SCIENTIFIC REPORT Nr. 12

Research Grant Nr. NGR-52-046-001

Principal investigator and contractor: Prof. F. Cap

"THE SOLUTION OF A SYSTEM OF n-th-ORDER DIFFERENTIAL EQUATIONS USING LIE SERIES"

by

F. CAP and D. FLORIANI

A) Definition of Lie Series

B) Properties of Lie Series

C) Construction of the Solutions

Abstract:

In the present work, the solution of a system of n-th order ordinary differential equations which is solved for $y_Q^{(n)} = f_Q(x,y,y',\ldots,y^{(n-1)})$ is obtained by means of the Lie series as introduced by Groebner. For this purpose, the concept of a "Lie series" is defined initially and some important properties are quoted. In the third part, the system of equations is solved.

In the following, an n-th order system of differential equations is solved by means of Lie series. These series prove to be very appropriate to this purpose since they permit the solution to be written down immediately in an explicit manner. General statements can be made on the convergence of these series (B).

A) Definition of Lie Series:

We shall introduce a linear differential operator in the following way:

$$D: = \sum_{0}^{n} F_{\varrho}(z_0, z_1, \ldots, z_n) \cdot \frac{\partial}{\partial z_{\varrho}} \tag{1}$$

The F_{ϱ} are assumed to be holomorphic functions of the complex variables $z_0, \ldots, z_{\varrho}$. If this operator is applied to another holomorphic function $f(z_0, \ldots, z_n)$, we have

$$g(z_0, \ldots, z_n) = Df(z_0, \ldots, z_n)$$

which is again holomorphic. The same holds, of course, if D is applied n times (in the same domain of holomorphy).

With the help of this operator, we may formally set up an infinite series

$$\sum_{0}^{\infty} \frac{t^k}{k!} \cdot D^k f(z_0, z_1, \ldots, z_n) \tag{2}$$

which will be written symbolically as

$$e^{tD} f(z) = (\exp tD)\, f(z) \tag{2a}$$

in the following. The series defined in this way have some properties which enable them to become valuable tools in several fields of mathematics.

B) Properties of Lie Series:

1) Absolute Convergence:

It is shown in Ref. 1, p. 7, theorem 2, that, if G is a finite closed domain of the z space in which $f(z_o,\dots,z_n)$ and D are holomorphic, a number $T>0$ can be found such that the Lie series (2) converge absolutely and uniformly in the whole of G. The function

$$g(t;z_o,\dots,z_n):=e^{tD}f(z_o,\dots,z_n)$$

is, therefore, holomorphic in $t,z_o,\dots,z_n$.

2) Differentiation:

By virtue of this convergence property we have

$$\frac{\partial}{\partial t}\,g(t;z)=\frac{\partial}{\partial t}\sum_0^{\infty}\frac{t^k}{k!}\cdot D^kf(z)=\sum_0^{\infty}\frac{t^k}{k!}\cdot D^{k+1}f(z) \qquad (3)$$

since the series (2) may be differentiated term by term with respect to t.

Furthermore, we have

$$\frac{\partial}{\partial z_Q}\,g(t;z)=\frac{\partial}{\partial z_Q}\sum_0^{\infty}\frac{t^k}{k!}\cdot D^kf(z)=\sum_0^{\infty}\frac{t^k}{k!}\cdot\frac{\partial}{\partial z_Q}\,D^kf(z) \qquad (4)$$

since the series on the right-hand converges uniformly (Proof: Ref. 1, p. 7).

3) Commutation theorem:

The proof of this theorem is here briefly sketched, on account of its significance.

It is easily shown that

$$D\Big(\sum_0^{n}a_Q\cdot f_Q(z_o,\dots,z_n)\Big)=\sum_0^{n}a_Q\cdot Df_Q(z_o,\dots,z_n) \qquad (5)$$

(with a_ϱ being constants) and, generally,

$$D^k\left(\sum_0^n a_\varrho . f_\varrho(z_o,\ldots,z_n)\right) = \sum_0^n a_\varrho . D^k f_\varrho(z_o,\ldots,z_n) \quad (6)$$

for any natural number n.

Furthermore, the validity of

$$D^k(f_1(z).f_2(z)) = \sum_0^k \binom{k}{\varrho}.(D^\varrho f_1(z)).(D^{k-\varrho} f_2(z)) \quad (7)$$

follows from the usual rules of differentiation, where again z is written instead of z_o, z_1, ..., z_n.

With this we have (see theorem 5 in Ref. 1):

$$e^{tD}\left(\sum_0^n a_\varrho . f_\varrho(z)\right) = \sum_0^n a_\varrho . e^{tD} f_\varrho(z) \quad (8)$$

$$e^{tD}(f_1(z).f_2(z)) = (e^{tD} f_1(z)).(e^{tD} f_2(z)). \quad (9)$$

In particular, it follows from (8) and (9) for a polynomial:

$$e^{tD}\left(\sum a_\varrho . z_o^\alpha . z_1^\beta . \ldots . z_r^\varrho\right) = \sum a_\varrho . (e^{tD} z_o)^\alpha . (e^{tD} z_1)^\beta . \ldots . (e^{tD} z_r)^\varrho$$

or, briefly: (1o)

$$e^{tD} P(z_o, z_1, \ldots, z_r) = P(e^{tD} z_o, \ldots, e^{tD} z_r).$$

As is whown in the general commutation theorem for Lie series, this equation holds for any functional relationship.

The functions

$$Z_\varrho(t; z_o, z_1, \ldots, z_n) := e^{tD} z_\varrho \quad (11)$$

that are holomorphic in t and $z_o,\ldots,z_n$ are introduced. From it follows:

$$Z_\varrho(t=0; z_o,\ldots,z_n) = z_\varrho . \quad (12)$$

We have then (theorem 6 in Ref. 1):

If for a holomorphic function $F(z)$ the power series expansion valid at the point $z_o, z_1, \ldots, z_n$ converges also in $Z_o, Z_1, \ldots, Z_n$ (which will certainly be the case for sufficiently small $|Z_\varrho - z_\varrho|$, i.e., for sufficiently small t), we have:

$$e^{tD}F(z_o,\ldots,z_n) = F(e^{tD}z_o,\ldots,e^{tD}z_n) = F(Z_o,\ldots,Z_n). \qquad (13)$$

This follows for polynomials from (1o). Let $F_n(z)$ be the portion of the power series for $F(z)$ up to the degree n. We then have because of the presupposed holomorphy:

$$\lim F_n(z) = F(z) \qquad \lim F_n(Z) = F(Z)$$
$$\lim \frac{\partial}{\partial z_\varrho} F_n(z) = \frac{\partial F(z)}{\partial z_\varrho} \qquad (14)$$
$$\lim D^k F_n(z) = D^k F(z)$$

for $n \to \infty$. Because of (1o) we have:

$$F_n(Z) = \sum^{\infty} \frac{t^k}{k!} \cdot D^k F_n(z). \qquad (15)$$

Since a majorant exists for $F(z)$ the right-hand series converge uniformly with respect to n, i.e., we have:

$$\lim \sum_0^{\infty} \frac{t^k}{k!} \cdot D^k F_n(z) = \sum_0^{\infty} \frac{t^k}{k!} \cdot \lim D^k F_n(z) \qquad (16)$$

and with (14) to (16):

$$F(Z) = \lim F_n(Z) = \lim \sum_0^{\infty} \frac{t^k}{k!} \cdot D^k F_n(z) = \sum_0^{\infty} \frac{t^k}{k!} \cdot \lim D^k F_n(z) =$$

$$= \sum_0^{\infty} \frac{t^k}{k!} \cdot D^k F(z) = e^{tD} F(z)$$

for $n \to \infty$.

C) Construction of the Solutions:

Let be given a system of differential equations:

$$y_\varrho^{(n)} = f_\varrho(x,y,y',\dots,y^{(n-1)}) \qquad (\varrho=1,\dots,r) \qquad (17)$$

with holomorphic functions f_ϱ. $y,y',\dots$ is here symbolic for y_σ, y'_σ with $\sigma = 1,\dots,r$. (17) can be represented by an equivalent system of first-order differential equations:

$$z_o := x \qquad z_{\varrho,\sigma} := y_\varrho^{(\sigma-1)} \qquad (\sigma=1,\dots,n) \qquad (18)$$

$$\begin{aligned} z'_{\varrho,\sigma} &= z_{\varrho,\sigma+1} \qquad \text{for } \sigma=1,\dots,n-1 \\ z'_{\varrho,n} &= f_\varrho(x,z_{\varrho\sigma}) \end{aligned} \qquad (19)$$

This system (19) is now solved by the Lie series which is formed by the operator

$$D := \frac{\partial}{\partial z_o} + \sum_\varrho \left[\sum_\sigma z_{\varrho,\sigma+1} \cdot \frac{\partial}{\partial z_{\varrho,\sigma}} + f_\varrho \cdot \frac{\partial}{\partial z_{\varrho,n}} \right] \qquad (20)$$

with $\sigma = 1,\dots,n-1$ and $\varrho = 1,\dots,r$:

$$\frac{\partial}{\partial t}(e^{tD}z_o) = \sum_o^\infty \frac{t^k}{k!} \cdot D^k(Dz_o) = 1$$

$$\frac{\partial}{\partial t}(e^{tD}z_{\varrho,\sigma}) = \sum_o^\infty \frac{t^k}{k!} \cdot D^k(Dz_{\varrho,\sigma}) = e^{tD}z_{\varrho,\sigma+1} \qquad (\sigma=1,\dots,n-1) \quad (21)$$

$$\frac{\partial}{\partial t}(e^{tD}z_{\varrho,n}) = \sum_o^\infty \frac{t^k}{k!} \cdot D^k(Dz_{\varrho,n}) = e^{tD}f_\varrho(z)$$

(follows from (3)). With

$$Z_o := e^{tD}z_o = z_o + t \qquad Z_{\varrho,\sigma} := e^{tD}z_{\varrho,\sigma} \qquad (22)$$

(21) can be rewritten in the form (because of (13)):

$$\frac{\partial}{\partial t} Z_o(t;z) = 1$$

$$\frac{\partial}{\partial t} Z_{\varrho,\sigma}(t;z) = Z_{\varrho,\sigma+1}(t;z)$$

$$\frac{\partial}{\partial t} Z_{\varrho,n}(t;z) = f_\varrho(Z).$$

or:

$$\frac{\partial}{\partial t} Z_o(t;z) = 1$$

$$\frac{\partial^n}{\partial t^n} Z_{\varrho,1}(t;z) = f_\varrho(Z_o, Z_{\varrho,\sigma})$$

or, in terms of the original variables:

$$Z_o = x = z_o + t, \qquad Z_{\varrho,\sigma} = y_\varrho^{(\sigma-1)}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \qquad \frac{\partial^n}{\partial x^n} y_\varrho(x;z) = f_\varrho(x,y,y',\ldots,y^{(n-1)}).$$

(21) is therefore identical with the original system (17), since $\frac{\partial^n}{\partial x^n}$ may also be written as $\frac{d^n}{dx^n}$, if the are understood to be parameters.

Consequently, the solution of (17) reads:

$$x(t) = z_o + t, \quad y_\varrho(t) = e^{tD} z_{\varrho,1} \quad \text{or} \quad y_\varrho(x) = e^{(x-z_o)\cdot D} z_{\varrho,1} \tag{23}$$

where (see (12)) the $z_{\varrho,\sigma}$ are the initial values of $y_\varrho^{(\sigma-1)}$ for $t = 0$:

$$y_\varrho^{(\sigma-1)}(t=0) = y_\varrho^{(\sigma-1)}(x=z_o) = z_{\varrho,\sigma} \tag{24}$$

Using (23) and (24) the problem of solving (17) with the initial conditions $y_\varrho^{(\sigma-1)}(0) = z_{\varrho,\sigma}$ has been accomplished.

References: (1) W. Groebner, "Die Lie-Reihen und ihre Anwendungen", VEB Deutscher Verlag der Wissenschaften, Berlin 1960