REMARKS ON A PAPER OF LEIGHTON[1]

by

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1. Introduction. Recently Leighton [1] obtained some interesting conditions for a solution to a differential system $y'' + py = 0$, $y(a) = 0$ to have a zero in an interval $(a,b]$. The primary purpose of this paper is to present counterexamples to Theorems 2 and 4 of [1] and to show how the hypotheses of those theorems may be strengthened so that the conclusions are valid. We also present a different proof of a part of Theorem 3 of [1] since the proof there relies on Theorem 2. At the same time, we present modest generalizations of those and other theorems of [1]; in particular, the requirement that the coefficient function p be of class C' and convex, or concave in one theorem, is replaced by the conditions that p be continuous and satisfy a certain integral inequality.

2. A generalization and extention of Sturm's Comparison Theorem. The following generalization of Sturm's Comparison Theorem was proved in [1] for the case $r(x) \equiv 1$.

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[1]This research was partially supported by NASA Grant NGR 43-001-029. Portions of this paper formed the basis of the second author's Masters Thesis at the University of Tennessee.

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THEOREM 2.1 *Suppose that* p, q, r, *are continuous on* $[a, b]$, $r(x) > 0$ *on* $[a, b]$, *and that* z *is a nonnull solution of the system*

$$(r(x)z')' + q(x)z = 0 \quad z(a) = 0 = z(b) .$$

If

$$\int_a^b \Big(p(x) - q(x)\Big) z^2(x)\, dx \geq 0 ,$$

then a nonnull solution y *of the system*

$$(r(x)y')' + p(x)y = 0$$

$$y(a) = 0$$

must have a zero on the interval $(a, b]$.

PROOF. First note that the limit of z/y exists at a, and that $(yz' - zy')/y$ is not the zero function on $(a, b]$. To verify this last statement suppose that $\Big([yz' - zy']/y\Big)(x) = 0$ for all x in $(a, b]$. This implies that $\Big([yz' - zy']/y^2\Big)(x) = 0$ for all x in $(a, b]$ and hence

$$0 = \int_b^x (yz' - zy')/y^2 = \int_b^x (z/y)' = 0 \text{ for all } x \text{ in } (a, b] ,$$

which implies that $z(x) = 0$ for all x in $(a, b]$. But this is contrary to the hypothesis.

Suppose that $y(x) \neq 0$ for all x in $(a, b]$. Then

$$(2.1) \qquad \int_a^b (z/y)\Big[y(rz') - z(ry')\Big]' = \int_a^b (p - q)z^2 \geq 0 .$$

Integrating the left member of (2.1) by parts yields the inequality

$$(z/y)\,(yrz' - zry')\Big|_a^b - \int_a^b (yrz' - zry')\,(yz' - zy')/y_2 \geq 0$$

which implies that $\int_a^b r\left[(yz' - zy')/y\right]^2 \leq 0$ which is impossible.

The next theorem and the lemma used to prove it are apparently sufficiently well-known that references to them in the literature are scarce; we include them here for completeness.

LEMMA 2.1. *Let* p *be continuous on* $(a, b]$, p_1 *continuous on* $[a, b]$ *and* $p_1(x) \geq p(x)$ *on* $(a, b]$, *let* u *be a solution to* $y'' + py = 0$ *on* $(a, b]$, $\lim_{t \to a} u(t) = 0 = u(b)$, *and suppose that there exists a deleted right neighborhood* N *of* a *such that* $u'(t)$ *is positive for all* t *in* N . *If* v *is a solution to the system* $y'' + p_1 y = 0$, $v(a) = 0$, *then there exists a* c *in* $(a, b]$ such that $v(c) = 0$.

PROOF. Suppose v is not zero on $(a, b]$, and note that

$$(v\,u' - uv')' = v\,u'' - u\,v'' = u\,v\,(p_1 - p) . \tag{2.2}$$

Assume $v(x) > 0$ for all x in $(a, b]$ and that $u(x) > 0$ for all x in (a, b) ; hence, $u'(b) < 0$ and $v'(a) > 0$. Integrating (2.2) from t to b one has

$$v(b)\,u'(b) - v(t)\,u'(t) + u(t)\,v'(t) = \int_t^b (p_1 - p)\,uv \quad \text{for} \quad t \text{ in } (a, b] . \tag{2.3}$$

For t sufficiently near a the left member of (2.3) is negative, which is impossible since the right member is non-negative. Hence v will vanish in $(a, b]$.

THEOREM 2.2. *Let* p *be continuous and decreasing on* $(x_1, x_4]$, *and let* u *be a solution to* $y'' + p(x)\, y = 0$ *with* $\lim_{t \to x_1} u(t) = 0 = u(x_2)$. *Suppose there exists a deleted right neighborhood* N *of* x_1 *such that* $u'(t)$ *is positive for all* t *in* N. *If* v *is a solution to* $y'' + p(x)y = 0$ *with consecutive zeros at* x_3, x_4, *where* $x_1 \leq x_3$, *then* $x_2 - x_1 \leq x_4 - x_3$.

PROOF. Let $t = x + x_1 - x_3$. Define z by $z(x) = u(t) = u(x + x_1 - x_3)$, and g by $g(x) = p(t) = p(x + x_1 - x_3) \geq p(x)$. For x in $(x_3, x_4]$, z is a solution to $z'' + g(x)z = 0$ with $\lim_{x \to x_3} z(x) = \lim_{t \to x_1} u(t) = 0$, and v is a solution to $v'' + p(x)\, v = 0$ with $v(x_3) = 0 = v(x_4)$. Since the first zero of $z(x)$ after $x = x_1$ is $x = x_2 - x_1 + x_3$, it follows from Lemma 2.1 that $x_2 - x_1 + x_3 \leq x_4$.

3. *Distribution of zeros.* Throughout this section $y = kx + m$ will be an equation of the line joining the points $(a, p(a))$ and $(b, p(b))$, where p is the coefficient function in the differential equation $y'' + py = 0$. The proof of the following generalization of Theorem 1 of [1] is accomplished using basically the same techniques as used in [1], but our proof does not rely on the concept of *principal* solution as does the proof of Leighton's theorem.

THEOREM 3.1. If p is positive and continuous on $[a, b]$, and y is a nonnull solution to the system

$$y'' + py = 0 , \; y(a) = 0 = y(b)$$

for which

$$\int_a^b (kx + m - p(x))\, y^2(x)\, dx \geq 0 ,$$

then

$$kp(b) \geq k \left[\left(p(a)\right)^{3/2} + 3\, \alpha k/_2 \right]^{2/3} ,$$

where α is the first positive zero of the Bessel's function $J_{1/3}$ (α is approximately 2.9).

PROOF. The conclusion is obvious if $p(a) = p(b)$, hence we assume that $p(a) \neq p(b)$. A nonnull solution to the differential system $z'' + (kx + m)z = 0$, $z(a) = 0$, must vanish again on $(a, b]$ by Theorem 2.1. Following Leighton, we make the change of variable $t = kx + m$ from which it follows that z is a solution to $z'' + (kx + m)z = 0$ if, and only if, y defined by $y(t) = z(x)$ is a solution to $y''(t) + t/k^2\, y(t) = 0$. Hence, any solution z to the equation $z'' + (kx + m)z = 0$ may be written as

$$z(x) = (kx + m)^{1/2} \left[c_1 J_{1_3}\left(\frac{2}{3k}(kx + m)^{3/2}\right) + c_2 J_{-1_3}\left(\frac{2}{3k}(kx + m)^{3/2}\right) \right]. \tag{3.1}$$

Suppose $a \geq 0$, let z be of the form (3.1) and suppose $z(a) = 0$. Suppose the next larger zero of $z(x)$ occurs at b_1.

Let α_1 and α_2 be given by

$$(3.2) \qquad \alpha_1 = \frac{2}{3k}\left(ka + m\right)^{\frac{3}{2}}, \quad \alpha_2 = \frac{2}{3k}\left(kb_1 + m\right)^{\frac{3}{2}} .$$

Since α_1 and α_2 have the same sign and since the zeros of $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$ are symmetric in the line $x = 0$, the assumption that $a \geq 0$ and α_1, α_2 are both positive does not sacrifice generality. It follows that α_1 and α_2 are consecutive zeros of a linear combination of $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$, since $kx + m \neq 0$ on $[a, b]$. Recall that $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$ are independent solutions to

$$(3.3) \qquad x^2 y'' + xy'' + \left(x^2 - \frac{1}{9}\right) y = 0$$

and that the transformation $y = u(x)/\sqrt{x}$ transforms (3.3) into

$$(3.4) \qquad u'' + \left(1 + \frac{5}{36x^2}\right) u = 0 .$$

For $x > 0$ the general solution of (3.4) may be written as

$$u(x) = x^{\frac{1}{2}}\left(c_1 J_{\frac{1}{3}}(x) + c_2 J_{-\frac{1}{3}}(x)\right) .$$

Since $u(x) = x^{\frac{1}{2}} J_{\frac{1}{3}}(x)$ is a solution of (3.4) satisfying $u(0) = 0 = u(\alpha)$, α_1 and α_2 are consecutive zeros of a solution of (3.4) and $0 < \alpha_1$, it follows from Theorem 2.2 that

$\alpha \leq \alpha_2 - \alpha_1$. From (3.2) one finds that

$$(3.5) \quad a = \frac{1}{k}\left(\frac{3k}{2}\ \alpha_1\right)^{\frac{2}{3}} - \frac{m}{k}, \quad b_1 = \frac{1}{k}\left(\frac{3k}{2}\ \alpha_2\right)^{\frac{2}{3}} - \frac{m}{k}.$$

Since $b_1 - a \leq b - a$, it follows from (3.5) that

$$b - a \geq \left(\frac{3}{2}\right)^{\frac{2}{3}} \cdot \frac{1}{k^{1/3}} \cdot \left(\alpha_2^{\frac{2}{3}} - \alpha_1^{\frac{2}{3}}\right)$$

or, since $\alpha + \alpha_1 \leq \alpha_2$ and since the assumption that α_1 and α_2 are positive implies $k > 0$,

$$b - a \geq \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{1}{k^{1/3}} \left[\left(\alpha_1 + \alpha\right)^{\frac{2}{3}} - \alpha_1^{\frac{2}{3}}\right].$$

After using α_1 as given by (3.2) and observing that $ka + m = p(a)$, a straight forward calculation yields

$$kp(b) \geq k \left[\left(p(a)\right)^{\frac{3}{2}} + \frac{3\alpha k}{2}\right]^{\frac{2}{3}} .$$

The following theorem gives a sufficient condition which is related to the necessary conditions given in Theorem 3.1.

THEOREM 3.2. <u>Let</u> p <u>be continuous on</u> [a, b] , <u>positive on</u> (a, b) , <u>let</u> $p(a) \geq 0$, $p(b) \geq 0$, <u>and</u> $p(c) \neq p(b)$. <u>If</u> z <u>is a solution to the system</u> $z'' + (kx + m)z = 0$, $z(a) = 0$ <u>for which</u>

$$\int_a^t \left[p(x) - (kx + m)\right] z^2(x)dx \geq 0 \text{ for all } t \text{ in } (a,b], \text{ if}$$

$$(3.6) \qquad kp(b) \geq k\left[\left(p(a)\right)^{\frac{3}{2}} + \frac{3\pi k}{2}\right]^{\frac{2}{3}}$$

<u>and</u>

$$(3.7) \qquad p^{\frac{3}{2}}(a) > \frac{-3\pi k}{2} ,$$

<u>then a solution to the system</u> $y'' + py = 0$, $y(a) = 0$ <u>will vanish again on</u> $(a, b]$.

PROOF. The conditions (3.6) and (3.7) imply that

$$\frac{2}{3k}\left(p(b)\right)^{\frac{3}{2}} - \frac{2}{3k}\left(p(a)\right)^{\frac{3}{2}} \geq \pi ,$$

which may be written as

$$\frac{2}{3k}(kb + m)^{\frac{3}{2}} - \frac{2}{3k}(ka + m)^{\frac{3}{2}} \geq \pi .$$

Thus, since the distance between consecutive zeros of solutions of Bessel's equation of order $\frac{1}{3}$ is less than π , the solution to the system $z'' + (kx + m)z = 0$, $z(a) = 0$ given by

$$z(x) = (kx + m)^{\frac{1}{2}}\left[c_1 J_{\frac{1}{3}}\left(\frac{2}{3k}(kx + m)^{\frac{3}{2}}\right) + c_2 J_{-\frac{1}{3}}\left(\frac{2}{3k}(kx + m)^{\frac{3}{2}}\right)\right]$$

$$z(a) = 0$$

will have a zero on $(a, b]$. Hence the conclusion follows from Theorem 2.1.

COROLLARY. *If* p *is positive, concave and of class* C' *on the interval* $[a, b]$, *if* $p(a) \neq p(b)$ *and conditions* (3.6) and (3.7) *hold, then a solution to the system* $y'' + py = 0$, $y(a) = 0$ *will vanish on the interval* $(a, b]$.

If the condition (3.7) is removed then the above corollary is the same as Theorem 2 in [1]. However, the following example shows that Leighton's Theorem 2 is false. (It should be remarked that the condition (3.7) automatically holds when $p(a) < p(b)$, so that in this case Theorem 2 in [1] is true). Let $p(x) = -2(x - 1)$ on the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$. Then p is concave, positive and of class C' and condition (3.6) holds. If a solution to the system $y'' + py = 0$, $y\left(\frac{1}{2}\right) = 0$ has a zero on $\left(\frac{1}{2}, \frac{3}{4}\right]$, then Sturm's comparison theorem implies that $\sin\left(x - \frac{1}{2}\right)$ has a zero on $\left(\frac{1}{2}, \frac{3}{4}\right]$. Therefore no solution of the system $y'' + py = 0$, $y\left(\frac{1}{2}\right) = 0$ can vanish on $\left(\frac{1}{2}, \frac{3}{4}\right]$ contrary to the conclusion of Theorem 2 of [1].

The next theorem appeared as Theorem 3 in [1]. Leighton's proof was based on Theorem 2 of the same paper; however, Theorem 2 cannot be used as stated since the hypothesis requires that $p(a) > 0$ and $p(b) > 0$. One also needs the condition (3.7) of Theorem 3.2 which is necessary in case k is negative.

The theorem depends on the following lemma.

LEMMA 3.1. *If* p *is positive, of class* C' *on* $[a, b]$ *and if either*

(3.8) $b - a > - p(a) / p'(a) > 0$ or

(3.9) $b - a > p(b) / p'(b) > 0$,

there exists a point c *in* (a, b) *at which the tangent line to the curve passes through the point* $(b, 0)$ *if* (3.8) *holds or through the point* $(a, 0)$ *if* (3.9) *holds. The number* c *is a solution of the equation* $p(c) = p'(c)(c - b)$ *or of* $p(c) = p'(c)(c - a)$ *according as* (3.8) *or* (3.9) *holds.*

PROOF. See [1, page 305].

THEOREM 3.3. *If* p *is positive, convex, and of class* C' *on* $[a, b]$ *and if the conditions*

(3.10) $b - a > p(b) / p'(b) > 0$, $(b - a)^3 \geq 9 \pi^2/4p'(c)$

or if the conditions

(3.11) $b - a > - p(a) / p'(a) > 0$, $(b - a)^3 \geq - 9 \pi^2/ 4 p'(c)$

hold, where c *is the point guaranteed by Lemma* 3.1, *then a nonnull solution of the differential system*

(3.12) $y'' + p(x)y = 0$, $y(a) = 0$

vanishes again on $(a, b]$.

PROOF. If conditions (3.10) hold, the tangent line assured by Lemma 3.1 has the equation $t(x) = p'(c)(x - a)$. Let $z(x)$ be a solution of the differential system $z'' + t(x)z = 0$, $z(a) = 0$. Since p is convex, if $z(x)$ has another zero on $(a, b]$, then by Sturm's Comparison Theorem a solution of system (3.12) must also have a zero on $(a, b]$. By hypothesis,

$$(3.13) \qquad (b - a)^3 \geq 9\pi^2 / 4p'(c) .$$

Multiplying (3.13) by $(p'(c))^3$, replacing $p'(c)(b - a)$ with $t(b)$ and using the fact that $t(a) = 0$ we obtain

$$p'(c)\left(t(b)\right)^3 \geq p'(c)\left[\frac{3\pi}{2}p'(c) + \left(t(a)\right)^{\frac{3}{2}}\right]^2 .$$

Therefore by Theorem 3.2, where $p(x)$ of that theorem is now $t(x)$ a solution to system (3.12) must have a zero on $(a, b]$, since a solution to the system $z'' + t(x)z = 0$, $z(a) = 0$ does.

If conditions (3.11) hold, the slope of the tangent line guaranteed by Lemma 3.1 is negative. However, condition (3.6) clearly holds since $t(b) = 0$ in this case, and we need only show that

$$\left(t(a)\right)^{\frac{3}{2}} + 3\pi p'(c) / 2 \geq 0 , \quad \text{where } t(x)$$

is the equation of the tangent line guaranteed by Lemma 3.1. By hypothesis,

$$(b - a)^3 \geq -9\pi^2 / 4p'(c) ,$$

from which it follows that

$$\left(p'(c)\right)^3 (a - b)^3 \geq 9\pi^2 \left(p'(c)\right)^2/4$$

or

$$\left(t(a)\right)^{\frac{3}{2}} \geq -3\pi\, p'(c) / 2$$

since $p'(c) < 0$. Thus the conclusion again follows from Theorem 3.2.

Finally, we give an alternate for Theorem 4 of [1].

THEOREM 3.4. *Let* p *be continuous,* $c \neq 0$, *and* d *real numbers. Suppose that* $cx + d \geq 0$ *for all* x *on* $[a, b]$ *and that* z *is a solution to the system*

$$z'' + (cx + d)z = 0 , \; z(a) = 0$$

with the property that

$$\int_a^t [p(x) - (cx + d)]\, z^2(x)dx \geq 0 \quad \text{for all } t \text{ in } (a , b] .$$

If

$$c(cb + d)^3 \geq c \left[\frac{3\pi c}{2} + (ca + d)^{3/2}\right]^2 \tag{3.14}$$

and

$$3\pi c /2 + (ca + d)^{3/2} \geq 0 , \tag{3.15}$$

then a solution to the system

$$y'' + py = 0 , \; y(a) = 0 \tag{3.16}$$

will have a zero on $(a , b]$.

PROOF. The proof is similar to that of Theorem 3.2.

COROLLARY. Let p be positive, convex and of class C' on $[a, b]$ with $p(a) \neq p(b)$. Let $y = cx + d$ be an equation of the line tangent to the curve $y = p(x)$ parallel to the chord joining the endpoints of the arc $y = p(x)$ $(a \leq x \leq b)$. If $ca + d \geq 0$, $cb + d \geq 0$, and if b_0 is the smallest value larger than a for which (3.14) and (3.15) hold, then a nonnull solution of the system (3.16) will vanish on the interval $(a, b_0]$.

If the requirement that condition (3.15) hold is removed from the hypothesis, then the above Corollary is the same as Theorem 4 of [1]. However, the conclusion is not then true if the function p is defined by $p(x) = -(x - 3)$ on the interval $[1, 2]$.

Bibliography

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