \frac{B_x}{B_r}, \quad \mu^*_{i,e} = \mu'_{i,e}(1 + {}'\mu^2_{i,e} B_r^2), \quad D^*_{i,e} = D'_{i,e}(1 + \mu^2_{i,e} B_r^2) \quad . \tag{44}$$

The radial component of the electric field can be eliminated by multiplying equation (42) by μ_i^* and equation (43) by μ_e^* and adding. The resulting equation takes the form

$$r \frac{d^2 n_o}{dr^2} + (1 + \beta r)\frac{dn_o}{dr} + (\beta + \alpha^2 r) n_o = 0 \tag{45}$$

where

$$\beta = \frac{(\delta^2 - 1)\bar{B}_x y \, E_{ox}}{(V_i + V_e)(1 + \delta y)(\delta + y)}$$

$$\alpha^2 = \frac{\xi(\mu_i^* + \mu_e^*)}{\mu_i^* \mu_e^* (V_i + V_e)} \tag{46}$$

Letting

$$\eta = \gamma_o \beta r \; , \quad n_o = e^{h_o \eta} M_o \; , \; \lambda_o^2 = \frac{\alpha^2}{\beta^2} = \frac{h_o(1+h_o)}{(1+2h_o)^2} \; , \; \gamma_o = - \frac{1}{1 + 2h_o} \; , \tag{47}$$

one finds that equation (45) reduces to

$$\eta \frac{d^2 M_o}{d\eta^2} + (1-\eta) \frac{dM_o}{d\eta} - (1+h_o)M_o = 0 \tag{48}$$

with the boundary conditions $n_o = n_{\mathfrak{E}}$ at $\eta = 0$ and $n_o = 0$ at $\eta = \eta_o$. Since n_o is finite at $\eta = 0$, the solution of equation (48) is the confluent hypergeometric function. The boundary condition at $\eta = \eta_o$ gives h_o and, hence, defines γ_o and λ_o^2. The value of h_o is always such that λ_o^2 is positive. The parameter η_o must be specified for the problem to be completely determined and requires the specification of βR. The analysis is carried out for a general η_o but specific values are assigned for numerical calculations.

The expression for the radial electric field may be derived by subtracting equations (42) and (43). The resulting expression for E_{or} is

$$E_{or} = \frac{q_1}{n_o} \frac{dn_o}{dr} - q_2 E_{ox} \tag{49}$$

$$q_1 = \frac{D_i^* - D_e^*}{\mu_i^* + \mu_e^*} \; , \quad q_2 = \frac{(1+\delta^2 \delta_1)y\bar{B}_x}{\delta + y + \delta\delta_1(1+\delta y)} \; , \quad \delta_1 = \frac{\mu_e'}{\mu_i'} = \frac{\delta + y(1+\bar{B}_x^2)}{1+\delta y(1+\bar{B}_x^2)} \; . \tag{50}$$

Since $\beta = \eta_o/\gamma_o R$, the following relation exists among the various parameters of the problem

$$E_x^* = \frac{E_{ox}R}{V_i + V_e} = \frac{\eta_o(1+\delta y)(\delta + y)}{\gamma_o(\delta^2 - 1)y\bar{B}_x} \tag{51}$$

The number density at the centerline may be evaluated by utilizing equation (23). However, in this case, $e(\Gamma_{ix} - \Gamma_{ex})$ is given by

$$e(\Gamma_{ix} - \Gamma_{ex}) = q_3 n_o + q_4 n_o E_{or} - q_5 \frac{dn_o}{dr} \; , \tag{52}$$

where

$$q_3 = e(\mu_i'' + \mu_e'')E_{ox}$$

$$q_4 = e(\mu_i^2\mu_i' + \mu_e^2\mu_e')B_r^2\bar{B}_x$$

$$q_5 = e(\mu_i^2 D_i' - \mu_e^2 D_e')B_r^2\bar{B}_x \tag{53}$$

$$\mu_{i,e}'' = \mu_{i,e}'(1 + \mu_{i,e}^2 B_r^2 \bar{B}_x^2) \quad .$$

Normal Mode Analysis

In this case, the perturbations n_1 and $\vec{E}_1$ are chosen as

$$n_1 = f(r) \exp [i(\omega t + m\Theta + kx)]$$

$$\vec{E}_1 = -\nabla U_1 \; ; \quad U_1 = g(r)\exp [i(\omega t + m\Theta + kx)] \quad . \tag{54}$$

The representation shown in equation (54) represents, essentially, a Fourier transform with respect to Θ, x, and t. Substituting equation (54) into equation (12) and introducing a new variable $\ell(r)$ defined by

$$n_o(r)g(r) = \frac{1}{\gamma_o^2\beta^2} \; \ell(r) - q_1 f(r) \quad , \tag{55}$$

the perturbed equation takes the form

$$(D_{i,e}' \mp q_1\mu_{i,e}') \left[\frac{1}{r}\frac{d}{dr}\left(r\,\frac{df}{dr}\right) - \frac{m^2}{r^2} f\right] + f \Big[\xi - i\omega \mp ikE_{ox}\mu_{i,e}''$$

$$- (D_{i,e}'' \mp q_1\mu_{i,e}'')k^2 - i\,\frac{m}{r}\,\mu_{i,e}\mu_{i,e}' B_r E_{ox}(1 + q_2\bar{B}_x) + ik\,\mu_{i,e}'\mu_{i,e}^2 B_r^2\bar{B}_x E_{ox} q_2\Big]$$

$$\pm \mu_{i,e}' q_2 E_{ox}\left[\frac{f}{r} + (1 + \mu_{i,e}^2 B_r^2)\,\frac{df}{dr}\right] + \Big[i2k\,\mu_{i,e}^2 B_r^2\bar{B}_x (D_{i,e}' \mp q_1\mu_{i,e}')$$

$$\mp \mu_{i,e}'\mu_{i,e}^2 B_r^2\bar{B}_x E_{ox}\Big]\frac{df}{dr} + \mu_{i,e}^2 B_r^2\,(D_{i,e}' \mp q_1\mu_{i,e}')\,\frac{d^2 f}{dr^2} \; =$$

$$\mu_{i,e}'\left\{\mp\left[\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\ell}{dr}\right) - \frac{m^2}{r^2}\,\ell\right] \pm (1 + \mu_{i,e}^2 B_r^2\bar{B}_x^2)k^2\ell \pm \frac{1}{r}\frac{d}{dr}\left(\frac{r}{n_o}\,\frac{dn_o}{dr}\,\ell\right)\right.$$

$$\mp \mu^2_{i,e} B^2_r \left[\frac{d^2 \ell}{dr^2} - \frac{d}{dr} \left(\frac{1}{n_o} \frac{dn_o}{dr} \ell\right)\right] \mp ik\mu^2_{i,e} B^2_r \bar{B}_x \left[2 \frac{d\ell}{dr} - \frac{1}{n_o} \frac{dn_o}{dr} \ell\right]$$

$$- im \, \mu_{i,e} B_r \bar{B}_x \frac{1}{n_o r} \frac{dn_o}{dr} \ell \Big\} \quad . \tag{56}$$

The coefficients containing the diffusion coefficients and mobilities may be simplified by utilizing the expression for ξ from equation (46) and the definition for q_1 from equation (50). The resulting expressions are

$$D'_{i,e} \mp q_1 \mu'_{i,e} = \frac{\xi}{\lambda_o^2 \beta^2} \; \frac{1}{1+\mu^2_{i,e} B^2_r}$$

$$D''_{i,e} \mp q_1 \mu''_{i,e} = \frac{\xi}{\lambda_o^2 \beta^2} \; \frac{1+\mu^2_{i,e} B^2_r \bar{B}^2_x}{1+\mu^2_{i,e} B^2_r} \tag{57}$$

The quantities β and ξ have the dimensions of cm^{-1} and sec^{-1} respectively. Choosing β^{-1} as a characteristic length and ξ^{-1} as a characteristic time, the following dimensionless quantities may be defined:

$$\bar{\omega} = \frac{\lambda_o^2 \omega}{\xi \, \gamma_o^2} = \bar{\omega}_1 + i\bar{\omega}_2$$

$$k_o = \frac{k}{\beta \gamma_o} \tag{58}$$

$$\bar{E}_x = \frac{\beta \, \lambda_o^2 \mu_i}{\gamma_o \xi} E_{ox} \quad .$$

Letting $\eta = \gamma_o \beta \, r$ and using equations (57) and (58), equation (56) may be written in the form

$$\frac{\xi \gamma_o^2}{\lambda_o^2} \Big\{ \frac{1}{1 + \mu^2_{i,e} B^2_r} \left[\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{df}{d\eta}\right) - \frac{m^2}{\eta^2} f\right] + \left[\frac{\lambda_o^2}{\gamma_o^2} - i\bar{\omega}_1 + \bar{\omega}_2\right.$$

$$\left. - \frac{1+\mu^2_{i,e} B^2_r \bar{B}_x}{1+\mu^2_{i,e} B^2_r} \, k_o^2 \pm ik_o \bar{E}_x \frac{\mu'_{i,e}}{\mu_i} \left(\mu^2_{i,e} B^2_r \bar{B}_x q_2 - \mu^2_{i,e} B^2_r \bar{B}^2_x - 1\right)\right] f$$

$$+ \bar{E}_x \frac{\mu'_{i,e}}{\mu_i}\left[\pm q_2 - im\,\mu_{i,e}B_r(q_2\bar{B}_x + 1)\right]\frac{f}{\eta} + \left[\frac{1}{\gamma_o} + \frac{2i\mu^2_{i,e}B^2_r\bar{B}_x}{1+\mu^2_{i,e}B^2_r}\right]\frac{df}{d\eta}$$

$$+ \frac{\mu^2_{i,e}B^2_r}{1+\mu^2_{i,e}B^2_r}\frac{d^2f}{d\eta^2}\Big\} = \mu'_{i,e}\Big\{ \mp\left[\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\ell}{d\eta} - \frac{m^2}{\eta^2}\ell\right]\right.$$

$$\pm \frac{1}{\eta}\frac{d}{d\eta}\left(\frac{\eta}{n_o}\frac{dn_o}{d\eta}\ell\right) \pm (1 + \mu^2_{i,e}B^2_r\bar{B}^2_x)k^2_o\ell \mp \mu^2_{i,e}B^2_r\left[\frac{d^2\ell}{d\eta^2}\right.$$

$$\left. - \frac{d}{d\eta}\left(\frac{1}{n_o}\frac{dn_o}{d\eta}\ell\right)\right] \mp i\,\mu^2_{i,e}B^2_r\bar{B}_x k_o\left[2\frac{d\ell}{d\eta} - \frac{1}{n_o}\frac{dn_o}{d\eta}\ell\right]$$

$$- i\,\mu_{i,e}B_r\bar{B}_x\frac{m}{\eta}\frac{1}{n_o}\frac{dn_o}{d\eta}\ell\Big\} \quad . \qquad (59)$$

Equation (59) represents a pair of coupled complex second order differential equations with variable coefficients. These equations do not have a closed form solution and, again, a transform technique will be employed.

It is shown in Appendix A that $\eta^{|m|}M_{mj}$ form an orthogonal set with respect to the weighting function $e^{-\eta}$. Proceeding formally, it is assumed that the dependent variables can be expanded in terms of the orthogonal set of confluent hypergeometric functions with the transformed functions defined as

$$F(h_j) \longleftrightarrow f(\eta)$$
$$L(h_j) \longleftrightarrow \ell(\eta) \quad . \qquad (60)$$

The details of the integral transformation are presented in Appendix A. Multiplying equation (59) by $\eta^{|m|}e^{-\eta}M_{mj}$ and integrating from 0 to η_o, an infinite set of algebraic equations in the transformed variables $F(h_j)$ and $L(h_j)$ is obtained. Leaving all definitions to Appendix A, these equations can be written as

$$\frac{\xi\gamma^2_o}{\lambda^2_o}\Big\{\frac{1}{1+\mu^2_{i,e}B^2_r}\sum_{p=1}^{\infty}D^*_{jp}F(h_p) + \left[\frac{\lambda^2_o}{\gamma^2_o} - i\bar{\omega}_1 + \bar{\omega}_2 - \frac{1+\mu^2_{i,e}B^2_r\bar{B}^2_x}{1+\mu^2_{i,e}B^2_r}k^2_o\right.$$

$$\pm ik_o\bar{E}_x \frac{\mu'_{i,e}}{\mu_i}(\mu^2_{i,e}B^2_r\bar{B}_x(q_2 - \bar{B}_x) - 1\Big] F(h_j)$$

$$+ \bar{E}_x \frac{\mu'_{i,e}}{\mu_i}\Big[\pm q_2 - im\,\mu_{i,e}B_r(q_2\bar{B}_x + 1)\Big] \sum_{p=1}^{\infty} a^*_{jp}F(h_p)$$

$$+\left[\frac{1}{\gamma_o} + i\frac{2\mu^2_{i,e}B^2_r\bar{B}_x}{1+\mu^2_{i,e}B^2_r}k_o\right]\sum_{p=1}^{\infty} C^*_{jp}F(h_p) + \frac{\mu^2_{i,e}B^2_r}{1+\mu^2_{i,e}B^2_r}\sum_{p=1}^{\infty} S^*_{jp}F(h_p)\Big\}$$

$$= \mu'_{i,e}\Big\{\mp \sum_{p=1}^{\infty} D^*_{jp}L(h_p) \pm (1+\mu^2_{i,e}B^2_r\bar{B}_x)k^2_oL(h_j) \pm \sum_{p=1}^{\infty} A^*_{jp}L(h_p)$$

$$+ \mu^2_{i,r}B^2_r \sum_{p=1}^{\infty} B^*_{jp}L(h_p) \mp ik_o\mu^2_{i,e}B^2_r\bar{B}_x \sum_{p=1}^{\infty} E^*_{jp}L(h_p)$$

$$- im\,\mu_{i,e}B_r\bar{B}_x \sum_{p=1}^{\infty} e^*_{jp}L(h_p)\Big\} \quad . \tag{61}$$

In principle it is possible to eliminate $F(h_p)$ (or $L(h_p)$) from the above equations and obtain an infinite set of equations in $L(h_p)$ or $F(h_p)$). In order for the infinite set of homogeneous equations to have non-trivial solutions, the determinant of the coefficients, which has infinite rows and columns, must equal zero. Setting the coefficient determinant equal to zero yields the general dispersion relation.

Application to a Helical Perturbation

It will be assumed that an approximate form of the dispersion relation can be obtained by assuming that the infinite determinant can be approximated by a two by two determinant. Thus, letting $p = j = 1$, the equations for the electrons and ions can be written in the form

$$\xi\ \frac{\gamma_o^2}{\lambda_o^2}(\Delta_1 + i\Delta_2)F(h_1) - \mu'_i(V_1 + iV_2)L(h_1) = 0$$

and

$$\xi\ \frac{\gamma_o^2}{\lambda^2}(\eta_1 + i\eta_2)F(h_1) - \mu'_i(\psi_1 - i\psi_2)L(h_1) = 0 \tag{62}$$

The mathematical steps needed to arrive at equation (62), together with the definitions of the coefficients of $L(h_1)$ and $F(h_1)$, are given in Appendix B. In order for the two simultaneous homogeneous equations to have a non-trivial solution, the determinant of the coefficients must equal zero. This yields the dispersion relation which gives the boundaries between the stable and unstable regions. Setting the determinant of the coefficients zero, i.e.,

$$\begin{vmatrix} \Delta_1 + i\Delta_2 & , & -V_1 - iV_2 \\ \eta_1 + i\eta_2 & , & -\psi_1 + i\psi_2 \end{vmatrix} = 0 \tag{63}$$

gives two equations, one from the real part and the other from the imaginary part. These equations relate the dimensionless wave number, k_o, the dimensionless electric field, $\bar{E}_x$, and the dimensionless real part of the frequency, $\bar{\omega}_1$. In order to obtain the boundary between the stable and unstable regions, the imaginary part of the frequency, $\bar{\omega}_2$, is set equal to zero.

The equations resulting from setting the real and imaginary parts of equation (63) equal to zero, with $\bar{\omega}_2 = 0$ are

$$M_1 k_o^4 + M_2 k_o^2 + M_3 k_o + M_4 + \bar{E}_x (M_5 k_o^2 + M_6 k_o + M_7) + \bar{\omega}_1 (M_8 k_o + M_9) = 0 \tag{64}$$

$$N_1 k_o^3 + N_2 k_o^2 + N_3 k_o + N_4 + \bar{E}_x (N_5 k_o^3 + N_6 k_o^2 + N_7 k_o + N_8) + \bar{\omega}_1 (N_9 k_o^2 + N_{10}) = 0 \tag{65}$$

The coefficients (M, N) are defined in Appendix B and are functions of δ, y, $\bar{E}_x$, and η_o.

The frequency, $\bar{\omega}_1$, follows from equation (64) as

$$\bar{\omega}_1 = - \frac{1}{M_8 k_o + M_9} \left[M_1 k_o^4 + M_2 k_o^2 + M_3 k_o + M_4 + \bar{E}_x (M_5 k_o^2 + M_6 k_o + M_7) \right] . \tag{66}$$

Substituting equation (66) into equation (65) results in a six degree polynomial in k_o, which is

$$P_1 k_o^6 + (P_2 + P_7 \bar{E}_z) k_o^4 + (P_3 + P_8 \bar{E}_x) k_o^3 + (P_4 + P_9 \bar{E}_x) k_o^2 + (P_5 + P_{10} \bar{E}_x) k_o + (P_6 + P_{11} \bar{E}_x) = 0 \tag{67}$$

where

$$P_1 = -N_9 M_1$$

$$P_2 = M_8N_1 - N_9M_2 - N_{10}M_1$$

$$P_3 = N_2M_8 + M_9N_1 - N_9M_3$$

$$P_4 = M_8N_3 + M_9N_2 - N_9M_4 - N_{10}M_2$$

$$P_5 = M_8N_4 + M_9N_3 - N_{10}M_3$$

$$P_6 = N_4M_9 - N_{10}M_4$$

$$P_8 = N_6M_8 + M_9N_5 - N_9M_6$$

$$P_9 = M_8N_7 + M_9N_6 - N_9M_7 - N_{10}M_5$$

$$P_{10} = M_8N_8 + M_9N_7 - N_{10}M_6$$

$$P_{11} = M_9N_8 - N_{10}M_7 \quad .$$

The $\bar{E}_x$ appearing in equation (67) is not independent of the other variables of the system, since by using the definitions of $\bar{E}_x$, ξ , and β_o , a relation between $\bar{E}_x$ and E_x^* is derived as

$$\bar{E}_x = \frac{(1+\delta^*)(1+\delta y(1+\bar{B}_x^2))}{\eta_o \delta(1+\delta y)} E_x^* \quad , \tag{69}$$

where

$$\delta^* = \frac{\mu_e^*}{\mu_i^*} = \delta_1 \left(\frac{1+\delta y}{1+y/\delta}\right) \quad , \tag{70}$$

and E_x^* is given by equation (51).

Equation (67) and its derivative with respect to k_o are used to establish the neutral stability boundary. The simultaneous solution of these equations establishes the value of y and k_o at the critical point for a given $\bar{B}_x$, and, from these, the other parameters are easily calculated. The pair of simultaneous non-linear algebraic equations are solved numerically using the Newton-Raphson method for the case of a right-handed screw $(m = -1)$.

RESULTS AND DISCUSSION

The analysis for a purely radial magnetic field shows that, even in the presence of both radial and axial density gradients, the perturbation behaves in accordance with the Simon-Hoh criterion which requires the product of the axial electric field and the axial density gradient to be positive for instability. Because the density gradient and the electric field are antiparallel over much of the accelerator length, it appears thet the origin of instability may not be dependent on the existence of an axial density gradient.

In seeking a possible explanation to the observed instability, the influence of the small but non-vanishing axial component of the magnetic field in the absence of an axial density gradient is investigated. It is shown that this can lead to a possible explanation of the observed instability.

The instability is found to exist for a right-handed helix ($m = -1$) only. This is in contrast to the left-handed helix required when the axial electric and magnetic fields are antiparallel as in the case of the positive column. The right-handed helix for parallel electric and magnetic fields results in a destabilizing $E_\theta \times B_x$ force in the direction of the containing walls and $E_\theta \times B_r$ drift in the axial direction. This drift in the axial direction is responsible for the anomalous diffusion observed by the various investigators (refs. 4, 5, 6, 7, and 8). The instability is present even for small values of B_x/B_r, however, when $B_x = 0$ the system is stable in the absence of an axial density gradient.

The stability boundary is established by solving equation (67) simultaneously with its derivative with respect to k_o to obtain y and k_o for specified η_o, δ and $\bar{B}_x$. Upon determination of y and k_o at the critical point, the results can be employed to calculate $\bar{\omega}_1$ and E_x^*. The electric field versus $\bar{B}_x$ curves given in figs. 3 and 4 show the regions of stable and unstable operations. Increasing B_r is stabilizing while increasing η_o is destabilizing to the extent that for some η_o between 1.5 and 2 the system becomes completely unstable for $\delta = 10$. This is in agreement with the experimental results of Hess et. al. (ref. 6) where it has been observed that, in some cases, the discharge becomes unstable almost immediately when the magnetic field is applied. The axial electric field provides the driving force for the instability (ref. 12) by rotating the electron helix with respect to the ion helix thus inducing an E_θ which, coupled with B_x drives the particles towards the wall and, coupled with B_r results in the anomalous diffusion in the axial direction. The stabilizing influence of B_r results from inhibiting the rotation of the electron helix relative to the ion helix. The destabilizing effect of increasing η_o is difficult to interpret since it is related to the radius, magnetic field, and ionization rate.

The frequency variation with $\bar{B}_x$ is shown in figs. 5 and 6. The frequency decreases with increasing δ and η_o but increases with increasing B_r. A plot of the wave length $\lambda = 2\pi/k$ versus $\bar{B}_x$ is shown in figs. 7 and 8; it is seen that it is practically independent of $\bar{B}_x$ and δ but increases with increasing η_o.

The critical value of y is plotted versus $\bar{B}_x$ in figs. 9 and 10. It is independent of $\bar{B}_x$ for $\bar{B}_x < 0.1$ and increases thereafter but decrease with increasing δ or decreasing η_o. The system moves from the unstable to the stable operating region with increasing y. This is a result of the consideration that, for constant $\bar{B}_x$, increasing y results from an increase in B_r which is stabilizing.

The results presented here are in qualitative agreement with the previously published experimental results (ref. 6). Thus, for $\eta_o = .01$ and $\delta = 1000$, the wavelength is approximately six times the radius which is approximately twice the value of 15 cm given by Hess et al. (ref. 6) and the frequency agrees favorably with their value of approximately 100 kc/sec. Using the wavelength of 36 cm and a frequency of 3.6×10^2 radians per second which is representative of the values at $\eta_o = .01$ and $\delta = 1000$, the propagation velocity is found to be of the order of 1.32×10^3 m/sec which is of the same magnitude (4×10^3 m/sec) given by Hess et al. The propogation velocity changes with the magnetic field in the ssme manner as the frequency.

Exact agreement between theoretical and experimental results cannot be expected because the parameters η_o and δ are not known for the conditions of the experiment and the experimental results are for the supercritical region. This is in addition to the fact that the dispersion relation is somewhat approximate. Neglect of the off diagonal terms in the dispersion determinant can lead to serious discrepancy if these are of the same order of magnitude as the diagonal terms. Further investigation is needed to verify the validity of this approximation. Ivash (ref. 13) has shown that for small steady state gradients and small degrees of ionization this approximation is valid for a plasma confined by parallel planes.

CONCLUDING REMARKS

The analysis presented here shows that a right-handed screw type perturbation becomes stable in a linear Hall current accelerator if the axial magnetic field is not identically zero. The driving force for the instability is the axial electric field which rotates the electron helix relative to the ion helix. This induces an azimuthal electric field which, when coupled with the axial magnetic field results in a destabilizing force in the direction of the containing walls and, when coupled with the radial magnetic field, results in anomalous diffusion in the axial direction. The instability is found to exist even for very small values of B_x/B_r. The variation of the various parameters with the magnetic field agrees favorably with the experimental observation. The favorable agreement of this model with experiment suggests that this mechanism is a possible explanation of the observed instability.

The axial magnetic field is necessary for the instability to exist since, it is shown that, for the conditions existing in a linear Hall accelerator, the system is stable in the absence of an axial magnetic field.

North Carolina State University

Raleigh, North Carolina

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APPENDIX A

The Confluent Hypergeometric Function and Related Transformation

Since the steady state solutions for n_o and E_{or} are expressible in terms of confluent hypergeometric functions, it is convenient to assume that the perturbed functions are expressible in terms of an infinite set of confluent hypergeometric functions. It is for this reason that equation (59) is transformed using an integral transform involving the confluent hypergeometric function. It is the purpose of this section to demonstrate the orthogonality of the set of functions used in the transform and carry out the integral transformation of equation (59).

Consider the following differential equation which, for $m = 0$ reduces to equation (45).

$$\frac{d^2n}{dr^2} + \left(\beta + \frac{1}{r}\right)\frac{dn}{dr} + \left(\alpha^2 + \frac{\beta}{r} - \frac{m^2}{r^2}\right) n = 0 \tag{1A}$$

Making transformations similar to that which led to equation (48), the resulting differential equation is

$$\frac{d^2M_{mj}}{d\eta^2} + \left(\frac{1 + 2|m|}{\eta} - 1\right)\frac{dM_{mj}}{d\eta} - \frac{(1+h_j + |m|)}{\eta} M_{mj} = 0 \; . \tag{2A}$$

The solution to equation (2A) is a confluent hypergeometric function given by

$$M_{mj} = M(1 + h_j + |m| \; , \; 1 + 2|m| \; , \; \eta) \tag{3A}$$

which has the infinite series representation (ref. 13)

$$M_{mj} = \sum_{k=o}^{\infty} \frac{\Gamma(1+h_j+|m|+k)\Gamma(1+2|m|)}{\Gamma(1+h_j+|m|)\Gamma(1+k+2|m|)} \frac{\eta^k}{k!} \tag{4A}$$

where Γ denotes the Gamma function.

The solution to equation (1A) is then

$$n = \eta^{|m|} e^{h_j \eta} M_{mj} \tag{5A}$$

with the boundary conditions

$$\begin{aligned} n &= o \qquad \eta = o \\ n &= o \qquad \eta = \eta_o \Rightarrow M_{mj}(\eta_o) = o \end{aligned} \tag{6A}$$

This last condition determines the value of h_j. It is now necessary to investigate the orthogonality of this set.

Consider the differential equation

$$\frac{d^2 n_j}{d\eta} + \left(\frac{1}{\eta} - 1 - 2h_j\right)\frac{dn_j}{d\eta} + \left(h_j + h_j^2 - \frac{1+2h_j}{\eta} - \frac{m^2}{\eta^2}\right)n_j = 0 \ . \qquad (7A)$$

For any two different value, i and j , two equations can be written in the form

$$\frac{d}{d\eta}\left(\eta e^{-(1+2h_j)\eta}\frac{dn_j}{d\eta}\right) + \left[-\frac{m^2}{\eta} - (1+2h_j) + h_j(1+h_j)\eta\right]e^{-(1+2h_j)\eta} n_j = 0 \qquad (8A)$$

$$\frac{d}{d\eta}\left(\eta e^{-(1+2h_i)\eta}\frac{dn_i}{d\eta}\right) + \left[-\frac{m^2}{\eta} - (1+2h_i) + h_i(1+h_i)\eta\right]e^{-(1+2h_i)\eta} n_i = 0 \ . \qquad (9A)$$

Multiply equation (8A) by $e^{(h_j-h_i)\eta} n_i$ and equation (9A) by $e^{(h_i-h_j)\eta} n_j$, subtract the resulting two equations and integrate from o to η_o to obtain

$$\int_o^{\eta_o}\left[e^{(h_i-h_j)\eta} n_j \frac{d}{d\eta}\left(\eta e^{-(1+2h_i)}\frac{dn_i}{d\eta}\right) - e^{(h_j-h_i)\eta} n_i \frac{d}{d\eta}\left(\eta e^{-(1+2h_j)\eta}\frac{dn_j}{d\eta}\right)\right]d\eta$$

$$+ 2(h_j-h_i)\int_o^{\eta_o} e^{-\eta}e^{-(h_j+h_i)\eta} n_i n_j d\eta + (h_i-h_j)(1+h_j+h_i)$$

$$\cdot \int_o^{\eta_o} \eta e^{-\eta} e^{-(h_j+h_i)\eta} n_i n_j d\eta = 0 \ . \qquad (10A)$$

Integrating the first integral of (10A) by parts twice and combining with the remaining terms of equation (10A), the following orthogonality condition is derived for $h_i \neq h_j$.

$$\int_o^{\eta_o} e^{-\eta} e^{(h_j+h_i)\eta} n_i n_j d\eta = 0 \ , \qquad j \neq i \ . \qquad (11A)$$

Substituting the expressions for n from equation (5A), it is seen that

$$\int_o^{\eta_o} e^{-\eta}\eta^{2|m|} M_{mj} M_{mi} d\eta = 0 \ . \qquad (12A)$$

Hence, the functions $\eta^{|m|}M_{mj}$ form an orthogonal set in the interval $[o,\eta_o]$ with respect to the weighting function $e^{-\eta}$. The same result is obtained starting with the differential equation

$$\eta \frac{d^2f}{d\eta^2} + (1-\eta)\frac{df}{d\eta} - (1+h_j + \frac{m^2}{\eta})f = 0 \quad . \tag{13A}$$

It is now assumed that the function f (and g) defined by equation (54) can be expressed as

$$f = \eta^{|m|} \sum_{j=1}^{\infty} A_j M_{mj} \tag{14A}$$

with the boundary conditions that $f(0) = f(\eta_o) = 0$, where the last condition determines h_j. Multiply equation (14A) by $\eta^{|m|}e^{-\eta}M_{mk}$, integrate over the interval $[o, \eta_o]$ and interchange summation and integration, the result can be expressed as

$$\int_o^{\eta_o} \eta^{|m|}e^{-\eta}f(\eta)M_{mk}d\eta = \sum_{j=1}^{\infty} A_j \int_o^{\eta_o} \eta^{2|m|}e^{-\eta}M_{mj}M_{mk}d\eta \quad . \tag{15A}$$

Since the integral on the right hand side of (15A) vanishes except when $j = k$, the coefficients of the series are determined uniquely. Define the transformed function $F(h_k)$ by

$$F(h_k) = \int_o^{\eta_o} \eta^{|m|}e^{-\eta}M_{mk}f(\eta)d\eta \quad . \tag{16A}$$

The coefficient A_k is given by

$$A_k = \frac{F(h_k)}{I_k} \tag{17A}$$

where

$$I_k = \int_o^{\eta_o} \eta^{2|m|}e^{-\eta}M_{mk}^2 d\eta \quad .$$

Thus the following transform pairs may be defined

$$f(\eta) = \eta^{|m|} \sum_{j=1}^{\infty} \frac{F(h_j)M_{mj}}{I_j}$$

$$F(h_j) = \int_0^{\eta_0} \eta^{|m|} M_{mj} f(\eta) d\eta \tag{18A}$$

$$\ell(\eta) = \eta^{|m|} \sum_{j=1}^{\infty} \frac{L(h_j) M_{mj}}{I_j}$$

$$L(h_j) = \int_0^{\eta_0} \eta^{|m|} e^{-\eta} M_{mj} \ell(\eta) d\eta \quad .$$

A few comments are in order concerning the series representation of $f(\eta)$ and $\ell(\eta)$ in the interval $[o, \eta_0]$. The expansion is in terms of an orthogonal set of characteristic functions which satisfy the differential equation

$$\eta \frac{d^2\phi}{d\eta^2} + (1-\eta)\frac{d\phi}{d\eta} - (1 + h_j + \frac{m^2}{\eta})\phi = 0$$

where

$$\phi = \eta^{|m|} M_{mj} \tag{19A}$$

and

$$\phi(0) = \phi(\eta_0) = 0 \quad .$$

The equation may be written in the form

$$\frac{d}{d\eta} (p^* \frac{d\phi_j}{d\eta}) + (q^* + \lambda_j^* r)\phi_j = 0$$

where

$$\begin{aligned} p^* &= \eta e^{-\eta} \\ q^* &= - \frac{m^2}{\eta} e^{-\eta} \\ r^* &= e^{-\eta} \\ \lambda_j^* &= -1 - h_j > 0 \quad . \end{aligned} \tag{20A}$$

It is noted that the differential equation has homogeneous boundary conditions and that $p^* \geq o$, $r^* \geq o$, $\lambda_j^* \geq o$, and $q^* < o$ for all η in the interval $[o, \eta_o]$. This is a proper Sturm-Liouville problem and it is known that all the eigenvalues are positive and real and the characteristic functions are real (ref. 14). Since the Sturm-Liouville problem is proper and since $p^*(\eta)$, $q^*(\eta)$, and $r^*(\eta)$ are regular (all derivatives exist) for $o < \eta < \eta_o$, it is known that a formal representation of a bounded piecewise differentiable function, $g(\eta)$, in terms of the characteristic function, $\phi_j(\eta)$, converges to $g(\eta)$ for all η in $[o, \eta_o]$ and converges to the mean value, $\frac{1}{2}(g(\eta^+) + g(\eta^-))$, where finite jumps occur. The series may or may not converge to the value of $g(\eta)$ at the end points (ref. 14).

After multiplying equation (59) by $\eta^{|m|} e^{-\eta} M_{mj}$, the integration is performed on each term separately. Due to the algebra involved all terms will not be shown but an example term is shown in detail with the overall results summarized. Consider the term

$$\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{df}{d\eta}\right) - \frac{m^2}{\eta^2} - \frac{m^2}{\eta^2} f \tag{21A}$$

which becomes

$$\int_o^{\eta_o} \left[\eta^{|m|-1} e^{-\eta} M_{mj} \frac{d}{d\eta} \left(\eta \frac{df}{d\eta}\right) - \eta^{|m|-2} m^2 e^{-\eta} M_{mj} f\right] d\eta \quad .$$

Integrating the first term by parts twice, the above integral becomes

$$\int_o^{\eta_o} f e^{-\eta} \eta \left\{\left[M''_{mj} + \left(\frac{1+2|m|}{\eta} - 1\right) M'_{mj} - \frac{1+h_j+|m|}{\eta} M_{mj}\right]\right.$$

$$\left. - \left(\frac{2}{\eta} + 1\right) M'_{mj} + \left(\frac{1-2|m|}{\eta^2} + \frac{2h_j - |m|}{\eta}\right) M_{mj} + M_{mj}\right\} d\eta \quad .$$

The term in square brackets is identically zero as seen from equation (2A), and the last term can be transformed directly. Hence, the expression (21A) transforms to

$$F(h_j) - \int_o^{\eta_o} f e^{-\eta} \eta^{|m|} \left(\frac{2}{\eta} + 1\right) M'_{mj} d\eta + \int_o^{\eta_o} f e^{-\eta} \eta^{|m|} \left(\frac{1-2|m|}{\eta^2} + \frac{2-|m|+h_j}{\eta}\right) M_{mj} d\eta \; . \tag{22A}$$

Substituting the expression for $f(\eta)$ from equation (18A), the transform for (21A) may be written as

$$\left[\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{df}{d\eta}\right) - \frac{m^2}{\eta^2}f\right] \longrightarrow \sum_{p=1}^{\infty} D^*_{jp}F(h_p) \qquad (23A)$$

where

$$D^*_{jp} = (1-2|m|)d^*_{jp} + (2-|m|+h_j)a^*_{jp} - 2c^*_{jp} - b^*_{jp} + \delta_{jp}$$

$$a^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|-1}e^{-\eta}M_{mj}M_{mp}d\eta$$

$$b^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|-1}e^{-\eta}M_{mp}M'_{mj}d\eta$$

$$c^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|-1}e^{-\eta}M_{mp}M'_{mj}d\eta$$

$$d^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|-2}e^{-\eta}M_{mp}M_{mj}d\eta$$

$$\delta_{jp} = \text{Kronecker delta} \quad = 1, \quad j = p$$
$$= 0, \quad j \neq p$$

In like manner the other terms of equation (59) transform as follows

$$\frac{f}{\eta} \longrightarrow \sum_{p=1}^{\infty} a^*_{jp}F(h_p)$$

$$\frac{df}{d\eta} \longrightarrow \sum_{p=1}^{\infty} C^*_{jp}F(h_p)$$

$$\frac{d^2f}{d\eta^2} \longrightarrow \sum_{p=1}^{\infty} S^*_{jp}F(h_p) \qquad (25A)$$

$$\frac{d}{d\eta}\left(\frac{1}{n_o}\frac{dn_o}{d\eta}\ell\right) \longrightarrow \sum_{p=1}^{\infty} U^*_{jp}L(h_p)$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\frac{\eta}{n_o}\frac{dn_o}{d\eta}\ell\right) \longrightarrow \sum_{p=1}^{\infty} A^*_{jp}L(h_p)$$

$$\frac{1}{n_o}\frac{dn_o}{d\eta}\ell \longrightarrow \sum_{p=1}^{\infty} s^*_{jp} L(h_p) \quad .$$

The integrals which appear, in addition to those in (24A), are

$$e^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|-1} e^{-\eta} \frac{1}{n_o}\frac{dn_o}{d\eta} M_{mj} M_{mp} d\eta$$

$$s^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|} e^{-\eta} \frac{1}{n_o}\frac{dn_o}{d\eta} M_{mj} M_{mp} d\eta \qquad (26A)$$

$$u^*_{jp} = \frac{1}{I_p}\int_0^{\eta_o} \eta^{2|m|} e^{-\eta} \frac{1}{n_o}\frac{dn_o}{d\eta} M_{mp} M'_{mj} d\eta$$

Note that the integrands of e^*_{jp}, s^*_{jp}, and u^*_{jp} take the form o/o at η_o. Utilizing L'Hospital's Rule, it is easily shown that all these functions have finite limits and the integrals are proper and can be evaluated easily using standard numerical techniques.

The coefficients of the transformed variable $L(h_p)$ and $F(h_p)$ are defined as follows

$$C^*_{jp} = \delta_{jp} - |m| a^*_{jp} - b^*_{jp}$$

$$S^*_{jp} = \delta_{jp} - b^*_{jp} - c^*_{jp} + |m|(|m|-1) d^*_{jp} + (1+h_j - |m|) a^*_{jp}$$

$$U^*_{jp} = s^*_{jp} - |m| e^*_{jp} - u^*_{jp} \qquad (27A)$$

$$A^*_{jp} = U^*_{jp} + e^*_{jp}$$

$$B^*_{jp} = S^*_{jp} - U^*_{jp}$$

$$E^*_{jp} = 2C^*_{jp} - s^*_{jp}$$

where B^*_{jp} and E^*_{jp} have been introduced in equation (59) for convenience. This completes the definitions needed to specify completely the integral transformation.

It is deemed necessary at this point to discuss briefly the numerical technique used to evaluate the previously defined integrals. Tables are available (ref. 13) for the confluent hypergeometric function but proved to be insufficient for our purposes. Therefore, the IBM 1410 digital computer was used to evaluate M_{mj} and the previously defined integrals.

In order to evaluate M_{mj} for given values of η , it is first necessary to find the value of h_j which satisfies the boundary condition at $\eta = \eta_o$, *i.e.*, find the value of "a" for which

$$M(a,b,\eta_o) = o \quad .$$

For computational purposes, $M(a,b,\eta)$ may be written as

$$M(a,b,\eta) = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} u_k$$

$$= 1 + u_1 + u_1u_2 + u_1u_2u_3 + \cdots \qquad (28A)$$

where

$$u_k = \frac{a+k-1}{b+k-1} \; \frac{\eta}{k} \quad .$$

Expressing $M(a,b,\eta)$ in this form is convenient for programming purposes. First the approximation for "a" to satisfy $M(a,b,\eta_o) = o$ is given by (ref. 15)

$$a_o = \frac{b}{2} - \frac{\pi^2(\frac{1}{4} + \frac{b}{2})^2}{4\eta_o} \quad . \qquad (29A)$$

Higher approximations are obtained using the Newton-Raphson method, *i.e.*,

$$a_n = a_{n-1} - \frac{M(a_{n-1},b,\eta_o)}{\frac{\partial}{\partial a} M(a_{n-1},b,\eta_o)}$$

where

$$\frac{\partial}{\partial a} M(a_{n-1},b,\eta_o) = \sum_{r=1}^{\infty} \frac{(a)_r \eta^r}{(b)_r r!} \sum_{n=1}^{r} \frac{1}{a+n-1} \qquad (30A)$$

and

$$(a)_r = a(a+1)(a+2)\ldots(a+r-1) \quad .$$

The recurrence relation of (30A) converges quite rapidly to the desired value with eight place accuracy. Knowing "a" , hence h_j, it is possible to tabulate M_{mj} to eight significant digits. The integrals are evaluated using the parabolic rule for numerical integration. The values of the various integrals for m = -1 are listed below.

$\eta_o = 0.01$

$h_o = -145.08020$ $\qquad h_1 = -659.86650$

$\gamma_o = .00345829$ $\qquad \gamma_1 = .00075830$

$\lambda_o = .49999600$ $\qquad \lambda_1 = .49999900$

$a^*_{11} = 325.10412$ $\qquad b^*_{11} = -325.12244$

$c^*_{11} = -93255.309$ $\qquad d^*_{11} = 186954.80$

$e^*_{11} = -114074.36$ $\qquad s^*_{11} = -386.99251$

$u^*_{11} = 140510.45$ $\qquad I_k = 1.0629 \times 10^{-8}$

$A^*_{11} = -140897.44$ $\qquad B^*_{11} = -94120.800$

$C^*_{11} = 1.0183200$ $\qquad D^*_{11} = -214318.28$

$E^*_{11} = 389.02915$ $\qquad S^*_{11} = -120943.88$

$U^*_{11} = -26823.080$

$\eta_o = 0.5$

$h_o = -3.4188112$ $\qquad h_1 = -13.738426$

$\gamma_o = .17130261$ $\qquad \gamma_1 = .03776884$

$\lambda_o = .49260900$ $\qquad \lambda_1 = .49964300$

$a^*_{11} = 6.5098743$ $\qquad b^*_{11} = -6.0203897$

$c^*_{11} = -34.223242$ $\qquad d^*_{11} = 75.005104$

$e^*_{11} = -45.973875$ $\qquad s^*_{11} = -7.7810131$

$u^*_{11} = 52.690467$ $\qquad I_k = 1.32118 \times 10^{-3}$

$A^*_{11} = -60.471480$ $\qquad B^*_{11} = -33.694189$

$C^*_{11} = .51051540$ $\qquad D^*_{11} = -82.462790$

$D^*_{11} = 8.8020439$ $\qquad S^*_{11} = -48.191794$

$U^*_{11} = -14.497605$

$\eta_o = 1.0$

$h_o = -2.0$

$\gamma_o = 1/3$

$\lambda_o = .47140400$

$h_1 = -7.1957702$

$\gamma_1 = .074674010$

$\lambda_1 = .49860400$

$a^*_{11} = 3.2666730$

$c^*_{11} = -7.8200312$

$e^*_{11} = -11.752077$

$u^*_{11} = 12.394485$

$b^*_{11} = -2.7721594$

$d^*_{11} = 18.919768$

$s^*_{11} = -3.9520591$

$I_k = .010394152$

$A^*_{11} = -16.346544$

$C^*_{11} = .50548640$

$E^*_{11} = 4.9630319$

$U^*_{11} = -4.5944607$

$B^*_{11} = -7.3195700$

$D^*_{11} = -19.747102$

$S^*_{11} = -11.914037$

APPENDIX B

Dispersion Relation

Consider equation (61) of the text and restrict attention to the $j = p = 1$ case. Equation (61) can be separated into an equation for the electrons and one for the ions. The equation for the ions is

$$\frac{\xi\gamma_o^2}{\lambda_o^2}\left\{\frac{\delta D_{11}}{\delta+y} + \frac{\lambda_o^2}{\gamma_o^2} + \frac{C_{11}}{\gamma_o} + \bar{\omega}_2 + \frac{yS_{11}}{\delta+y} - \frac{\delta+y\bar{B}_x^2}{\delta+y}\, k_o^2\right.$$

$$+ \frac{q_2 a_{11}\delta}{\delta+y(1+\bar{B}_x^2)}\ \bar{E}_x + i\left[-\bar{\omega}_1 + (q_2 y\bar{B}_x - \delta - y\bar{B}_x^2)\frac{k_o\bar{E}_x}{\delta+y(1+\bar{B}_x^2)}\right.$$

$$\left.\left.- \frac{m\delta a_{11}(q_2\bar{B}_x + 1)\sqrt{y/\delta}}{\delta+y(1+\bar{B}_x^2)}\ \bar{E}_x + \frac{2C_{11}y\bar{B}_x}{\delta+y}\, k_o\right]\right\} F(h_1) \ =$$

$$\mu_i'\left\{- D_{11} + A_{11} - \frac{y}{\delta} B_{11} + \left(1 + \frac{y}{\delta}\bar{B}_x^2\right)k_o^2\right.$$

$$\left.- i\left[\frac{y\bar{B}_x E_{11}}{\delta}\, k_o + m\bar{B}_x e_{11}\sqrt{y/\delta}\right]\right\} L(h_1) \tag{1B}$$

The equation may be written in the form

$$\frac{\xi\gamma_o^2}{\lambda_o^2}\ (\eta_i + i\eta_2)F(h_1) - \mu_i'(\psi_1 - i\psi_2)L(h_1) = 0 \tag{2B}$$

where

$$\begin{aligned}
\eta_1 &= b_1 + b_2\bar{E}_x - b_3k_o^2\\
\eta_2 &= b_4k_o\bar{E}_x - b_5\bar{E}_x - \bar{\omega}_1 + b_6k_o\\
\psi_1 &= b_7 + b_8k_o^2\\
\psi_2 &= b_9k_o + b_{10}
\end{aligned} \tag{3B}$$

and

$$b_1 = \frac{\delta D_{11}}{\delta+y} + \frac{\lambda_o^2}{\gamma_o^2} + \frac{C_{11}}{\gamma_o} + \bar{\omega}_2 + \frac{yS_{11}}{\delta+y}$$

$$b_2 = \frac{\delta q_2 a_{11}}{\delta+y(1+\bar{B}_x^2)}$$

$$b_3 = \frac{\delta+y\bar{B}_x^2}{\delta+y}$$

$$b_4 = \frac{q_2 y\bar{B}_x - \delta-y\bar{B}_x^2}{\delta+y(1+\bar{B}_x^2)}$$

$$b_5 = \frac{m\delta a_{11}\sqrt{y/\delta}\,(q_2\bar{B}_x+1)}{\delta+y(1+\bar{B}_x^2)} \tag{4B}$$

$$b_6 = \frac{2\bar{B}_x y C_{11}}{\delta+y}$$

$$b_7 = -D_{11} + A_{11} - \frac{y}{\delta} B_{11}$$

$$b_8 = \frac{\delta+y\bar{B}_x^2}{\delta}$$

$$b_9 = \frac{y\bar{B}_x E_{11}}{\delta}$$

$$b_{10} = m\bar{B}_x e_{11}\sqrt{y/\delta}$$

From equation (61), the equation for the electrons is

$$\frac{\xi\gamma_o^2}{\lambda_o^2}\left\{ \frac{D_{11}}{1+\delta y} + \frac{\lambda_o^2}{\gamma_o^2} + \frac{C_{11}}{\gamma_o} + \bar{\omega}_2 + \frac{\delta y S_{11}}{1+\delta y} - \frac{\delta q_2 a_{11}}{1+\delta y(1+\bar{B}_x^2)} \quad \bar{E}_x\right.$$

$$- \frac{1+\delta y\bar{B}_x^2}{1+\delta y}\, k_o^2 + i\left[-\bar{\omega}_1 + \frac{\delta(1+\delta y\bar{B}_x(\bar{B}_x-q_2))}{1+\delta y(1+\bar{B}_x^2)} \quad k_o\bar{E}_x\right.$$

$$- \frac{m\delta a_{11}(q_2\bar{B}_x+1)\ /\delta y}{1+\delta y(1+\bar{B}_x^2)}\ \bar{E}_x + \frac{2C_{11}\delta y\bar{B}_x}{1+\delta y}\ k_o \Big]\Big\}\ F(h_1) =$$

$$\mu_i' \Big\{\delta_1 D_{11} - \delta_1 A_{11} - \delta_1(1+\delta y\bar{B}_x^2)k_o^2 + \delta_1 \delta y B_{11}$$

$$+ \ i\Big[\delta_1 \delta y \bar{B}_x E_{11} k_o - m\delta_1 \bar{B}_x e_{11}\ /\delta y\ \Big]\Big\}\ L(h_1) \quad . \qquad (5B)$$

The equation may be written in the form

$$\frac{\xi\gamma_o^2}{\lambda_o^2}\ (\Delta_1 + i\Delta_2)F(h_1) - \mu_i'(V_1 + iV_2)L(h_1) = 0 \qquad (6B)$$

where

$$\Delta_1 = t_1 - t_2\bar{E}_x - t_3 k_o^2$$

$$\Delta_2 = t_4 k_o \bar{E}_x - t_5\ \bar{E}_x - \bar{\omega}_1 + t_6 k_o \qquad (7B)$$

$$V_1 = t_7 - t_8 k_o^2$$

$$V_2 = t_9 k_o - t_{10}$$

and

$$t_1 = \frac{D_{11}}{1+\delta y} + \frac{\lambda_o^2}{\gamma_o^2} + \frac{C_{11}}{\gamma_o} + \bar{\omega}_2 + \frac{\delta y S_{11}}{1+\delta y}$$

$$t_2 = \frac{a_{11}\delta q_2}{1+\delta y(1+\bar{B}_x^2)}$$

$$t_3 = \frac{1+\delta y\bar{B}_x^2}{1+\delta y}$$

$$t_4 = \frac{\delta(\bar{B}_x\delta y\ (\bar{B}_x - q_2) + 1)}{1+\delta y(1+\bar{B}_x^2)} \qquad (8B)$$

$$t_5 = \frac{m\delta a_{11}(q_2\overline{B}_x + 1)\sqrt{\delta y}}{1+\delta y(1+\overline{B}_x^2)}$$

$$t_6 = \frac{2\overline{B}_x \delta y C_{11}}{1+\delta y}$$

$$t_7 = \delta_1(D_{11} - A_{11} + \delta y B_{11})$$

$$t_8 = \delta_1(1+\delta y \overline{B}_x^2)$$

$$t_9 = \delta_1 \delta y \overline{B}_x E_{11}$$

$$t_{10} = m\delta_1 \overline{B}_x e_{11} \sqrt{\delta y}$$

In order for the two homogeneous equations, (2B) and (6B), to have non-trivial solutions, the determinant of the coefficients must equal zero, i.e.,

$$\begin{vmatrix} \Delta_1 + i\Delta_2 \ , & -V_1 - iV_2 \\ \eta_i + i\eta_2 \ , & -\psi_i + i\psi_2 \end{vmatrix} = 0 \qquad (9B)$$

which becomes on expanding

$$-\Delta_1\psi_1 - \Delta_2\psi_2 + \eta_1 V_1 - \eta_2 V_2 + i(\Delta_1\psi_2 - \Delta_2\psi_1 + \eta_1 V_2 + \eta_2 V_1) = 0 \quad . \qquad (10B)$$

This gives the stability boundary when $\bar{\omega}_2 = 0$. The real part of (10B) leads to equation (64) of the text and the imaginary part results in equation (65). The coefficients of these two equations are defined as follows

$$M_1 = t_3 b_8 + b_3 t_8$$

$$M_2 = t_3 b_7 - t_1 b_8 - t_6 b_9 - b_1 t_8 - b_3 t_7 - b_6 t_9$$

$$M_3 = t_{10} b_6 - b_{10} t_6$$

$$M_4 = b_1 t_7 - t_1 b_7$$

$$M_5 = t_2 b_8 - t_4 b_9 - b_2 t_8 - b_4 t_9$$

$$M_6 = t_5 b_9 - t_9 b_{10} + b_4 t_{10} + b_5 t_9$$

$$M_7 = t_2 b_7 + t_5 b_{10} + b_2 t_7 - b_5 t_{10}$$

$$M_8 = b_9 + t_9$$

$$M_9 = b_{10} - t_{10}$$

$$N_1 = -t_3 b_9 - t_6 b_8 - t_9 b_3 - b_6 t_8$$

$$N_2 = -t_3 b_{10} + t_{10} b_3$$

$$N_3 = t_1 b_9 - t_6 b_7 + b_1 t_9 + b_6 t_7$$

$$N_4 = t_1 b_{10} - b_1 t_{10} \qquad (11B)$$

$$N_5 = -t_4 b_8 - b_4 t_8$$

$$N_6 = t_5 b_8 + b_5 t_8$$

$$N_7 = -t_2 b_9 - t_4 b_7 + b_2 t_9 + b_4 t_7$$

$$N_8 = -t_2 b_{10} + t_5 b_7 - b_2 t_{10} - b_5 t_7$$

$$N_9 = b_8 + t_8$$

$$N_{10} = b_7 - t_7$$

This completes the derivation of the dispersion relation.

APPENDIX C

Coefficients of Equations (39) and (40)

$$A_1 = (1+\delta)\,[k_o^2\,(1+y) - d_1(1+y) - y\,d_2(y+\Delta)]$$

$$A_2 = (1+\delta)\,[m\sqrt{y\delta}\,A_{11}\,(k_o^2 - d_1 - y\,d_2) - y\,d_2(\delta-1)k_o]$$

$$A_3 = -k_o^3\,[1 + \overline{D}_e + \delta(1+\overline{D}_i) + 1+\delta] - k_o^2\,m\sqrt{y\delta}\,A_{11}\,[\overline{D}_i - \overline{D}_e + \delta - \frac{1}{\delta}$$

$$+ k_o[d_1(1 + \overline{D}_e + \delta(1+\overline{D}_i)) + y\,d_2\,(\delta^2(1+\overline{D}_i) + \frac{1+\overline{D}_e}{\delta})$$

$$+ (b_1 + yb_2)(1+\delta)] + m\sqrt{y\delta}\,A_{11}\,[(\overline{D}_i - \overline{D}_e)(d_1 + yd_2) + b_1\,(\delta - \frac{1}{\delta})]$$

$$a_6 = (1+\delta)^2(1+y)$$

$$a_4 = (1+\delta)^2[(1+y)H^2 - (1+y)(2d_1 + b_1 + yb_2 + d_2y\Delta) - y\,d_2(y+\Delta)]$$

$$a_3 = -m\sqrt{y\delta}\,A_{11}(1+\delta)H^2\,[\overline{D}_i(1+\delta y) - \overline{D}_e(1+y/\delta) - 2(\delta - \frac{1}{\delta})]$$

$$a_2 = (1+\delta)\,\{(1+\delta)(1+y)[d_1b_1 + yd_1b_2 + yb_1d_2\Delta + y^2b_2d_2]$$

$$+ (1+\delta)[d_1(1+y) + yd_2(y+\Delta)][d_1 + b_1 + yb_2 + d_2y\Delta]$$

$$- H^2\,[yd_2(\delta-1)((1+\overline{D}_i)(1+\delta y) - (1+\overline{D}_e)(1+y/\delta)) + (1+\delta)(1+y)(b_1 + yb_2)$$

$$- m^2y\,A_{11}^2\,(\delta-1)(\overline{D}_i - \overline{D}_e + \delta - \frac{1}{\delta})]\}$$

$$a_1 = -k_om\sqrt{\delta y}\,A_{11}(1+\delta)H^2\,\{d_1[y(-\delta(1+\overline{D}_i) + \frac{1}{\delta}\,(1+\overline{D}_e)) + \overline{D}_e - \overline{D}_i]$$

$$+ yd_2\,[-\overline{D}_i(\Delta+\delta y) + \overline{D}_e(\Delta + \frac{y}{\delta}) - y\,\frac{(\delta^2-1)}{\delta}\,]$$

$$+ (\frac{\delta^2-1}{\delta})\,[b_1(2+y) + yb_2]\}$$

$$a_o = -(1+\delta)\left\{(1+\delta)[d_1(1+y) + yd_2(y+\Delta)][d_1b_1 + yd_1b_2 + y\Delta b_1d_2 + y^2b_2\, d_2]\right.$$

$$\left. + m^2H^2(\delta-1)y\, A_{11}^2\, [(\overline{D}_i - \overline{D}_e)(d_1 + yd_2) + b_1\, (\delta - \frac{1}{\delta})]\right\}$$

$$c_3 = m\sqrt{y\delta}\, (1+\delta)^2(1+y)A_{11}$$

$$c_2 = (1+\delta)^2(\delta-1)[(1+y)yd_2 - m^2y\, A_{11}^2]$$

$$c_1 = -m\sqrt{\delta y}\, A_{11}(1+\delta)^2[yd_2(y+\Delta) + d_1(1+y)]$$

$$c_0 = m^2(1+\delta)^2(\delta-1)(d_1 + yd_2)y\, A_{11}^2$$

where

$$\Delta = \frac{1}{\delta} - 1+\delta\ , \quad \overline{D}_i = \frac{\overline{\Theta}}{\delta}\ \frac{1+\delta}{1+\overline{\Theta}}\ , \quad \overline{D}_e = \frac{1+\delta}{1+\overline{\Theta}}\ , \quad \overline{\Theta} = V_i/V_e\ .$$

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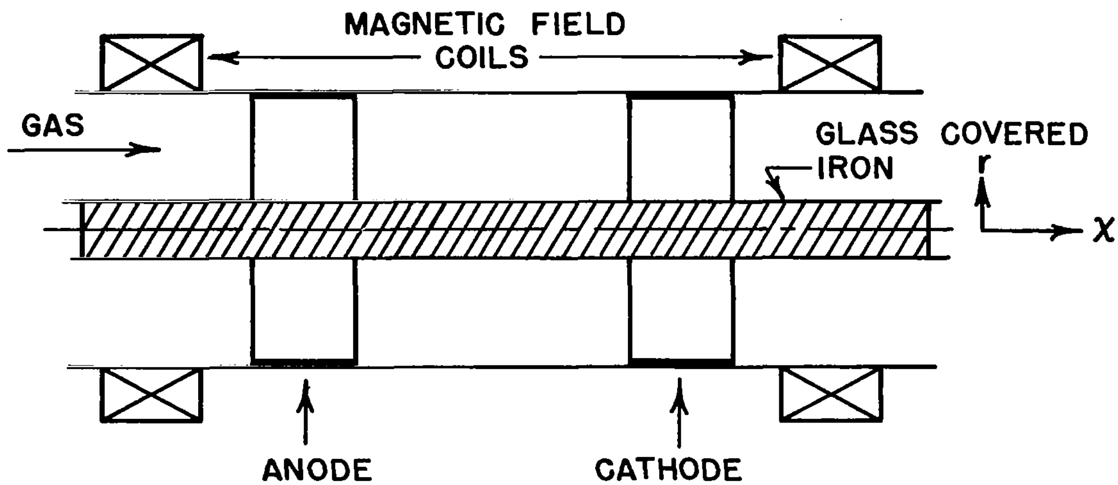

Figure 1. Schematic of linear Hall current ion accelerator

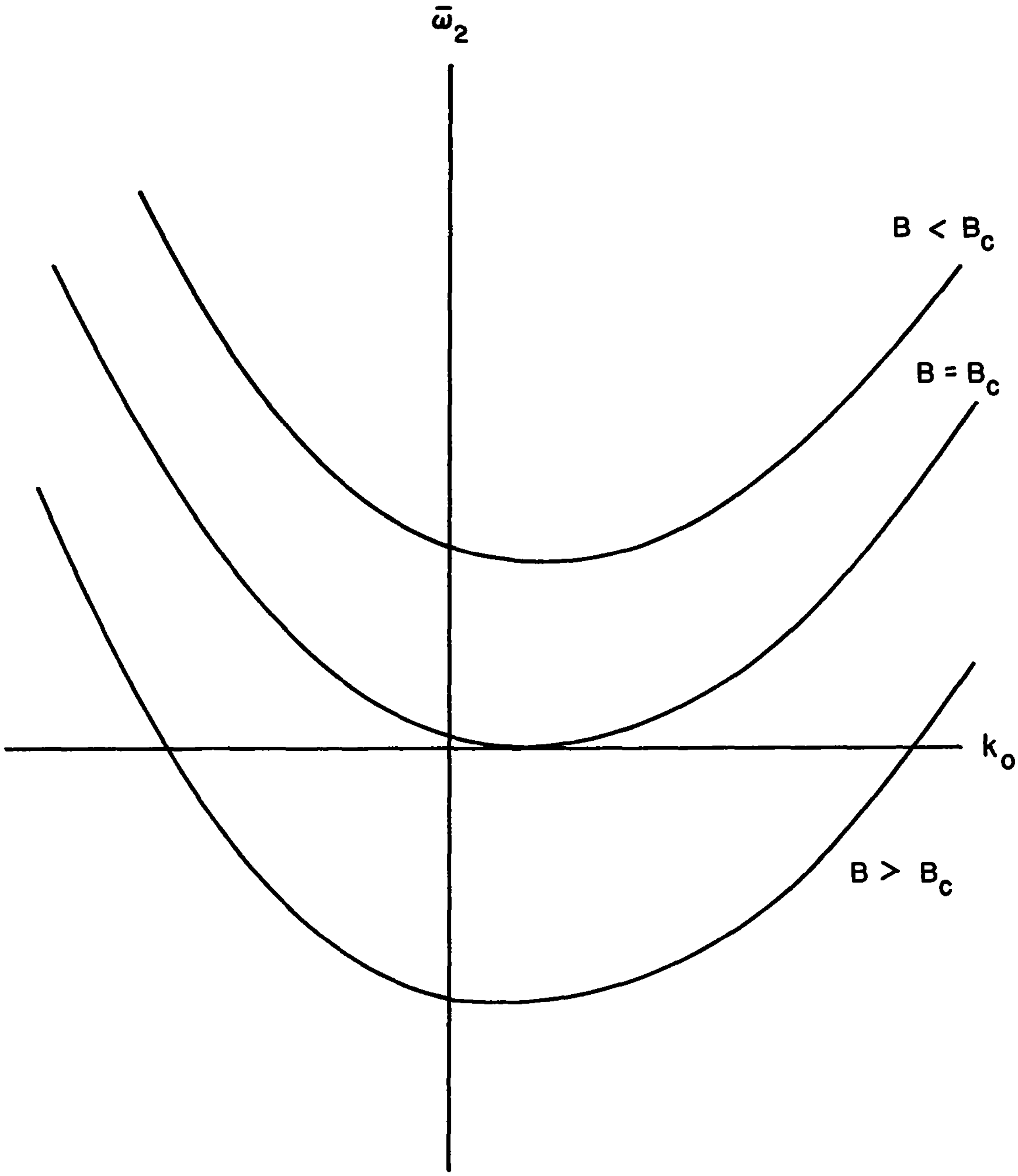

Figure 2. Growth rate, $\bar{\omega}_2$, versus wave number, k_o

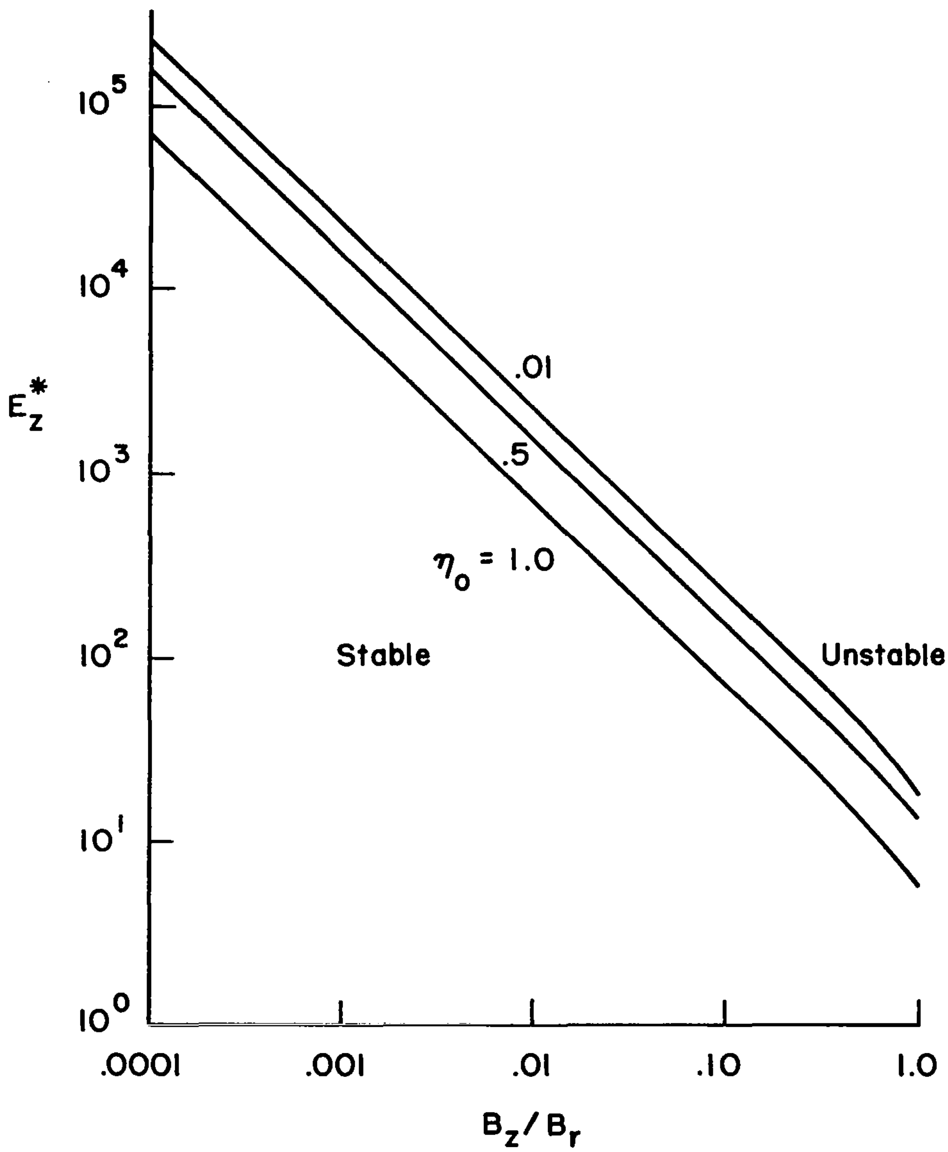

Figure 3. Electric field versus magnetic field ratio, $\delta = 10$

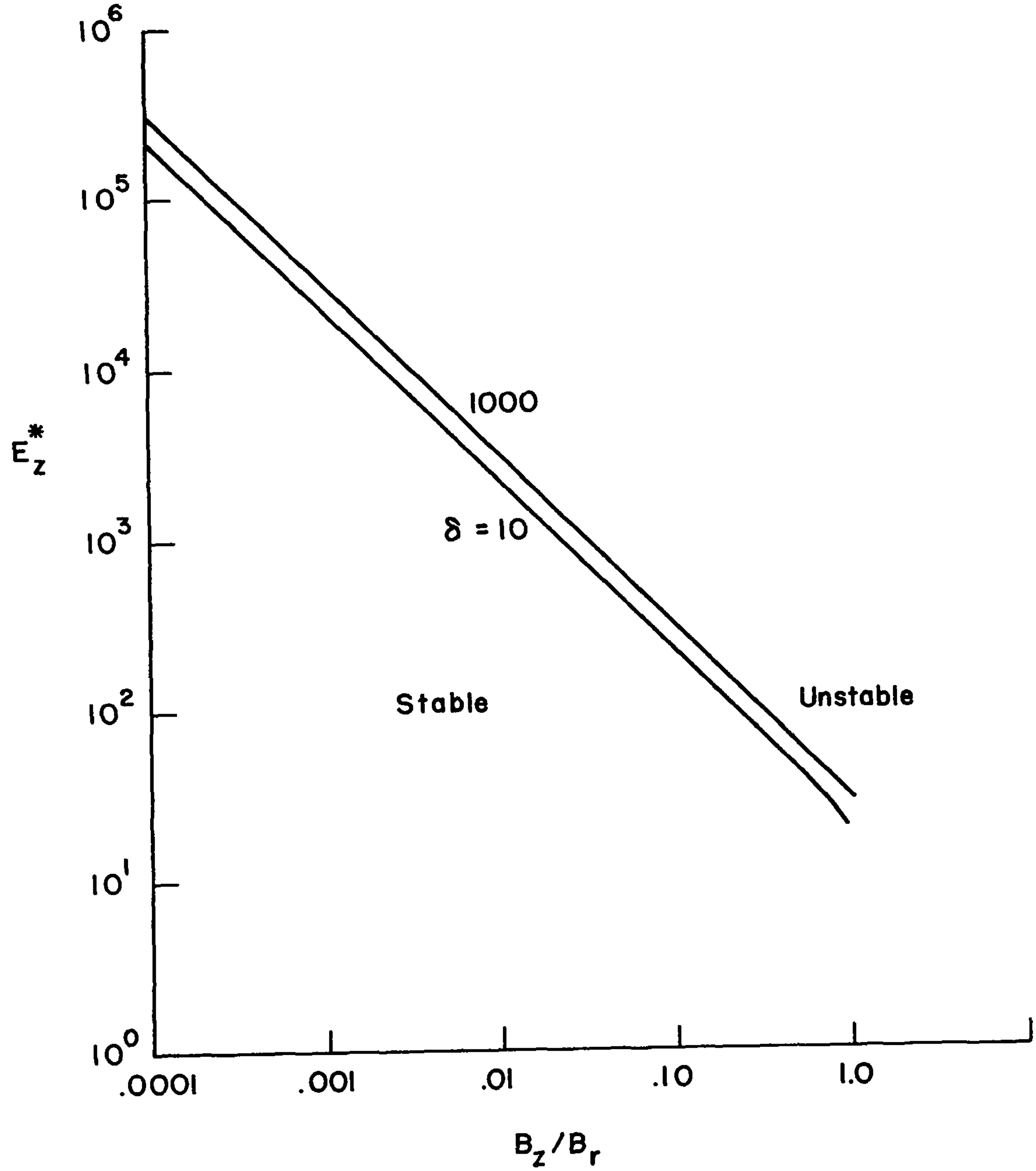

Figure 4. Electric field versus magnetic field ratio, $\eta_o = .01$

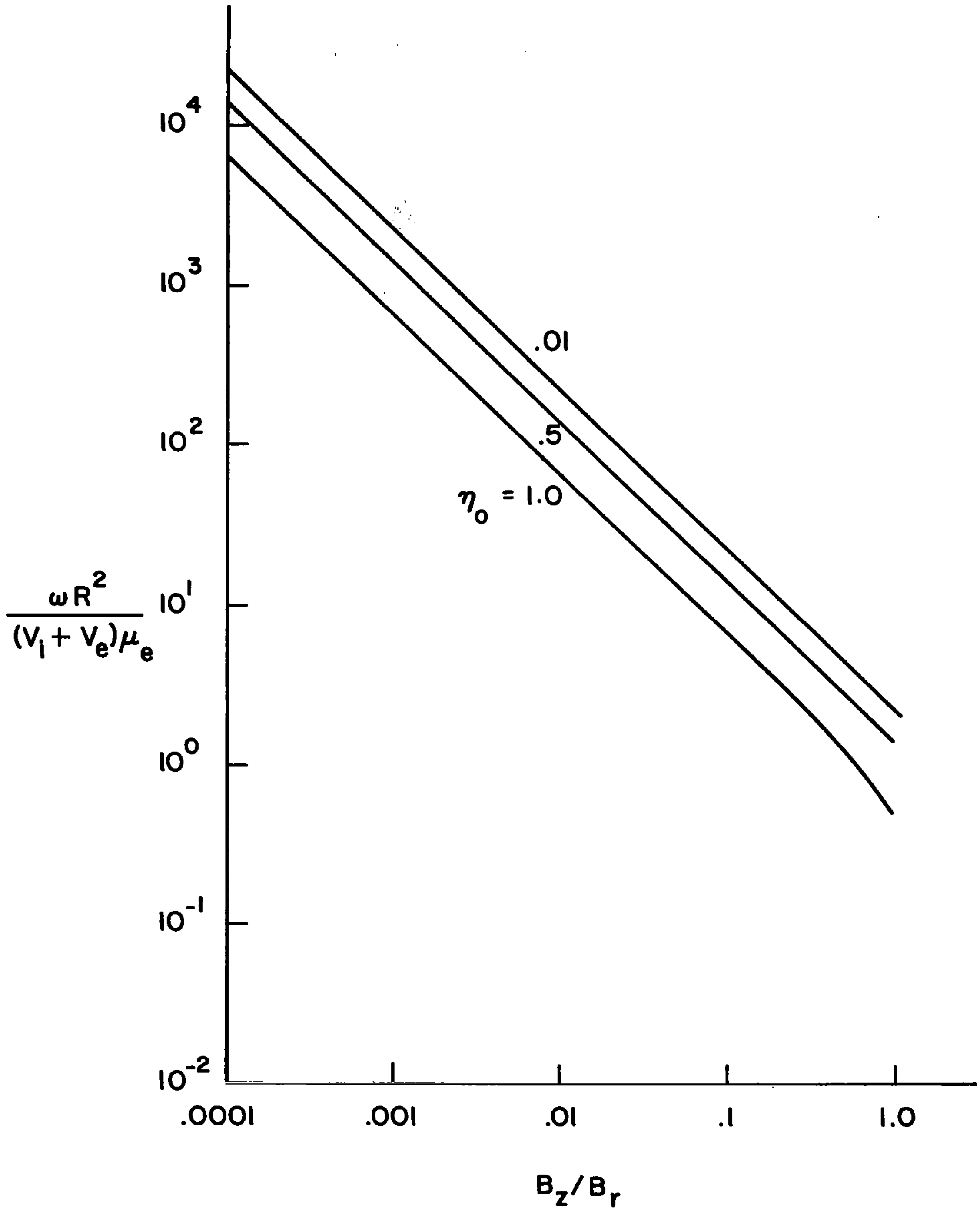

Figure 5. Frequency versus magnetic field ratio, $\delta = 10$

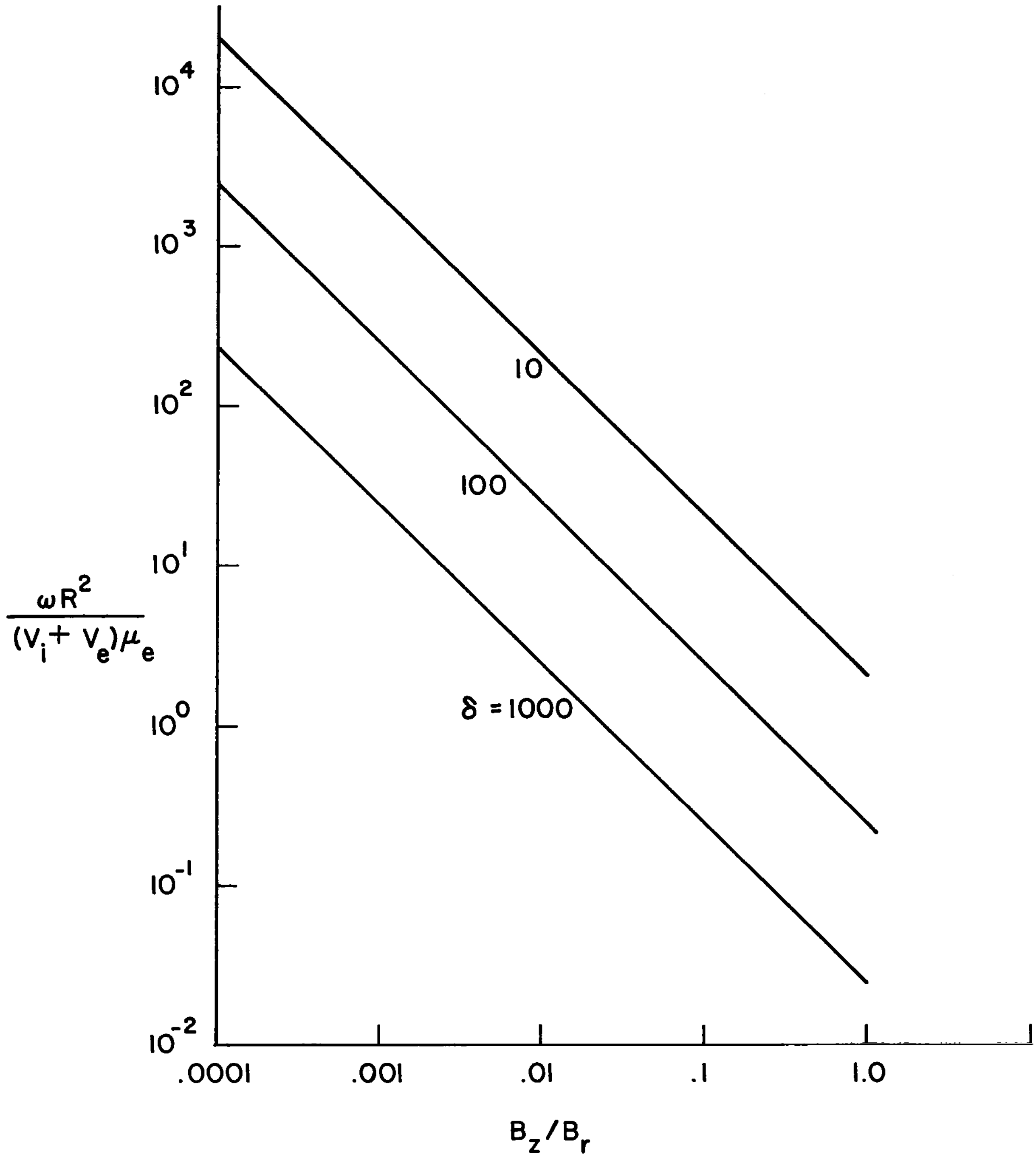

Figure 6. Frequency versus magnetic field ratio, η_o = .01

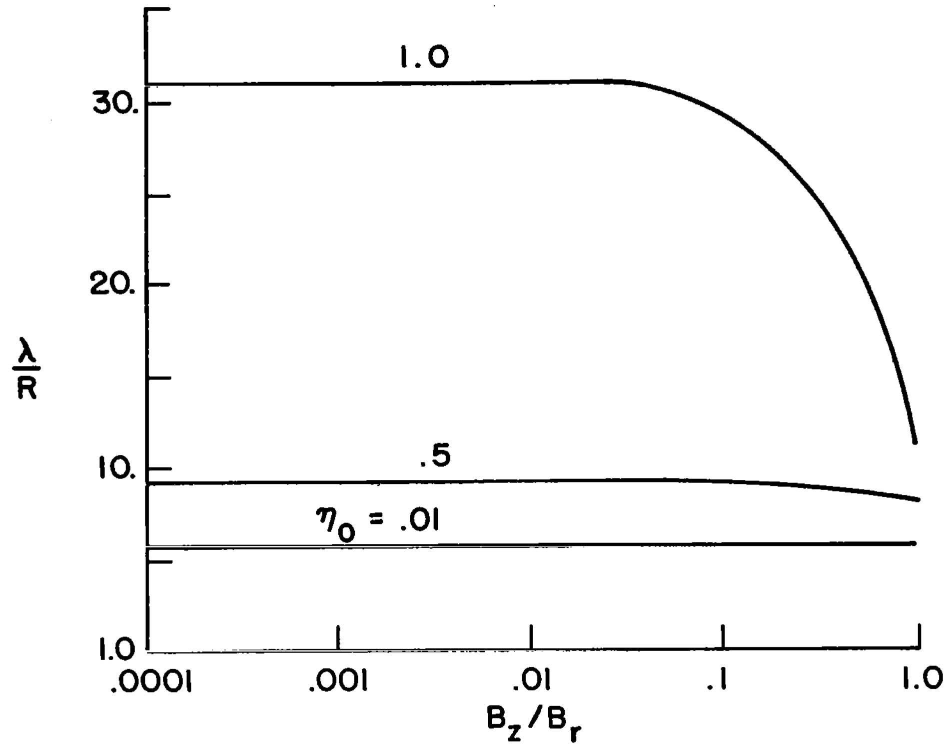

Figure 7. Wavelength versus magnetic field ratio, $\delta = 10$

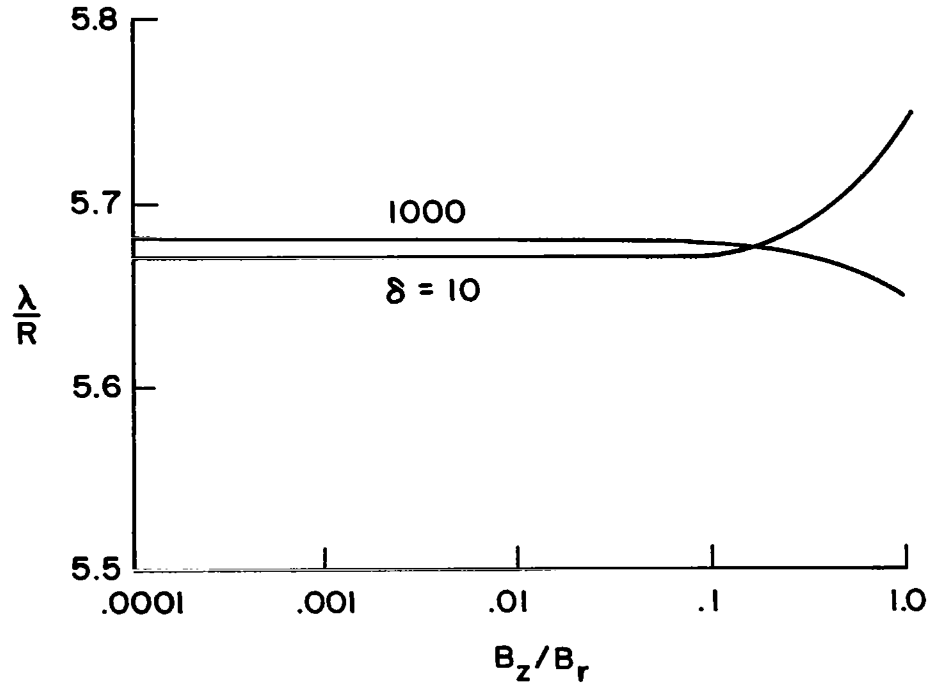

Figure 8. Wavelength versus magnetic field ratio, $\eta_0 = .01$

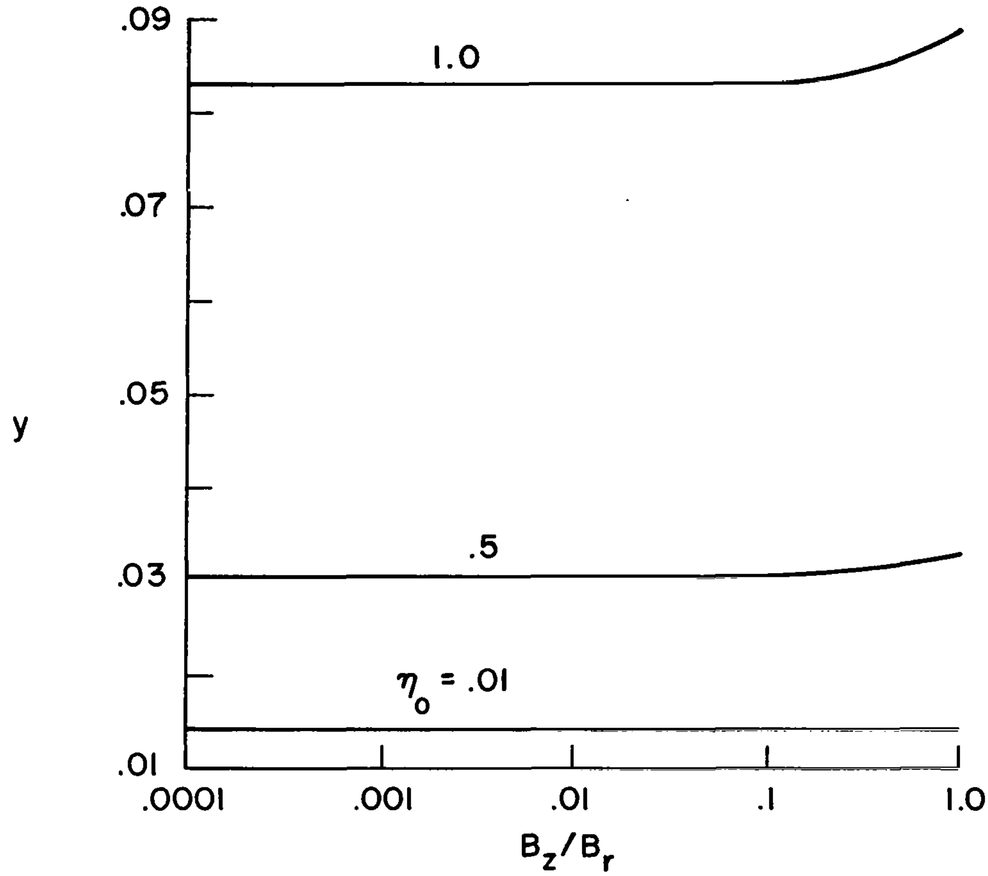

Figure 9. Critical magnetic field versus magnetic field ratio, δ = 10

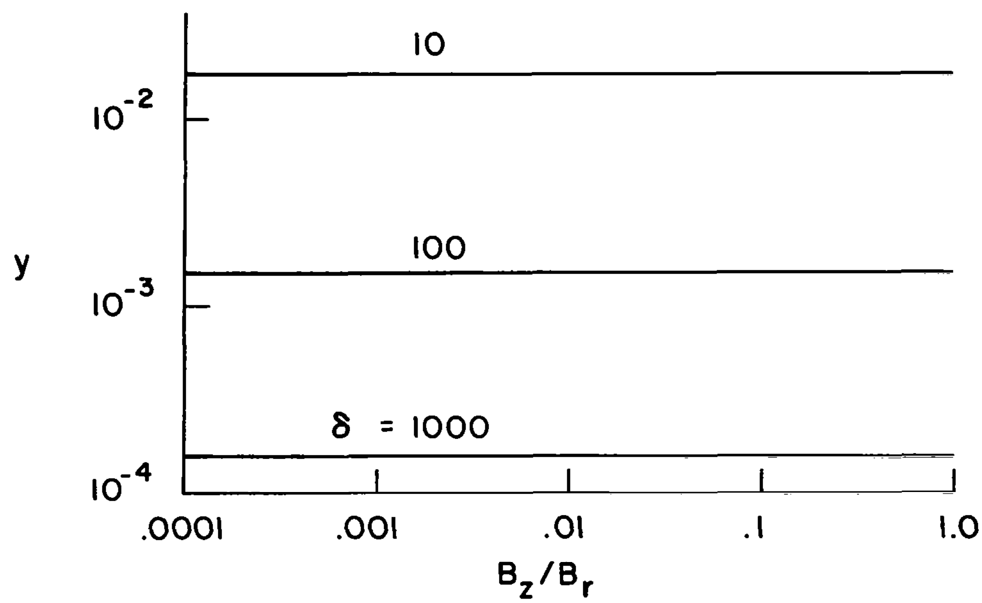

Figure 10. Critical magnetic field versus magnetic field ratio, η_o = .01

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