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BASIC ANALOG CIRCUITS FOR TWO-DIMENSIONAL DISTRIBUTED ELASTIC STRUCTURES

by R. L. Barnoski

Prepared by
MEASUREMENT ANALYSIS CORPORATION
Los Angeles, Calif.
for George C. Marshall Space Flight Center

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1. INTRODUCTION

Strictly speaking, the analog circuits derived in this discussion describe the static elastic behavior for various distributed elastic structures. Linear elasticity theory and strain energy procedures form the basis of the derivations as depicted by the flow diagram of Figure 1. The strain energy is expressed in terms of the displacements (and their spatial derivatives) although equivalent circuits can be developed by expressing the elastic energy in terms of forces and moments.

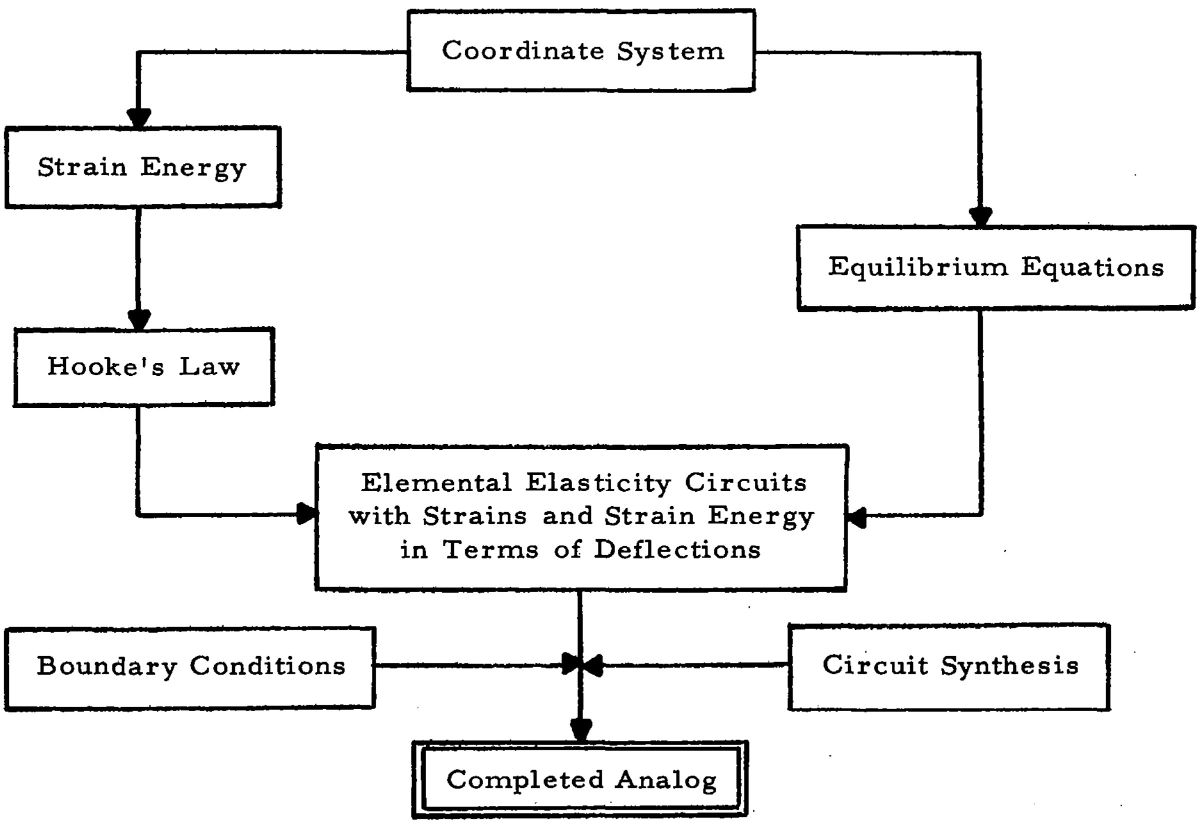


Figure 1. Strain Energy-Compatibility Approach for a Passive Electrical Analog of an Elastic Structural System

By defining the displacements and their spatial derivatives as voltages, the strain energy can be expressed such that each of the individual terms correspond electrically to power dissipation across resistors. Transformers are then used to construct the spatial geometry described by the displacements and their spatial derivatives while resistors are added to account for the strain energy. The magnitudes of the resistors are inversely proportional to the magnitudes of the coefficients in the strain energy expression and the current flows through these resistors are proportional to internal forces or moments. These circuits are force-current, displacement-voltage analogs or static-mobility analogs. In the discussions which follow, static-mobility analogs are derived for (1) a flat rectangular plate, (2) a flat circular plate and (3) a cylindrical shell.

Such "static" circuits can be readily converted to describe the dynamic behavior of a specific elastic structure (References 1 and 3). The "dynamic" circuit is called a mobility analog and is obtained from the "static" circuits in the following way:

1. the voltages are redefined as spatial velocities
2. inductors are substituted for resistors
3. capacitors are added at appropriate nodes .

The inductors account for the strain energy while the capacitors account for inertial forces associated with lumped masses. The transformer interconnections (which, in general, are the most difficult tasks in the analog development) remain unchanged. Thus, knowing the analog circuit for the static elastic behavior of a specific distributed structure, the dynamic circuit (or mobility analog) is ensured.

2. ELASTIC BEHAVIOR OF A FLAT RECTANGULAR PLATE

The physical system is shown in Figure 2 as a flat rectangular differential plate segment of dimensions dx, dy and of thickness h. The bending moments per unit length are denoted as M_{xx} and M_{yy} while the twisting moments per unit length are shown by M_{xy} and M_{yx}. The shear forces per unit length are shown as Q_x and Q_y. Deflections are assumed to be small in comparison to the plate thickness and strain in the middle plane of the plate is assumed negligible.

The equation of motion can be written as

$$D\nabla^4(w) + m\ddot{w} = 0 \tag{2.1}$$

where m is the mass per unit area, w the lateral deflection from the static equilibrium position, ν the Poisson's ratio and

$$\ddot{(\ )} = \frac{d^2(\ )}{dt^2}$$

$$D = \frac{Eh^3}{12(1 - \nu^2)} \tag{2.2}$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$$

The coefficient D is the flexural rigidity of the plate, E is Young's modulus, and ∇ denotes the dell operator which is used in this section to define the spatial derivatives for rectangular geometry.

Expressed as functions of the strains (Reference 4, page 46), the strain energy per unit **area** for a differential segment of rectangular plate in bending becomes

$$2V_0 = D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^4 w}{\partial x^2\, \partial y^2}\right] + 2D(1 - \nu)\left[\frac{\partial^2 w}{\partial x\, \partial y}\right]^2 \qquad (2.3)$$

In alternate form, Eq. (2. 3) may be rewritten as

$$2V_0 = D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]^2 + D(1 - \nu^2)\left[\frac{\partial^2 w}{\partial y^2}\right]^2 + 2D(1 - \nu)\left[\frac{\partial^2 w}{\partial x\, \partial y}\right]^2 \qquad (2.4)$$

where the first two terms account for the strain energy in bending and the remaining term specifies the strain energy due to twisting.

Multiplying Eq. (2. 4) by the plate dimensions (Δx, Δy,) and expressing the spatial derivatives as spatial first order difference expressions, the strain energy for a difference segment of rectangular plate becomes

$$2V = D\frac{\Delta y}{\Delta x}\left[\Delta_x(\theta_x) + \nu \frac{\Delta x}{\Delta y}\, \Delta_y(\theta_y)\right]^2 + D\frac{\Delta x}{\Delta y}(1 - \nu^2)\left[\Delta_y(\theta_y)\right]^2$$

$$+ 2D\frac{\Delta y}{\Delta x}(1 - \nu)\left[\Delta_x(\theta_y)\right]^2 \qquad (2.5)$$

where the slopes in bending are defined by

$$\theta_x = \frac{\partial w}{\partial x} \qquad \theta_y = \frac{\partial w}{\partial y} \tag{2.6}$$

and the difference operator $\Delta_j(\ )$ denotes the first order difference of the quantity ( ) with respect to the jth direction. By interpreting the bracketed terms as voltages, Eq. (2.5) is expressed in terms of electrical power dissipation by

$$2V = P = \sum_{j=1}^{3} \frac{E_j^2}{R_j} \tag{2.7}$$

where E_j is the voltage drop across the R_j resistor. In expanded form, Eq. (2.7) becomes

$$2V = P = \frac{E_1^2}{R_1} + \frac{E_2^2}{R_2} + \frac{E_3^2}{R_3} \tag{2.8}$$

where the voltages are defined by

$$\begin{aligned} E_1 &= \Delta_x(\theta_x) + \nu \frac{\Delta x}{\Delta y} \Delta_y(\theta_y) \\ E_2 &= \Delta_y(\theta_y) \\ E_3 &= \Delta_x(\theta_y) \end{aligned} \tag{2.9}$$

and the resistors are of magnitudes

$$R_1 = \frac{\Delta x}{\Delta y} \frac{1}{D}$$

$$R_2 = \frac{\Delta y}{\Delta x} \frac{1}{D(1 - \nu^2)} \tag{2.10}$$

$$R_3 = \frac{\Delta x}{\Delta y} \frac{1}{2D(1 - \nu)}$$

By constructing a two dimensional rectangular grid in the coordinates x and y, the voltages can be formed electrically to produce the circuits of Figures 2. These circuits are electrically equivalent to the strain energy given by Eq. (2. 5) with the voltages equivalent to lateral and angular displacements and the currents equivalent to shear forces and moments. The finite difference grid shows nine rectangular plate segments where the x-difference positions are given by capital letters and the y-difference positions as numbers. Thus, the numerical difference between two consecutive letters is the difference length Δx whereas the difference between two consecutive digits is the difference length Δy. The positive signs indicate the transformer polarity and define the manner in which the transformers must be interconnected to form the proper spatial geometry.

Although sketched as three distinct circuits: (1) the lateral deflection circuit, (2) the θ_x slope circuit, and (3) the θ_y circuit, these circuits are magnetically coupled by the transformers. Transformers 2 and 3 couple the lateral deflection with the θ_x and θ_y slope circuits, respectively. Transformer 1 accounts for the Poisson coupling in the first bracketed term of Eq. (2.4) by constraining the θ_x and θ_y circuits. Resistors R_1 and R_2 account for the bending strain energy while the R_3 resistor accounts for the twisting strain energy. To include the effects of lateral loading, current generators are added to the nodes in the deflection circuit with the current input being equivalent to the magnitude of the lateral loading acting over a difference segment of plate.

Since the magnitudes and directions of forces and moments are required to calculate stresses in various sections of the plate, it is necessary to know the mechanical equivalents of the currents through the resistors. The relationships between the moments and curvature are (Section 21 of Reference 4)

$$M_{xx} = -D\left[\frac{\partial(\theta_x)}{\partial x} + \nu \frac{\partial(\theta_y)}{\partial y}\right]$$

$$M_{yy} = -D\left[\frac{\partial(\theta_y)}{\partial y} + \nu \frac{\partial(\theta_x)}{\partial x}\right] \tag{2.11}$$

$$M_{xy} = -M_{yx} = D(1 - \nu)\frac{\partial(\theta_y)}{\partial x}$$

where $\frac{\partial^2 w}{\partial x\, \partial y}$ is arbitrarily expressed as a partial derivative of the slope θ_y.

Consider the calculation of current flow through R_2. To form the E_2 voltage from the expressions in (2.11), the M_{xx} bending moment is multiplied by Poisson's ratio, then subtracted from M_{yy} producing the result

$$-(M_{yy} + \nu M_{xx}) = D(1 - \nu^2)\left[\frac{\partial^2 w}{\partial y^2}\right] \tag{2.12}$$

As a difference equation, Eq. (2.12) appears as

$$- \Delta x (M_{yy} - \nu M_{xx}) = \frac{\Delta x}{\Delta y} D(1 - \nu^2) \Delta_y(\theta_y) \tag{2.13}$$

In form, Eq. (2.13) is similar to Ohm's law for current flow through a resistor

$$I_j = \frac{E_j}{R_j} \tag{2.14}$$

where

$$E_2 = \Delta_y(\theta_y)$$

$$R_2 = \frac{\Delta y}{\Delta x} \cdot \frac{1}{D(1 - \nu^2)} \tag{2.15}$$

$$I_2 \equiv I(R_2) = - \Delta x (M_{yy} - \nu M_{xx})$$

In a similar manner, current flows through R_1 and R_3 are determined as

$$I(R_1) = - M_{xx} \Delta y$$

$$I(R_3) = (M_{xy} - M_{yx}) \Delta y \tag{2.16}$$

By convention Eq. (2.14) implies positive current flows through the resistors; i.e. flows from a higher potential to a lower potential.

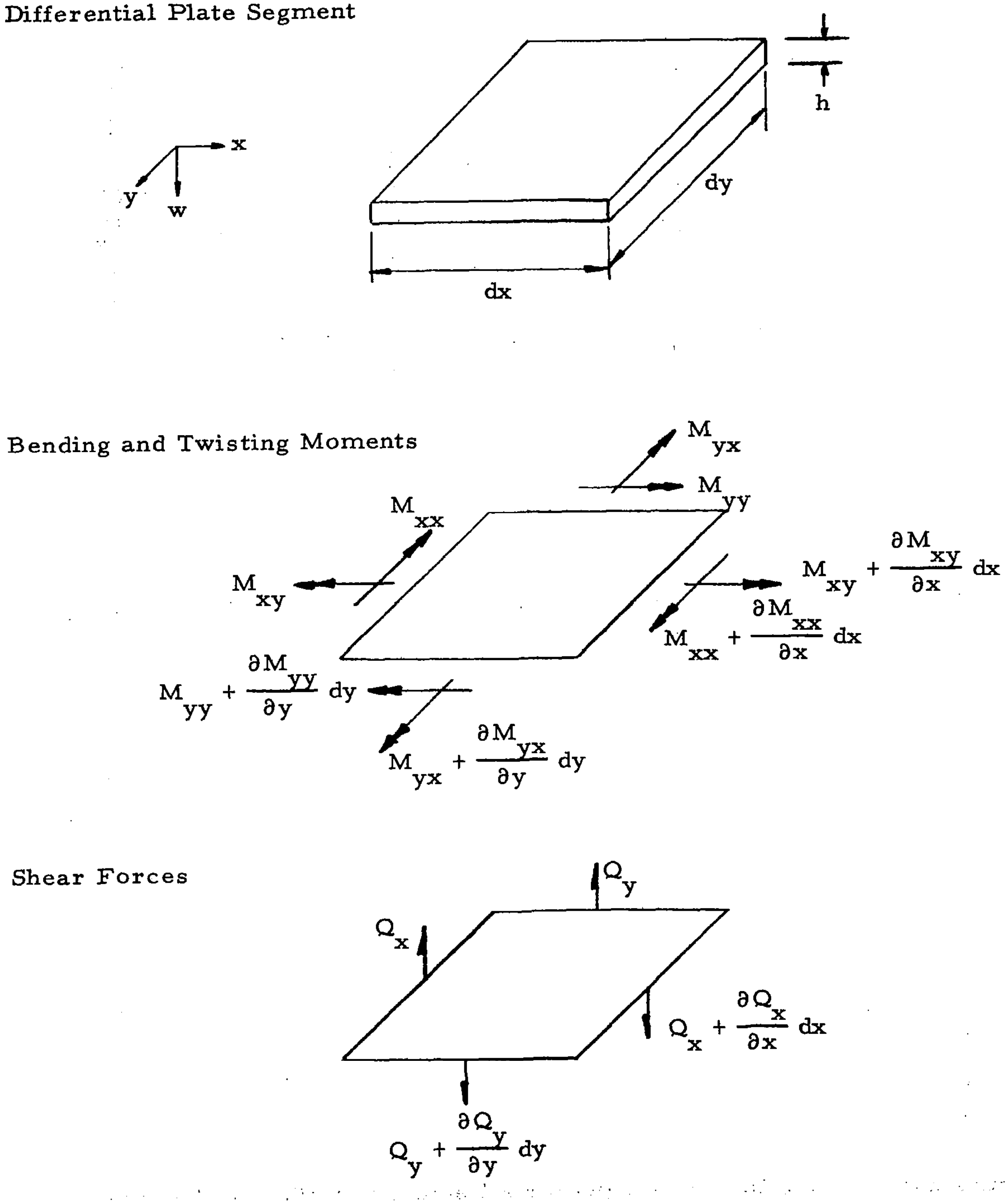


Figure 2. Differential Segment of a Rectangular Plate in Bending

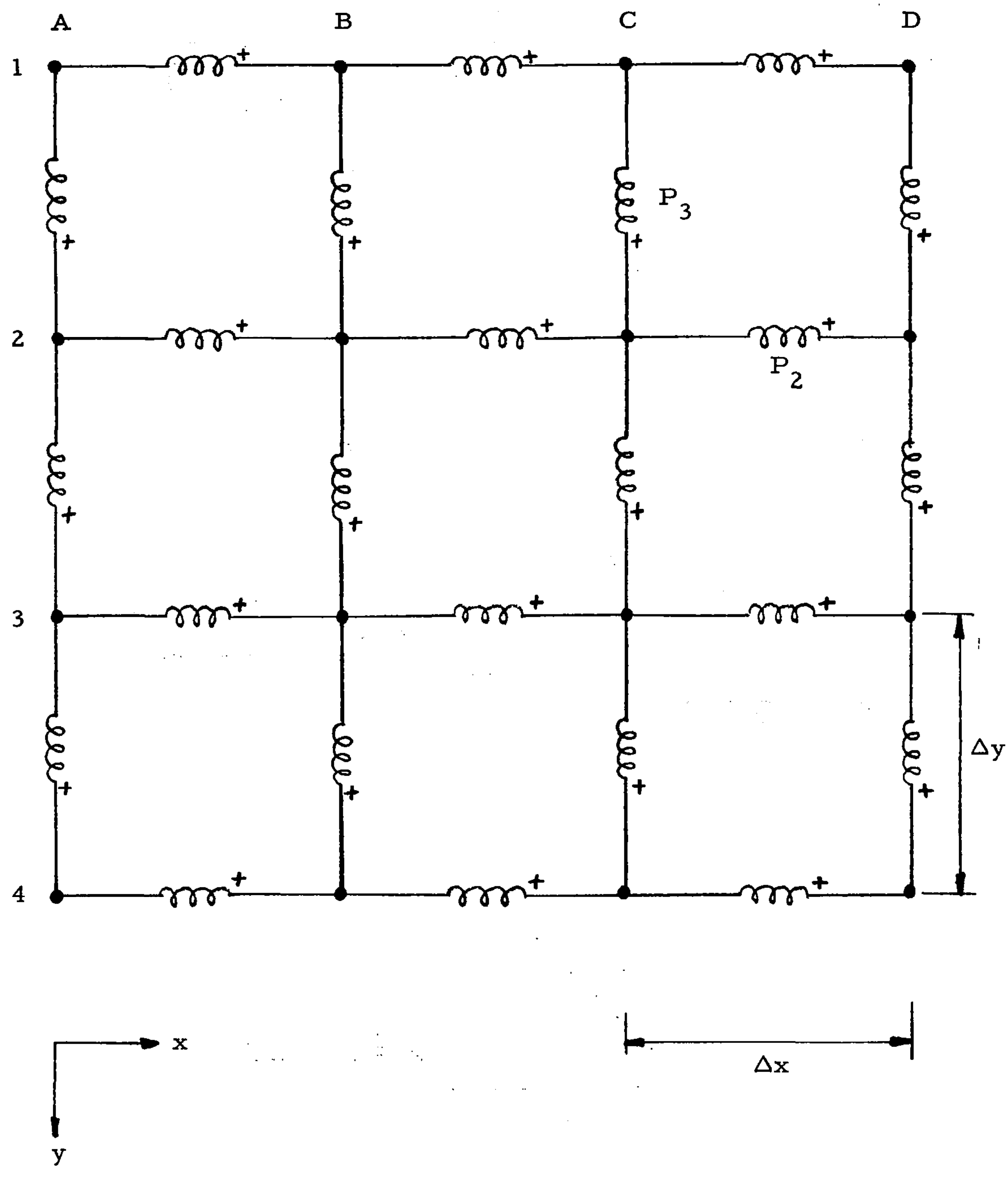


Figure 2-a. Deflection Circuit (w) for a Rectangular Plate in Bending Assuming Small Deflection Theory

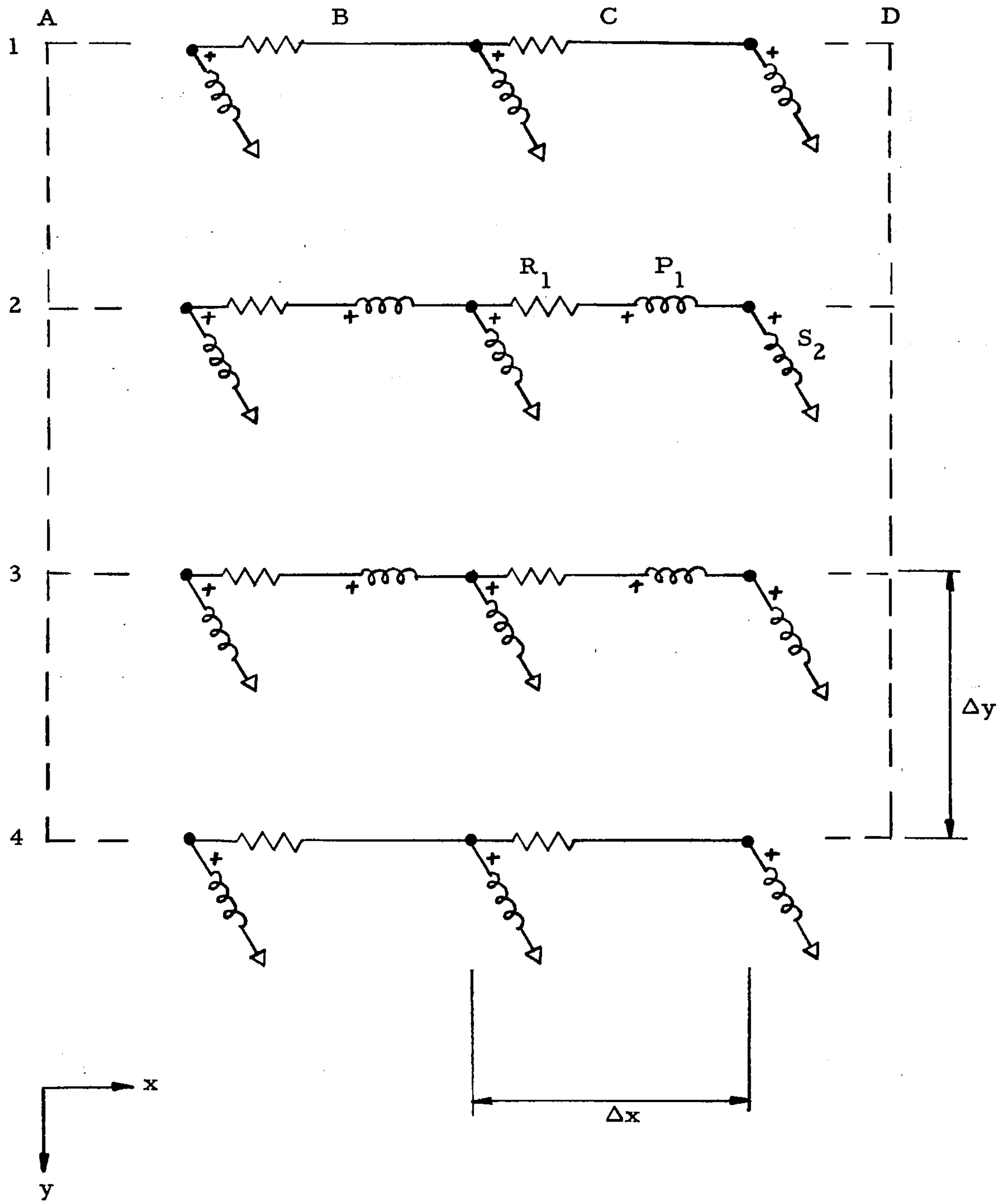


Figure 2-b. θ_x Circuit for a Rectangular Plate in Bending Assuming Small Deflection Theory

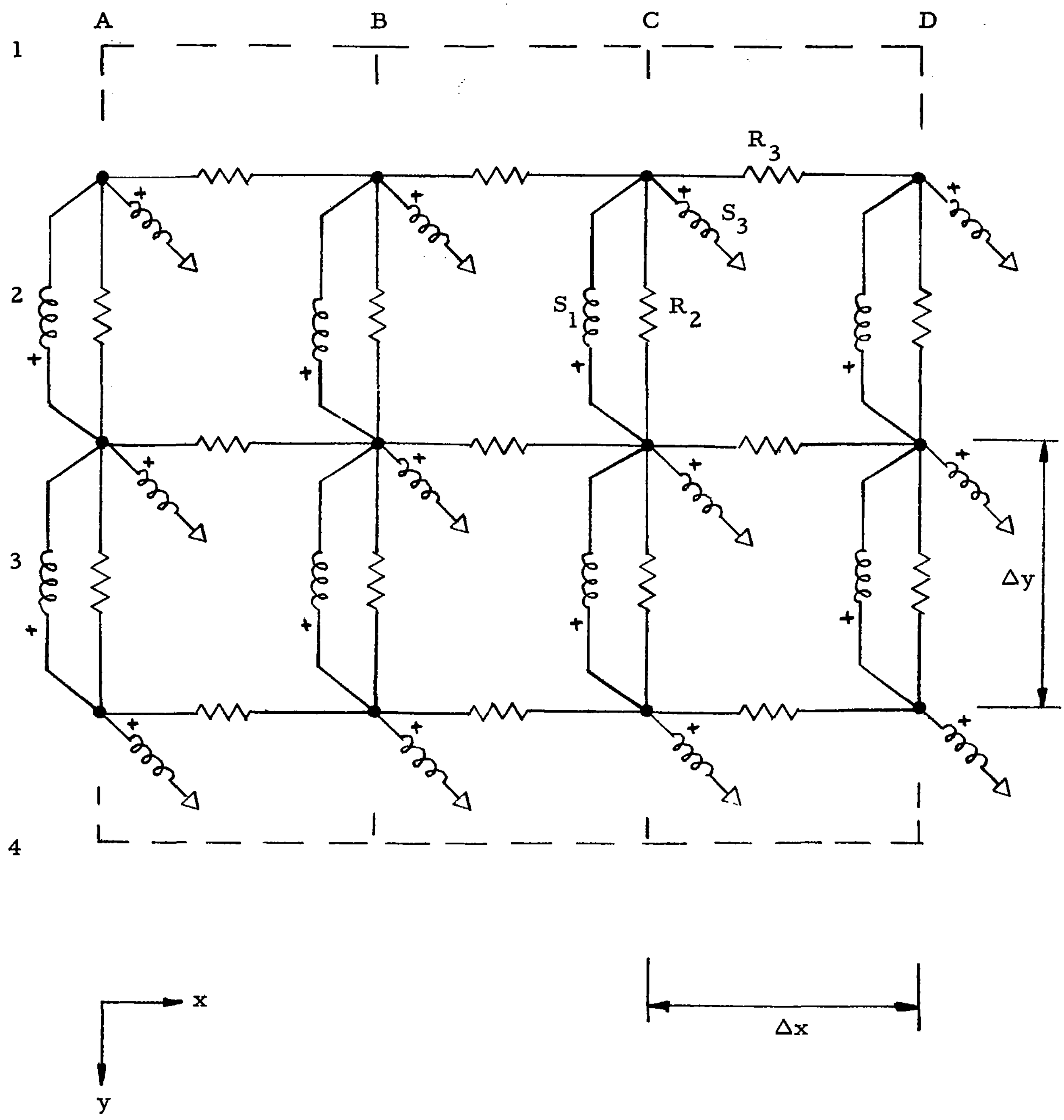


Figure 2-c. θ_y Circuit for a Rectangular Plate in Bending Assuming Small Deflection Theory

Circuit Elements	
Resistors	**Transformers**
$R_1 = \frac{\Delta x}{\Delta y} \cdot \frac{1}{D}$	$\frac{P_1}{S_1} = \nu \frac{\Delta x}{\Delta y}$
$R_2 = \frac{\Delta y}{\Delta x} \cdot \frac{1}{D(1 - \nu^2)}$	$\frac{P_2}{S_2} = \Delta x$
$R_3 = \frac{\Delta x}{\Delta y} \cdot \frac{1}{2D(1 - \nu)}$	$\frac{P_3}{S_3} = \Delta y$

$D = \frac{Eh^3}{12(1 - \nu^2)}$ = flexural rigidity of the plate

E = Young's modulus

h = thickness of the plate

ν = Poisson's ratio

Δx = incremental x distance of the plate grid

Δy = incremental y distance of the plate grid

Figure 2-d. Element Values of the Circuits for a Rectangular Plate in Bending

3. LATERAL ELASTIC BEHAVIOR OF A FLAT CIRCULAR PLATE

The structure considered here is of two-dimensions and is described in terms of cylindrical coordinates as contrasted with the rectilinear coordinates for the rectangular plate. Linear elastic behavior is assumed so that the deflections are considered small compared with the thickness of the plate.

A differential segment of thickness h for a circular plate and the accompanying moments and shear forces is shown as Figure 3. The bending moments per unit length are given as M_r and M_ϕ, the twisting moments per unit length by $M_{r\phi}$ and $M_{\phi r}$, and the shear forces per unit length by Q_r and Q_ϕ.

The equation of motion for the lateral vibration of a flat circular plate is

$$D\nabla^4 w + m\ddot{w} = 0 \tag{3.1}$$

where the spatial operator for cylindrical coordinates appears as

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)^2 \tag{3.2}$$

and w denotes the lateral deflection (in the z direction) from the static equilibrium position of the plate.

In form, Eq. (3.1) is similar to Eq. (2.1); and differs only in the spatial geometry defined by the respective dell operator.

The strain energy for a differential section of circular plate may be expressed symbolically as

$$2V = 2V_b + 2V_s \tag{3.3}$$

where V_b is the strain energy in bending and V_s is the shear strain energy. In terms of the curvilinear deflections (and derivatives), the finite difference form of (3.3) becomes

$$\begin{aligned} 2V_b = {} & \frac{D\nu^2 \Delta r}{r^3 \Delta\phi}\left[\frac{r^2\Delta\phi}{\nu\Delta r}\Delta_r(\theta_r) + r\Delta\phi\,\theta_r + \Delta_\phi(\theta_\phi)\right]^2 \\ & + \frac{D\Delta r}{r^3\Delta\phi}\left[r\Delta\phi\,\theta_r + \frac{\nu r^2\Delta\phi}{\Delta r}\Delta_r(\theta_r) + \Delta_\phi(\theta_\phi)\right]^2 \\ & + \frac{D\Delta r\Delta\phi}{r^3}\left[\frac{r}{\Delta r}\Delta_r(\theta_\phi) - \theta_\phi\right]^2 \end{aligned} \tag{3.4}$$

and

$$2V_s = \frac{D\Delta\phi}{r\Delta r}\left[\Delta_r\left\{\frac{r}{\Delta r}\Delta_r(\theta_r) + \theta_r + \frac{1}{r\Delta\phi}\Delta_\phi(\theta_\phi)\right\}\right]^2$$

$$+ \frac{D\Delta r}{r^3\Delta\phi}\left[\Delta_\phi\left\{\frac{r}{\Delta r}\Delta_r(\theta_r) + \theta_r + \frac{1}{r\Delta\phi}(\theta_\phi)\right\}\right]^2 \qquad (3.5)$$

where

$$\theta_r \equiv \frac{1}{\Delta r}\Delta_r(w)$$

$$\theta_\phi \equiv \frac{1}{\Delta\phi}\Delta_\phi(w) \qquad (3.6)$$

Consistent in form with energy dissipation by resistors, Eq. (3.3) appears as

$$2V = P = \sum_{j=1}^{5} \frac{E_j^2}{R_j} \qquad (3.7)$$

where the voltages across the resistors are defined by

$$E_1 = \frac{r^2\Delta\phi}{\nu\Delta r}\Delta_r(\theta_r) + r\Delta\phi\,\theta_r + \Delta_\phi(\theta_\phi)$$

$$E_2 = r\Delta\phi\,\theta_r + \frac{\nu r^2\Delta\phi}{\Delta r}\Delta_r(\theta_r) + \Delta_\phi(\theta_\phi)$$

$$E_3 = \frac{r}{\Delta r}\Delta_r(\theta_\phi) - \theta_\phi \tag{3.8}$$

$$E_4 = \Delta_r\left(\frac{r}{\Delta r}\Delta_r(\theta_r) + \theta_r + \frac{1}{r\Delta\phi}\Delta_\phi(\theta_\phi)\right)$$

$$E_5 = \Delta_\phi\left(\frac{r}{\Delta r}\Delta_r(\theta_r) + \theta_r + \frac{1}{r\Delta\phi}\Delta_\phi(\theta_\phi)\right)$$

and the associated resistors are of magnitude

$$R_1 = \frac{r^3\Delta\phi}{\nu^2\Delta r}\cdot\frac{1}{D}$$

$$R_2 = \frac{r^3\Delta\phi}{\Delta r}\cdot\frac{1}{D}$$

$$R_3 = \frac{r^3}{\Delta r\,\Delta\phi}\cdot\frac{1}{D} \tag{3.9}$$

$$R_4 = \frac{r\Delta r}{\Delta\phi}\cdot\frac{1}{D}$$

$$R_5 = \frac{r^3\Delta\phi}{\Delta r}\cdot\frac{1}{D}$$

The circuits displayed as Figures 3 are electrically equivalent to the strain energy given by Eq. (3.3). The difference grid is described using an r-ϕ plane where the ϕ difference positions are noted by capital letters and the r difference positions by numbers. The nodal voltages in the θ_r circuits correspond to the slope θ_r, the nodal voltages in the θ_ϕ circuits correspond to the slope θ_ϕ, and the nodal voltages in the Q circuit correspond to the quantity

$$\frac{r}{\Delta r} \Delta_r(\theta_r) + \theta_r + \frac{r}{\Delta\phi} \Delta_\phi(\theta_\phi) \quad .$$

The transformers couple the various circuits and serve only to form the required geometry; and the resistors R_1, R_2 and R_3 account for the strain energy in bending while R_4 and R_5 account for the shear strain energy.

To determine the magnitude of positive circuit flows through the various resistors, the relationships between shear forces, moments and curvature are required. In finite difference form, these relationships are

$$M_r = \frac{D\nu}{r^2\Delta\phi}\left[\frac{r^2\Delta\phi}{\nu\Delta r}\Delta_r(\theta_r) + r\Delta\phi\,\theta_r + \Delta_\phi(\theta_\phi)\right]$$

$$M_\phi = \frac{D}{r^2\Delta\phi}\left[\frac{\nu r^2\Delta\phi}{\Delta r}\Delta_r(\theta_r) + r\Delta\phi\,\theta_r + \Delta_\phi(\theta_\phi)\right]$$

$$M_{r\phi} = M_{r\phi} = \frac{D(1-\nu)}{r^2}\left[\frac{r}{\Delta r}\Delta_r(\theta_\phi) - \theta_\phi\right] \qquad (3.10)$$

$$Q_r = \frac{D}{r\Delta r}\Delta_r\left[\frac{r}{\Delta r}\Delta_r(\theta_r) + \theta_r + \frac{1}{r\Delta\phi}\Delta_\phi(\theta_\phi)\right]$$

$$Q_\phi = \frac{D}{r^2\Delta\phi}\Delta_\phi\left[\frac{r}{\Delta r}\Delta_r(\theta_r) + \theta_r + \frac{1}{r\Delta\phi}\Delta_\phi(\theta_\phi)\right]$$

By comparing the form of Eqs. (3. 4) and (3. 5) with (3. 10), the positive currents are of values

$$I\,(R_1) = \frac{\nu \Delta r}{r} M_r$$

$$I\,(R_2) = \frac{\Delta r}{r} M_\phi$$

$$I\,(R_3) = \frac{\Delta r\, \Delta \phi}{r\,(1 - \nu)} M_{r\phi} \tag{3.11}$$

$$I\,(R_4) = \Delta \phi\, Q_r$$

$$I\,(R_5) = \frac{\Delta r}{r} Q_\phi$$

By assuming axial symmetry, the spatial derivatives become independent of ϕ and no deflections in the ϕ dimension are permitted. For this condition, the strain energy terms of (3. 4) and (3. 5) reduce to

$$2V_b = \frac{D\nu^2 \Delta r \Delta \phi}{r} \left[\frac{r}{\nu \Delta r} \Delta_r(\theta_r) + \theta_r \right]^2$$

$$+ \frac{D \Delta r \Delta \phi}{r} \left[\frac{\nu r}{\Delta r} \Delta_r(\theta_r) + \theta_r \right]^2 \tag{3.12}$$

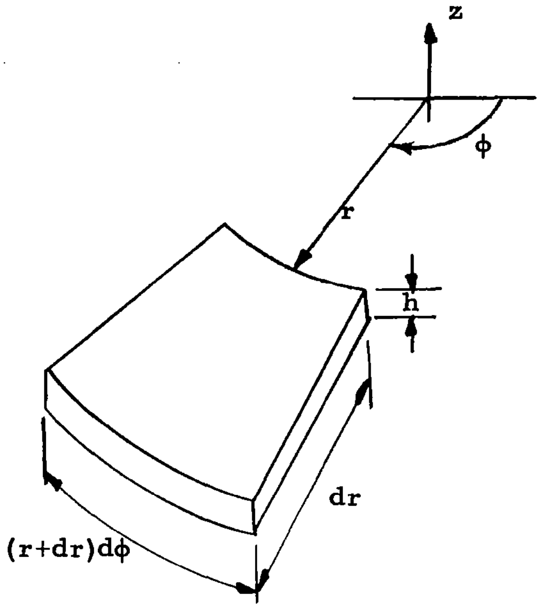
$$2V_s = \frac{D\Delta\phi}{r\Delta r}\left[\Delta_r\left(\frac{r}{\Delta r}\,\Delta_r(\theta_r) + \theta_r\right)\right]^2 \tag{3.13}$$

and circuits simulating these equations are shown as Figure 4. From the stress-strain relationships given by Eq. (3.10), positive current flows through the resistors are

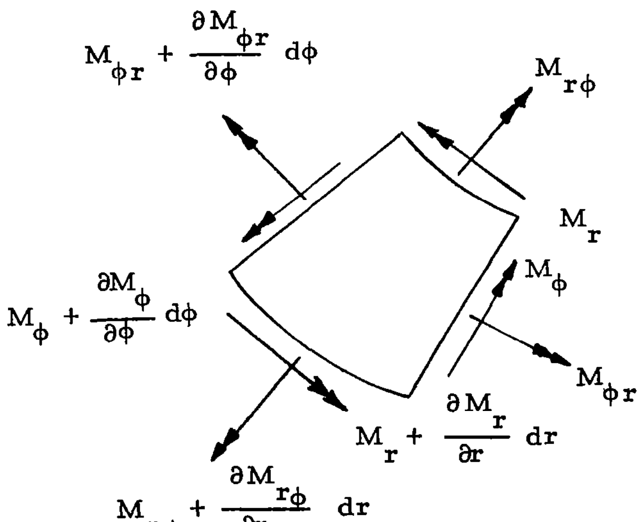
$$I(R_1) = \nu\Delta r\,\Delta\phi\,M_r$$

$$I(R_2) = \Delta r\,\Delta\phi\,M_\phi \tag{3.14}$$

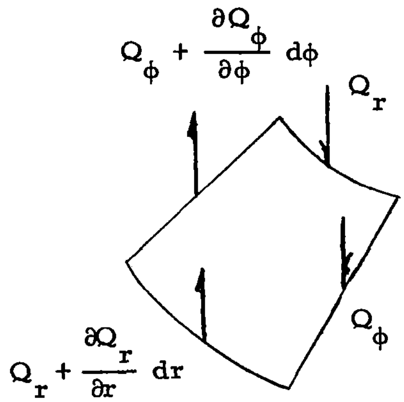
$$I(R_3) = \Delta\phi\,Q_r$$



Differential Segment



Bending and Twisting Moments



Shear Forces

Figure 3. Differential Segment of a Circular Plate and the Associated Forces

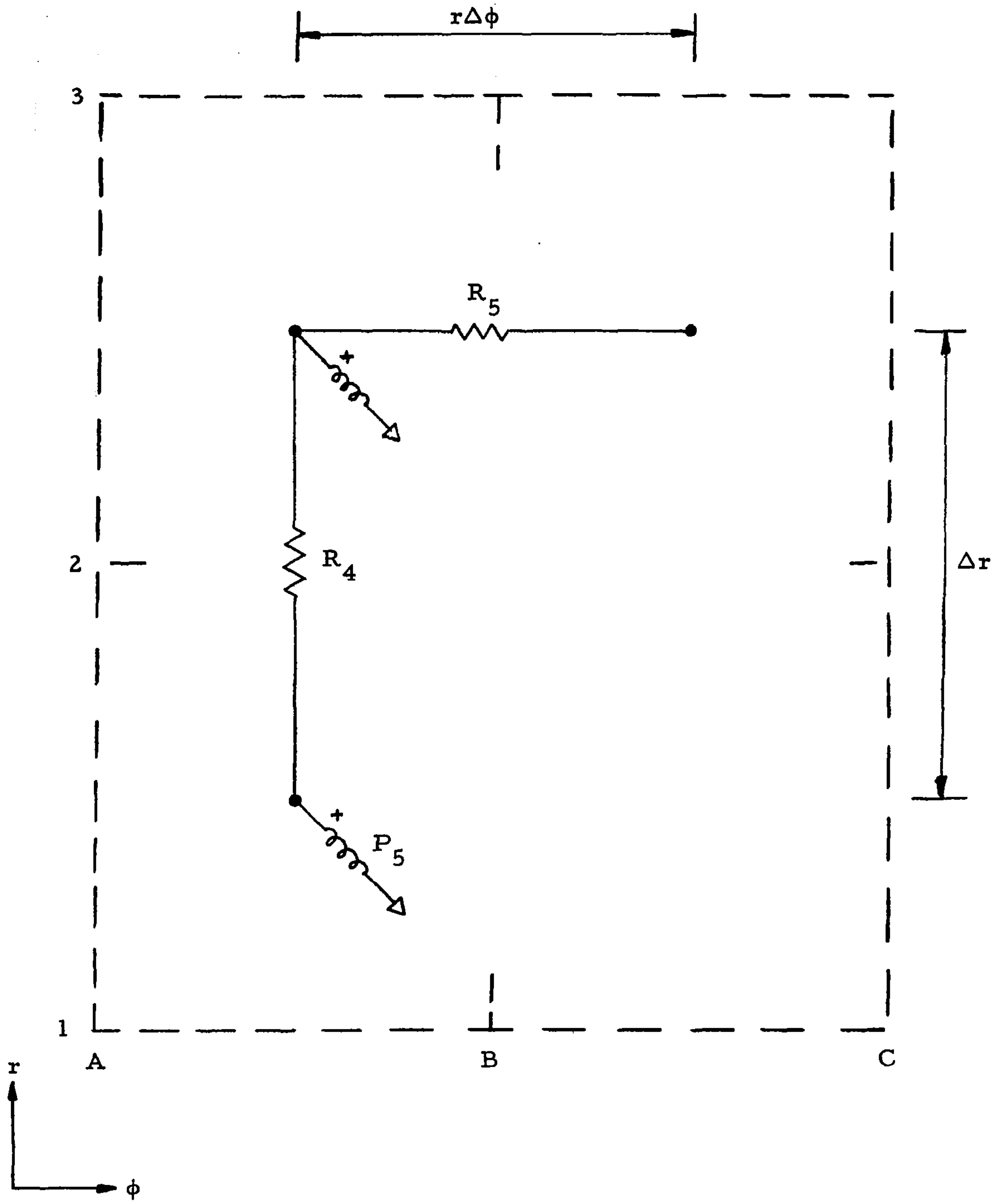
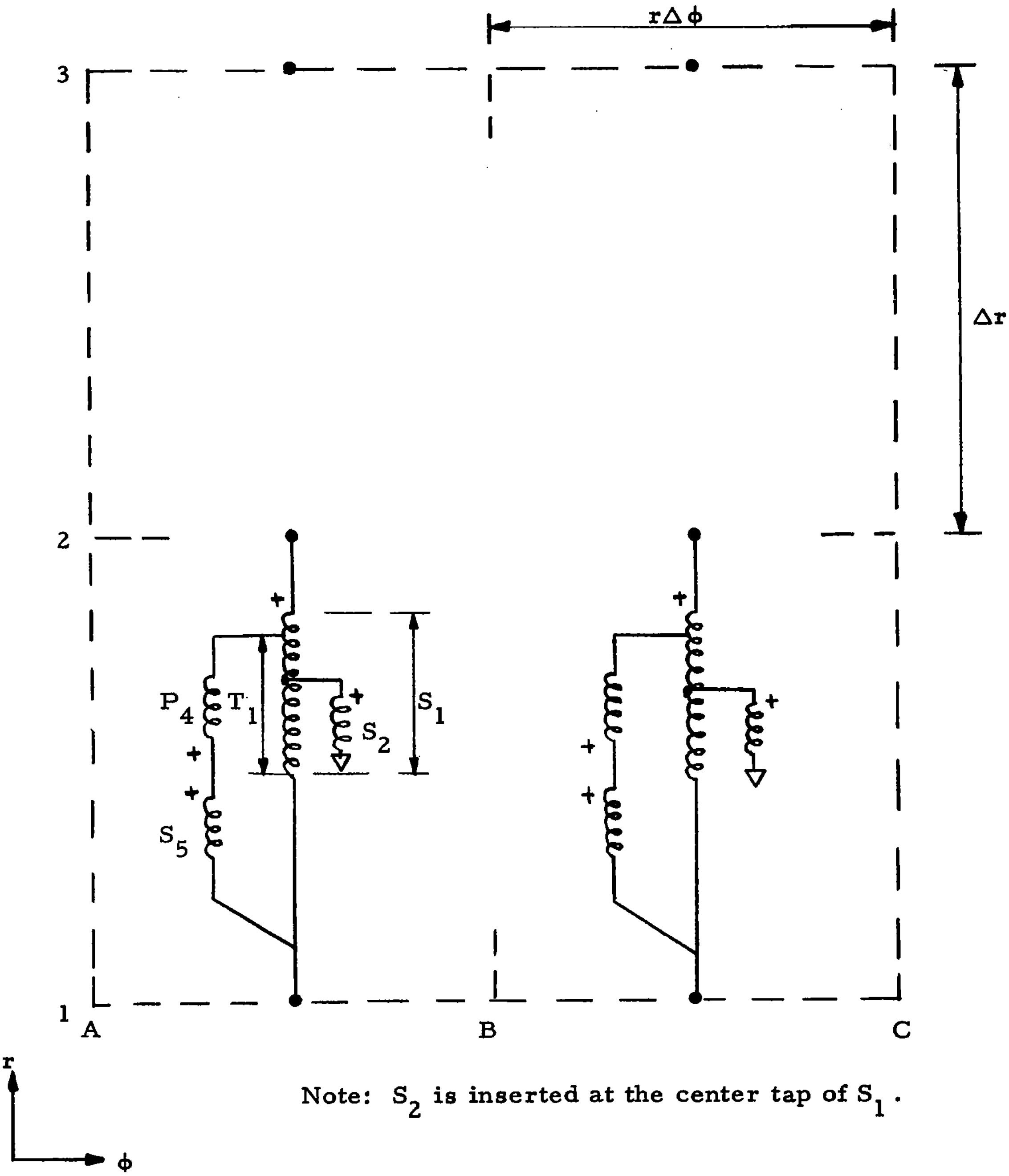


Figure 3-a. Q Circuit for an Arbitrary Section of Flat Circular Plate



Note: S_2 is inserted at the center tap of S_1.

Figure 3-b. θ_r Circuit for an Arbitrary Section of Flat Circular Plate

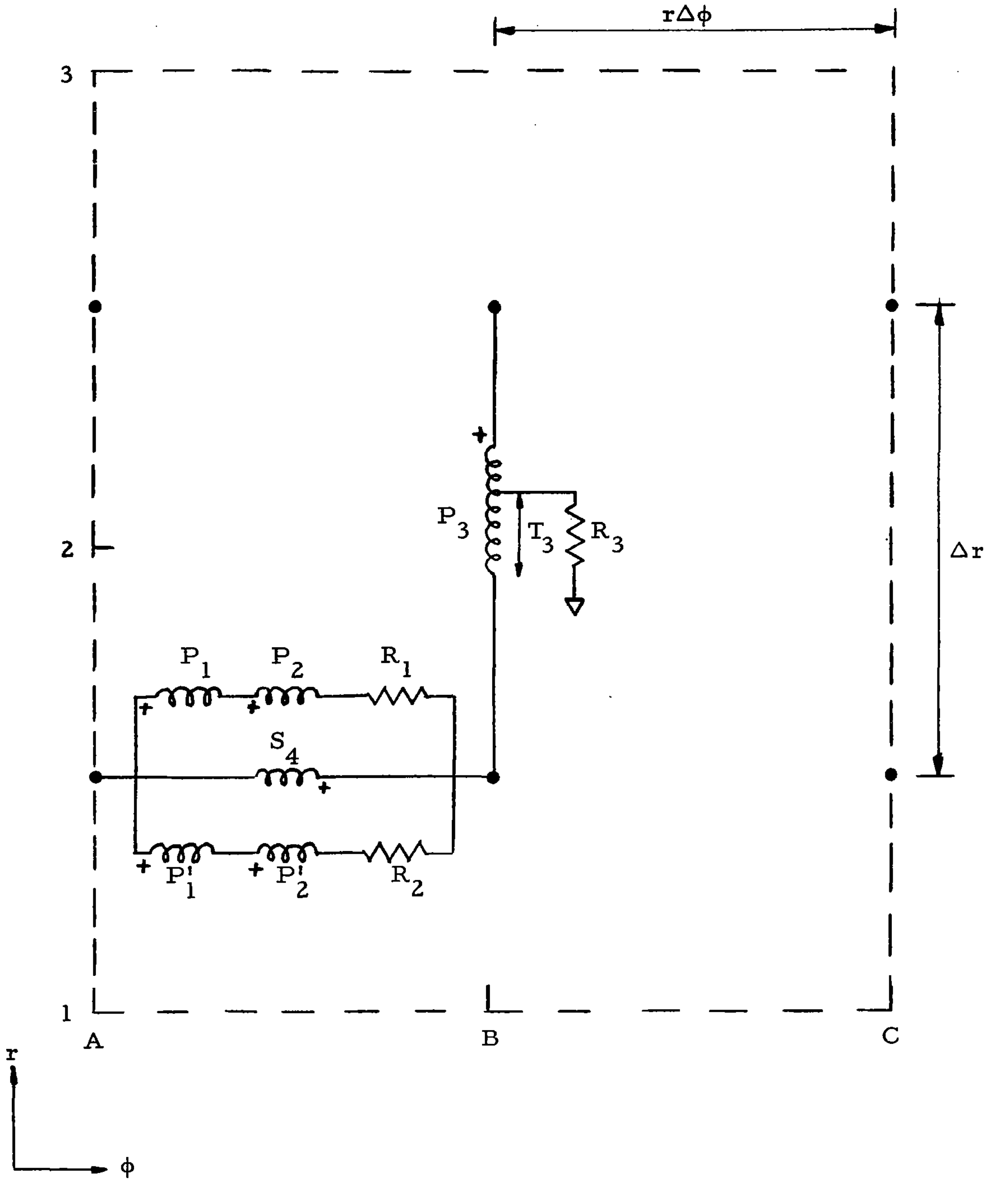


Figure 3-c. θ_ϕ Circuit for an Arbitrary Section of Flat Circular Plate

Circuit Elements	
Resistors	**Transformers**
$R_1 = \frac{r^3 \Delta\phi}{\Delta r} \cdot \frac{1}{D\nu^2}$	$\frac{T_1}{S_1} = \frac{r}{\Delta r} + \frac{1}{2}$
$R_2 = \frac{r^3 \Delta\phi}{\Delta r} \cdot \frac{1}{D}$	$\frac{P_1}{S_1} = \frac{r^2 \Delta\phi}{\nu \Delta r}$; $\frac{P'_1}{S_1} = \frac{\nu r^2 \Delta\phi}{\Delta r}$
$R_3 = \frac{r^3}{\Delta r \Delta\phi} \cdot \frac{1}{D}$	$\frac{P_2}{S_2} = \frac{P'_2}{S_2} = r\Delta\phi$
$R_4 = \frac{r\Delta r}{\Delta\phi} \cdot \frac{1}{D}$	$\frac{T_3}{S_3} = \frac{r}{\Delta r} - \frac{1}{2}$
$R_5 = \frac{r^3 \Delta\phi}{\Delta r} \cdot \frac{1}{D}$	$\frac{P_5}{S_5} = 1$; $\frac{P_4}{S_4} = \frac{1}{r\Delta\phi}$

$$D = \frac{Eh^3}{12(1 - \nu^2)} = \text{flexural rigidity}$$

E = Young's modulus

h = plate thickness

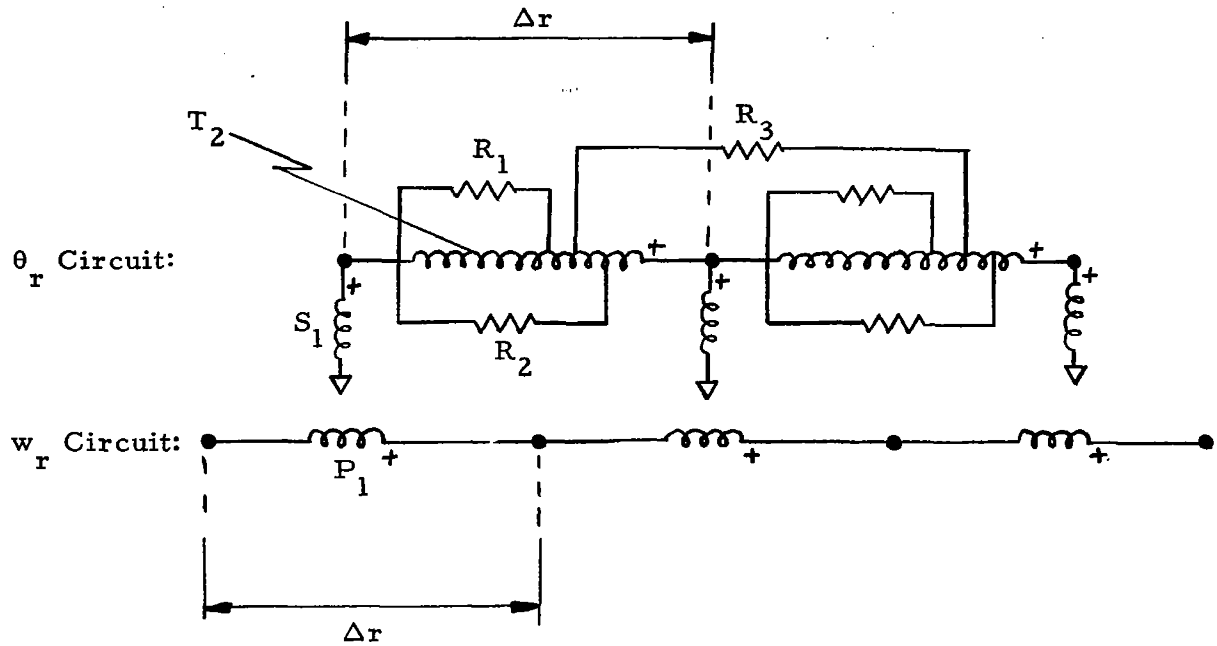
r = radial distance

ν = Poisson/s ratio

Δr = radial distance

$r\Delta\phi$ = angular distance

Figure 3-d. Circuit Element Values for an Arbitrary Section of Flat Circular Plate



Circuit Elements:

$$R_1 = \frac{r}{\nu^2 D \Delta r \Delta \phi}$$

$$\frac{P_1}{S_1} = \Delta r$$

$$R_2 = \frac{r}{D \Delta r \Delta \phi}$$

$$T_2 = \Delta r$$

$$R_3 = \frac{r \Delta r}{D \Delta \phi}$$

Note: T_2 is an autotransformer where R_1 is tapped at position $\frac{r}{\nu \Delta r} + \frac{1}{2}$; R_2 at position $\frac{\nu r}{\Delta r} + \frac{1}{2}$; R_3 at $\frac{r}{\Delta r} + \frac{1}{2}$

Figure 4. Circuit for the Elastic Behavior of a Circular Plate with Axial Symmetry

4. ELASTIC BEHAVIOR OF A CYLINDRICAL SHELL

Based upon a general theory of circular cylindrical shells (Reference 4, pg 342), the equations for the elastic behavior appear as

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x \partial \phi} - \frac{v}{a}\frac{\partial w}{\partial x} + \frac{p_1(1-\nu^2)}{Eh} = 0$$

$$\frac{1+\nu}{2a}\frac{\partial^2 u}{\partial x \partial \phi} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} - \frac{1}{a^2}\frac{\partial w}{\partial \phi}$$

$$+ \frac{h^2}{12a^2}\frac{\partial^3 w}{\partial x^2 \partial \phi} + \frac{\partial^3 w}{a^2 \partial \phi^3} + \frac{h^2}{12a^2}\left(\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{a^2 \partial \phi^2}\right) + \frac{p_2(1-\nu^2)}{Eh} = 0$$

(4. 1)

$$\nu\frac{\partial u}{\partial x} + \frac{\partial v}{a \partial \phi} - \frac{w}{a} - \frac{h^2}{12}\left(a\frac{\partial^4 w}{\partial x^4} + \frac{2}{a}\frac{\partial^4 w}{\partial x^2 \partial \phi^2} + \frac{\partial^4 w}{a^3 \partial \phi^4}\right)$$

$$- \frac{h^2}{12}\left(\frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \phi} + \frac{1}{a^3}\frac{\partial^3 v}{\nu \phi^3}\right) + \frac{a p_3(1-\nu^2)}{Eh} = 0$$

where u, v and w denote deflections in the x, y and z directions, a the radius of the cylindrical shell, p_1, p_2 and p_3 external pressure loadings directed in the x, ϕ and radial directions, h the shell thickness, ν Poisson's ratio and E Young's modulus. By adding the proper inertial force (i. e. mass times a second time derivative to each of the above equations), Eqs. (4. 1) can be resolved to equations of motion for a differential section of cylindrical shell.

The total strain energy for a differential section of cylindrical shell is expressed as

$$2V = 2V_m + 2V_b \tag{4.2}$$

where V_m is the strain energy due to membrane action and V_b is the strain energy due to bending. In expanded differential form

$$\begin{aligned} 2V_m &= [N_x \epsilon_x + N_\phi \epsilon_\phi + N_{x\phi} \gamma_{x\phi}] \; a \, d\phi \, dx \\ 2V_b &= [M_x \chi_x + M_\phi \chi_\phi + M_{x\phi} \chi_{x\phi}] \; a \, d\phi \, dx \end{aligned} \tag{4.3}$$

where N_x and N_ϕ are extensional forces per unit length, $N_{x\phi}$ the shear force per unit length, M_x and M_ϕ the bending moments per unit length and $M_{x\phi}$ the twisting moment per unit length. The strains associated with the forces and moments are shown as ϵ_x, ϵ_ϕ, $\gamma_{x\phi}$, χ_x, χ_ϕ and $\chi_{x\phi}$ where the subscripts relate the strains with the appropriate forces and moments.

By Hooke's law, the stress-strain relationships for the extensional and shear forces are

$$N_x = \frac{Eh}{1 - \nu^2}\left[\epsilon_x + \nu\epsilon_\phi\right]$$

$$N_\phi = \frac{Eh}{1 - \nu^2}\left[\epsilon_\phi + \nu\epsilon_x\right] \tag{4.4}$$

$$N_{x\phi} = N_{\phi x} = \frac{Eh}{2(1 + \nu)}\gamma_{x\phi}$$

and for the bending and twisting moments

$$-M_x = D\left[\chi_x + \nu\chi_\phi\right]$$

$$-M_\phi = D\left[\chi_\phi + \nu\chi_x\right] \tag{4.5}$$

$$M_{x\phi} = M_{\phi x} = D(1 - \nu)\chi_{x\phi}$$

where D is the conventional rigidity

$$D = \frac{Eh^3}{12(1 - \nu^2)} \tag{4.6}$$

In terms of the shell deflections and derivatives, the strains appear as

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_\phi = \frac{1}{a}\left[\frac{\partial v}{\partial \phi} - w\right]$$

$$\gamma_{x\phi} = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \phi}$$

$$\chi_x = \frac{\partial^2 w}{\partial x^2} \qquad \chi_\phi = \frac{1}{a^2}\left[\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial v}{\partial \phi}\right] \tag{4.7}$$

$$\chi_{x\phi} = \frac{1}{a}\left[\frac{\partial^2 w}{\partial x \partial \phi} + \frac{1}{2}\frac{\partial v}{\partial x}\right]$$

By expressing the strain energy in terms of the strains, Eqs. (4.3) can be restated in the form

$$2V_m = \frac{Eh}{1 - \nu^2}\left[\epsilon_x + \nu\, \epsilon_\phi\right]^2 a\, d\phi\, dx + Eh\left[\epsilon_\phi\right]^2 a\, d\phi\, dx$$

$$+ \frac{Eh}{1 + \nu}\left[\gamma_{x\phi}\right]^2 a\, d\phi\, dx$$

$$2V_b = D\left[\chi_x + \nu\chi_\phi\right]^2 a\, d\phi\, dx + \frac{Eh^3}{12}\left[\chi_\phi\right]^2 a\, d\phi\, dx \tag{4.8}$$

$$+ \frac{Eh^3}{6\,(1 + \nu)}\left[\chi_{x\phi}\right]^2 a\, d\phi\, dx$$

In finite difference form, the strains appear as

$$\epsilon_x = \frac{\Delta_x(u)}{\Delta x} \qquad \epsilon_\phi = \frac{1}{a}\left[\frac{\Delta_\phi(v)}{\Delta\phi} - w\right]$$

$$\gamma_{x\phi} = \frac{1}{\Delta x}\left[\Delta_x(v) + \frac{\Delta x}{a\,\Delta\phi}\,\Delta_\phi(u)\right]$$

$$\chi_x = \frac{\Delta_x(\theta_x)}{\Delta x} \qquad \chi_\phi = \frac{1}{a^2\,\Delta\phi}\left[\Delta_\phi(\theta_\phi) + \Delta_\phi(v)\right] \tag{4.9}$$

$$\chi_{x\phi} = \frac{1}{a\,\Delta x}\left[\Delta_x(\theta_\phi) + \frac{1}{2}\,\Delta_x(v)\right]$$

By substituting (4. 9) into (4. 8) and assuming $\Delta\phi$ as unity, the strain energy expressions become

$$\begin{aligned} 2V_m = {} & \frac{Eha}{\Delta x\,(1 - \nu^2)}\left[\Delta_x(u) + \frac{\nu\Delta x}{a}\left(\Delta_\phi(v) - w\right)\right]^2 \\ & + \frac{Eh\,\Delta x}{a}\left[\Delta_\phi(v) - w\right]^2 \\ & + \frac{Eha}{\Delta x\,(1 + \nu)}\left[\Delta_x(v) + \frac{\Delta x}{a}\,\Delta_\phi(u)\right]^2 \end{aligned} \tag{4.10}$$

$$
2V_b = \frac{Da}{\Delta x}\left[\Delta_x(\theta_x) + \frac{\nu \Delta x}{a^2}\left(\Delta_\phi(\theta_\phi) + \Delta_\phi(v)\right)\right]^2
$$

$$
+ \frac{E\,\Delta x}{12}\left(\frac{h}{a}\right)^3\left[\Delta_\phi(\theta_\phi) + \Delta_\phi(v)\right]^2 \tag{4.11}
$$

$$
+ \frac{Eh^3}{6\,(1+\nu)\,a\,\Delta x}\left[\Delta_x(\theta_\phi) + \frac{1}{2}\,\Delta_x(v)\right]^2
$$

where the directional slopes are defined as

$$
\theta_x = \frac{\partial w}{\partial x} = \frac{\Delta_x(w)}{\Delta x}
$$

$$
\bar{\theta}_\phi = \frac{\partial w}{\partial \phi} = \frac{\Delta_\phi(w)}{\Delta \phi} \tag{4.12}
$$

In Eqs. (4.10) and (4.11), the $\Delta\phi$ term is assumed equal to one radian as a matter of convenience. The form of the strain energy expressions are equivalent to

$$
2V = P = \sum_{j=1}^{6} \frac{E_j^2}{R_j} \tag{4.13}
$$

where the spatial derivatives are defined as voltages and the reciprocal of the coefficients as resistors.

Figures 5 are electrically equivalent to Eqs. (4.10), (4.11) and (4.12) and consist of five distinct circuits (u, v, w, θ_x and θ_ϕ) coupled by the seven transformers. Compared with the membrane and bending strain energy expressions, the voltages across each of the resistors correspond to the spatial derivatives and the resistor magnitudes are the reciprocals of the coefficients. Resistors R_1 and R_2 account for the extensional strain energy, R_3 for the shear strain energy, R_4 and R_5 for the bending strain energy and R_6 for the twisting strain energy. The coordinate locations are consistent with difference geometry and shown as Figure 5. By the stress-strain relationships of Eqs. (4.4), (4.5) and (4.6) and the strain expressions of Eq. (4.7), positive current flows through the six resistors are

$$
\begin{aligned}
I(R_1) &= a N_x \\
I(R_2) &= \Delta x (N_\phi - \nu N_x) \\
I(R_3) &= N_{x\phi} \\
I(R_4) &= -a M_x \\
I(R_5) &= \frac{\Delta x}{a}(-M_\phi + \nu M_x) \\
I(R_6) &= 2 M_{x\phi}
\end{aligned}
\qquad (4.14)
$$

By assuming axial symmetry, the strain energy expression becomes very much simplified as the spatial derivatives become independent of ϕ and v equals zero. For this assumption, the strains reduce to

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_\phi = -\frac{w}{a}$$

$$\gamma_{x\phi} = 0 \tag{4.15}$$

$$\chi_x = \frac{\partial^2 w}{\partial x^2} \qquad \chi_\phi = 0$$

$$\chi_{x\phi} = 0 \tag{4.16}$$

since $\frac{\partial}{\partial \phi} = 0$. In finite-difference form, the strain energy given by Eqs. (4.8) resolve to

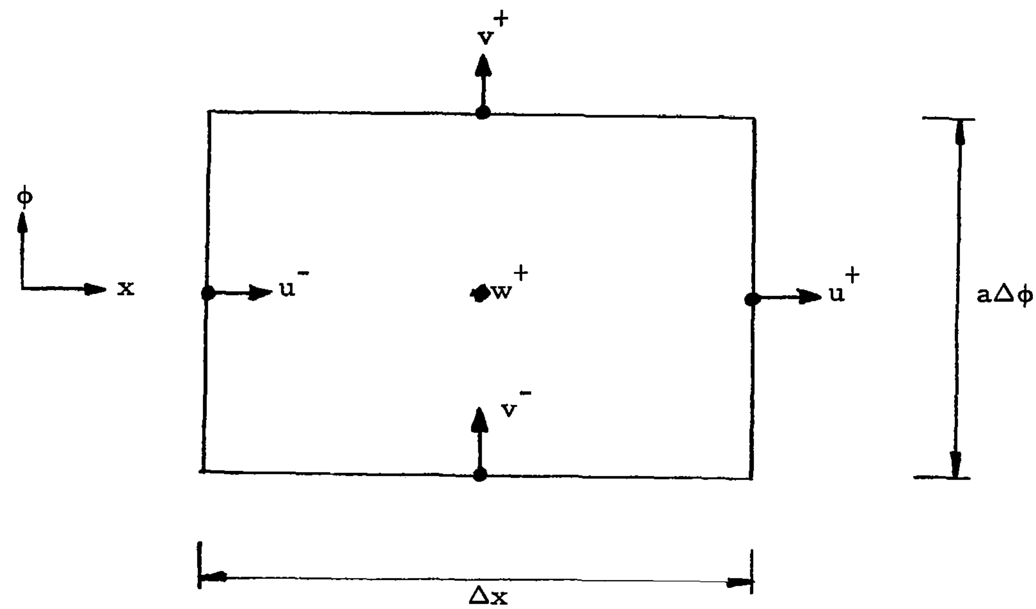
$$2V_m = \frac{Eha}{\Delta x(1 - \nu^2)}\left[\Delta_x(u) - \frac{\nu \Delta x}{a} w\right]^2 + \frac{Eh\Delta x}{a}\left[w\right]^2$$

$$2V_b = \frac{Da}{\Delta x}\left[\Delta_x(\theta_x)\right]^2 \tag{4.17}$$

An analog circuit simulating Eq. (4.17) is sketched as Figure 6. Three distinct circuits (u, w and θ_x) are shown for a difference segment of dimension Δx; and are coupled by transformers 1 and 2. Resistors R_1 and R_2 account for the strain energy in extension whereas R_3 accounts for the strain energy in bending. By the stress-strain relationships of Eqs. (4.4) and (4.5), the positive currents through the three resistors are

$$
\begin{aligned}
I(R_1) &= a N_x \\
- I(R_2) &= \Delta x (N_\phi - \nu N_x) \\
I(R_3) &= -a M_x
\end{aligned}
\tag{4.18}
$$

Difference Section:



Differential Section:

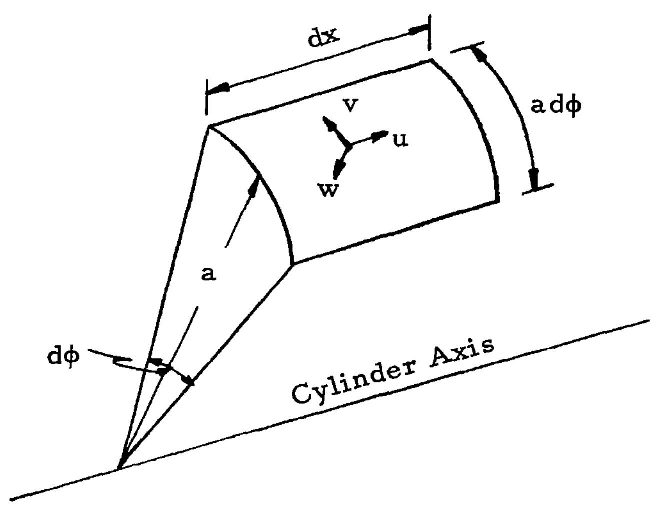


Figure 5. Geometry for a Cylindrical Shell

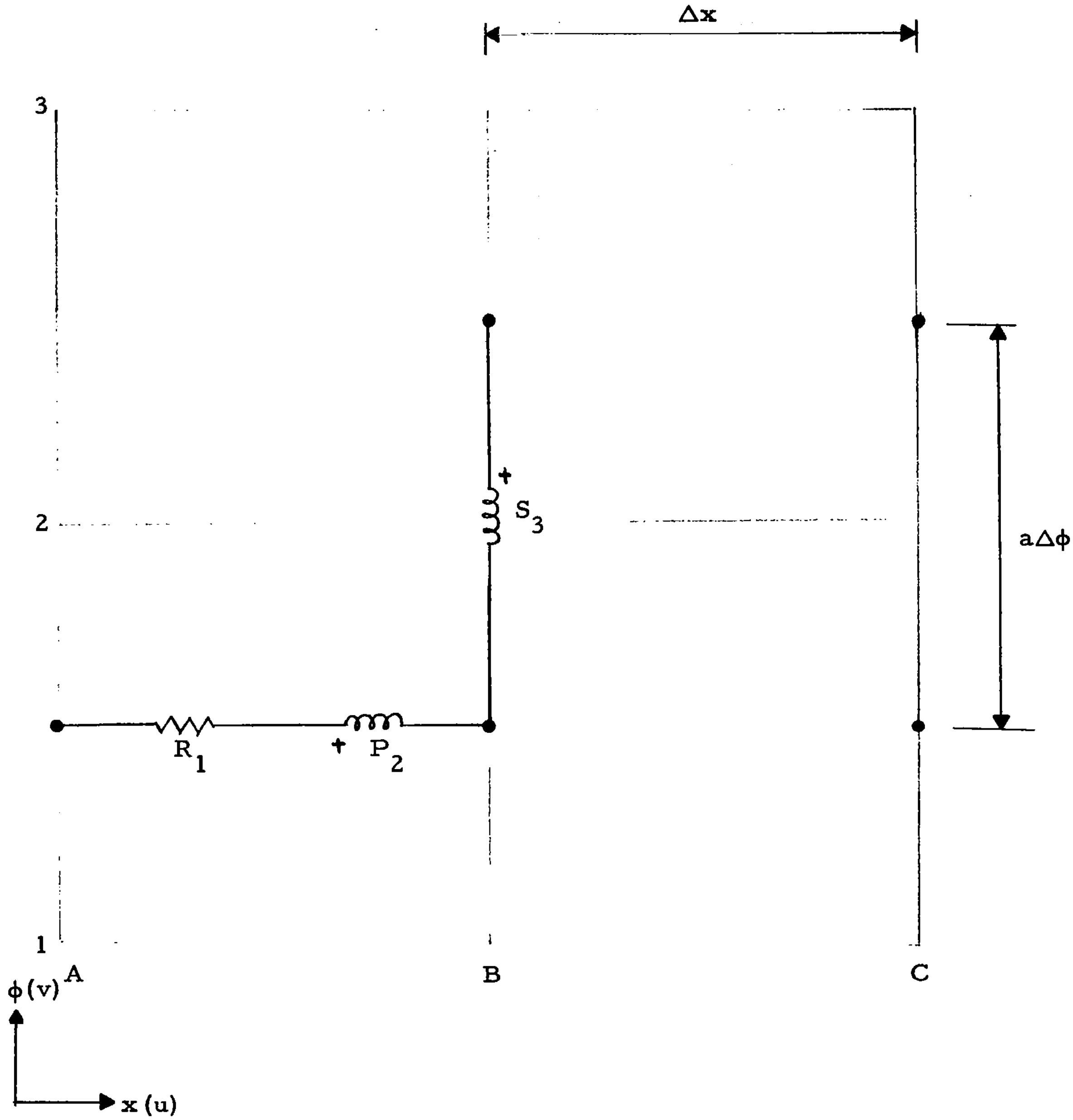


Figure 5-a. u Circuit for an Arbitrary Section of Cylindrical Shell

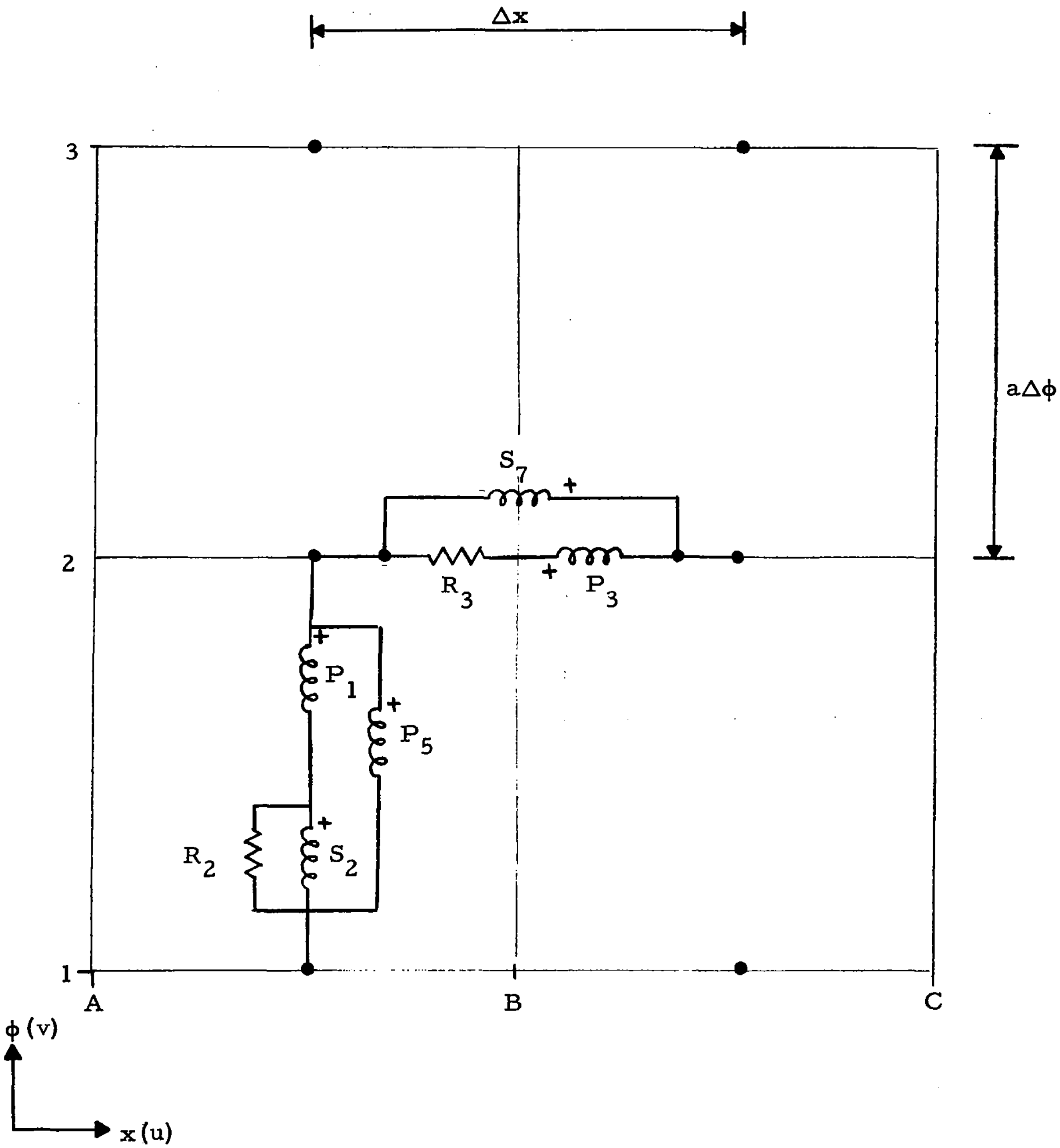


Figure 5-b. v Circuit for an Arbitrary Section of Cylindrical Shell

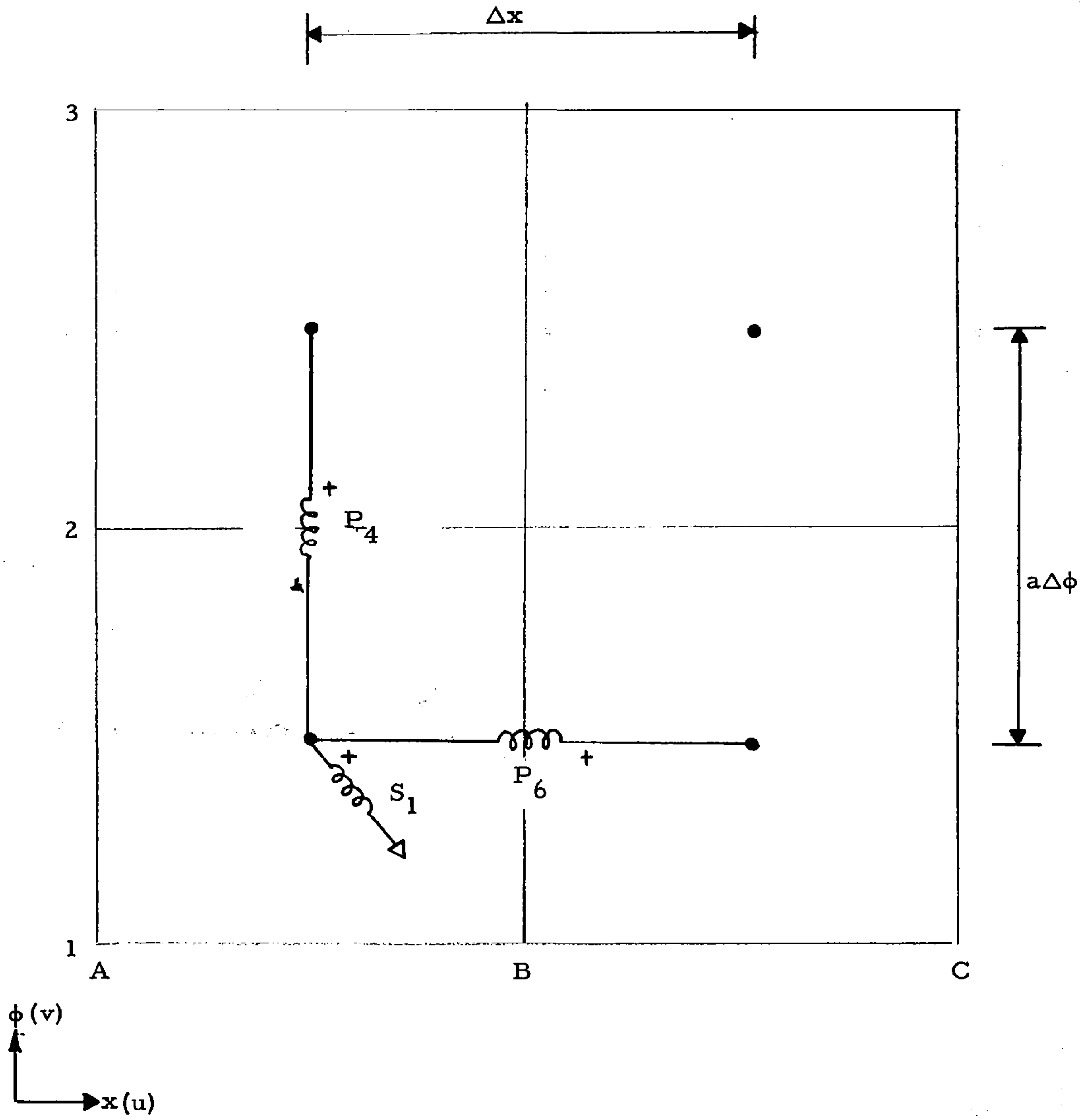


Figure 5-c. w Circuit for an Arbitrary Section of Cylindrical Shell

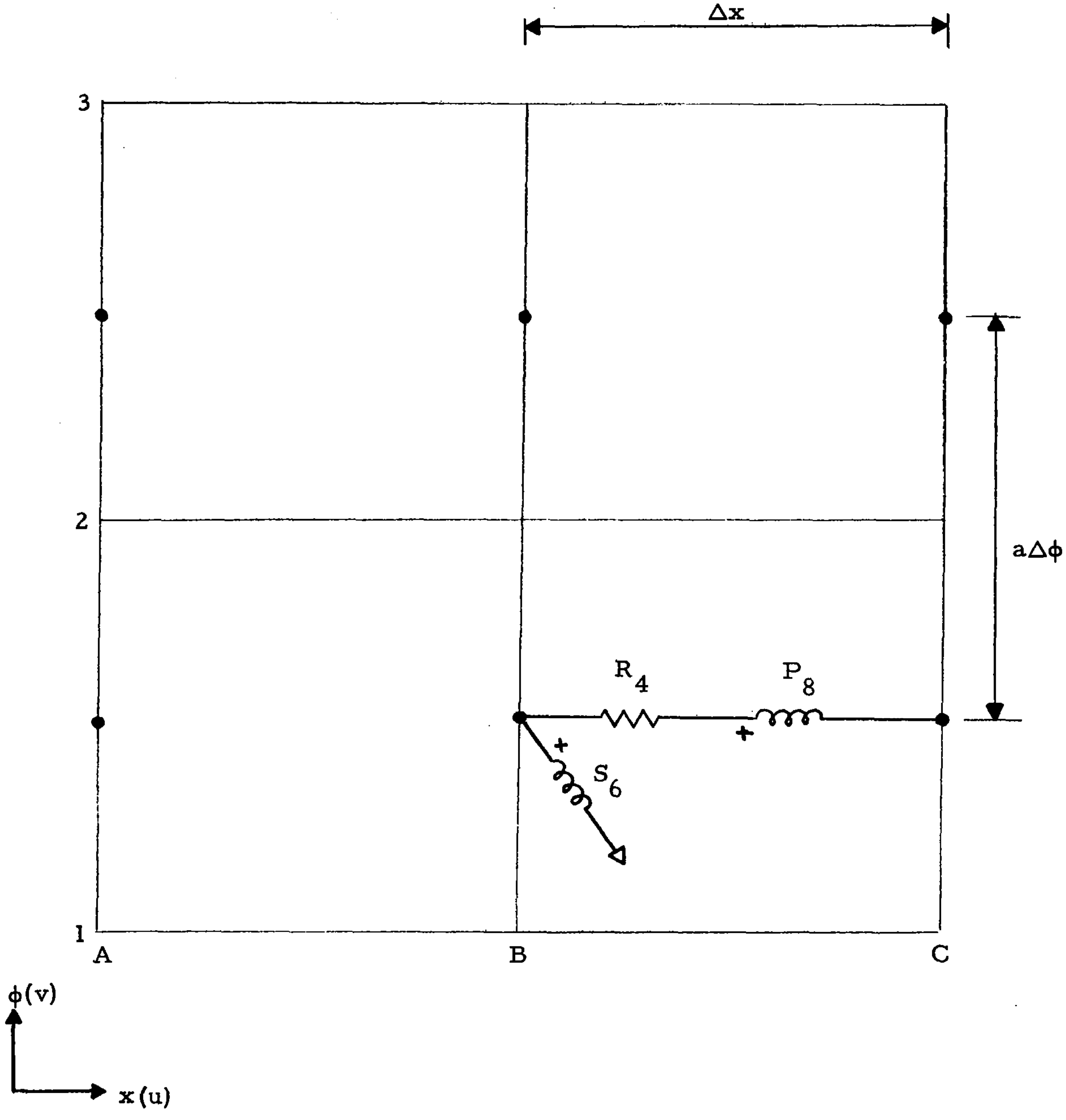


Figure 5-d. θ_x Circuit for an Arbitrary Section of Cylindrical Shell

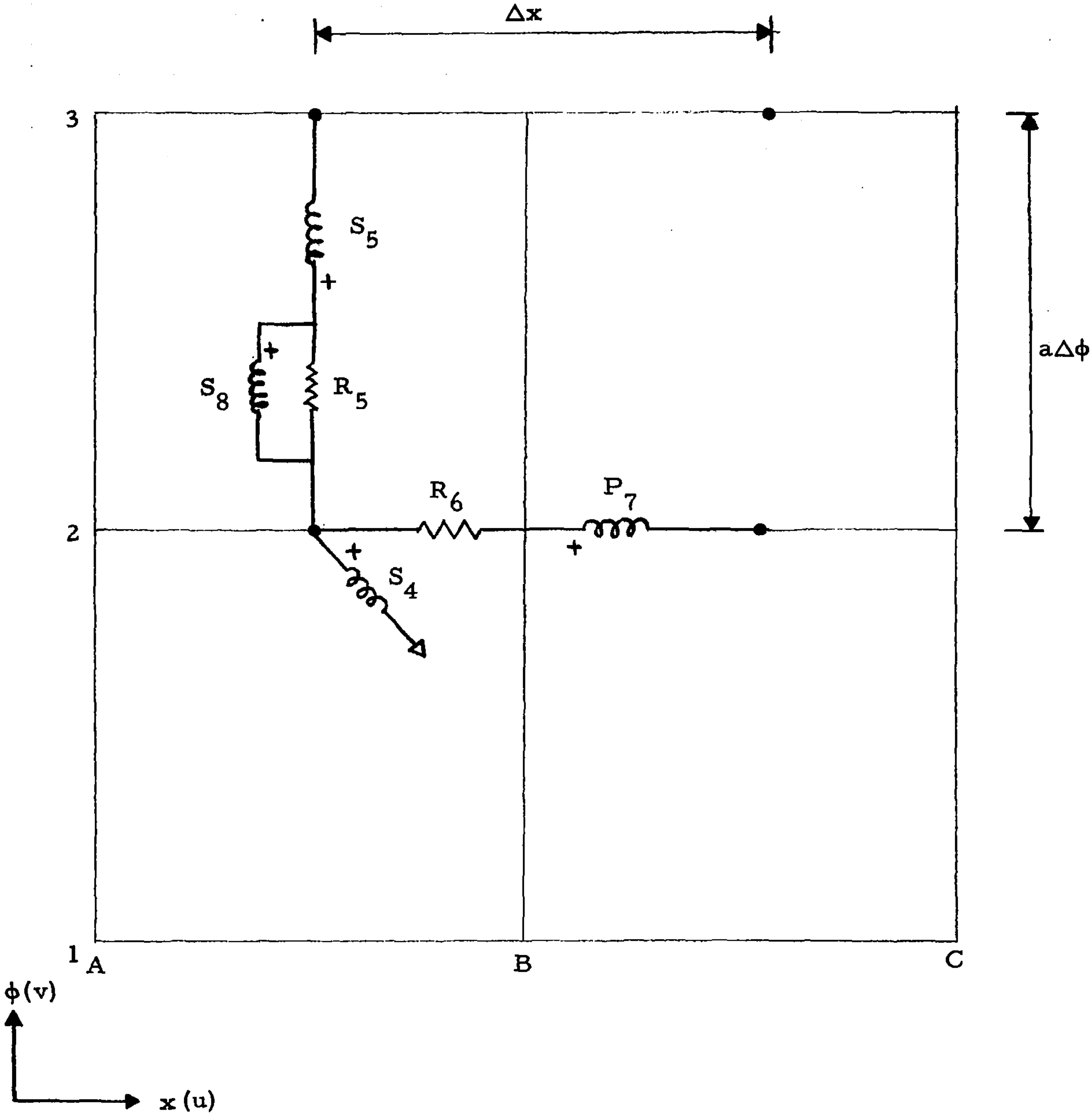


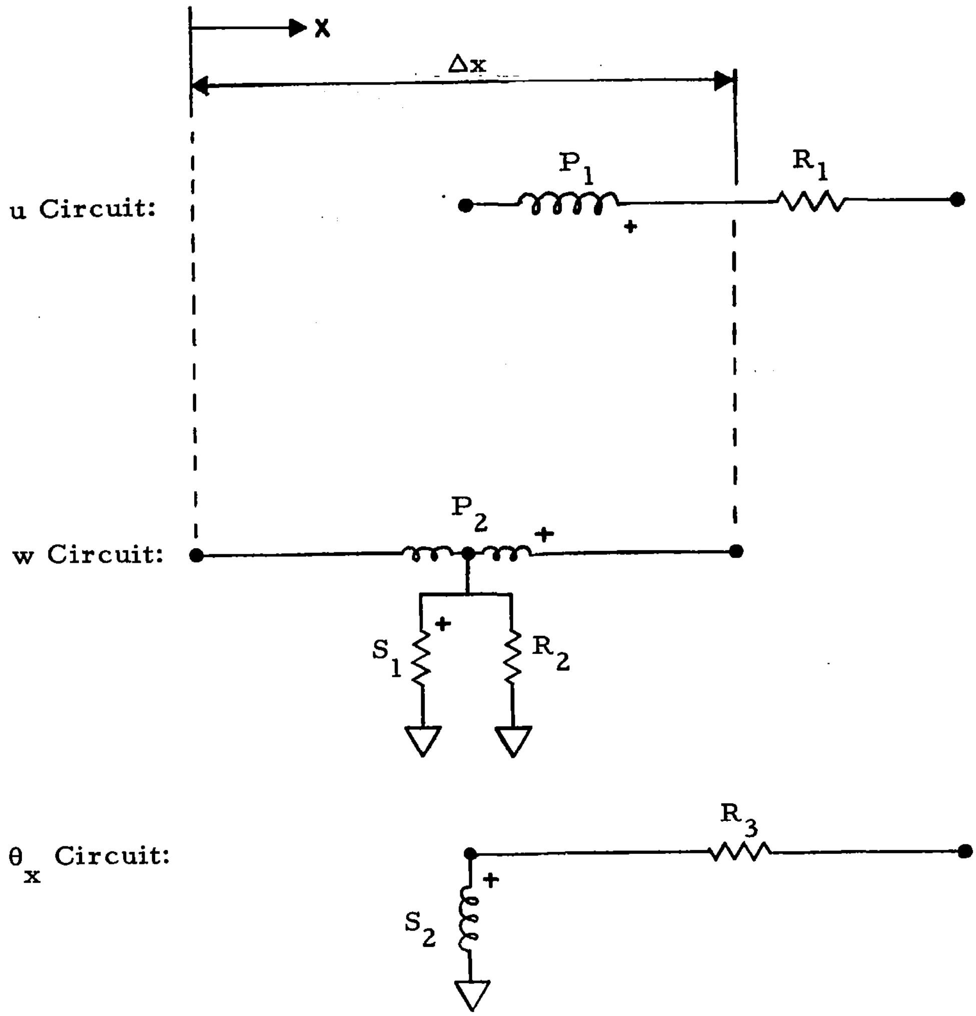
Figure 5-e. θ_ϕ Circuit for an Arbitrary Section of Cylindrical Shell

Circuit Elements	
Resistors	**Transformers**
$R_1 = \frac{(1 - \nu^2)\Delta x}{Eh\ a}$	$\frac{P_1}{S_1} = 1$; $\frac{P_2}{S_2} = \frac{\nu \Delta x}{a}$
$R_2 = \frac{a}{Eh\ \Delta x}$	$\frac{P_3}{S_3} = \frac{\Delta x}{a}$; $\frac{P_4}{S_4} = 1$
$R_3 = \frac{(1 + \nu)\ \Delta x}{Eh\ a}$	
$R_4 = \frac{\Delta x}{Da}$	$\frac{P_5}{S_5} = 1$; $\frac{P_6}{S_6} = \Delta x$
$R_5 = \frac{12}{E\ \Delta x}\left(\frac{a}{h}\right)^3$	$\frac{P_7}{S_7} = \frac{1}{2}$; $\frac{P_8}{S_8} = \frac{\nu\Delta x}{a^2}$
$R_6 = \frac{6\,(1 + \nu)\ a\ \Delta x}{Eh^3}$	

$$D = \frac{Eh^3}{12\,(1 - \nu^2)} = \text{flexural rigidity}$$

E = Young's modulus
a = radius of cylinder
h = thickness
ν = Poisson's ratio
Δx = axial distance
$\Delta\phi$ = angular section (assumed as one radian)

Figure 5-f. Circuit Element Values for an Arbitrary Section of Cylindrical Shell



Circuit Elements:

$$R_1 = \frac{1 - \nu^2}{E} \cdot \frac{\Delta x}{ha} \qquad \frac{P_1}{S_1} = \frac{\nu \Delta x}{a}$$

$$R_2 = \frac{1}{E} \cdot \frac{a}{h\Delta x}$$

$$R_3 = \frac{1}{D} \cdot \frac{\Delta x}{a} \qquad \frac{P_2}{S_2} = \Delta x$$

Figure 6. Circuit for the Elastic Behavior of a Cylindrical Shell with Axial Symmetry.

5. SUMMARY REMARKS

The derivations discussed herein are consistent with the basic theory presented in References 1 and 3. The analog circuits describe the static elastic behavior of difference segments of (1) a flat rectangular plate, (2) a flat circular plate and (3) a cylindrical shell. These circuits are mathematically equivalent to finite-difference models and physically equivalent to lumped parameter models. Such difference segments can be considered as elemental building blocks with which to synthesize a complete electrical model of a physical system.

As shown, the various analog circuits describe the static behavior of three specific elastic structures. These circuits can be routinely converted to describe the dynamic behavior of the three elastic structures as mentioned in the Introduction. Although developed in terms of uniform physical properties (that is, uniform mass and stiffness distributions), these analogs can directly accommodate nonuniform physical properties. The boundary conditions can be arbitrary and the applied external loading can be any arbitrary deterministic or random function of both space and time.

Of no less importance in this derivation is the procedure used to derive the analog models. The strain energy—electrical power equivalence used here is summarized, then applied to elasticity theory in Reference 2. Although considered in terms of structural applications, these same techniques can be applied to any physical system described as a function of space and time (i. e., a partial differential equation).

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